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On chordal and bilateral SLE in multiply connected domains

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Abstract We discuss the possible candidates for conformally invariant random non-self-crossing curves which begin and end on the boundary of a multiply connected planar domain, and which satisfy a Markovian-type property. We consider both, the case when the curve connects a boundary component to itself (chordal), and the case when the curve connects two different boundary components (bilateral). We establish appropriate extensions of Loewner's equation to multiply connected domains for the two cases. We show that a curve in the domain induces a motion on the boundary and that this motion is enough to first recover the motion of the moduli of the domain and then, second, the curve in the interior. For random curves in the interior we show that the induced random motion on the boundary is not Markov if the domain is multiply connected, but that the random motion on the boundary together with the random motion of the moduli forms a Markov process. In the chordal case, we show that this Markov process satisfies Brownian scaling and discuss how this limits the possible conformally invariant random non-self-crossing curves. We show that the possible candidates are labeled by two functions, one homogeneous of degree zero, the other homogeneous of degree minus one, which describes the interaction of the random curve with the boundary. We show that the random curve has the locality property for appropriate choices of the interaction term.

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1 Introduction

In this paper we discuss the possible candidates for a mathematically rigorous notion of conformally invariant random non-self-crossing curves which begin and end on the boundary of a multiply connected planar domain, and which satisfy a Markovian-type property. The Markovian-type property means that the random curves can be developed dynamically as a (locally) growing family of random compacts. We aim to proceed in the spirit of Schramm, who deduced that, under an additional reflection symmetry, there is only a one parameter family of such random curves in simply connected domains, which he termed Stochastic Loewner Evolutions (SLE), see [27]. As such conformally invariant random growing compacts are conjectured to arise as scaling limits of interfaces of 2-dimensional statistical mechanical systems at criticality, Schramm had with one stroke identified what those limits can be. This has many consequences and applications, see [17–19,28], and references therein.

Statistical mechanical systems have been studied in discrete approximations of multiply connected domains and Riemann surfaces, see [2,15], and the connections with conformal field theory (CFT) indicate that the SLE should also extend to multiply connected domains and Riemann surfaces.

For multiply connected domains the situation is already more subtle when compared to the simply connected case, because moduli spaces enter the picture and, as we will show, one has to consider interactions with these moduli.

Families of random compacts from the boundary to the boundary now come in two flavors, as the random compact may grow to either connect a boundary component to itself (the chordal case) or it may grow to connect two different boundary components. We call the latter the *bilateral* case.

The radial case, treated in [5], where the random compact grows from the boundary to an interior point, can be considered as a limit of the bilateral case, when the boundary component the random compact grows towards shrinks to a point. This can be made precise, see [12].

Our procedure rests on an appropriate extension of Loewner's equation to the multiply connected case. In the simply connected case, Loewner's equation allows to encode a simple curve in a domain D which has one endpoint on the boundary ∂D by a continuous motion on the boundary, see [20]. In the multiply connected case, we show in Theorem 3.1 and Theorem 3.2 that a simple curve induces a motion on the boundary of the domain. To recover the curve inside the domain requires also the knowledge of the moduli $\mathbf{M}$ (which describe the conformal equivalence class), as the curve grows. We show in Theorem 4.1 and Theorem 4.2 that these moduli can be recovered from the boundary motion and thus, once these moduli have been obtained, the curve in the interior itself.

A growing random non-self-crossing curve in a multiply connected domain can then also be encoded into a random motion $\xi(t)$ on the boundary. However, if the connectivity is greater than one, then ξ cannot be Markov. We show in Sect. 5 that in the chordal case the boundary motion ξ together with the motion of moduli $\mathbf{M}$ is a Markov process, and that it satisfies Brownian scaling.

These facts dramatically reduce the number of possible diffusions. Indeed, one is only free to choose two function A and B in the variables ξ and $\mathbf{M}$, which are homogeneous of degree minus one and zero, respectively. These terms measure the interaction of the random growing compact with the boundary (for example if it is desired that the random set avoids the interior boundary components). In the simply connected case the variable $\mathbf{M}$ disappears. Since the only homogeneous function of degree zero in one variable is a constant, it follows $B = \sqrt{\kappa}$.

SLE(κ, ρ), see [10,18], also fits naturally into this framework. There, the random compact grows into the upper half-plane, the boundary is the real axis, and the interaction is with a finite number of points on the real axis and given in terms of the simplest homogeneous function of degree minus one, $1/x$. Even though the upper half-plane is simply connected, the marked points on the boundary can serve as moduli and then SLE(κ, ρ) is given by a particular moduli diffusion.

For multiply connected domains it is natural to look for an interaction A which is expressed in terms of domain functionals such as the Green function. Appropriate combinations of derivatives of the Green function are homogeneous of degree minus one in ξ and the moduli. The 'harmonic random Loewner chains' studied in [29] are a particular example of this. A version of Loewner's equation and equations for the moduli motion equivalent to ours have also been established in [29] by different techniques. However, there the solvability of the moduli equation was not established, as pointed out by the author. Establishing the Lipschitz property of the vector field in the moduli equation is the key step in our proof of Theorem 4.1 and Theorem 4.2. Based on recent work of Lawler, the harmonic random Loewner chains of Zhan appear to be the scaling limit of Laplacian-b random walk in multiply connected domains, see [16].

In our opinion the only further reduction in possible diffusions $(\xi, \mathbf{M})$ are regularity requirements on the homogenous functions A and B. The main challenge is to identify which physical model corresponds to which choice of functions. In this paper we do this for percolation ($A = 0, B = \sqrt{6}$). In [30], Zhan shows that "annulus SLE$_6$" satisfies locality. This is the same process we define in Sect. 5 in the special case of connectivity two. Our results show that it is in fact the only process which is conformally invariant, satisfies the Markovian type property, and locality, see Remark 5.1.

For random curves connecting a boundary point to an interior point, the *radial* case, we obtained results similar to many obtained in this paper, see [5]. A main difference is that in the radial case the scaling property is not present and thus one cannot conclude that the interaction functions A and B are homogeneous. The results of that paper were announced in [4].

Several physical aspects related to this paper, such as connections to CFT vertex operators and the Coulomb gas formalism, are discussed in [8,11,14] .

2 Bilateral and Chordal standard domains

2.1 Harmonic measures

Denote D a region of connectivity $n > 1$ in the complex plane. The components of the complement in the extended complex plane are denoted by $E_1, E_2, \dots, E_n$. We assume that no E_k reduces to a point and that there is a unique unbounded component E_n. By applying preliminary conformal maps, we may assume that D is bounded by an outer contour C_n and $n-1$ inner contours $C_1, \dots, C_{n-1}$, where the contours are oriented such that D lies to the left in the direction of the contour. Denote $\omega_k(z)$ the solution to the Dirichlet problem in D with the boundary values 1 on C_k and 0 on the other contours. We have $0 < \omega_k(z) < 1$ in D and

$$\omega_1(z) + \omega_2(z) + \cdots + \omega_n(z) = 1. \tag{1}$$

$\omega_k(z)$ is called the harmonic measure of C_k in z. The conjugate harmonic differential of ω_k has periods

$$\alpha_{kj} = \int_{C_j} *d\omega_k = \int_{C_j} \frac{\partial \omega_k}{\partial n}\, \mathrm{d}s \tag{2}$$

along C_j. Here, $\partial/\partial n$ denotes the normal derivative to the right of the direction of the contour, and ds stands for arc-length measure. It is well known, [21], that the $(n-1) \times (n-1)$ matrix $\boldsymbol{\alpha}$ with entries α_{kj}, $1 \le k, j \le n-1$, is positive definite and symmetric. In particular, the linear system

$$\begin{aligned} \lambda_1\alpha_{11} + \lambda_2\alpha_{21} + \cdots + \lambda_{n-1}\alpha_{n-1,1} &= 2\pi \\ \lambda_1\alpha_{12} + \lambda_2\alpha_{22} + \cdots + \lambda_{n-1}\alpha_{n-1,2} &= 0 \\ &\vdots \\ \lambda_1\alpha_{1,n-1} + \lambda_2\alpha_{2,n-1} + \cdots + \lambda_{n-1}\alpha_{n-1,n-1} &= 0 \end{aligned} \tag{3}$$

has a unique solution. It follows from (1) that any solution of (3) also solves

$$\lambda_1\alpha_{1n} + \lambda_2\alpha_{2n} + \cdots + \lambda_{n-1}\alpha_{n-1,n} = -2\pi.$$

Thus there is a multiple-valued integral $F(z)$ with periods $\pm 2\pi i$ along C_1 and C_n and all other periods equal to zero, the real part being constant equal to λ_k on C_k (we set $\lambda_n = 0$). The function $f(z) = \mathrm{e}^{-F(z)}$ is then single-valued and one can show, [1], that f maps D conformally onto the annulus $\mathrm{e}^{-\lambda_1} < |w| < 1$ minus $n-2$ concentric arcs situated on the circles $|w| = \mathrm{e}^{-\lambda_k}$, $k = 2, \ldots, n-1$. We call such a circularly slit annulus a *bilateral standard domain*. By adding an imaginary constant to $F(z)$ we obtain another map onto a bilateral standard domain and we may normalize the map f by requiring $f(z_0) = \mathrm{e}^{-\lambda_1}$ for some $z_0 \in C_1$. With this normalization we call f the *canonical map* for (D, z_0, C_n).

2.2 Green function

Denote D again a region of finite connectivity which is bounded by contours $C_1, \ldots, C_n$; this time the case $n = 1$ is included.

We consider a point $z_0 \in D$ and solve the Dirichlet problem in D with the boundary values $\ln|\zeta - z_0|$. The solution is denoted by $h(z)$. The function

$$G(z) = G_D(z, z_0) = h(z) - \ln|z - z_0|$$

is the Green function in D with pole at z_0. It is the unique function which is harmonic in D except at z_0, where it differs from $\ln|z - z_0|$ by a harmonic function, and which vanishes on the boundary of D. The Green function is conformally invariant in the sense that if $f : D \to D'$ is conformal, then

$$G_D(z, z_0) = G_{D'}(f(z), f(z_0)). \tag{4}$$

The conjugate harmonic function of $G(z, z_0)$ is multiple-valued. It has the period 2π along a small circle about z_0, and the periods

$$p_k(z_0) = \int_{C_k} *dG(z, z_0), \quad k = 1, \ldots, n.$$

It can be shown that $p_k(z_0) = -2\pi\omega_k(z_0)$, [1]. Let now $z_0 \in C_n$. By linearity, $u(z) = \partial G(z, z_0)/\partial n_{z_0}$ is a harmonic function in z. Its conjugate differential has periods

$$A_k(z_0) = \int\limits_{C_k} *du = \frac{\partial}{\partial n_{z_0}} \int\limits_{C_k} *d_z G(z, z_0)$$
$$= -2\pi \frac{\partial}{\partial n_{z_0}} \omega_k(z_0). \quad (5)$$

Thus the linear combination $u + \lambda_1\omega_1 + \cdots + \lambda_{n-1}\omega_{n-1}$ is free from periods provided that

$$\lambda_1\alpha_{1k} + \lambda_2\alpha_{2k} + \cdots + \lambda_{n-1}\alpha_{n-1,k} = -A_k, \quad k = 1, \ldots, n-1. \quad (6)$$

If we write $\mathbf{P}$ for the matrix $\boldsymbol{\alpha}/2\pi$, $\boldsymbol{\lambda}^{\mathrm{T}} = (\lambda_1, \ldots, \lambda_{n-1})$, and

$$\partial\boldsymbol{\omega}(z_0)^{\mathrm{T}}/\partial n = (\partial\omega_1(z_0)/\partial n, \ldots, \partial\omega_{n-1}(z_0)/\partial n),$$

then the solution to (6) is given by

$$\boldsymbol{\lambda} = \mathbf{P}^{-1} \frac{\partial\boldsymbol{\omega}(z_0)}{\partial n}.$$

Hence

$$-\mathrm{i}\left(\frac{\partial G(z, z_0)}{\partial n_{z_0}} + \boldsymbol{\omega}(z)^{\mathrm{T}}\mathbf{P}^{-1}\frac{\partial\boldsymbol{\omega}(z_0)}{\partial n}\right)$$

is the imaginary part of a single-valued analytic function $\Psi(z)$. It can be shown that Ψ maps D conformally onto the upper half-plane $\Im(w) > 0$ minus $n-1$ horizontal slits with imaginary parts

$$\Im(w) = -[\mathbf{P}^{-1}\partial\boldsymbol{\omega}(z_0)/\partial n]_j, \quad j = 1, \ldots, n-1.$$

Under this map, $\Psi(C_n) = \mathbb{R}$, and $\Psi(z_0) = \infty$. We call the upper half-plane minus a finite number of horizontal slits a *chordal standard domain*. If D is contained in the upper half-plane and for some $x > 0$ we have $(\mathbb{R}\backslash[-x, x]) \subset C_n$ and $\zeta = \infty$, then, by adding an appropriate real constant, we may assume that $g \equiv \Psi/2$ satisfies the hydrodynamic normalization at infinity,

$$\lim_{z\to\infty} (g(z) - z) = 0. \quad (7)$$

With this normalization, we call g the canonical mapping for D.

3 Evolution of slit mappings

The evolution of slit mappings in multiply connected domains was first studied by Komatu in [12] for the doubly connected case, and in [13] for general finite connectivity. Komatu treated this case by considering circular slit annuli. We obtain analogs of his main result in Theorems 3.1 and 3.2. Since we consider a different family of standard domains in the chordal case from the one Komatu considered, and, in the bilateral case, consider a normalization different from the one employed in [13], we give, for the sake of completeness and the convenience of the reader, proofs of these results. A further reason for including the proofs is that the original reference [13] is not widely available.

However, Komatu [13] does not study the question of how to recover a slit in the interior from a motion on the boundary, which is our ultimate goal. This requires knowledge of the motion of the moduli, and in particular the Lipschitz property of the vector field driving that motion. We will find the vector field and establish the Lipschitz property in Sect. 4. This difficulty does not arise in the simply connected case. There, due to the absence of moduli, all one needs is Loewner's equation.

3.1 Chordal Loewner equation

Consider a chordal standard domain D. Let $\gamma : [0, t_\gamma] \to \overline{D}$ be a Jordan arc such that $\gamma(0) \in \mathbb{R}$, and $\gamma(0, t_\gamma] \subset D$. Let g_t be the canonical mapping from $D\backslash\gamma[0,t]$ with the normalization (7), and denote D_t the chordal standard domain $g_t(D\backslash\gamma[0,t])$. It is well known, see [21], that g_t solves the extremal problem

$$a_1 = \max$$

among all univalent functions on $D\backslash\gamma[0,t]$ with expansion

$$z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \quad a_k \in \mathbb{R},$$

near infinity. In particular, if $g_t(z) = z + a_t/z + o(1/|z|)$, then $a_{t^*} \leq a_t$ whenever $0 < t^* < t < t_\gamma$. In fact, a simple argument shows that

$$a_{t^*} < a_t \quad \text{if } t^* < t. \tag{8}$$

Thus we may assume that $a_t = 2t$. We wish to find a differential equation for the family $\{g_t : t \in [0, t_\gamma]\}$.

Denote $C_j(t), j = 1, \ldots, n$, the boundary components of D_t. We always have $C_n(t) = \mathbb{R}$. For $j = 1, \ldots, n-1$, let $y_j(t)$ be the imaginary part of (points on) the slit $C_j(t)$. Denote $\xi(t)$ the starting point on $\mathbb{R}$ of the Jordan arc $g_t(\gamma[t, t_\gamma])$ in D_t, i.e. $g_t(\gamma_t)$. For $0 < t^* < t < t_\gamma$, set

$$g_{t,t^*} = g_{t^*} \circ g_t^{-1}.$$

Then g_{t,t^*} is a conformal map from D_t onto $D_{t^*}\backslash g_{t^*}(\gamma[t^*, t])$. The point $\xi(t^*) = g_{t^*}(\gamma_{t^*})$ corresponds to two prime ends in $D_{t^*}\backslash g_{t^*}(\gamma[t^*, t])$. Denote $\beta_0(t, t^*)$ and $\beta_1(t, t^*)$, with $\beta_0(t, t^*) < \beta_1(t, t^*)$, the pre-images of these prime ends under g_{t,t^*}, i.e.

$$g_{t,t^*}(\beta_0(t, t^*)) = g_{t,t^*}(\beta_1(t, t^*)) = g_{t^*}(\gamma_{t^*}).$$

Then, if $x \in \mathbb{R}\backslash[\beta_0(t, t^*), \beta_1(t, t^*)]$,

$$g_{t,t^*}(x) \in \mathbb{R}.$$

Consider the analytic function

$$z \mapsto g_{t,t^*}(z) - z,$$

which satisfies

$$g_{t,t^*}(z) - z = \frac{2(t^* - t)}{z} + o(1/|z|), \tag{9}$$

and note that $z \mapsto \Im(g_{t,t^*}(z) - z)$ is harmonic and constant on each boundary component. By Poisson's formula

$$\Im(g_{t,t^*}(z) - z) = -\frac{1}{2\pi} \int_{\partial D_t} \Im(g_{t,t^*}(\zeta) - \zeta) \frac{\partial G_t(\zeta, z)}{\partial n_1} \, \mathrm{d}s, \tag{10}$$

where $G_t(\zeta, z)$ is the Green function for D_t with pole at z. Note that there is no problem with integrability in (10) because

$$\Im(g_{t,t^*}(\zeta) - \zeta) = \frac{-y}{x^2 + y^2} + O(1/|\zeta|^2), \quad \zeta = x + \mathrm{i}y,$$

and

$$\sup\{\Im(\zeta) : \zeta \in \partial D_t\} < \infty. \tag{11}$$

Since $\Im(g_{t,t^*}(z) - z)$ has a single-valued harmonic conjugate, it is orthogonal to the real part of any Abelian differential of the first kind, see [4], and we have

$$\begin{aligned} &\Im(g_{t,t^*}(z) - z) \\ &= -\frac{1}{2\pi} \int_{\partial D_t} \Im(g_{t,t^*}(\zeta) - \zeta) \left(\frac{\partial G_t(\zeta, z)}{\partial n_1} + \boldsymbol{\omega}_t(z)^{\mathrm{T}} \boldsymbol{P}_t^{-1} \frac{\partial \boldsymbol{\omega}_t(\zeta)}{\partial n} \right) \mathrm{d}s. \end{aligned} \tag{12}$$

It follows from Sect. 2 that

$$\int_{C_k(t)} \left(\frac{\partial G_t(\zeta, z)}{\partial n_1} + \boldsymbol{\omega}_t(z)^{\mathrm{T}} \boldsymbol{P}_t^{-1} \frac{\partial \boldsymbol{\omega}_t(\zeta)}{\partial n} \right) \mathrm{d}s = 0, \quad k = 1, \ldots, n-1,$$

and also that

$$z \mapsto -\mathrm{i} \left(\frac{\partial G_t(\zeta, z)}{\partial n_1} + \boldsymbol{\omega}_t(z)^{\mathrm{T}} \boldsymbol{P}_t^{-1} \frac{\partial \boldsymbol{\omega}_t(\zeta)}{\partial n} \right)$$

is the imaginary part of a single-valued analytic function $\Psi_t(z) = \Psi_t(z, \zeta)$. Thus, since $\Im(g_{t,t^*}(\zeta) - \zeta)$ is constant on each $C_k(t)$, $k = 1, \ldots, n-1$, and identically zero on $\mathbb{R} \setminus [\beta_0(t, t^*), \beta_1(t, t^*)]$,

$$g_{t,t^*}(z) - z = \frac{1}{2\pi} \int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \Im(g_{t,t^*}(\zeta) - \zeta) \Psi_t(z, \zeta) \, \mathrm{d}\zeta + \mathrm{i}c, \tag{13}$$

where c is a real constant. Note that if $z \mapsto \tilde{\Psi}_t(z, \zeta)$ is another analytic function with the same imaginary part as Ψ_t, then

$$\Psi_t(z, \zeta) - \tilde{\Psi}_t(z, \zeta) = a(\zeta),$$

where a is real and depends only on ζ. We fix a normalization by requiring that

$$\lim_{z \to \infty} \Psi_t(z, \zeta) = 0. \tag{14}$$

If we let $z \to \infty$, then $g_{t,t^*}(z) - z \to 0$. By bounded convergence, the integral in (13) converges to zero as well and it follows that $c = 0$. Next,

$$2(t^* - t) = \lim_{z \to \infty} z(g_{t,t^*}(z) - z) = f(0),$$

where

$$w \mapsto f(w) \equiv \frac{1}{w}\left[g_{t,t^*}(1/w) - (1/w)\right]$$

is regular near zero. By the Schwarz reflection principle g_{t,t^*} extends to the entire complex plane minus the slits $C_1(t), \ldots, C_{n-1}(t)$, their conjugates, and the real interval $[\beta_0(t,t^*), \beta_1(t,t^*)]$. Denote C the collection of these $2n-1$ finite slits. Then f also extends to a corresponding domain with boundary $\tilde{C}$. From Cauchy's integral formula we have

$$\begin{aligned} 2(t^* - t) &= \frac{1}{2\pi i}\int_{\tilde{C}} \frac{f(\zeta)}{\zeta}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\tilde{C}} \frac{g_{t,t^*}(1/\zeta) - 1/\zeta}{\zeta^2}\,\mathrm{d}\zeta \\ &= -\frac{1}{2\pi i}\int_{C} (g_{t,t^*}(\eta) - \eta)\,\mathrm{d}\eta = -\frac{1}{2\pi}\int_{C} \Im(g_{t,t^*}(\eta) - \eta)\,\mathrm{d}\eta, \end{aligned} \tag{15}$$

where the final equality uses the fact that $d\eta$ is real for horizontal slits. The slits $C_1(t), \ldots, C_{n-1}(t)$ and their conjugates do not contribute to the last integral since $\Im(g_{t,t^*}(\eta) - \eta)$ takes the same value on both "sides" of a given slit. For the slit $[\beta_0(t,t^*), \beta_1(t,t^*)]$, $\Im(g_{t,t^*}(\eta) - \eta)$ takes opposite values on the upper and lower "side" of the slit and, since the direction of integration is reversed, we finally get

$$t^* - t = -\frac{1}{2\pi}\int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \Im(g_{t,t^*}(\eta))\,\mathrm{d}\eta. \tag{16}$$

Setting $z = g_t(w)$ in (13) we have

$$g_{t^*}(w) - g_t(w) = \frac{1}{2\pi}\int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \Im(g_{t,t^*}(\eta))\Psi_t(z,\eta)\mathrm{d}\eta.$$

We are now ready to let $t^* \nearrow t$ in (17). Note first that, for $\eta \in [\beta_0(t,t^*), \beta_1(t,t^*)]$, $\eta \mapsto \Im(g_{t,t^*}(\eta))$ is continuous and non-negative and that also $\eta \mapsto A(\eta) := \Psi_t(z,\eta)$ is continuous. Thus it follows from the mean-value theorem of integration and (16) that

$$\begin{aligned} &\frac{1}{2\pi}\int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \Im(g_{t,t^*}(\eta))A(\eta)\,\mathrm{d}\eta \\ &\quad = \frac{\Re(A(\eta')) + \mathrm{i}\Im(A(\eta''))}{2\pi}\int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \Im(g_{t,t^*}(\eta)\,\mathrm{d}\eta \\ &\quad = -\left[\Re(A(\eta')) + \mathrm{i}\Im(A(\eta''))\right](t^* - t), \end{aligned} \tag{17}$$

for some $\eta', \eta'' \in [\beta_0(t,t^*), \beta_1(t,t^*)]$. Hence

$$\lim_{t^* \nearrow t} \frac{g_{t^*}(w) - g_t(w)}{t^* - t} = -\Psi_t(z, \xi(t)).$$

By the same argument we may let $t \searrow t^*$. On the right-hand side above we then only need to change t to t^* and introduce an overall minus sign. Thus we have established the following

Theorem 3.1 (Chordal Loewner equation) *If γ is a Jordan arc in a standard domain D starting on $\mathbb{R}$ with the parametrization from above, and if g_t is the canonical map for $D\backslash\gamma[0,t]$, then, using the notation from above, the family $\{g_t : t \in [0,t_\gamma]\}$ satisfies the equation*

$$\partial_t g_t(z) = -\Psi_t(g_t(z), \xi(t)), \tag{18}$$

with initial condition $g_0(z) = z$, and where $\Psi_t(z,\zeta)$ is the analytic function in z with imaginary part

$$-\frac{\partial G_t(z,\zeta)}{\partial n_\zeta} - \boldsymbol{\omega}_t(z)^{\mathrm{T}} \boldsymbol{P}_t^{-1} \frac{\partial \boldsymbol{\omega}_t(\zeta)}{\partial n},$$

normalized by $\lim_{z\to\infty} \Psi_t(z,\zeta) = 0$.

Remark 3.1 In the simply connected case, when $D = \mathbb{H}$ is the upper half-plane, the Green function is given by

$$G(z,w) = \Re\left(\ln \frac{z-\overline{w}}{z-w}\right).$$

Thus, if $w = x + \mathrm{i}y$,

$$-\Psi(z,w) = -i\frac{\partial}{\partial y}|_{y=0} \ln \frac{z-x+\mathrm{i}y}{z-x-\mathrm{i}y} = \frac{2}{z-w},$$

and (18) reduces to the well known chordal Loewner equation.

3.2 Bilateral Komatu–Loewner equation

We consider a bilateral standard domain D with inner radius Q. Let $\gamma : [0,t_\gamma] \to \overline{D}$ be a Jordan arc such that $\gamma(0) \in S^1$, and $\gamma(0,t_\gamma] \subset D$. Let f_t be the canonical mapping from $D\backslash\gamma[0,t]$ with the normalization $f_t(Q) > 0$, and denote D_t the chordal standard domain $f_t(D\backslash\gamma[0,t])$. If $Q_t = f_t(Q)$, then it can be shown that $t \in [0,t_\gamma] \mapsto Q_t \in [Q,1]$ is continuous and strictly increasing, [13]. Thus we may assume that γ is parametrized such that $t = \ln Q_t$. For this parameter it is shown in [13] that $t \mapsto f_t(z)$ is differentiable. An expression for the derivative is also given. However, the expression given there is not explicit enough for the purposes we have in mind. In particular, we will need to know that the vector field is itself a Lipschitz function in the moduli of the domain. We sketch a proof of what we call the bilateral Komatu–Loewner equation, leading to an expression of the derivative $\partial_t f_t$ in terms of the Green function, harmonic measures, their derivatives and harmonic conjugates. The argument is similar to the radial case, [5]. In fact, the radial case can be obtained as a limiting case from the bilateral case when $Q \to 0$, [12].

Denote $C_j(t), j = 1,\ldots,n$, the boundary components of D_t. We always have $C_n(t) = S^1$, and $C_1(t) = \{|z| = e^t\}$. For $j = 2,\ldots,n-1$, let $m_j(t)$ be the radial distance of the circular slit $C_j(t)$ from the origin. Denote $\xi(t)$ the starting point on S^1 of the Jordan arc $g_t(\gamma[t,t_\gamma])$ in D_t, i.e. $g_t(\gamma_t)$. For $\ln Q < t^* < t < t_\gamma \le 0$, set

$$g_{t,t^*} = g_{t^*} \circ g_t^{-1}.$$

Then g_{t,t^*} is a conformal map from D_t onto $D_{t^*}\backslash g_{t^*}(\gamma[t^*,t])$. The point $\xi(t^*) = g_{t^*}(\gamma_{t^*})$ corresponds to two prime ends in $D_{t^*}\backslash g_{t^*}(\gamma[t^*,t])$. Denote $\exp(i\beta_0(t,t^*))$ and

$\exp(i\beta_1(t,t^*))$, with $\beta_0(t,t^*) < \beta_1(t,t^*)$, the pre-images of these prime ends under g_{t,t^*}, i.e.

$$g_{t,t^*}(\exp(\mathrm{i}\beta_0(t,t^*))) = g_{t,t^*}(\exp(\mathrm{i}\beta_1(t,t^*))) = g_{t^*}(\gamma_{t^*}).$$

Then, if $|z| = 1$ and $\beta_1(t,t^*) \le \arg z \le \beta_0(t,t^*) + 2\pi$,

$$|g_{t,t^*}(z)| = 1.$$

The function

$$z \mapsto \ln \frac{g_{t,t^*}(z)}{z}$$

is analytic and single-valued throughout D_t. By Poisson's formula

$$\ln\left|\frac{g_{t,t^*}(z)}{z}\right| = -\frac{1}{2\pi}\int_{\partial D_t} \ln\left|\frac{g_{t,t^*}(\zeta)}{\zeta}\right| \frac{\partial G_t(\zeta,z)}{\partial n_1}\,\mathrm{d}s, \tag{19}$$

where $G_t(\zeta, z)$ is the Green function for D_t with pole at z. Using orthogonality and the period relations as we did in the chordal case, it follows that

$$\ln\frac{g_{t,t^*}(z)}{z} = -\frac{i}{2\pi}\int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \ln\left|\frac{g_{t,t^*}(\zeta)}{\zeta}\right| \Psi_t(z,\zeta)\,\mathrm{d}s + \mathrm{i}c, \tag{20}$$

for some real constant c. To eliminate c, we evaluate the identity (20) at $z = q = e^t$ and then take the difference:

$$\ln\frac{g_{t,t^*}(z)}{z} - \ln\frac{q^*}{q} = -\frac{i}{2\pi}\int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \ln\left|\frac{g_{t,t^*}(\zeta)}{\zeta}\right| \big[\Psi_t(z,\zeta) - \Psi_t(q,\zeta)\big]\,\mathrm{d}s. \tag{21}$$

By Cauchy's integral formula,

$$0 = \frac{1}{2\pi i}\int_{\partial D_t} \ln\left(\frac{g_{t,t^*}(\zeta)}{\zeta}\right)\frac{\mathrm{d}\zeta}{\zeta}. \tag{22}$$

In particular, the right-hand side of (22) is real. Since all boundary components are concentric circular arcs, $\mathrm{d}\zeta/\zeta$ is purely imaginary along ∂D_t, i.e.

$$\frac{\mathrm{d}\zeta}{\zeta} = i\,d\arg\zeta, \quad \zeta \in \partial D_t.$$

Hence

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_{\partial D_t} \ln\left|\frac{g_{t,t^*}(\zeta)}{\zeta}\right| \,\mathrm{d}\arg\zeta \\ &= \frac{1}{2\pi}\int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \ln\left|g_{t,t^*}(\mathrm{e}^{\mathrm{i}\varphi})\right|\mathrm{d}\varphi - \frac{1}{2\pi}\int_0^{2\pi} \ln\frac{q^*}{q}\,\mathrm{d}\varphi \\ &\quad + \frac{1}{2\pi}\sum_{j=2}^{n-1}\int_{C_j(t)} \ln\frac{m_j(t^*)}{m_j(t)}\,\mathrm{d}\arg\zeta. \end{aligned} \tag{23}$$

Since the two "sides" of $C_j(t)$ make opposite contributions,

$$\int_{C_j(t)} \mathrm{d}\arg\zeta = 0, \quad j = 2, \ldots, n-1,$$

and we finally get

$$t^* - t = \frac{1}{2\pi} \int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \ln\left|g_{t,t^*}(\mathrm{e}^{\mathrm{i}\varphi})\right| \, \mathrm{d}\varphi. \tag{24}$$

Letting $z = g_t(w)$ in (21), we have

$$\begin{aligned} &\ln \frac{g_{t^*}(w)}{g_t(w)} - (t^* - t) \\ &= -\frac{i}{2\pi} \int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \ln|g_{t,t^*}(\mathrm{e}^{\mathrm{i}\varphi})| \left[\Psi_t(z, \mathrm{e}^{\mathrm{i}\varphi}) - \Psi_t(q, \mathrm{e}^{\mathrm{i}\varphi})\right] \mathrm{d}s. \end{aligned} \tag{25}$$

We now wish to let $t^* \nearrow t$ in (25). Note first that, for $\varphi \in [0, 2\pi]$, $\varphi \mapsto \ln|g_{t,t^*}(\mathrm{e}^{\mathrm{i}\varphi})|$ is continuous and non-positive and that also

$$\varphi \mapsto A(\varphi) := \Psi_t(z, \mathrm{e}^{\mathrm{i}\varphi}) - \Psi_t(q, \mathrm{e}^{\mathrm{i}\varphi})$$

is continuous. Thus it follows from the mean-value theorem of integration that

$$\begin{aligned} &\frac{1}{2\pi(t^* - t)} \int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \ln\left|g_{t,t^*}(\mathrm{e}^{\mathrm{i}\varphi})\right| A(\varphi) \, \mathrm{d}\varphi \\ &= \frac{\Re(A(\varphi')) + \mathrm{i}\Im(A(\varphi''))}{2\pi(t^* - t)} \int_{\beta_0(t,t^*)}^{\beta_1(t,t^*)} \ln\left|g_{t,t^*}(\mathrm{e}^{\mathrm{i}\varphi})\right| \, \mathrm{d}\varphi \\ &= \Re(A(\varphi')) + \mathrm{i}\Im(A(\varphi'')), \end{aligned} \tag{26}$$

for some $\varphi', \varphi'' \in [\beta_0(t, t^*), \beta_1(t, t^*)]$. Hence

$$\lim_{t^* \nearrow t} \frac{\ln g_{t^*}(w) - \ln g_t(w)}{t^* - t} = 1 + \mathrm{i}[\Psi_t(z, \xi_t) - \Psi_t(e^t, \xi(t))]. \tag{27}$$

By the same argument we may let $t \searrow t^*$. On the right-hand side above we then only need to change t to t^* and introduce an overall minus sign. Thus we have established the following

Theorem 3.2 (Bilateral Komatu–Loewner equation) *If γ is a Jordan arc in a standard domain D starting on S^1 with the parametrization from above, and if g_t is the canonical map for $D \backslash \gamma[0, t]$, then, using the notation from above, the family $\{g_t : t \in [\ln Q, t_\gamma]\}$ satisfies the equation*

$$\partial_t \ln g_t(z) = 1 + \mathrm{i}[\Psi_t(g_t(z), \xi_t) - \Psi_t(e^t, \xi(t))], \tag{28}$$

with initial condition $g_{\ln Q}(z) = z$.

4 Motion of moduli

4.1 Chordal case

The right-hand side of the chordal Loewner equation, at time t, involves the Green function of the domain D_t, and also various functions derived from the Green function. Consequently, it does not make sense to ask for the solution of (18) for a given continuous curve $t \mapsto \xi(t)$, since the vector-field on the right-hand side of (18) is not specified by giving that information alone. To specify the Green function of D_t we also need the *moduli* of the domain D_t. We will now consider what the appropriate moduli space is for our purposes and find a system of equations these moduli satisfy. Once this system is found, we can solve it for a given input $t \mapsto \xi(t)$, and then, in a second step, solve the radial Komatu–Loewner equation using ξ and the moduli.

The geometric description of D_t requires $3n-3$ real parameters, three for each (interior) slit, given, for example, by the imaginary components of the slits, i.e $y_j(t)$, $j = 1, \ldots, n-1$, and the real components

$$x_j(t) < x'_j(t), \quad j = 1, \ldots, n-1,$$

determining the endpoints of the slit $C_j(t)$, $j = 1 \ldots, n-1$. On the other hand, it is well known that two n-connected domains with non-degenerate boundary continua are conformally equivalent if $3n-6$ real parameters agree for $n > 2$. If $n = 2$ then there is only one real parameter describing the conformal class, and if $n = 1$, then all such domains are conformally equivalent.

The slits we wish to grow mark two points on one of the boundary continua, the beginning ($t = 0$) and end point ($t = \infty$) of the slit. Any n-connected planar domain with two marked boundary points on one boundary component is conformally equivalent to the upper half-plane with $n-1$ horizontal slits and such that the marked boundary points are mapped to 0 and ∞. However, there is a one-parameter group of automorphisms, namely multiplication by $a > 0$, which maps the slit upper half-plane onto a slit upper half-plane, while fixing 0 and ∞. It is now easy to see that the moduli space of n-connected planar domains with two marked boundary points on one of the boundary components is $3n-4$ dimensional for all $n \geq 2$, and zero dimensional if $n = 1$. Nonetheless, we will take $\mathbf{y}(t) = (y_1(t), \ldots, y_{n-1}(t))$, $\mathbf{x}(t) = (x_1(t), \ldots, x_{n-1}(t))$, and $\mathbf{x}'(t) = (x'_1(t), \ldots, x'_{n-1}(t))$ as the moduli of the domain D_t and write $\mathbf{M}(t) := (\mathbf{y}(t), \mathbf{x}(t), \mathbf{x}'(t))$. To obtain the conformal equivalence classes from this $3n-3$ dimensional parameter space, we need to identify $(\mathbf{y}(t), \mathbf{x}(t), \mathbf{x}'(t))$ and $(\tilde{\mathbf{y}}(t), \tilde{\mathbf{x}}(t), \tilde{\mathbf{x}}'(t))$, whenever there exists an $a > 0$ such that $\mathbf{y} = a\tilde{\mathbf{y}}$, $\mathbf{x} = a\tilde{\mathbf{x}}$, and $\mathbf{x}' = a\tilde{\mathbf{x}}'$. The extra parameter $\mathbf{M}$ keeps track of will be reflected in a symmetry (invariance) of the moduli diffusion. For a standard domain the marked points are 0 and ∞. For a point $\mathbf{M}$ in the "moduli space" we denote by $D = D(\boldsymbol{M})$ the corresponding standard domain.

By boundary correspondence, if $z \in C_j$, then $g_t(z) \in C_j(t)$ and

$$\Im(g_t(z)) = y_j(t).$$

Thus, by considering the imaginary part of the chordal Loewner equation,

$$\partial_t y_j(t) = -\Im(\Psi_t(g_t(z), \xi(t))). \tag{29}$$

Further, if

$$z_j(t) = x_j(t) + \mathrm{i}y_j(t), \quad z_j'(t) = x_j'(t) + \mathrm{i}y_j(t)$$

are the endpoints of the slit $C_j(t)$, then

$$z_j(t) = g_t(\eta_j(t) + \mathrm{i}y_j(0)), \quad z_j'(t) = g_t(\eta_j'(t) + \mathrm{i}y_j(0)),$$

where $x_j(0) < \eta_j(t), \eta_j'(t) < x_j'(0)$. Indeed, the pre-images of the tips of $C_j(t)$, that is $\eta_j(t) + \mathrm{i}y_j(0)$ and $\eta_j'(t) + \mathrm{i}y_j(0)$, are the solutions to the equation

$$\frac{\partial}{\partial z} g_t(z) = 0,$$

on the set of prime-ends corresponding to $C_j \backslash \{z_j(0), z_j'(0)\}$. A tip of $C_j(t)$ cannot be the image of a tip of C_j because then the analytic function $\partial g_t/\partial z$ would not have the required number of zeroes, $2n-2$.

Lemma 4.1 (Motion of moduli – chordal case) *The moduli*

$$\mathbf{M}(t) = (\mathbf{y}(t), \mathbf{x}(t), \mathbf{x}'(t))$$

satisfy the system of equations

$$\begin{aligned} \partial_t y_j(t) &= \left[\boldsymbol{P}_t^{-1} \frac{\partial \boldsymbol{\omega}_t(\xi(t))}{\partial n} \right]_j, \\ \partial_t x_j(t) &= -\Re\left(\Psi_t\left(x_j(t) + iy_j(t), \xi(t)\right)\right), \\ \partial_t x_j'(t) &= -\Re\left(\Psi_t\left(x_j'(t) + iy_j(t), \xi(t)\right)\right), \end{aligned} \tag{30}$$

for $j = 1, \ldots, n-1$.

Proof We note that $\partial g_t/\partial z$ and $\partial^2 g_t/(\partial z)^2$ are analytic functions that extend analytically to the prime-ends corresponding to $C_1, \ldots, C_{n-1}$ with the endpoints of the slits removed. By the implicit function theorem,

$$t \mapsto \eta_j(t) + \mathrm{i}y_j(0)$$

is differentiable with derivative

$$\mathrm{DER}_t := \left[\frac{\partial^2 g_t}{(\partial z)^2}(\eta_j(t) + \mathrm{i}y_j(0)) \right]^{-1} \frac{\partial^2 g_t}{\partial t \partial z}(\eta_j(t) + \mathrm{i}y_j(0)).$$

By counting zeroes we find that

$$\frac{\partial^2 g_t}{(\partial z)^2}(\eta_j(t) + \mathrm{i}y_j(t)) \neq 0$$

and so DER_t is finite. Hence

$$\begin{aligned} \partial_t x_j(t) &= \partial_t \Re(g_t(\eta_j(t) + \mathrm{i}y_j(0))) \\ &= -\Re\left(\Psi_t\left(x_j(t) + \mathrm{i}y_j(t), \xi(t)\right)\right) + \Re\left((\partial_z g_t)(z_j(t)) \times \mathrm{DER}_t\right) \\ &= -\Re\left(\Psi_t\left(x_j(t) + \mathrm{i}y_j(t), \xi(t)\right)\right). \end{aligned} \tag{31}$$

In a similar way we obtain the derivative of $x_j'(t)$. It remains to check that (29) agrees with the first equation in (30). To this end we note that

$$\Im(\Psi_t(z,\zeta)) = \frac{\partial G_t(\zeta,z)}{\partial n_1} + \boldsymbol{\omega}_t(z)^{\mathrm{T}} \boldsymbol{P}_t^{-1} \frac{\partial \boldsymbol{\omega}_t(\zeta)}{\partial n}.$$

From the boundary behavior of the Green function and the harmonic measures, it follows that for $z \in C_j(t)$

$$\frac{\partial G(\zeta,z;t)}{\partial n_1} = 0, \quad \text{and} \quad \omega_k(z) = \delta_{jk}.$$

The lemma follows. □

We now have our main existence statement.

Theorem 4.1 *Given a continuous function $t \in [0,\infty) \mapsto \xi(t) \in \mathbb{R}$ and the moduli* $\mathbf{M}$ *of a standard domain D, there exists a unique solution $\mathbf{M}(t)$ to the system (30) on an interval $[0,t_\xi)$ with $\mathbf{M}(0) = \mathbf{M}$, and where t_ξ is characterized by*

$$t_\xi = \inf\{\tau : \lim_{t \nearrow \tau} y_j(t) = 0 \textit{ for some } j \in \{1,\ldots,n-1\}\}.$$

Further, if D_t is the standard domain determined by $\mathbf{M}(t)$, and if $\Psi_t(z,\zeta)$ is the holomorphic vector field associated to D_t (cf. Section 2.2), then, for any $z \in D$, the equation

$$\partial_t g_t^D(z) = \Psi_t(g_t^D(z),\xi(t)), \quad g_0^D(z) = z,$$

has a unique solution on $[0,t_z)$, where

$$t_z = \sup\{t \le t_\xi : \inf_{s\in[0,t]} |g_s^D(z) - \xi(s)| > 0\}.$$

Finally, for $t < t_\xi$ set $K_t = \{z \in D : t_z \le t\}$. Then g_t^D is the canonical conformal map from $D\backslash K_t$ onto D_t with hydrodynamic normalization at infinity.

Proof For the existence of the solution to the moduli equations (30) on $[0,t_\xi)$ we need to know that the vector field in (30) is Lipschitz as a function of $\mathbf{M}$, with a Lipschitz constant that only depends on distance to $\xi(t)$ of the slit (or slits) nearest to $\xi(t)$. Let $\mathbf{M}$ and $\tilde{\mathbf{M}}$ be two points in moduli space with corresponding standard domains D and $\tilde{D}$, such that

$$|y_j - \tilde{y}_j|, |x_j - \tilde{x}_j|, |x_j' - \tilde{x}_j'| < \epsilon.$$

We assume that ϵ is so small that

$$C_j \cap \tilde{C}_k = \emptyset, \quad \text{whenever } j \neq k.$$

Denote z_j, z_j' the endpoints of the slit C_j and $\tilde{z}_j, \tilde{z}_j'$ the corresponding endpoints of $\tilde{C}_j$. Denote Ψ the canonical map for D and $\tilde{\Psi}$ the canonical map for $\tilde{D}$. Then we need to show that

$$\tilde{\Psi}(\tilde{z}_j) - \Psi(z_j), \tilde{\Psi}(\tilde{z}_j') - \Psi(z_j') = O(\epsilon), \quad j = 1,\ldots,n-1. \tag{32}$$

This can be shown as in the radial case by the use of an interior variation that induces a smooth mapping $z \mapsto \tilde{z}$ from D to $\tilde{D}$ which maps slit-endpoints to corresponding slit-endpoints, see [4]. The non-compactness of the upper half-plane is of no concern as the mapping from D to $\tilde{D}$ may be assumed to be the identity outside of a compact.

The second part of the theorem now follows from general results about ordinary differential equations, exactly as in the simply connected case. □

4.2 Bilateral case

As we mentioned before, the bilateral case is similar to the radial case. The geometric description of a bilateral standard domain with n boundary components requires $1+3(n-2)$ real parameters: one for the radius Q of the inner circle, and three for each concentric circular slit. If C_j is one of the interior slits, then $C_j = \{r_j e^{i\theta}, \theta_j \leq \theta \leq \theta'_j\}$, and we will take $m_j = \ln r_j$, and θ_j, θ'_j as parameters to identify C_j.

If, in an arbitrary n-connected domain D, where $n \geq 2$, we choose a boundary point w and a boundary component that does not contain w, then there is a unique conformal map from D onto a bilateral standard domain, which sends w to 1, and the other distinguished boundary component to the inner boundary circle of the standard domain. Thus the conformal equivalence classes of n-connected domains with one marked boundary point and one distinguished boundary component which does not include the marked point are given by $1 + 3(n-2) = 3n - 5$ parameters. We call the parameters

$$(\ln Q, m_2, \ldots, m_{n-1}, \theta_2, \ldots, \theta_{n-1}, \theta'_2, \ldots, \theta'_{n-1})$$

the *moduli* of the domain. Note that, unlike in the chordal case, these are true moduli, in the sense that different sets of parameters correspond to different conformal equivalence classes.

In the bilateral case it was natural to choose the parameter $t = \ln Q$ as time. For a bilateral standard domain D_t, where $t = \ln Q$, we let

$$\mathbf{M}(t) = (m_2(t), \ldots, m_{n-1}(t), \theta_2(t), \ldots, \theta_{n-1}(t), \theta'_2(t), \ldots, \theta'_{n-1}(t)).$$

We then can obtain the following results in the same way as in the chordal case.

Lemma 4.2 (Motion of moduli – bilateral case) *The moduli* $\mathbf{M}(t)$ *satisfy the system*

$$\begin{aligned}
\partial_t m_j(t) &= 1 - \Im[\Psi_t(m_j(t)e^{i\theta_j(t)}, \xi(t)) - \Psi_t(e^t, \xi(t))],\\
\partial_t \theta_j(t) &= \Re[\Psi_t(m_j(t)e^{i\theta_j(t)}, \xi(t)) - \Psi_t(e^t, \xi(t))],\\
\partial_t \theta'_j(t) &= \Re[\Psi_t(m_j(t)e^{i\theta'_j(t)}, \xi(t)) - \Psi_t(e^t, \xi(t))],
\end{aligned} \tag{33}$$

where $j = 2, \ldots, n-1$.

As in the radial case, it can be shown that the vector field appearing on the right above is Lipschitz in the moduli and we obtain

Theorem 4.2 *Given a continuous function* $t \in [0, \infty) \mapsto \xi(t) \in S^1$ *and the moduli* $\mathbf{M}$ *of a bilateral standard domain D with interior boundary circle of radius Q, there exists a unique solution* $\mathbf{M}(t)$ *to the system* (30) *on an interval* $[\ln Q, t_\xi)$ *with* $\mathbf{M}(0) = \mathbf{M}$*, and where* t_ξ *is characterized by*

$$t_\xi = \inf\{\tau : \lim_{t \nearrow \tau} m_j(t) = 0 \text{ for some } j \in \{2, \ldots, n-1\}\}.$$

Further, if D_t is the bilateral standard domain determined by $\mathbf{M}(t)$*, and if* $\Psi_t(z, \zeta)$ *is the holomorphic vector field associated to* D_t *(cf. Sect. 2.2), then, for any* $z \in D$*, the equation*

$$\partial_t \ln g_t^D(z) = 1 + [\Psi_t(g_t^D(z), \xi(t)) - \Psi_t(e^t, \xi(t))], \quad g_{\ln Q}^D(z) = z,$$

has a unique solution on $[\ln Q, t_z)$*, where*

$$t_z = \sup\{t \leq t_\xi : \inf_{s \in [\ln Q, t]} |g_s^D(z) - \xi(s)| > 0\}.$$

Finally, for $t < t_\xi$ *set* $K_t = \{z \in D : t_z \leq t\}$*. Then* g_t^D *is the canonical conformal map from* $D \backslash K_t$ *onto* D_t *with* $g_t^D(Q) = e^t$*.*

5 Chordal SLE in multiply connected domains

5.1 Conformal invariance and Markovian-type property

The purpose of this paper is (1) to give a "natural" construction of conformally invariant measures on "simple curves" in multiply connected domains, and (2) to study some of the properties of these random curves. We will now motivate, using informal arguments, our particular construction of conformally invariant measures on simple curves. The arguments lead to a small class of processes which contains chordal SLE_κ in multiply connected domains.

For a domain D with n non-degenerate boundary continua and two boundary points (or, more generally, prime ends) z and w lying on the same boundary continuum, let $W(D, z, w)$ be the set of Jordan arcs in D with endpoints z and w. Denote $\{\mathcal{L}_{D,z,w}^{\mathbf{M}}\}_{D,z,w}$ a family of probability measures on Jordan arcs in the complex plane such that

$$\mathcal{L}_{D,z,w}^{\mathbf{M}}(W(D, z, w)) = 1,$$

and where $\mathbf{M} = M(D)$. Such families arise, or are conjectured to arise, as distributions of interfaces of statistical mechanical systems at criticality. Based on these models, e.g. percolation, one expects that the distributions describing the interfaces in different domains with different marked points are related by a Markovian-type property and conformal invariance. Denote γ a random Jordan arc with law $\mathcal{L}_{D,z,w}^{\mathbf{M}}$. The Markovian-type property says that if γ' is a sub-arc of γ which has z as one endpoint and whose other endpoint we denote by z', and if $\mathbf{M}' = M(D \backslash \gamma')$, then the conditional law of γ given γ' is

$$\mathrm{law}(\gamma | \gamma') = \mathcal{L}_{D \backslash \gamma', z', w}^{\mathbf{M}'}. \tag{34}$$

Conformal invariance means that if $f : D \to D'$ is conformal, $z' = f(z)$, $w' = f(w)$, then

$$\mathcal{L}_{D',z',w'}^{\mathbf{M}} = f_* \mathcal{L}_{D,z,w}^{\mathbf{M}}. \tag{35}$$

If (35) holds, then to understand the family $\{\mathcal{L}_{D,z,w}^{\mathbf{M}}\}$ it is enough to consider standard domains D, take $w = \infty$, $z = 0$, and, by the identification of standard domains with their moduli, we may write

$$\mathcal{L}_{D,0,\infty}^{\mathbf{M}} = \mathcal{L}^{\mathbf{M}}.$$

In this case there is a natural parametrization of the Jordan arcs we consider. Let

$$s \in [0, \infty) \mapsto \gamma(s) \in \overline{D}$$

be a Jordan arc in a standard domain D such that

$$\gamma(0) \in \mathbb{R},\ \gamma(0, \infty) \subset D, \quad \text{and} \quad \lim_{t \to \infty} \gamma(t) = \infty.$$

Denote $\mathbf{M} = M(D)$ the point in the moduli space corresponding to D and let $g_t^{\mathbf{M}}$ be the canonical mapping from $D \backslash \gamma[0, t]$ onto a standard domain $D_t := g_t^{\mathbf{M}}(D \backslash \gamma[0, t])$. Then

$$g_t^{\mathbf{M}}(z) = z + \frac{a_t}{z} + o(1/|z|), \quad z \to \infty, \tag{36}$$

where a_t is called the half-plane capacity. The function $t \mapsto a_t$ is continuous, strictly increasing, starts at zero and satisfies $a_t \to \infty$ as $t \to \infty$ (this final statement is not true if the curve creeps along to infinity very close to the real axis and we exclude this case for the purpose of this argument). Thus we may and always will assume that γ is parametrized by half-plane capacity, i.e. so that $a_t = 2t$. This parametrization is natural in the following sense. If $t \geq 0$, $\mathbf{M}(t) = M(D_t)$, and $\tilde{\gamma}$ is the curve defined by

$$s \in [0, \infty) \mapsto \tilde{\gamma}(s) = g_t^{\mathbf{M}}(\gamma(t+s)),$$

then the canonical mapping $g_s^{\mathbf{M}(t)}$ from $D_t \backslash \tilde{\gamma}[0, s]$ is given by

$$g_s^{\mathbf{M}(t)} = g_{t+s}^{\mathbf{M}} \circ \left(g_t^{\mathbf{M}}\right)^{-1},$$

and so $g_s^{\mathbf{M}(t)}(D_t \backslash \tilde{\gamma}[0, s]) = D_{t+s}$. In particular, it is easy to see that

$$g_s^{\mathbf{M}(t)}(z) = z + \frac{2s}{z} + o(1/|z|), \quad z \to \infty,$$

i.e. $\tilde{\gamma}$ is also parametrized by half-plane capacity.

Let now $\{g_s^{\mathbf{M}} : s \geq 0\}$ be the random family of canonical maps corresponding to the random Jordan arcs $\{\gamma[0, s] : s \geq 0\}$ in a standard domain D, and denote

$$\mathcal{L}^{\mathbf{M}} = \text{law}(\{g_s^{\mathbf{M}} : s \geq 0\}).$$

Then, applying first the Markovian-type property and then conformal invariance, (34), (35), we find

$$\text{law}(\{g_{t+s}^{\mathbf{M}} : s \geq 0\} | g_t^{\mathbf{M}}) = \left(g_t^{\mathbf{M}}\right)_*^{-1} \mathcal{L}^{\mathbf{M}(t)}.$$

Equivalently,

$$\text{law}\left(\{g_{t+s}^{\mathbf{M}} \circ \left(g_t^{\mathbf{M}}\right)^{-1} : s \geq 0\} | g_t^{\mathbf{M}}\right) = \text{law}(\{g_s^{\mathbf{M}(t)} : s \geq 0\}). \tag{37}$$

By the chordal Loewner equation, (18), for each $t \geq 0$, the σ-field generated by $g_t^{\mathbf{M}}$ is equal to $\sigma((\xi(r), \mathbf{M}(r)) : r \in [0, t])$, where $\xi(0) = 0$. Similarly, it is easy to see that we can reconstruct $g_{t+s}^{\mathbf{M}} \circ \left(g_t^{\mathbf{M}}\right)^{-1}$ from $\{(\xi(t+r) - \xi(t), \mathbf{M}(t+r)) : r \in [0, s]\}$, using Theorem 4.1. Thus (37) implies

$$\begin{aligned} &\text{law}(\{(\xi(t+s) - \xi(t), \mathbf{M}(t+s)) : s \geq 0\} | \{(\xi(r), \mathbf{M}(r)) : r \in [0, t]\}) \\ &\quad = \text{law}(\{(\tilde{\xi}(s), \tilde{\mathbf{M}}(s)) : s \geq 0\}), \end{aligned} \tag{38}$$

where $\tilde{\mathbf{M}}(s) = M(D_t \backslash \tilde{\gamma}[0,s])$, for a random Jordan arc $\tilde{\gamma}$ with law $\mathcal{L}^{\mathbf{M}(t)}$. The equality (38) is precisely the statement that $\{(\xi(t), \mathbf{M}(t)) : t \geq 0\}$ is a Markov process. In the doubly connected case this had already been stated in [6]. Since in the doubly connected case there is only one conformal invariant, this invariant was taken as 'time' in [6] and thus the difficulty of the existence of the moduli diffusion, which we address in Theorem 4.1, can be avoided in the case of connectivity two. We note that in the simply connected case ($n = 1$), (38) reduces to

$$\text{law}(\{\xi(t+s) - \xi(t) : s \geq 0\} | \{\xi(r) : r \in [0,t]\}) = \text{law}(\{\tilde{\xi}(s) : s \geq 0\}),$$

from which it follows that ξ is a process with independent, and identically distributed increments. From this, continuity, and the symmetry $\text{law}(\xi) = \text{law}(-\xi)$, Schramm derived in [27] that $\xi(t) = \sqrt{\kappa} B_t$ for a standard one-dimensional Brownian motion and a positive constant κ. The continuity follows from the continuity of the Jordan arcs, and the symmetry is actually observed in various discrete models, such as the percolation exploration process.

5.2 Scaling

For chordal SLE in the upper half-plane $\mathbb{H}$ the scaling property is usually arrived at as a consequence of the scaling property of the driving function, Brownian motion. Indeed, denote

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z,$$

chordal SLE in $\mathbb{H}$ and let K_t be its hull at time t, i.e. g_t maps $\mathbb{H} \backslash K_t$ conformally onto $\mathbb{H}$. If $c > 0$, then h_t defined by

$$h_t(z) = \frac{1}{c} g_{c^2 t}(cz)$$

is the normalized conformal map from $\mathbb{H} \backslash \frac{1}{c} K_{c^2 t}$ onto $\mathbb{H}$ and

$$\partial_t h_t(z) = \frac{2}{h_t(z) - \sqrt{\kappa} \frac{1}{c} B_{c^2 t}}, \quad h_0(z) = z.$$

Since $\frac{1}{c} B_{c^2 t}$ is also a standard Brownian motion, it follows that

$$\text{law}\left(\frac{1}{c} K_{c^2 t} : t \geq 0\right) = \text{law}(K_t : t \geq 0). \tag{39}$$

However, we can also turn the argument around and ask for a law on growing compacts K_t in the upper half-plane which is conformally invariant, the parameter t being the half-plane capacity as above. For the conformal map $z \mapsto cz$, this implies (39), as the half-plane capacity scales quadratically. Denote γ_t the tip of the curve generating K_t. Then the driving function for the Loewner equation is given by $w_t = g_t(\gamma_t)$, and (39) implies

$$\text{law}\left(\frac{1}{c} w_{c^2 t} : t \geq 0\right) = \text{law}(w_t : t \geq 0),$$

i.e. the driving function has Brownian scaling. Examples of diffusion processes with Brownian scaling are multiples of Brownian motion but also Bessel processes. More generally, if w satisfies the stochastic differential equation

$$\mathrm{d}w_t = \sigma(w_t)\,\mathrm{d}B_t + b(w_t)\,\mathrm{d}t,$$

then w has Brownian scaling if

$$\sigma(cx) = \sigma(x), \quad cb(cx) = b(x),$$

see [23]. If we assume that the coefficients σ and b are continuous, then this is saying that σ is constant, and b homogeneous of degree minus one.

In the multiply connected case we can argue similarly. Denote $g_t^{\mathbf{M}}$ the normalized conformal map from $D(\mathbf{M})\backslash K_t^{\mathbf{M}}$ onto D_t. The superscript $\mathbf{M}$ indicates that the random compact set is a hull in the domain $D(\mathbf{M})$. Conformal invariance of the growing random compacts $K_t^{\mathbf{M}}$ requires that

$$\mathrm{law}\left(\frac{1}{c}K_{c^2t}^{c\mathbf{M}} : t \geq 0\right) = \mathrm{law}(K_t^{\mathbf{M}} : t \geq 0). \tag{40}$$

Let $w_t^{\mathbf{M}} = g_t^{\mathbf{M}}(\gamma_t)$, where γ_t is the tip of the curve generating $K_t^{\mathbf{M}}$. Then (40) implies

$$\mathrm{law}\left(\frac{1}{c}w_{c^2t}^{c\mathbf{M}}, \frac{1}{c}\mathbf{M}_{c^2t}^{c\mathbf{M}} : t \geq 0\right) = \mathrm{law}(w_t^{\mathbf{M}}, \mathbf{M}_t^{\mathbf{M}} : t \geq 0),$$

where the superscript $\mathbf{M}$ indicates that $\mathbf{M}_0 = \mathbf{M}$. Thus, the moduli diffusion $(w_t, \mathbf{M}_t)$ also satisfies Brownian scaling. As in the one dimensional (simply connected) case, this implies that the coefficient of the martingale part of the stochastic differential equation is homogeneous of degree zero, and the drift coefficients all homogeneous of degree minus one. The drift coefficients of $d\mathbf{M}_t$ are given in (30) and we check immediately that they are indeed homogeneous of degree minus one. On the other hand, a homogeneous function of degree zero, which is a function of more than one variable does not have to be constant. Constancy of the martingale coefficient now is a property in addition to conformal invariance and the Markovian-type property.

One reasonable property to ask for is that the law of the random curve γ, at least until it hits one of the interior boundary components, be absolutely continuous with respect to an SLE-type curve in the corresponding simply connected domain obtained by filling the holes. This is reasonable on heuristic grounds, as SLE curves should occur as scaling limits of 2-dimensional discrete models, whose local behavior should not be affected by the topology of the domain. If we make this assumption of absolute continuity, then the driving functions will have to be absolutely continuous as well. As SLE-type driving functions in simply connected domains have constant diffusion coefficient, see above, it follows from the Girsanov's theorem that the diffusion coefficient in the multiply connected case is constant as well. More precisely, we have the following.

Proposition 5.1 *Let* $(\xi, \mathbf{M}) \mapsto A(\xi, \mathbf{M})$, $(\xi, \mathbf{M}) \mapsto B(\xi, \mathbf{M})$ *be smooth and homogeneous of degree 0 and -1, respectively, and let* $(\xi(t), \mathbf{M}(t))$ *be the diffusion which solves*

$$\mathrm{d}\xi(t) = A(\xi(t), \mathbf{M}(t))\,\mathrm{d}B(t) + B(\xi(t), \mathbf{M}(t))\,\mathrm{d}t$$

and (30)*, where* $B(t)$ *is a standard Brownian motion. Denote* g_t *the solution to the chordal Loewner equation* (18) *in the chordal standard domain* D *defined on* $[0, t_\xi)$

(cf. Theorem 4.1), and K_t its hull in D. Denote f_t the unique conformal equivalence from $\mathbb{H}\backslash K_t$ onto $\mathbb{H}$, such that $\lim_{z\to\infty} f_t(z) - z = 0$. *Then $h_t \equiv f_t \circ g_t^{-1}$ is smooth at $z = \xi(t)$, and if $W(t) = h_t(\xi(t))$, then, on $t < t_\xi$,*

$$\begin{aligned}\mathrm{d}W(t) &= h_t'(\xi(t))A(\xi(t),\mathbf{M}(t))\mathrm{d}B(t) + \frac{A^2(\xi(t),\mathbf{M}(t)) - 6}{2} h_t''(\xi(t))\mathrm{d}t \\ &\quad + h_t'(\xi(t))[B(\xi(t),\mathbf{M}(t)) + k_t(\xi(t))]\mathrm{d}t. \end{aligned} \tag{41}$$

Proof K_t is parametrized by half-plane capacity in D and not in $\mathbb{H}$. However, it is well known that $\partial_t f_t(z) = 2h_t'(\xi(t))^2/(f_t(z) - W(t))$. Thus

$$\begin{aligned}\partial_t h_t(z) &= (\partial_t f_t)(g_t^{-1}(z)) + f_t'(g_t^{-1}(z))\partial_t g_t^{-1}(z) \\ &= \frac{2[h_t'(\xi(t))]^2}{h_t(z) - h_t(\xi(t))} + h_t'(z)\Psi_t(z,\xi(t)). \end{aligned} \tag{42}$$

If k_t is defined via Ψ_t as in (44), then

$$\begin{aligned}\lim_{z\to\xi(t)} \partial_t h_t(z) &= \lim_{z\to\xi(t)} \left[\frac{2[h_t'(\xi(t))]^2}{h_t(z) - h_t(\xi(t))} - \frac{2h_t'(z)}{z - \xi(t)}\right] + h_t'(\xi(t))k_t(\xi(t)) \\ &= -3h_t''(\xi(t)) + h_t'(\xi(t))k_t(\xi(t)), \end{aligned}$$

and (41) follows from an appropriate version of Itô's formula. □

A similar calculation will lead to the locality property discussed in the final section.

Corollary 5.1 *For there to exist two increasing sequences of stopping times $\{T_n\}$, $\{S_n\}$, such that*

- *almost surely, as $n \to \infty$, T_n increases to t_ξ;*
- *almost surely, as $n \to \infty$, S_n increases to the exit time of D by a chordal SLE in $\mathbb{H}$;*
- *for every n, the law of $\{K_t : t < T_n\}$ is absolutely continuous with respect to a progressively measurable time change of chordal SLE in $\mathbb{H}$ stopped at S_n;*

it is necessary and sufficient that $A^2(\xi,\mathbf{M}) \equiv \kappa$ for some $\kappa \geq 0$.

Proof We will only show necessity. Denote K_s a parametrization of K_t by half-plane capacity. Then $\mathrm{d}s = [h'(\xi(t))]^2\mathrm{d}t$, and, under this time-change, (41) becomes

$$\begin{aligned}\mathrm{d}W(s) &= A(\xi(s),\mathbf{M}(s))\mathrm{d}B(s) + \frac{A^2(\xi(s),\mathbf{M}(s)) - 6}{2} \cdot \frac{h_s''(\xi(s))}{[h'(\xi(s))]^2}\mathrm{d}s \\ &\quad + \frac{B(\xi(s),\mathbf{M}(s)) + k_s(\xi(s))}{h'(\xi(s))}\mathrm{d}s. \end{aligned} \tag{43}$$

Assume now that $\{T_n\}$ and $\{S_n\}$ are sequences of stopping times with the properties as stated in the corollary. Then $S_n = \int_0^{T_n}[h_t'(\xi(t))]^2\mathrm{d}t$. Without loss of generality, we may assume that the coefficients of the stochastic differential equation (43) are bounded on $[0, S_n]$ for each n. By Girsanov's theorem, $\{W(s), s < S_n\}$ is absolutely continuous with respect to $\{\tilde{W}(s), s < S_n\}$ satisfying $d\tilde{W}(s) = A(\xi(s),\mathbf{M}(s))\mathrm{d}B(s)$, which in turn is absolutely continuous with respect to a multiple of a stopped 1-dimensional standard Brownian motion if and only if $A^2(\xi,\mathbf{M}) \equiv \kappa$ for some $\kappa \geq 0$. Finally, the map which associates to a continuous path $t \mapsto \xi(t)$ the solution $t \mapsto f_t$ of the chordal Loewner equation is continuous if both spaces are equipped with the topology of uniform convergence on compacts, [3]. Thus the absolute continuity of the law of the hull K_s with respect to the hull of a chordal Loewner evolution in $\mathbb{H}$ is equivalent to the absolute continuity of the laws of the driving functions. □

In the following sections we will focus on the case of constant diffusion coefficient.

5.3 Moduli diffusion and interactions with the boundary

For the purposes of this subsection a different normalization of mappings on standard domains is useful. We will change the normalization of the maps g_t by changing the vector field in the chordal Loewner equation (18). For a chordal standard domain D and $w \in \mathbb{R}$, define the real function $k(w)$ by

$$k(w) = \lim_{z \to w} \left(\Psi(z, w) + \frac{2}{z - w} \right), \tag{44}$$

and the conformal map $\Psi^0(z) = \Psi^0(z, w)$ by

$$\Psi(z, w) = \Psi^0(z, w) + k(w).$$

Then $\Psi^0(z, w) = \Psi^0_D(z, w)$ is the unique conformal map from D onto the upper half-plane with a finite number of horizontal slits which sends w to ∞ and satisfies

$$\lim_{z \to w} \left(\Psi^0(z, w) + \frac{2}{z - w} \right) = 0.$$

Consider the modified chordal Loewner equation

$$\partial_t g^0_t(z) = -\Psi^0(g^0_t(z), \xi^0(t)), \quad g^0_0(z) = z. \tag{45}$$

This is the normalization used in [29]. Geometrically, this normalization means that if g^0 removes a small vertical slit from the boundary of the upper half-plane, then the images of the two sides of this slit under g^0 have the same length up to first order, see [5].

Let κ be a positive real number and $A = A_\kappa(w, \mathbf{M})$ a function homogeneous of degree minus one in the variables $w \in \mathbb{R}$, and $\mathbf{M}$ in an open subset of $\mathbb{R}^{3n-3}$. Consider the system of stochastic differential equations

$$\begin{aligned}
\mathrm{d}\xi(t) &= \sqrt{\kappa} dB_t + A_\kappa(\xi(t), \mathbf{M}_t)\, \mathrm{d}t, \\
\mathrm{d}y_j(t) &= \Im \left(\Psi^0_t \left(x_j(t) + i y_j(t), \xi(t) \right) \right), \\
\mathrm{d}x_j(t) &= \Re \left(\Psi^0_t \left(x_j(t) + i y_j(t), \xi(t) \right) \right), \\
\mathrm{d}x'_j(t) &= \Re \left(\Psi^0_t \left(x'_j(t) + i y_j(t), \xi(t) \right) \right), \quad j = 1, \ldots, n-1,
\end{aligned} \tag{46}$$

where $\mathbf{M}_t = (y_1(t), \ldots, y_{n-1}(t), x_1(t), \ldots, x_{n-1}(t), x'_1(t), \ldots, x'_{n-1}(t))$. In particular, we assume that the coefficient of the martingale part is constant. If A is Lipschitz, this system has a unique solution. Then we can solve the modified chordal Loewner equation (45) for $(\xi(t), \mathbf{M}_t)$. Denote K_t the random compact such that g^0_t maps the complement of K_t in D conformally onto the standard domain D_t.

We can interpret the term A as an interaction of the random growing compact set K_t with the boundary components, and it may be possible to choose A so that the set K_t will avoid these interior boundary components. A similar situation arises for $\mathrm{SLE}_{\kappa,\rho}$, see [10]. In that case, a random growing compact set in a simply connected domain interacts with a finite number n of boundary points, the interaction strength at point j being given by a real constant ρ_j. Then the driving function for the chordal Loewner equation is given by the diffusion

$$\begin{aligned} \mathrm{d}\nu(t) &= \sqrt{\kappa}\mathrm{d}B_t + \sum_{j=1}^{n} \frac{\rho_j}{\nu(t) - Z^j(t)} \,\mathrm{d}t \\ \mathrm{d}Z^j(t) &= \frac{2}{Z^j(t) - \nu(t)} \,\mathrm{d}t, \quad j = 1, \ldots, n, \end{aligned} \tag{47}$$

a system with drift coefficients homogeneous of degree minus one similar to (46).

There are many possible candidates for the homogeneous function $A(w, \mathbf{M})$. If it is to be a domain functional of the domain $D = D(\mathbf{M})$, then natural candidates arise from derivatives of the Green function. Indeed, if $G(z, w, \mathbf{M})$ is the Green function for the domain $D = D(\mathbf{M})$ and $c > 0$, then

$$G(z, w, \mathbf{M}) = G(cz, cw, c\mathbf{M})$$

by conformal invariance and so

$$\frac{\partial_z^k \partial_w^l G(z, w, \mathbf{M})}{\partial_z^m \partial_w^n G(z, w, \mathbf{M})}$$

is homogeneous of degree minus one whenever

$$k + l = m + n + 1, \quad k, l, m, n \in \mathbb{N}.$$

The "harmonic random Loewner chains" Zhan studies in his thesis, see [29], correspond to the choice $k = l = m = 1$, $n = 0$. Via integration, or directly by conformal invariance, we also see that

$$\frac{\partial_z^{k+1} \omega_j(z, \mathbf{M})}{\partial_z^k \omega_j(z, \mathbf{M})}$$

is homogeneous of degree minus one.

5.4 Chordal SLE, percolation, and locality

The case of percolation is an example where there is no interaction, that is $A \equiv 0$. For the following calculation we return to the original chordal Loewner equation (18). Then ξ in (46) has a nonzero drift coming from changing back the normalization.Thus, to model cluster-boundaries of percolation in a multiply connected domain D we make the ansatz

$$\mathrm{d}\xi(t) = -k_t(\xi(t)) + \sqrt{\kappa}\, \mathrm{d}B_t, \tag{48}$$

where the subscript t refers to the domain D_t, k_t to (44), and where $\mathbf{M}(t)$ satisfies (30).

This choice of drift reflects that the exploration process for percolation is as likely to turn right as it is to turn left. Other discrete models lead to different drifts. In this section we show that the ansatz (48) leads to random growing compacts satisfying the locality property if $\kappa = 6$.

Denote $\{g_t^E, t \geq 0\}$ the solution of the chordal Loewner equation in a standard domain E starting at $z = 0$ for the diffusion (48). Denote $\{K_t, t \geq 0\}$ the associated growing compacts. Let A be a hull in E that does not contain zero. For the following calculations we restrict to the event $\{t < \tau\}$, where $\tau := \inf\{t : K_t \cap A \neq \emptyset\}$. Let Φ_A be the canonical mapping from $E \backslash A$, g_t^* the canonical mapping from $\Phi_A(E \backslash (A \cup K_t))$, and h_t the canonical mapping from $g_t(E \backslash (A \cup K_t))$. Since the canonical mapping for $E \backslash (A \cup K_t)$ is unique, we have

$$h_t \circ g_t = g_t^* \circ \Phi_A. \tag{49}$$

Furthermore, up to a time change, the family $\{g_t^*\}$ also satisfies a chordal Loewner equation beginning with the standard domain $E^* := \Phi_A(E \backslash A)$. In fact, reasoning as in [5], it follows that

$$\partial_t g_t^*(z) = -|h_t'(\xi(t))|^2 \Psi_t^*(\xi^*(t), w_t^*), \tag{50}$$

where $w_t^* = g_t^*(z)$, and $\xi^*(t) = h_t(\xi(t))$. The question we are interested in is whether $(\xi^*, \mathbf{M}^*)$ is a time change of $(\xi, \mathbf{M})$. Since $h_t = g_t^* \circ \Phi_A \circ g_t^{-1}$, we have

$$\partial_t h_t(z) = \left[\partial_t g_t^*\right](\Phi_A(g_t^{-1}(z))) + (g_t^* \circ \Phi_A)'(g_t^{-1}(z))(\partial_t g_t^{-1}(z)), \tag{51}$$

and we note that

$$\partial_t g_t^{-1}(z) = (g_t^{-1})'(z) \Psi_t(\xi(t), z). \tag{52}$$

Then (51),(50), and (52) imply

$$\partial_t h_t(z) = -h_t'(\xi(t))^2 \Psi_t^*(\xi^*(t), h_t(z)) + h_t'(z) \Psi_t(\xi(t), z). \tag{53}$$

Hence the stochastic differential

$$\partial_t h_t(z)\, \mathrm{d}t + h_t'(\xi(t))\, \mathrm{d}\xi(t)$$

has martingale part $h_t'(\xi(t))\sqrt{\kappa}\, dB_t$ and its drift part can be grouped into the three components

$$\begin{aligned} I :&= -h_t'(\xi(t))^2 [\Psi_t^*(\xi^*(t), h_t(z)) - k_t^*(\xi^*(t))]\, \mathrm{d}t \\ &\quad + h_t'(z)[\Psi_t(\xi(t), z) - k_t(\xi(t))]\, \mathrm{d}t, \\ II :&= -h_t'(\xi(t))^2 k_t^*(\xi^*(t))\, \mathrm{d}t, \\ III :&= -[h_t'(\xi(t)) - h_t'(z)] k_t(\xi(t))\, \mathrm{d}t. \end{aligned} \tag{54}$$

When $z \to \xi(t)$, then part III converges to zero, and part II, together with the martingale part, converges to a time-change of (48) starting at E^*. Finally, for part I, by the definition of $k(\xi; t)$ a double application of l'Hôpital's rule gives

$$\lim_{z \to \xi} \left(\frac{2h'(\xi)^2}{h(z) - h(\xi)} - \frac{2h'(z)}{z - \xi} \right) = -3h''(\xi). \tag{55}$$

Thus, by Itô's formula,

$$\mathrm{d}h_t(\xi(t)) = -h_t'(\xi(t))^2 k_t^*(\xi^*(t))\, \mathrm{d}t + \frac{\kappa - 6}{2} h_t''(\xi(t))\, \mathrm{d}t + h_t'(\xi(t))\sqrt{\kappa}\, \mathrm{d}B_t, \tag{56}$$

which is indeed a time-change of (48) if and only if $\kappa = 6$. From (50) it follows immediately that the equations for $\mathbf{M}^*$ are given by the same time change of the equations for $\mathbf{M}$.

Theorem 5.1 (Chordal SLE_6) *The solution to the chordal Loewner equation based on the diffusion* (48) *satisfies the locality property if and only if* $\kappa = 6$.

Remark 5.1 The ansatz with constant diffusion coefficient is the only one which will lead to locality. Indeed, if we replace $\sqrt{\kappa}$ by a smooth (C^2 is sufficient) homogeneous function of degree zero $(\xi, \mathbf{M}) \mapsto A(\xi, \mathbf{M})$, then (56) becomes

$$\mathrm{d}h_t(\xi(t)) = -h_t'(\xi(t))^2 k_t^*(\xi^*(t))\,\mathrm{d}t + h_t'(\xi(t)) A(\xi(t), \mathbf{M}(t))\,\mathrm{d}B_t + \frac{A^2(\xi(t), \mathbf{M}(t)) - 6}{2} h_t''(\xi(t))\mathrm{d}t, \quad (57)$$

which is a time-change of the original motion if and only if $A(\xi, \mathbf{M}) \equiv \pm\sqrt{6}$.

References

1. Ahlfors, L.: Complex Analysis, 2nd edn. McGraw-Hill, New York (1966)
2. Aizenman, M.: The geometry of critical percolation and conformal invariance. Stat. Phys. **19**, 104–120 (1996)
3. Bauer, R.: Discrete Löwner evolution. Ann. Fac. Sci. Toulouse Math. (6); **12**(4), 433–451 (2003)
4. Bauer, R., Friedrich, R.: Stochastic Loewner evolution in multiply connected domain. C. R. Acad. Sci. Paris, Ser. I **339**, 579-584 (2004)
5. Bauer, R., Friedrich, R.: On radial stochastic Loewner evolution in multiply connected domains. J. Funct. Anal. **237**, 565–588 (2006)
6. Beffara, V.: Mouvement Brownien plan, SLE, invariance conforme et dimension fractales. Thesis, Univeristy of Paris, Orsay (2003)
7. Belavin, A.A., Polyakov, A.M., Zamolodchikov, A.B.: Infinite conformal symmetry in two-dimensional quantum field theory. Nuclear Phys. B **241**, 333–380 (1984)
8. Cardy, J.: SLE(kappa,rho) and Conformal Field Theory. arXiv math-ph/0412033
9. Courant, R.: Dirichlet's Principle, with an appendix by M. Schiffer. Interscience, New York (1950)
10. Dubedat, J.: Some remarks on commutation relations for SLE. arXiv, math.PR/0411299.
11. Friedrich, R., Kalkkinen, J.: On conformal field theory and stochastic Loewner evolution, Prépublications de l'IHÉS P/03/28, (2003), which appeared inÊ Nuclear Phys. B **687**(3), 279–302 (2004)
12. Komatu, Y.: Untersuchungen über konforme Abbildung von zweifach zusammenhängenden Gebieten. Proc. Phys. Math. Soc. Jpn. **25**(3), 1–42 (1943)
13. Komatu, Y.: On conformal slit mapping of multiply-connected domains. Proc. Jpn. Acad. **26**(7), 26–31 (1950)
14. Kontsevich, M.: Arbeitstagung 2003 - CFT, SLE and phase boundaries. MPIM (2003)
15. Langlands, R., Pouillot, Y., Saint-Aubin, Y.: Conformal invariance in two-dimensional percolation. Bull. A.M.S. **30**, 1–61 (1994)
16. Lawler, G.F.: The Laplacian-b random walk and the Schramm–Loewner evolution. Illinois J. Math. **50**, 701–746 (2006)
17. Lawler, G.F., Schramm, O., Werner, W.: Values of Brownian intersection exponents. I. Half-plane exponents, Acta Math. **187**(2), 275–308 (2001)
18. Lawler, G.F., Schramm, O., Werner, W.: Conformal restriction: the chordal case. J. Amer. Math. Soc. **16**, 917–955 (2003)
19. Lawler, G.F., Schramm, O., Werner, W.: Conformal invariance of planar loop-erased random walks and uniform spanning trees. Ann. Probab. **32**(1B), 939–995 (2004)
20. Löwner, K.: Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I. Math. Ann. **89**, 103–121 (1923)
21. Nehari, Z.: Conformal Mapping. McGraw-Hill, New York (1952)
22. Pommerenke, C.: Boundary behaviour of conformal maps. Grund. Math. Wiss. 299. Springer, New York (1992)
23. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, Grundlehren der mathematischen Wissenschaften, vol. 293, 3rd edn. Springer, Berlin Heidelberg New York (1999)
24. Rohde, S., Schramm, O.: Basic properties of SLE. Ann. Math. **161**, 879–920 (2005)
25. Schiffer, M.: Hadamard's formula and variation of domain-functions. Amer. J. Math. **68**, 417–448 (1946)
26. Schiffer, M., Spencer, D.: Functionals of Finite Riemann Surfaces. Princeton University Press, Princeton (1954)
27. Schramm, O.: Scaling limits of loop-erased random walks and uniform spanning trees. Israel J. Math. **118**, 221–288 (2000)

28. Werner, W.: Random planar curves and Schramm-Loewner evolutions, lecture notes from the 2002 St. Flour summer school. Springer, Berlin Heidelberg, New York (2003)
29. Zhan, D.: Random Loewner chains in Riemann surfaces. Thesis, California Institute of Technology (2004)
30. Zhan, D.: Stochastic Loewner evolution in doubly connected domains. Probab. Theory Related Fields **129**(3), 340–380 (2004)