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Ozsváth-Szabó invariants and fillability of contact structures

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Abstract Recently Ozsváth and Szabó defined an invariant of contact structures with values in the Heegaard-Floer homology groups. They also proved that a version of the invariant with twisted coefficients is non trivial for weakly symplectically fillable contact structures. In this article we show that their non vanishing result does not hold in general for the contact invariant with untwisted coefficients. As a consequence of this fact Heegaard-Floer theory can distinguish between weakly and strongly symplectically fillable contact structures.

1 Introduction

Recently Ozsváth and Szabó showed how to associate to any contact manifold (Y, ξ) an isotopy invariant $c(\xi) \in \widehat{HF}(-Y)/\pm 1$ in the Heegaard-Floer homology of $-Y$ reduced modulo ± 1. They also proved that $c(\xi) = 0$ if ξ is an overtwisted contact structure, and $c(\xi)$ is a primitive element of $\widehat{HF}(-Y)/\pm 1$ if ξ is Stein fillable, [18]. One can get rid of the sign indeterminacy in the definition of $c(\xi)$ by working with the Heegaard–Floer homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. This is the choice we will do throughout this article. The Ozsváth-Szabó contact invariant has already been useful in proving tightness of contact structures which resisted to all previously known techniques: see for example [17,16,15]

In this article we study the relation between the Ozsváth-Szabó contact invariant and the symplectic fillability of contact structures. There are two different notions of symplectic fillability. A contact manifold (Y, ξ) is said to be *weakly symplectically fillable* if Y oriented by ξ is the oriented boundary of a symplectic 4-manifold

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(X, ω) such that $\omega|_\xi > 0$. A contact manifold (Y, ξ) is said to be *strongly symplectically fillable* if ξ is the kernel of a 1-form α such that $d\alpha = \omega|_Y$. Strong fillability implies weak fillability, but the converse is not true. The first example of a weakly but not strongly fillable contact manifold was discovered on T^3 by Eliashberg [2], and more examples were constructed by Ding and Geiges [1] on torus bundles over S^1 building on Eliashberg's.

We will construct infinitely many weakly fillable contact structures whose contact invariant is trivial. These are the first examples of tight contact structures with vanishing Ozsváth–Szabó invariant over $\mathbb{Z}/2\mathbb{Z}$. More precisely, let

$$M_0 = T^2 \times [0,1]/(\mathbf{v}, 1) = (A\mathbf{v}, 0)$$

be the mapping torus of the map $A : T^2 \to T^2$ induced by the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. Giroux constructed a family of weakly symplectically fillable contact structures ξ_n on M_0 for $n \in \mathbb{N}^+$ as follows. Put coordinates (x, y, t) on $T^2 \times \mathbb{R}$ and fix a function $\phi : \mathbb{R} \to \mathbb{R}$. For any $n > 0$ the 1-form

$$\alpha_n = \sin(\phi(t))dx + \cos(\phi(t))dy$$

on $T^2 \times \mathbb{R}$ defines a contact structure ξ_n on M_0 provided that

(1) $\phi'(t) > 0$ for any $t \in \mathbb{R}$
(2) α_n is invariant under the action $(\mathbf{v}, t) \mapsto (A\mathbf{v}, t-1)$
(3) $(2n-1)\pi \leqslant \sup_{t \in \mathbb{R}}(\phi(t+1) - \phi(t)) < 2n\pi$.

The main result of this article is the following theorem.

Theorem 1.1 *If n is even, then the Ozsváth–Szabó contact invariant $c(\xi_n)$ is trivial.*

Theorem 1.1 should be contrasted with a recent non vanishing result for the contact invariant with twisted coefficients proved by Ozsváth and Szabó. Associated to any module A over the group ring $\mathbb{Z}[H^1(M, \mathbb{Z})]$ of $H^1(M, \mathbb{Z})$ there is a Heegaard–Floer homology group "with twisted coefficients" $\underline{\widehat{HF}}(M; A)$. The ordinary "untwisted" Heegaard–Floer group is a particular case of this construction with $A = \mathbb{Z}/2\mathbb{Z}$. See [19], Section 8. In this setting the contact invariant $c(\xi)$ can be generalised to an invariant $\underline{c}(\xi; A)$ with values in $\underline{\widehat{HF}}(-M; A)/\mathbb{Z}[H^1(M, \mathbb{Z})]^\times$, where $\mathbb{Z}[H^1(M, \mathbb{Z})]^\times$ denotes the multiplicative group of the invertible elements in $\mathbb{Z}[H^1(M, \mathbb{Z})]$.

Let (W, ω) be a weak symplectic filling of the contact manifold (M, ξ). Following [23], we define a $\mathbb{Z}[H^1(M, \mathbb{Z})]$-module structure on $\mathbb{Z}[\mathbb{R}]$ via the ring homomorphism $H^1(M, \mathbb{Z}) \to \mathbb{Z}[\mathbb{R}]$ defined as

$$\gamma \mapsto T^{\int_M \gamma \wedge \omega}$$

where T^r denotes the group-ring element associated to the real number r. The Heegaard-Floer homology group with twisted coefficients in the module $\mathbb{Z}[\mathbb{R}]$ will be denoted by $\underline{HF}(M; [\omega])$. The contact invariant with twisted coefficients of weakly symplectically fillable contact structures satisfies the following non vanishing theorem.

Theorem 1.2 ([23], Theorem 4.2) *Let* (W, ω) *be a weak symplectic filling of* (M, ξ). *Then the associated contact invariant* $\underline{c}(\xi, [\omega]) \in \underline{\widehat{HF}}(M; [\omega])/\mathbb{Z}[H^1(M, \mathbb{Z})]^\times$ *is non torsion and primitive.*

Theorem 1.2 implies that the "untwisted" Ozsváth–Szabó invariant of a strongly symplectically fillable contact structure is non trivial, therefore the contact manifolds (M_0, ξ_n) are not strongly symplectically fillable if n is even. Theorem 1.1 shows that, in general, the use of twisted coefficients in the non triviality theorem for weakly symplectically fillable contact structures cannot be avoided, and that the Heegaard-Floer theory is subtle enough to distinguish between weakly and strongly symplectically fillable contact structures.

2 Contact Ozsváth–Szabó invariants

2.1 Heegaard–Floer homology

Heegaard–Floer homology is a family of topological quantum field theories for $Spin^c$ three–manifolds introduced by Ozsváth and Szabó in [20,19,21]. They associate $\mathbb{Z}/2\mathbb{Z}$–graded Abelian groups $\widehat{HF}(Y, \mathfrak{t})$, $HF^\infty(Y, \mathfrak{t})$, $HF^-(Y, \mathfrak{t})$, and $HF^+(Y, \mathfrak{t})$ to any closed oriented $Spin^c$ 3–manifold $(Y, \mathfrak{t})$, and homomorphisms

$$F^\circ_{W,\mathfrak{s}} : HF^\circ(M, \mathfrak{t}_1) \to HF^\circ(M, \mathfrak{t}_2)$$

to any oriented $Spin^c$ cobordism $(W, \mathfrak{s})$ between two $Spin^c$ manifolds $(M, \mathfrak{t}_1)$ and $(M, \mathfrak{t}_2)$. Here HF° denotes any of the four functors $\widehat{HF}$, HF^+, HF^-, and HF^∞. We write $HF^\circ(Y)$ for the direct sum $\bigoplus_{\mathfrak{t}\in Spin^c(Y)} HF^\circ(Y)$ and F°_W for the sum $\sum_{\mathfrak{s}\in Spin^c(W)} F^\circ_{W,\mathfrak{s}}$. F°_W is a well defined map because $F^\circ_{W,\mathfrak{s}} \neq 0$ only for finitely many $Spin^c$–structures on W. The homomorphisms between Heegaard–Floer homology groups satisfy the following composition rule.

Theorem 2.1 ([21], Theorem 3.4) *Let* $(W_1, \mathfrak{s}_1)$ *be a* $Spin^c$*–cobordism between* $(Y_1, \mathfrak{t}_1)$ *and* $(Y_2, \mathfrak{t}_2)$, *and let* $(W_2, \mathfrak{s}_2)$ *be a* $Spin^c$*–cobordism between* $(Y_2, \mathfrak{t}_2)$ *and* $(Y_3, \mathfrak{t}_3)$. *Denote by* W *the cobordism between* Y_1 *and* Y_2 *obtained by gluing* W_1 *and* W_2 *along* Y_2. *Then*

$$F^\circ_{W_2,\mathfrak{s}_2} \circ F^\circ_{W_1,\mathfrak{s}_1} = \sum_{\substack{\mathfrak{s} \in Spin^c(W) \\ \mathfrak{s}|_{W_i} = \mathfrak{s}_i}} F^\circ_{W,\mathfrak{s}}.$$

The groups $HF^\circ(Y, \mathfrak{t})$ are linked to each other by the exact triangles

$$\begin{tikzcd} HF^{-}(Y,\mathfrak{t}) \arrow[r] & HF^{\infty}(Y,\mathfrak{t}) \arrow[r] & HF^{+}(Y,\mathfrak{t}) \arrow[ll, bend left] \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} \widehat{HF}(Y,\mathfrak{t}) \arrow[r] & HF^{+}(Y,\mathfrak{t}) \arrow[r] & HF^{+}(Y,\mathfrak{t}) \arrow[ll, bend left] \end{tikzcd} \tag{2}$$

These exact triangles are natural, in the sense that they commute with the maps induced by cobordisms.

The Heegaard-Floer homology groups $HF^{\circ}(Y,\mathfrak{t})$ have a natural $\mathbb{Z}/\mathrm{div}(\mathfrak{t})$ relative grading, where $\mathrm{div}(\mathfrak{t})$ is the divisibility of $c_1(\mathfrak{t})$ in $H^2(Y,\mathbb{Z})$. it was shown in [22] that, when $c_1(\mathfrak{t})$ is a torsion element, the relative $\mathbb{Z}$–grading admits a natural lift to an absolute $\mathbb{Q}$–grading. In conclusion, for a torsion $Spin^c$–structure $\mathfrak{t}$ on Y the Ozsváth–Szabó homology groups $HF^{\circ}(Y,\mathfrak{t})$ split as

$$HF^{\circ}(Y,\mathfrak{t}) = \bigoplus_{d\in\mathbb{Q}} HF^{\circ}_{d}(Y,\mathfrak{t}).$$

When $\mathfrak{t} \in Spin^c(Y)$ has torsion first Chern class, there is an isomorphism between the homology groups $\widehat{HF}_d(Y,\mathfrak{t})$ and $\widehat{HF}_{-d}(-Y,\mathfrak{t})$.

Proposition 2.2 (See [22], Theorem 7.1) *Let $(W,\mathfrak{s})$ be a $Spin^c$ cobordism between two $Spin^c$ manifolds $(Y_1,\mathfrak{t}_1)$ and $(Y_2,\mathfrak{t}_2)$. If the $Spin^c$ structures $\mathfrak{t}_i$ have both torsion first Chern class and $x \in HF^{\circ}(Y_1,\mathfrak{t}_1)$ is a homogeneous element of degree $d(x)$, then $F_{W,\mathfrak{s}}(x) \in HF^{\circ}(Y_2,\mathfrak{t}_2)$ is also homogeneous of degree*

$$d(x) + \frac{1}{4}(c_1^2(\mathfrak{s}) - 3\sigma(W) - 2\chi(W)).$$

Notice that F°_W might map a homogeneous element $x \in HF^{\circ}_d(Y_1,\mathfrak{t}_1)$ into a non homogeneous element $F^{\circ}_W(x) \in HF^{\circ}(Y_2)$.

2.2 Definition of the contact invariants.

The Ozsváth–Szabó contact invariant is defined using the correspondence between contact structures and open book decompositions of three–manifolds recently discovered by Giroux. An *open book decomposition* of a 3–manifold Y is a fibred link $B \subset Y$ together with a fibration $\pi : Y \setminus B \to S^1$. The link B is called the *binding* of the open book decomposition and the union of a fibre of $\pi : Y \setminus B \to S^1$ with B is called a *page*.

Definition 2.3 ([9], Definition 1) Let (Y,ξ) be a contact 3–manifold. An open book decomposition (B,π) of Y is said to be adapted to ξ if:

(1) B is transverse to ξ,
(2) ξ is defined by a contact form α such that $d\alpha$ is a symplectic form on any fibre of π,
(3) the orientation of B induced by the contact structure coincides with the orientation as boundary of the fibres of π oriented by $d\alpha$.

By [9] Theorem 3 any contact structure on a three manifold admits an adapted open book decomposition. This open book decomposition is not unique, in fact two open book decompositions which differ by the positive plumbing of an annulus are adapted to isotopic contact structures. See [9] Section B. After positive plumbing, we can assume that the binding is connected and pages have genus $g \geq 2$. Adding a 2–handle along B with the framing induced by a page we form a cobordism V between Y and Y_0, where Y_0 is a 3-manifold fibred over S^1 with fibres of genus $g \geq 2$. On Y_0 there is a canonical $Spin^c$–structure $\mathfrak{t}_0$ induced by the fibration. $\widehat{HF}(-Y_0, \mathfrak{t}_0) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with the summands lying in different degrees for the absolute $\mathbb{Z}/2\mathbb{Z}$ grading, while $HF^+(-Y_0, \mathfrak{t}_0) = \mathbb{Z}/2\mathbb{Z}$. See [18] Section 3. We fix a distinguished element $c_0 \in \widehat{HF}(-Y_0, \mathfrak{t}_0)$ as the homogeneous element of $\widehat{HF}(-Y_0, \mathfrak{t}_0)$ which is mapped to the non zero element of $HF^+(-Y_0, \mathfrak{t}_0)$ by the natural map $\widehat{HF}(-Y_0, \mathfrak{t}_0) \to HF^+(-Y_0, \mathfrak{t}_0)$. We denote by $\overline{V}$ the cobordism V turned upside–down, so that $\overline{V}$ is a cobordism between $-Y_0$ and $-Y$.

Definition 2.4 The *Ozsváth–Szabó contact invariant* of a contact 3–manifold (Y, ξ) is the element $c(\xi) \in \widehat{HF}(-Y)$ defined by

$$c(\xi) = \widehat{F}_{\overline{V}}(c_0).$$

By [18] Theorem 1.3 $c(\xi)$ is independent of the choice of the open book decomposition adapted to ξ and is an isotopy invariant. The Ozsváth–Szabó contact invariant is non trivial and detects important topological properties of the contact structures, in fact

Theorem 2.5 ([18], Theorem 1.4 and Theorem 1.5) *If (Y, ξ) is overtwisted, than $c(\xi) = 0$. If (Y, ξ) is Stein fillable, then $c(\xi) \neq 0$.*

The Ozsváth–Szabó contact invariant $c(\xi)$ encodes the homotopy invariants of ξ, see [18], Proposition 4.6. Any contact structure ξ on a 3–manifold Y determines a $Spin^c$–structure $\mathfrak{t}_\xi$ on Y, then $c(\xi) \in \widehat{HF}(-Y, \mathfrak{t}_\xi)$. If the first Chern class of ξ is torsion, by [10] Theorem 4.16 the homotopy type of ξ is determined by the $Spin^c$–structure $\mathfrak{t}_\xi$ and by the $\mathbb{Q}$–valued Gompf invariant $d_3(\xi)$ defined as follows.

Definition 2.6 (See [10], Definition 4.2) Let ξ be an oriented tangent plane field on the 3–manifold Y with torsion first Chern class, and let (X, J) be a almost complex 4–manifold such that Y is the boundary of X and $\xi = TY \cap J(TY)$ is the field of complex lines in TY. Then we define

$$d_3 = \frac{1}{4}(c_1(J)^2 - 2\chi(X) - 3\sigma(X))$$

where χ denote the Euler characteristic, σ the signature, and $c_1(J)^2$ is defined because $c_1(\xi) = c_1(J)|_Y$ is torsion.

By [18], Proposition 4.6, if $c_1(\xi)$ is a torsion element of $H^2(Y, \mathbb{Z})$, then $c(\xi)$ is an homogeneous element of degree $-d_3(\xi) - \frac{1}{2}$.

Theorem 2.7 ([18], Theorem 4.2 and [17], Theorem 2.3) *If the contact manifold (Y', ξ') is obtained from the contact manifold (Y, ξ) by Legendrian surgery along a Legendrian knot L, and W is the cobordism between Y and Y' obtained by adding a 2–handle to $Y \times [0, 1]$ along $L \times \{1\}$ with framing -1 with respect to the contact framing, then*

$$\widehat{F}_{\overline{W}}(c(\xi')) = c(\xi)$$

where $\overline{W}$ denotes the cobordism W turned upside–down.

The space of oriented contact structures on Y has a natural involution.

Definition 2.8 For any contact structure ξ on a 3–manifold Y we denote by $\overline{\xi}$ the contact structure on Y obtained from ξ by inverting the orientation of the planes.

This operation is compatible with the conjugation of the $Spin^c$-structure defined by the contact structure, in fact $\mathfrak{t}_{\overline{\xi}} = \overline{\mathfrak{t}_{\xi}}$. There is an isomorphism $\mathfrak{J}$: $HF^\circ(-Y, \mathfrak{s}) \to HF^\circ(-Y, \overline{\mathfrak{s}})$ defined in [19], Theorem 2.4. We recall that the isomorphism $\mathfrak{J}$ preserves the $\mathbb{Z}/2\mathbb{Z}$–grading of the Heegaard–Floer homology groups and is a natural transformation in the following sense.

Proposition 2.9 ([21], Theorem 3.6) *Let $(W, \mathfrak{s})$ be a $Spin^c$-cobordism between $(Y_1, \mathfrak{t}_1)$ and $(Y_2, \mathfrak{t}_2)$. Then the following diagram*

$$\begin{array}{ccc}
HF^\circ(Y_1, \mathfrak{t}_1) & \xrightarrow{F^\circ_{W,\mathfrak{s}}} & HF^\circ(Y_2, \mathfrak{t}_2) \\
\downarrow \mathfrak{J} & & \downarrow \mathfrak{J} \\
HF^\circ(Y_1, \overline{\mathfrak{t}}_1) & \xrightarrow{F^\circ_{W,\overline{\mathfrak{s}}}} & HF^\circ(Y_2, \overline{\mathfrak{t}}_2)
\end{array}$$

commutes.

The isomorphism $\mathfrak{J}$ commutes also with the maps in the exact triangles (1) and (2) relating the different Heegaard–Floer homology groups.

Theorem 2.10 *Let (Y, ξ) be a contact manifold, then*

$$c(\overline{\xi}) = \mathfrak{J}(c(\xi)).$$

Proof If (B, π) is an open book decomposition adapted to ξ, then the open book decomposition $(-B, \overline{\pi})$, where $-B$ denotes the binding B with opposite orientation and $\overline{\pi}$ is the composition of π with the complex conjugation on S^1, is adapted to $\overline{\xi}$. The pages of $(-B, \overline{\pi})$ are the pages of (B, π) with opposite orientation, so the fibration on Y_0 induced by $(-B, \overline{\pi})$ differs from the fibration induced by

(B, π) for the orientation of the fibres, therefore its canonical $Spin^c$–structure is the conjugate of $\mathfrak{t}_0$. The commutative diagram

$$\begin{array}{ccc}
\widehat{HF}(-Y_0', \mathfrak{t}_0) & \longrightarrow & HF^+(-Y_0, \mathfrak{t}_0) \\
\mathfrak{J}\big\downarrow & & \mathfrak{J}\big\downarrow \\
\widehat{HF}(-Y_0', \bar{\mathfrak{t}}_0) & \longrightarrow & HF^+(-Y_0, \bar{\mathfrak{t}}_0)
\end{array}$$

together with the fact that $\mathfrak{J}$ is an isomorphism and preserves the $\mathbb{Z}/2\mathbb{Z}$–grading of the Heegaard–Floer homology groups shows that the distinguished element of $\widehat{HF}(-Y_0', \bar{\mathfrak{t}}_0)$ is $\bar{c}_0 = \mathfrak{J}(c_0)$, therefore

$$c(\bar{\xi}) = \widehat{F}_{\overline{V}}(\bar{c}_0) = \widehat{F}_{\overline{V}}(\mathfrak{J}(c_0)) = \mathfrak{J}(\widehat{F}_{\overline{V}}(c_0)) = \mathfrak{J}(c(\xi)). \qquad \square$$

2.3 Ozsváth–Szabó contact invariants of strongly symplectically fillable contact structures

In this section we prove a non vanishing theorem for the Ozsváth–Szabó contact invariant of strongly symplectically fillable contact structures. This theorem can be easily derived as a corollary of the more general non vanishing Theorem 1.2 proved by Ozsváth and Szabó using the twisted coefficients, however it is also possible to adapt the proof of Theorem 1.2, so that we do not need to use Heegaard-Floer homologies with twisted coefficients. We choose this second option, but the proof requires some more Heegaard–Floer machinery.

From the exact triangle (1) we define a fifth group $HF^{red}(Y, \mathfrak{t})$ as the kernel of the map

$$HF^-(Y, \mathfrak{t}) \to HF^\infty(Y, \mathfrak{t})$$

or, equivalently, as the cokernel of the map

$$HF^\infty(Y, \mathfrak{t}) \to HF^+(Y, \mathfrak{t}).$$

The group $HF^{red}(Y, \mathfrak{t})$ is always finitely generated. Let W be an oriented cobordism between the 3–manifolds Y_1 and Y_2. An *admissible cut* of W ([21], Definition 8.3) is a 3–manifold $N \subset W$ which divides W into two pieces W_1 and W_2 such that $b_2^+(W_i) > 0$ for $i = 1, 2$, and the connecting homomorphism $\delta : H^1(N, \mathbb{Z}) \to H^2(W, \partial W)$ of the Meyer–Vietoris sequence of the pair (W_1, W_2) is trivial. It is shown in [21], Example 8.4 that an admissible cut of W always exists if $b_2^+(W) > 1$. By [21] Lemma 8.2 the maps

$$F^\infty_{W_1,\mathfrak{s}} : HF^\infty(Y_1, \mathfrak{s}|_{Y_1}) \to HF^\infty(N, \mathfrak{s}|_N)$$
$$F^\infty_{W_1,\mathfrak{s}} : HF^\infty(N, \mathfrak{s}|_N) \to HF^\infty(Y_2, \mathfrak{s}|_{Y_2})$$

vanish for any $Spin^c$–structure $\mathfrak{s}$ on W, therefore an easy diagram chase on the exact triangle (1) allows us to define a "mixed" homomorphism $F^{mix}_{W,\mathfrak{s}} : HF^-(Y_1, \mathfrak{t}_1) \to HF^+(Y_2, \mathfrak{t}_2)$ which factors through $HF^{red}(N, \mathfrak{s})$. By [21], Theorem 8.5 the mixed map $F^{mix}_{W,\mathfrak{s}}$ does not depend on the particular admissible cut used to define it.

The mixed map can be used to define a numerical invariant of smooth four–manifolds with $b_2^+ > 1$ which is conjecturally equal to the Seiberg–Witten invariant. If X is a closed oriented 4–manifold, after removing two balls we can view it as a cobordism from S^3 to S^3. The groups $HF^+(S^3)$ and $HF^-(S^3)$ have distinguished elements Θ^+ and Θ^- which are the non trivial elements in minimal (resp. maximal) degree. See [19], Section 3 for the computation of the Heegaard–Floer homology groups of S^3. The four–dimensional invariant of X is the map

$$\Phi_X : Spin^c(X) \to \mathbb{Z}/2\mathbb{Z}$$

where $\Phi_X(\mathfrak{s})$ is defined as the coefficient of Θ^+ in $F^{mix}_{X,\mathfrak{s}}(\Theta^-)$.

We denote by $c^+(\xi)$ the image of $c(\xi)$ in $HF^+(-Y)$. Theorem 2.7 can be refined in the following way.

Lemma 2.11 *Suppose that (Y', ξ') is obtained from (Y, ξ) by Legendrian surgery on a Legendrian link L and that (W, ω) is the symplectic cobordism from (Y, ξ) to (Y', ξ') induced by this surgery. Then we have*

$$F^+_{\overline{W},\mathfrak{k}}(c^+(\xi')) = c^+(\xi)$$

for the canonical spin^c-structure $\mathfrak{k}$ associated to the symplectic structure on W, and

$$F^+_{\overline{W},\mathfrak{s}}(c^+(\xi')) = 0$$

for any spin^c-structure $\mathfrak{s}$ on W with $\mathfrak{s} \neq \mathfrak{k}$.

Proof As in the proof of [17] Theorem 2.3 there exists an open book decomposition of Y adapted to the contact structure ξ so that the surgery link lies on a page. We can also assume that the binding is connected and the pages have genus $g > 1$. An open book decomposition adapted to ξ' is obtained from the open book decomposition adapted to ξ by composing the monodromy with right–handed Dehn twists along the surgery link. Let Y_0 and Y_0' be the 3-manifolds obtained from Y and Y' respectively by 0-surgery on the binding, and let V, V' be the induced cobordisms. The surgery on L induces cobordisms W between Y and Y' and W_0 from Y_0 to Y_0'. Both Y_0 and Y_0' are surface bundles over S^1, and W_0 admits a Lefschetz fibration over the annulus. Let $\mathfrak{t}_0$ and $\mathfrak{t}_0'$ be the Spin^c-structures on Y_0 and Y_0' respectively determined by the fibration, and let $\mathfrak{k}_0$ be the canonical Spin^c-structure on W_0 determined by the Lefschetz fibration. By [24], Theorem 5.3,

$$F^+_{\overline{W_0},\mathfrak{k}_0} : HF^+(-Y_0', \mathfrak{t}_0') \to HF^+(-Y_0, \mathfrak{t}_0)$$

is an isomorphism, while the maps

$$F^+_{\overline{W_0},\mathfrak{s}} : HF^+(-Y_0', \mathfrak{t}') \to HF^+(-Y_0, \mathfrak{t})$$

are trivial when $\mathfrak{s} \neq \mathfrak{k}_0$.

Let W' be the cobordism $W' = W_0 \cup_{Y_0'} V' = V \cup_Y W$ from Y_0 to Y'. Since the cobordism V' is obtained by adding a unique 2-handle along a homologically non trivial curve, the restriction map $H^2(W', \mathbb{Z}) \to H^2(W_0, \mathbb{Z})$ is an isomorphism, therefore there is a unique Spin^c-structure $\mathfrak{k}_0'$ on W which extends $\mathfrak{k}_0$. By the composition formula [21] Theorem 3.4 $F^+_{W',\mathfrak{k}'_0} = F^+_{V'} \circ F^+_{W_0,\mathfrak{k}_0}$ and for any other Spin^c-structure $\mathfrak{s} \neq \mathfrak{k}'_0$ the map $F^+_{X,\mathfrak{s}}$ is trivial. Let $\mathfrak{s}'$ be the restriction of $\mathfrak{k}'_0$ to W, then the diagram

$$\begin{tikzcd} HF^+(-Y'_0, \mathfrak{t}'_0) \arrow[r, "F^+_{W_0,\mathfrak{k}_0}"] \arrow[d, "F^+_{V'}"'] & HF^+(-Y_0, \mathfrak{t}_0) \arrow[d, "F^+_{V}"'] \\ HF^+(-Y', \mathfrak{t}_{\xi'}) \arrow[r, "F^+_{W,\mathfrak{s}'}"] & HF^+(-Y, \mathfrak{t}_\xi) \end{tikzcd}$$

commutes and $F^+_{W,\mathfrak{s}} = 0$ for any $\mathfrak{s} \neq \mathfrak{s}'$. To finish the proof, we have to identify $\mathfrak{s}'$ with $\mathfrak{k}$.

By [3], Theorem 1.1, the symplectic structure induced by the Lefschetz fibration on W_0 extends over the 2-handle V', thus we obtain a symplectic structure ω' on W' with canonical Spin^c-structure $\mathfrak{k}'_0$. The restriction of ω' to W coincides with the symplectic structure on W induced by the Legendrian surgery, therefore $\mathfrak{s}' = \mathfrak{k}$. □

We have stated Lemma 2.11 in the form in which we are going to use it, however it can be proved in the same way for the stronger contact invariant in $\widehat{HF}(-Y)$ with integer coefficients.

Lemma 2.12 *Let (Y, ξ) be a contact manifold, then there exists a concave symplectic filling $(W', \omega_{W'})$ of (Y, ξ) with canonical Spin^c-structure $\mathfrak{k}_{W'}$ such that $b_2^+(W') > 1$ and*

$$c^+(\xi) = F^{mix}_{W',\mathfrak{k}_{W'}}(\Theta^-).$$

Proof Combining [6], Theorem 1.1 and [5] Lemma 3.1 there is a Stein fillable contact manifold (Y', ξ') and a symplectic cobordism (V_1, ω_{V_1}) from (Y, ξ) to (Y', ξ') so that Y' is a rational homology sphere and V_1 is composed by 2-handles attached in a Legendrian way. By [25] Lemma 1 there is a concave filling (V_2, ω_{V_2}) of (Y', ξ') with canonical *Spin*^{c}-structure $\mathfrak{k}_{V_2}$ such that $b_2^+(V_2) > 1$ and $c^+(\xi') = F^{mix}_{V_2,\mathfrak{k}_{V_2}}(\Theta^-)$.

Let $(W', \omega_{W'})$ be the concave filling of (Y, ξ) obtained by gluing (V_1, ω_{V_1}) and (V_2, ω_{V_2}) along (Y', ξ'), and let $\mathfrak{k}_{V_1}$ be the canonical *Spin*^{c}-structures of (V_1, ω_{V_1}). Since Y is a rational homology sphere, $H^2(W', \mathbb{Z}) = H^2(V_1, \mathbb{Z}) \oplus H^2(V_2, \mathbb{Z})$ therefore there exists a unique *Spin*^{c}-structure $\mathfrak{k}_{W'}$ on W' which restricts to $\mathfrak{k}_{V_1}$ on V_1 and to $\mathfrak{k}_{V_2}$ on V_2. The composition formula [21], Theorem 3.4, together with Lemma 2.11, yields

$$c^+(\xi) = F^+_{V_1,\mathfrak{k}_{V_1}} \circ F^{mix}_{V_2,\mathfrak{k}_{V_2}}(\Theta^-) = F^{mix}_{W',\mathfrak{k}_{W'}}(\Theta^-). \qquad \square$$

Theorem 2.13 *Let (Y, ξ) be a strongly symplectically fillable contact manifold, then $c(\xi) \neq 0$.*

Proof Let (W_1, ω_1) be a strong symplectic filling of (Y, ξ), and let (W_2, ω_2) be the concave symplectic filling considered in Lemma 2.12. Gluing (W_1, ω_1) and (W_2, ω_2) we obtain a closed symplectic manifold (X, ω) with $b_2^+(X) > 1$. The composition formula [21] Theorem 3.4 gives

$$F^+_{W_2,\mathfrak{k}_{W_2}}(c^+(\xi)) = F^+_{W_2,\mathfrak{k}_{W_2}} \circ F^{mix}_{W_1,\mathfrak{k}_{W_1}}(\Theta^-)$$
$$= \sum_{\substack{\mathfrak{s} \in Spin^c(X) \\ \mathfrak{s}|_{W_i} = \mathfrak{k}_{W_i}}} F^{mix}_{X,\mathfrak{s}}(\Theta^-) = \sum_{\substack{\mathfrak{s} \in Spin^c(X) \\ \mathfrak{s}|_{W_i} = \mathfrak{k}_{W_i}}} \Phi_X(\mathfrak{s}).$$

One of the $Spin^c$–structures in the sum is the canonical one $\mathfrak{k}_X$ coming from the symplectic structure on X. For any other $Spin^c$–structure $\mathfrak{s}$ in the sum we have $c_1(\mathfrak{s}) - c_1(\mathfrak{k}_X) \in \delta(\alpha(\mathfrak{s}))$ for $\alpha(\mathfrak{s}) \in H^1(Y, \mathbb{Z})$, where δ is the homomorphism $H^1(Y) \to H^2(X)$ in the Meyer–Vietoris exact sequence for the pair (W_1, W_2), therefore

$$\langle c_1(\mathfrak{s}) - c_1(\mathfrak{k}_X), [\omega] \rangle_X = \langle \alpha(\mathfrak{s}), [\omega|_Y] \rangle_Y = 0$$

in fact $\omega|_Y$ is exact because W_1 is a strong filling.

By [24] Theorem 1.1 the only non zero term in the sum is $\Phi_X(\mathfrak{k}_X) = 1$, therefore $F^+_{W_1,\mathfrak{k}_{W_1}}(c^+(\xi)) = \Theta^+$ which implies that $c^+(\xi) \neq 0$. In turn, this implies that $c(\xi) \neq 0$. □

Remark 2.14 Actually the proof of Theorem 2.13 proves the stronger fact that, if we see (W_1, ω_1) as a symplectic cobordism between the standard (S^3, ξ_0) and (Y, ξ), then

$$F^+_{W_1,\mathfrak{k}_{W_1}}(c(\xi)) = c(\xi_0)$$

for the canonical $Spin^c$–structure of (W_1, ω_1) and

$$F^+_{W_1,\mathfrak{s}}(c(\xi)) = 0$$

for any other $Spin^c$–structure on (W_1, ω_1) with

$$\langle c_1(\mathfrak{s}), [\omega_1] \rangle_{W_1} = \langle c_1(\mathfrak{k}_{W_1}), [\omega_1] \rangle_{W_1}.$$

3 Weakly fillable contact structures with trivial untwisted $\mathbb{Z}/2\mathbb{Z}$ Ozsváth-Szabó contact invariant

3.1 Tight contact structures on M_0

Let M_0 be the T^2-bundle over S^1 with monodromy map $A : T^2 \times \{1\} \to T^2 \times \{0\}$ given by $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

Put coordinates (x, y, t) on $T^2 \times \mathbb{R}$ and fix a function $\phi : \mathbb{R} \to \mathbb{R}$. For any $n > 0$ the 1-form

$$\alpha_n = \sin(\phi(t))dx + \cos(\phi(t))dy$$

on $T^2 \times \mathbb{R}$ defines a contact structure ξ_n on M_0 provided that

(1) $\phi'(t) > 0$ for any $t \in \mathbb{R}$
(2) α_n is invariant under the action $(\mathbf{v}, t) \mapsto (A\mathbf{v}, t-1)$
(3) $(2n-1)\pi \leqslant \sup_{t\in\mathbb{R}}(\phi(t+1) - \phi(t)) < 2n\pi$

The main results about this family of contact structures are the following.

Theorem 3.1 ([8], Proposition 2 and Theorem 6) *The contact structures ξ_n do not depend on the function ϕ up to isotopy, and are all universally tight and distinct.*

Theorem 3.2 ([11], Theorem 0.1) *The tight contact structures ξ_n are the only tight contact structures on M_0 up to isotopy.*

Theorem 3.3 ([1], Theorem 1) *For any $n \in \mathbb{N}$, ξ_n is weakly symplectically fillable. There is a number n_0 such that, for any $n > n_0$, ξ_n is not strongly symplectically fillable.*

The fibration on M_0 admits a transverse 1–dimensional foliation induced by the foliation by segments on $T^2 \times [0, 1]$. Let F be the image of $\{0\} \times [0, 1]$ in M_0, then F is Legendrian with respect to the contact structure ξ_n for all n.

The manifold M_0 has a presentation as 0-surgery on the right-handed trefoil knot K, in fact the complement of K in S^3 fibres over S^1 with fibre the holed torus and the monodromy acts on the homology of the fibre as $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ for some choice of coordinates in the fibre. Moreover the identification between M_0 and the 0–surgery on K can be chosen so that the complement of a tubular neighbourhood of K in S^3 is mapped diffeomorphically into the complement of a tubular neighbourhood of F in M_0 and the meridian of K is mapped to a longitude of F.

We perform a change of coordinates in a neighbourhood of F to determine what longitude of F corresponds to the meridian of K and to compute the twisting number of ξ_n along F induced by this longitude.

Lemma 3.4 *Let $R = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ be the rotation by angle $-\frac{\pi}{3}$. Then A is conjugate to R in $GL^+(2, \mathbb{R})$.*

Proof A and R are conjugated in $GL(2, \mathbb{C})$ because they have the same characteristic polynomial with distinct roots, therefore they are conjugate also in $GL(2, \mathbb{R})$ because they are both real. Let $B \in GL(2, \mathbb{R})$ be a matrix such that $BAB^{-1} = R$. For any $x \in \mathbb{R}^2 \setminus \{0\}$ we have $x \wedge Ax \neq 0$ because A has no real eigenvalues, therefore, after identifying $\bigwedge^2 \mathbb{R}^2$ to $\mathbb{R}$ using the canonical basis, $x \wedge Ax$ has constant sign as a function $\mathbb{R}^2 \to \mathbb{R}$. A direct computation at $x = \binom{0}{1}$ shows that $x \wedge Ax$ is negative. For the same reason, $x \wedge Rx$ is also negative, therefore $\det B > 0$ because $x \wedge Rx = B^{-1}Bx \wedge B^{-1}ABx = (\det B)^{-1} Bx \wedge ABx$. □

Lemma 3.5 *The twisting number of ξ_n along the Legendrian curve F is $tn(F, \xi_n) = -n$*

Proof Let U be a small A-invariant neighbourhood of $(0, 0)$ in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ so that

$$V = U \times [0, 1]/(\mathbf{v}, 1) = (A\mathbf{v}, 0)$$

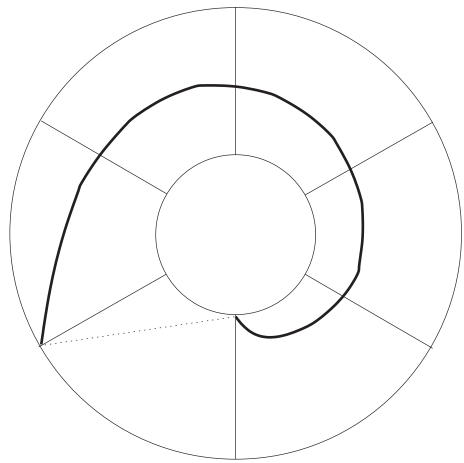

Fig. 1 The boundary of V. The inner circle is glued to the outer one after a rotation of $-\frac{\pi}{3}$. The dotted line closes to a longitude of V, the radial lines close to a leaf of the transverse foliation and the bold line closes to a dividing curve for ξ_0

is a standard neighbourhood of F. Then B^{-1} is defined on U and $U_0 = B^{-1}(U)$ is a R-invariant neighbourhood of (0, 0), i. e. a disc centred in (0, 0). In the coordinates (x', y', t) of $U_0 \times \mathbb{R}$ the 1-form α_n can be written as

$$\alpha_n = \sin(2\pi(n + \frac{5}{6})t)dx' + \cos(2\pi(n + \frac{5}{6})t)dy'.$$

By Lemma 3.4 the leaves of the transverse foliation in the boundary of the neighbourhood of K have slope $-\frac{1}{6}$, therefore they intersect the meridian of K once. If we put coordinates (θ, t) on $\partial U_0 \times I$, then the longitude of F corresponding to the meridian of K is the image in ∂V of the arc $t \mapsto (e^{it\frac{\pi}{3}}, t)$ (the dotted curve in Figure 1) because it intersect the leaves of the transverse foliation only once. A dividing curve of ξ_n is isotopic to the image of the arc $t \mapsto (e^{-2\pi(n+\frac{5}{6})t}, t)$ therefore the twisting number of ξ_n along F, which is the algebraic intersection of a dividing curve with the longitude, is $-n$. Figure 1 shows what happens for $n = 1$.

Lemma 3.6 *If $L \subset M_0$ is a Legendrian curve which is smoothly isotopic to F, then $tn(L, \xi_n) \leqslant tn(F, \xi_n)$.*

Proof Since $A^6 = I$, M_0 has a six-fold cover with total space T^3 induced by a cover of S^1. Let $\widehat{F}_3$ and $\widehat{L} \subset T^3$ be the pre-images of F and L respectively. By [12], Theorem 7.6, $\widehat{F}_3$ maximises the twisting number in its smooth isotopy class. The lemma follows from the obvious monotonicity of the twisting number under finite coverings. □

Since the right-handed trefoil can be put in Legendrian form with Thurston-Bennequin invariant 1, this surgery presentation yields a Stein fillable contact structure on M_0.

Proposition 3.7 *The Stein fillable contact structure on M_0 described by the presentation of M as 0-surgery on the right-handed trefoil knot K is ξ_1.*

Proof By Theorem 3.2, the Stein fillable contact structure on M_o is isotopic to ξ_k for some $k \in \mathbb{N}$.

It is easy to make the meridian of K Legendrian with Thurston–Bennequin invariant -1 in the standard tight contact structure of S^3, therefore $tn(F, \xi_k) \geq -1$ because the image of the meridian of K is isotopic to F as a framed knot in M_0. By Lemma 3.5 and Lemma 3.6 this is possible only if $k = 1$. □

3.2 Tight contact structures on $-\Sigma(2, 3, 6n + 5)$

The manifold $-\Sigma(2, 3, 6n + 5)$ is obtained from M_0 by $-(n + 1)$–surgery on F. For any $n \in \mathbb{N}$ and $n \geq 2$ we define $\mathcal{P}_n^* = \{-n + 1, -n + 3, \ldots, n - 3, n - 1\}$. If n is even, then $0 \notin \mathcal{P}_n^*$ and we define $\mathcal{P}_n = \mathcal{P}_n^* \cup \{0\}$. In the following we will always consider n even, although some of the facts that we are going to prove are true for any n.

Let S_+ and S_- denote the operations of positive and negative stabilisation defined, for example, in [4], Section 2.7. Given $i \in \mathcal{P}_n^*$, denote the contact structure on $-\Sigma(2, 3, 6n + 5)$ obtained by Legendrian surgery on (M_0, ξ_1) along the Legendrian knot $S_+^{(n-1+i)/2} S_-^{(n-1-i)/2}(F)$ by η_i. We denote the tight contact structure on $-\Sigma(2, 3, 6n + 5)$ obtained by Legendrian surgery on (M_0, ξ_n) along F by η_0.

The contact manifolds $(-\Sigma(2, 3, 6n + 5), \eta_i)$ for $i \in \mathcal{P}_n^*$ are the Stein fillable contact manifolds considered in [13], in fact (M_0, ξ_1) is the Stein fillable contact manifold obtained by Legendrian surgery on a positive trefoil knot in S^3 with Thurston-Bennequin invariant 0 by Proposition 3.7, and performing Legendrian surgery on a stabilisation of F is equivalent to performing Legendrian surgery on a stabilisation of a meridian of the trefoil knot.

Proposition 3.8 *Let $\overline{\eta}_i$ be the contact structure obtained from η_i by reversing the orientation of the contact planes. Then $\overline{\eta}_i$ is isotopic to η_{-i}.*

Proof For any $n \in \mathbb{N}^+$ (M_0, ξ_n) is isotopic to $(M_0, \overline{\xi}_n)$. The isotopy is induced by a translation in the t direction in the cover $T^2 \times \mathbb{R}$, therefore it fixes F. We denote $S_+^{(n-1+i)/2} S_-^{(n-1-i)}(F)$ thought of as a Legendrian knot in $(M_0, \overline{\xi}_n)$ by $\overline{S_+^{(n-1+i)/2} S_-^{(n-1-i)/2}}(F)$. Since changing the orientation of the planes changes positive stabilisations into negative ones and vice versa, $\overline{S_+^{(n-1+i)/2} S_-^{(n-1-i)/2}(F)}$ is Legendrian isotopic to $S_+^{(n-1-i)/2} S_-^{(n-1+i)/2}(F)$, therefore inverting the orientation of the planes transforms Legendrian surgery on $S_+^{(n-1+i)/2} S_-^{(n-1-i)}(F)$ into Legendrian surgery on $S_+^{(n-1-i)/2}\ S_-^{(n-1+i)/2}(F)$. □

Theorem 3.9 *The contact structures η_i on $-\Sigma(2, 3, 6n + 5)$, with $i \in \mathcal{P}_n$, are all pairwise non isotopic.*

Proof By [13], Theorem 4.2, and [14], Corollary 4.2, the contact structures η_i with $i \in \mathcal{P}_n^*$ are pairwise non isotopic. In particular, since we are considering n even, η_i

is never isotopic to η_{-i} if $i \in \mathcal{P}_n^*$ because $0 \notin \mathcal{P}_n^*$. Suppose by contradiction that η_0 is isotopic to η_i for some $i \in \mathcal{P}_n^*$. Inverting the orientation of the contact planes and applying Proposition 3.8, we obtain that η_0 is also isotopic to η_{-i}. From this it would follow that η_i is isotopic to η_{-i}. □

Remark 3.10 Using methods from [7] one can prove that $-\Sigma(2, 3, 17)$ admits at most three tight contact structures up to isotopy, therefore Proposition 3.8 gives the classification of the tight contact structures on $-\Sigma(2, 3, 17)$.

3.3 Computation of the homotopy invariants

In this subsection we will compute the Gompf's three-dimensional homotopy invariant $d_3(\eta_i)$. This computation will show that all η_i are homotopic and therefore all their Ozsváth–Szabó contact invariants belong to the same factor of $\widehat{HF}(-M)$.

By [10], Theorem 4.5 (for an easy proof of this theorem for integer homology spheres see also [13], Proposition 2.2), η_{i_1} is homotopic to η_{i_2} as a plane field if and only if $d_3(\eta_{i_1}) = d_3(\eta_{i_2})$, where

$$d_3(\eta_i) = \frac{1}{4}(c_1^2(J_i) - 2\chi(X_i) - 3\sigma(X_i))$$

and (X_i, J_i) is an almost complex manifold such that $\partial X_i = M$ and $\eta_i = TM \cap J(TM)$.

As almost complex manifold for the computation of $d_3(\eta_i)$ we will take symplectic fillings of (M, η_i) endowed with an adapted almost complex structure. More precisely, let (X_0, ω) be the weak symplectic filling of (M_0, ξ_n) for any $n \in \mathbb{N}$ constructed in [1] Proposition 15. If $T \subset M_0$ is a fibre of the torus bundle $M_0 \to S^1$, then we can assume that $\int_T \omega = 1$. In the setting of symplectic fillings Legendrian surgery corresponds to adding symplectic 2–handles, so adding symplectic 2-handles to (X_0, ω) as explained in the definition of (M, η_i), we obtain symplectic manifolds (X, ω_i) which fill (M, η_i) for $i \in \mathcal{P}_n$. We choose almost complex structures J_i adapted to ω_i so that the contact structure η_i is J_i–invariant for any $i \in \mathcal{P}_n$, all J_i coincide on X_0 and the fibre T in $M_0 = \partial X_0$ are quasi-complex submanifolds.

In M_0, the homology class represented by F is Poincaré dual of $[\omega_0|_{M_0}]$, because $F \cdot T = 1 = \int_T \omega_0$ and $[T]$ generates $H^2(M_0)$, therefore F bounds a surface $\Sigma \subset X_0$ which represents the Poincaré dual of $[\omega]$. Applying the homology long exact sequence to the pair (X, X_0) we obtain $H_2(X) = H_2(X_0) \oplus \mathbb{Z}[\overline{\Sigma}]$, where $\overline{\Sigma} \subset X$ is the surface obtained by capping Σ with the core of the 2-handle attached along F^3. Analogously, the cohomology exact sequence yields $H^2(X) \cong H^2(X_0) \oplus \mathbb{Z}$, where the isomorphism is given by $\alpha \mapsto (\iota^*\alpha, \langle \alpha, [\overline{\Sigma}] \rangle)$.

Lemma 3.11 *Let $\alpha \in H^2(X)$ be the 2-dimensional cohomology class determined by $\iota^*(\alpha) = 0$ and $\langle \alpha, [\overline{\Sigma}] \rangle = 1$. Then, up to torsion, α is the Poincaré dual of $[T] \in H_2(X) \cong H_2(X, \partial X)$.*

Proof Any 2-dimensional homology class can be represented as a closed, oriented embedded surface. Let K be a surface representing a homology class in $H_2(X_0)$, then $K \cdot T = 0$ because K can be made disjoint from $\partial X_0 = M_0$ and $\langle \alpha, [K] \rangle = \langle \iota^*\alpha, [K] \rangle = 0$. On the other hand, $\overline{\Sigma} \cdot T = F \cdot T = 1 = \langle \alpha, [\overline{\Sigma}] \rangle$. □

Theorem 3.12 *The contact structures η_i with $i \in \mathcal{P}_n$ are pairwise homotopic and $d_3(\eta_i) = -\frac{3}{2}$.*

Proof To prove that the contact structures are homotopic we will show that they have the same three dimensional invariant d_3. Since in the computation of $d_3(\eta_i)$ we use the almost complex manifolds (X, J_i) which are smoothly diffeomorphic, it is enough to prove that $c_1^2(J_i)$ does not depend on i. Given $i_1, i_2 \in \mathcal{P}_n$ we can decompose

$$c_1^2(J_{i_1}) - c_1^2(J_{i_2}) = \langle (c_1(J_{i_1}) + c_1(J_{i_2})), PD(c_1(J_{i_1}) - c_1(J_{i_2}))\rangle.$$

By the functoriality of the Chern classes for any $i \in \mathcal{P}_n$ we have $\iota^*(c_1(J_i)) = c_1(J_i|_{X_0})$, then $\iota^*(c_1(J_{i_1}) - c_1(J_{i_2})) = 0$, because all J_i agree on X_0. Lemma 3.11 implies that $PD(c_1(J_{i_1}) - c_1(J_{i_2}))$ is a multiple of $[T]$. Since T is a complex submanifold of (X, J_i), the adjunction equality gives $\langle c_1(J_i), [T]\rangle = \chi(T) + T \cdot T = 0$, then $c_1^2(J_{i_1}) - c_1^2(J_{i_2}) = 0$.

$d_3(\eta_i)$ can be computed for any of the Stein fillable contact structures η_i with $i \in \mathcal{P}_n^*$ using the Stein filling (W, J_i) described in [13], Figure 2. One can immediately check that $c_1^2(J_i) = 0$, $\chi(W) = 3$ and $\sigma(W) = 0$. □

We stress the point that the Stein manifolds (W, J_i) used to compute $d_3(\eta_i)$ are different from the almost complex manifolds (X, J_i) used in the first part of Theorem 3.12 to show that all η_i are homotopic.

3.4 Computation of the Ozsváth-Szabó invariants

In [22], Section 8, $HF^+(\Sigma(2, 3, 6n + 5))$ is computed. Applying the long exact sequence relating HF^+ and $\widehat{HF}$ and the isomorphism between $\widehat{HF}_d(Y)$ and $\widehat{HF}_{-d}(-Y)$ it is easy to show that $\widehat{HF}(-\Sigma(2, 3, 6n + 5)) = (\mathbb{Z}/2\mathbb{Z})^{n+1}_{(+2)} \oplus (\mathbb{Z}/2\mathbb{Z})^2_{(+1)}$. The degree of $c(\xi)$ is $+1$ because $d_3(\eta_i) = -\frac{3}{2}$. By [25], Section 4 $\widehat{HF}_{(+1)}(-\Sigma(2, 3, 6n + 5))$ is freely generated by the elements $c(\eta_i)$ for $i \in \mathcal{P}_n^*$.

Proof of Theorem 1.1. The fix space $Fix(\mathfrak{J}) \subset \widehat{HF}_{(+1)}(-\Sigma(2, 3, 6n+5))$ is generated by elements of the form $c(\eta_i) + c(\eta_{-i})$ for $i \in \mathcal{P}_n^*$. Let W be the smooth cobordism between M_0 and $-\Sigma(2, 3, 6n + 5)$ constructed by attaching a 2-handle to M_0 along F, then by [18], Theorem 4.2

$$\widehat{F}_{\overline{W}}(c(\eta_i) + c(\eta_{-i})) = \widehat{F}_{\overline{W}}(c(\eta_i)) + \widehat{F}_{\overline{W}}(c(\eta_{-i})) = 2c(\xi_1) = 0.$$

Consequently $Fix(\mathfrak{J}) \subset \ker \widehat{F}_{\overline{W}}$, in particular

$$c(\xi_n) = \widehat{F}_{\overline{W}}(\eta_0) = 0$$

because $c(\eta_0) \in Fix(\mathfrak{J})$ by Proposition 3.8 and Theorem 2.10. □

In view of Theorem 2.13 we have the following corollary.

Corollary 3.13 *The contact manifolds (M_0, ξ_n) are not strongly symplectically fillable if n is even.*

This is a new non fillability result, because the integer n_0 in Theorem 3.3 is not given explicitly.

4 A remark on integer coefficients

Unfortunately Theorem 1.1 does not imply that the Ozsváth–Szabó contact invariants $c(\xi_n)$ for n even with untwisted integer coefficients are zero, but only that they are the double of some elements of $HF^+(-M_0)/\pm 1$. Fix an open book decomposition of M_0 adapted to ξ_n for an even n. We denote by M_0' the 3–manifold obtained by 0–surgery on the binding and by M_0'' the 3–manifold obtained by 1–surgery on the binding. Of course the manifolds M_0' and M_0'' depend on n. By [19], Theorem 9.1 there is a surgery exact triangle

$$\begin{tikzcd}
HF^+(-M_0') \arrow[rr, "F^+"] & & HF^+(-M_0) \arrow[dl] \\
& HF^+(-M_0'') \arrow[ul] &
\end{tikzcd}$$

The group $HF^+(-M_0')$ is generated by c_0^+, therefore if $F^+(c_0^+) = c^+(\xi_2) \neq 0$, the exact triangle becomes a short exact sequence

$$0 \to HF^+(-M_0') \to HF^+(-M_0) \to HF^+(-M_0'') \to 0$$

If $c^+(\xi_n)$ is non primitive there are torsion elements in $HF^+(-M_0'')$. Since all Heegaard–Floer homology groups known so far are free, it is reasonable to expect that $c^+(\xi_n) = 0$ also in the Heegaard–Floer homology group with integer coefficients.

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