Math. Z. (2006) 253: 97–134
DOI 10.1007/s00209-005-0889-3

Mathematische Zeitschrift

Marco Schlichting

Negative K-theory of derived categories

Received: 1 March 2004 / Accepted: 12 May 2005 / Published online: 16 February 2006

Abstract We define negative K-groups for exact categories and for "derived categories" in the framework of Frobenius pairs, generalizing definitions of Bass, Karoubi, Carter, Pedersen-Weibel and Thomason. We prove localization and vanishing theorems for these groups. *Dévissage* (for noetherian abelian categories), additivity, and resolution hold. We show that the first negative K-group of an abelian category vanishes, and that, in general, negative K-groups of a noetherian abelian category vanish. Our methods yield an explicit non-connective delooping of the K-theory of exact categories and chain complexes, generalizing constructions of Wagoner and Pedersen-Weibel. Extending a theorem of Auslander and Sherman, we discuss the K-theory homotopy fiber of $\mathcal{E}^{\oplus} \to \mathcal{E}$ and its implications for negative K-groups. In the appendix, we replace Waldhausen's cylinder functor by a slightly weaker form of non-functorial factorization which is still sufficient to prove his approximation and fibration theorems.

Mathematics Subject Classification (2000) 18E30·19D35·55P47

Contents

M. Schlichting
Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA
E-mail: mschlich@math.lsu.edu

1 Introduction

In his lecture at the International Congress in Vancouver 1974, Quillen [Qui75, 7] writes: "At the moment a theory of negative K-groups for exact categories has not been developed." It seems that, up to now, such a theory has not been developed. The purpose of this article is to define negative K-groups for exact categories (or more generally for "derived categories") and to prove localization and vanishing theorems for these groups. Our negative K-groups generalize definitions of Bass, Karoubi, Pedersen-Weibel, Thomason, Carter and Yao.

To motivate the need of a theory of negative K-groups, recall that if X is a scheme, $U \subset X$ an open subscheme with complement Z, then the restriction of vector bundles from X to U induces a map of Grothendieck groups $K_0(X) \to K_0(U)$ which is not surjective, in general. For this reason, Thomason had to introduce negative K-groups for schemes in order to prove his homotopy fibration of K-theory spectra [TT90]

$$K_Z^B(X) \to K^B(X) \to K^B(U). \tag{1}$$

He did so by mimicking Bass' definition.

It is known that negative K-groups vanish for a noetherian regular separated scheme. However, only a few calculations are known in the non-regular case. A conjecture of Weibel [Wei80] states that $K_i(R) = 0$ for $i < -d$ and R a noetherian ring of Krull dimension d. It has been verified for $d = 0, 1, 2$ [Wei01]. Hsiang conjectured that $K_i(\mathbb{Z}G) = 0$ for $i < -1$ and G a finitely presented group [Hsi84]. For recent progress on this conjecture, we refer the reader to [BFJR04].

Let $\mathcal{E}$ be an exact category and $D^b(\mathcal{E})$ its bounded derived category. It is known that $K_0(\mathcal{E}) = K_0(D^b(\mathcal{E}))$. So instead of defining negative K-groups for exact categories, we attempt to define negative K-groups for triangulated categories. We think of negative K-groups for triangulated categories as obstruction groups in the following sense. For an idempotent complete triangulated category $\mathcal{A}$, the vanishing of $K_{-1}(\mathcal{A})$ is exactly the obstruction for the Verdier quotient $\mathcal{B}/\mathcal{A}$ to be idempotent complete for all full triangle embeddings $\mathcal{A} \subset \mathcal{B}$ with $\mathcal{B}$ idempotent complete (Remark 1).

Unfortunately, we can only define negative K-groups for triangulated categories "which admit models". It is known that models are essential in the definition of higher algebraic K-theory [Sch02]. So working with models should not be too inconvenient.

In section 2 we explain the framework for defining negative K-groups (section 2.2). We need a category of "models" $\mathfrak{M}$ together with some extra data including a functor D from $\mathfrak{M}$ to small triangulated categories. Here the word "models" does not refer to "Quillen model categories". The category of models could be the category of non-unital rings, the category of small exact categories, the category of Frobenius pairs (see below), etc.

We define negative K-groups for objects in $\mathfrak{M}$ (definition 2). In order to describe our Localization Theorem (theorem 1) we define a sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathfrak{M}$ to be exact if in the sequence $D\mathcal{A} \to D\mathcal{B} \to D\mathcal{C}$ of triangulated categories the composition is trivial, the first functor is fully faithful, and the induced functor from the Verdier quotient $D\mathcal{B}/D\mathcal{A}$ to $D\mathcal{C}$ is an equivalence (up to direct factors). Then the Localization Theorem (theorem 1) states that an exact sequence in $\mathfrak{M}$ induces a long exact sequence of negative K-groups for $i \leq 0$,

$$\cdots \mathbb{K}_i(\mathcal{A}) \to \mathbb{K}_i(\mathcal{B}) \to \mathbb{K}_i(\mathcal{C}) \stackrel{\delta}{\to} \mathbb{K}_{i-1}(\mathcal{A}) \to \mathbb{K}_{i-1}(\mathcal{B}) \to \cdots \tag{2}$$

The proof is a simple diagram chase. We chose an axiomatic approach to defining negative K-groups to possibly allow non-linear models, which, however, are not dealt with in this article.

Section 3 gives the background on triangulated categories which we need in section 5. The results in this section are not new, and all ideas are contained in [Nee92] and [Nee01]. Nevertheless, we give complete proofs because we don't know of any reference which covers exactly our situation.

In section 4 and 5 we show that we can take as a category of models $\mathfrak{M}$ the category of Frobenius pairs (theorem 3). Recall that a Frobenius category is an exact category with enough injectives and projectives, and where injectives and projectives coincide. Its stable category is a triangulated category. A Frobenius pair is a Frobenius category together with a full Frobenius subcategory whose projective-injective objects are also projective-injective in the ambient Frobenius category (definition 5). We define the derived category of a Frobenius pair to be the Verdier quotient of the associated stable categories. Any small triangulated category we know of which comes from algebraic geometry, or some other additive situation, arises as the derived category of a Frobenius pair. For example (see section 6), any small triangulated subcategory of the derived category of a Grothendieck abelian category, or of the derived category of modules over a differential graded algebra, or of the derived category of an exact category, or of the derived category of Thomason's complicial BiWaldhausen categories [TT90], arises as the derived category of a Frobenius pair. In particular, our definition yields a definition of negative $\mathbb{K}$-groups and a localization exact sequence in these situations.

Section 7 proves additivity for $\mathbb{K}_i$ and the fact that the functors $\mathbb{K}_i$ commute with filtered colimits.

The results of sections 8, 9, 10 and 11 deal with exact categories and are described at the end of the introduction.

Frobenius pairs can be seen as Waldhausen categories when we declare cofibrations to be inflations (*i.e.,* admissible monomorphisms) and weak equivalences to be maps which yield an isomorphism in the derived category. In this way, one can attach a K-theory space to a Frobenius pair. In section 12, we provide an explicit delooping of this K-theory space which defines a spectrum $\mathbb{K}$ whose negative homotopy groups are the groups $\mathbb{K}_i$, $i \leq 0$, defined in section 2 (theorem 8). This delooping construction generalizes constructions of Wagoner [Wag72], Pedersen-Weibel [PW89] and the author [Sch04]. An exact sequence of Frobenius pairs induces a homotopy fibration of non-connective $\mathbb{K}$-theory spectra (theorem 9). The long exact sequence (2) becomes the negative part of the long exact sequence of homotopy groups of spectra associated to this homotopy fibration. The long exact sequence (2) is known for $i \geq 0$ by the work of Thomason [TT90], at least if the Frobenius pair has a cylinder functor. The point of this article is that the sequence is also exact in negative degrees. We note that (1) is the homotopy fibration associated to an exact sequence of Frobenius pairs. The homotopy fibration (theorem 9) is used in the author's work with Hornbostel on hermitian K-theory of rings [HS04].

Since the Waldhausen categories attached to a Frobenius pair don't, in general, have cylinder functors, we give a proof of Waldhausen's fibration and approximation theorems [Wal85] and of Thomason's cofinality theorem [TT90] where we replace the existence of a cylinder functor by the existence of a factorization of every map into a cofibration followed by a weak equivalence. The factorization is not required to be functorial. This is done in the appendix.

For an exact category $\mathcal{E}$, the results of the article read as follows. There are groups $\mathbb{K}_i(\mathcal{E})$, $i \leq 0$ such that $\mathbb{K}_0$ is K_0 of the idempotent completion of $\mathcal{E}$ (remark 3). The negative K-groups of $\mathcal{E}$ are Bass' groups $K_i(R)$ when $\mathcal{E}$ is the category of finitely generated projective R-modules (theorem 5). They are Thomason's groups $K_i^B(X)$ when $\mathcal{E}$ is the category of vector bundles of finite rank on a quasi-compact, quasi-separated scheme X which admits an ample family of line bundles (remark 6). Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence of exact functors between exact categories such that $D^b(\mathcal{A}) \to D^b(\mathcal{B}) \to D^b(\mathcal{C})$ is an "exact sequence of triangulated categories". Then there is an exact sequence as in (2), see remark 3. In particular, if $\mathcal{A} \to \mathcal{B}$ is such that $D^b(\mathcal{A}) \to D^b(\mathcal{B})$ is an equivalence, *e.g.* $\mathcal{A} \subset \mathcal{B}$ satisfies resolution, then $\mathbb{K}_i(\mathcal{A}) \to \mathbb{K}_i(\mathcal{B})$ is an isomorphism for $i \leq 0$.

The groups $\mathbb{K}_i(\mathcal{E})$ are the negative homotopy groups of a spectrum $\mathbb{K}(\mathcal{E})$ whose positive homotopy groups are the Quillen K-groups of $\mathcal{E}$ (section 12.2). The sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ above induces a homotopy fibration of $\mathbb{K}$-theory spectra yielding a long exact sequence (2) for $i \in \mathbb{Z}$.

In section 9 we give a presentation of $\mathbb{K}_{-1}(\mathcal{E})$. Let $D(\mathcal{E})$ be the unbounded derived category of $\mathcal{E}$ as defined in [Nee90]. Then the group $\mathbb{K}_{-1}(\mathcal{E})$ is the quotient of the abelian monoid of isomorphism classes of idempotents of $D(\mathcal{E})$ under direct sum operation, modulo the submonoid of those idempotents which split in $D(\mathcal{E})$ (corollary 6).

We conjecture that negative $\mathbb{K}$-groups vanish for $\mathcal{A}$ a small abelian category. As evidence, we show that $\mathbb{K}_{-1}(\mathcal{A}) = 0$ for any small abelian category $\mathcal{A}$ (theorem 6). We also show that $\mathbb{K}_i(\mathcal{A}) = 0$, $i < 0$, for any small noetherian abelian category (theorem 7). In particular, negative G-theory for a noetherian scheme is trivial. This also explains the fact that negative K-groups of noetherian regular separated schemes vanish, as the inclusion of vector bundles of finite rank into coherent modules is a derived equivalence. Moreover, *devissage* trivially holds for negative K-groups of noetherian abelian categories as these groups vanish.

If the conjecture is true, then we have the following rather surprising consequence. Write $\mathcal{E}^{\oplus}$ for the split exact category, which, as an additive category, is $\mathcal{E}$. We show that the identity functor $\mathcal{E}^{\oplus} \to \mathcal{E}$ induces a map $\mathbb{K}(\mathcal{E}^{\oplus}) \to \mathbb{K}(\mathcal{E})$ of spectra whose homotopy fiber is the $\mathbb{K}$-theory spectrum of an abelian category (proposition 2, theorem 9). This generalizes a theorem of Auslander and Sherman [She89]. If the conjecture is true, then the long exact sequence (2) implies isomorphisms $\mathbb{K}_i(\mathcal{E}^{\oplus}) \to \mathbb{K}_i(\mathcal{E})$, $i < 0$.

2 Negative K-theory of derived categories

In this section we define negative K-theory of "triangulated categories which admit models". Let $\mathfrak{T}$ be the category of small triangulated categories with triangle functors as morphisms. We refer the reader to [Kel96], [Nee01] for the definition and basic properties of triangulated categories.

Definition 1 Call a sequence of small triangulated categories $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ *exact* if the composition is zero, the map $\mathcal{A} \to \mathcal{B}$ is fully faithful, and the induced map from the Verdier quotient [Ver96, II 2] $\mathcal{B}/\mathcal{A}$ to $\mathcal{C}$ is *cofinal*, *i.e.,* it is fully faithful, and every object of $\mathcal{C}$ is a direct summand of an object of $\mathcal{B}/\mathcal{A}$.

Recall that if $\mathcal{A}$ is idempotent complete, then $\mathcal{A}$ is *épaisse*,*i.e.,* $\mathcal{A}$ is equivalent to the full subcategory of $\mathcal{B}$ of objects sent to zero in $\mathcal{C}$ [Nee01, 2.1.10].

2.1 Some facts

We need three elementary facts about triangulated categories. The first is that the idempotent completion of a triangulated category is again a triangulated category (for a proof, see [BS01]). Write $\tilde{\mathcal{T}}$ for the idempotent completion of $\mathcal{T}$ equipped with its canonical structure of a triangulated category. We define $I\!K_0(\mathcal{T}) = K_0(\tilde{\mathcal{T}})$. The second fact we need is that for a small triangulated category $\mathcal{T}$, the functor K_0 sets up a one-to-one correspondence between the set of dense triangulated subcategories of $\mathcal{T}$ and the set of subgroups of $K_0(\mathcal{T})$ (see [Tho97]). Recall that a subcategory of $\mathcal{T}$ is dense if it is cofinal and closed under isomorphisms in $\mathcal{T}$. The third fact is that an exact sequence of triangulated categories $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ induces an exact sequence $I\!K_0\mathcal{A} \to I\!K_0(\mathcal{B}) \to I\!K_0(\mathcal{C})$ of abelian groups. This follows from [SGA5, VIII 3.1] and the injection $K_0(\mathcal{T}) \to K_0(\tilde{\mathcal{T}})$ (fact 2 above). In general, the first map of the exact sequence is not injective, nor is the second map surjective. The first part of the paper addresses the question of how to extend the sequence to the right. Given the three facts above, the rest of the definition of the functors $I\!K_i$, $i < 0$, and the extension of the exact sequence to the right are elementary.

2.2 Set-up

The set-up for this section is a category $\mathfrak{M}$ and a functor $D : \mathfrak{M} \to \mathfrak{T}$. Call a sequence in $\mathfrak{M}$ exact if it becomes exact after applying D. Moreover, write $I\!K_0(\mathcal{M}) = I\!K_0(D\mathcal{M})$ for $\mathcal{M}$ an object of $\mathfrak{M}$. We suppose that there are two endofunctors $\mathcal{F}, \mathcal{S} : \mathfrak{M} \to \mathfrak{M}$ and natural transformations $id \to \mathcal{F} \to \mathcal{S}$ such that

1. $\mathcal{F}, \mathcal{S}$ preserve exact sequences in $\mathfrak{M}$,
2. $I\!K_0(\mathcal{F}\mathcal{M}) = 0$ for every object $\mathcal{M}$ of $\mathfrak{M}$ and
3. $\mathcal{M} \to \mathcal{F}\mathcal{M} \to \mathcal{S}\mathcal{M}$ is exact for any object $\mathcal{M}$ of $\mathfrak{M}$.

The functor $\mathcal{S}$ is called suspension, and $\mathcal{F}$ stands for "flasque".

Ideally, we would like to take $\mathfrak{M} = \mathfrak{T}$ and $D = id$, but we don't know how to construct the functors $\mathcal{F}$ and $\mathcal{S}$ in this case. Instead, we will see in section 5 that we can take $\mathfrak{M}$ to be the category of Frobenius pairs and D to be the functor that associates to a Frobenius pair its derived category. Then the functors $\mathcal{F}$ and $\mathcal{S}$ of the set-up do exist. Moreover, there are functors from the category of rings (schemes, small exact categories, complicial BiWaldhausen categories [TT90]) to the category of Frobenius pairs such that D corresponds to the bounded derived category of finitely generated projective R-modules for a ring R (bounded derived category of locally free sheaves of O_X-modules for X a scheme, the bounded derived category of an exact category, the derived category of a complicial BiWaldhausen category, respectively). This is made precise in section 6. The following definition will yield a definition of lower algebraic K-groups in these cases.

Definition 2 Let $\mathcal{M}$ be an object of $\mathfrak{M}$. We define its negative $I\!\!K$-groups by

$$I\!\!K_{-n}(\mathcal{M}) = I\!\!K_0(\mathcal{S}^n\mathcal{M}), \quad n \in \mathbb{N}.$$

2.3 The connecting homomorphism

Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be an exact sequence in $\mathfrak{M}$. Idempotent completion in $\mathfrak{T}$ preserves exact sequences of triangulated categories. It follows from the assumptions of the set-up (section 2.2) that there is a natural diagram of idempotent complete triangulated categories

$$\begin{tikzcd}
\widetilde{D\mathcal{A}} \arrow[r, "i"] \arrow[d, "i_\mathcal{A}"'] & \widetilde{D\mathcal{B}} \arrow[r, "p"] \arrow[d, "i_\mathcal{B}"] & \widetilde{D\mathcal{C}} \arrow[d, "i_\mathcal{C}"] \\
D\mathcal{F}\mathcal{A} \arrow[r, "i_\mathcal{F}"] \arrow[d, "p_\mathcal{A}"'] & D\mathcal{F}\mathcal{B} \arrow[r, "p_\mathcal{F}"] \arrow[d, "p_\mathcal{B}"] & D\mathcal{F}\mathcal{C} \arrow[d, "p_\mathcal{C}"] \\
\widetilde{D\mathcal{S}\mathcal{A}} \arrow[r, "i_\mathcal{S}"'] & \widetilde{D\mathcal{S}\mathcal{B}} \arrow[r, "p_\mathcal{S}"'] & \widetilde{D\mathcal{S}\mathcal{C}}
\end{tikzcd}$$

with exact rows and columns. The categories $D\mathcal{F}\mathcal{M}$ are idempotent complete for $\mathcal{M}$ in $\mathfrak{M}$ because the map $D\mathcal{F}\mathcal{M} \to \widetilde{D\mathcal{F}\mathcal{M}}$ is an equivalence of categories. This follows from fact 2 in section 2.1 and $K_0(\widetilde{D\mathcal{F}\mathcal{M}}) = 0$. To simplify notation we assume that a subcategory is closed under isomorphisms in its ambient category.

We define a map $\delta : I\!\!K_0(\mathcal{C}) \to I\!\!K_{-1}(\mathcal{A}) = I\!\!K_0(\mathcal{S}\mathcal{A})$ as follows. First, define $\delta : \mathrm{Ob}\widetilde{D\mathcal{C}} \to I\!\!K_0(\mathcal{S}\mathcal{A})$. Let c be an object of $\widetilde{D\mathcal{C}}$. Since $p_\mathcal{F}$ is a localization (essential surjectivity on objects follows from $I\!\!K_0\mathcal{F} = 0$ and section 2.1), there is an object B of $D\mathcal{F}\mathcal{B}$ such that $p_\mathcal{F}(B) \cong i_\mathcal{C}(c)$. Calculating $p_\mathcal{S} p_\mathcal{B}(B) \cong p_\mathcal{C}(c) = 0$, we see that $p_\mathcal{B}(B) \in \widetilde{D\mathcal{S}\mathcal{A}}$, and we set $\delta(c) = [p_\mathcal{B}(B)] \in I\!\!K_0(\mathcal{S}\mathcal{A})$.

Let B' be another object of $D\mathcal{F}\mathcal{B}$ such that $p_\mathcal{F}(B') \cong i_\mathcal{C}(c)$. Then there are maps $\beta : B \to B''$, $\beta' : B' \to B''$ with cones A, A' in $D\mathcal{F}\mathcal{A}$ whose classes $[A]$, $[A']$ in $I\!\!K_0(\mathcal{F}\mathcal{A})(= 0$, see section 2.2), and hence in $I\!\!K_0(\mathcal{S}\mathcal{A})$, are trivial. It follows that $[p_\mathcal{B}(B)] = [p_\mathcal{B}(B'')] - [p_\mathcal{A}(A)] = [p_\mathcal{B}(B'')] = [p_\mathcal{B}(B'')] - [p_\mathcal{A}(A')] = [p_\mathcal{B}(B')]$ in $I\!\!K_0(\mathcal{S}\mathcal{A})$. Hence δ yields a well defined map from objects of $\widetilde{D\mathcal{C}}$ to

$I\!K_0(\mathcal{SA})$. The argument also shows that δ is well-defined on isomorphism classes of objects of $\widetilde{D\mathcal{C}}$. Given a distinguished triangle $c \to c' \to c'' \to \Sigma c$ in $\widetilde{D\mathcal{C}}$, its image under $i_{\mathcal{C}}$ is isomorphic to the image under $p_{\mathcal{F}}$ of a distinguished triangle $B \to B' \to B'' \to \Sigma B$ in $D\mathcal{FB}$. Its image under $p_{\mathcal{B}}$ yields an exact triangle in $\widetilde{D\mathcal{SA}}$ giving rise to the relation $\delta(c) - \delta(c') + \delta(c'')$ in $I\!K_0(\mathcal{SA})$. Applying the above argument to the i-th suspension, we have the following.

Lemma 1 *The map δ described in section 2.3 yields a well defined map $\delta : I\!K_i(\mathcal{C}) \to I\!K_{i-1}(\mathcal{A})$ of abelian groups, $i \leq 0$.*

Theorem 1 *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a short exact sequence in $\mathfrak{M}$. Then the sequence of abelian groups*

$$\begin{aligned} \cdots \to I\!K_i(\mathcal{A}) \to I\!K_i(\mathcal{B}) \to I\!K_i(\mathcal{C}) \\ \xrightarrow{\delta} I\!K_{i-1}(\mathcal{A}) \to I\!K_{i-1}(\mathcal{B}) \to I\!K_{i-1}(\mathcal{C}) \to \cdots \end{aligned}$$

is exact, $i \leq 0$.

Proof We only need to show the case $i = 0$. The general case follows by applying the case $i = 0$ to the i-th suspension of the given exact sequence in $\mathfrak{M}$. Exactness at $I\!K_0(\mathcal{B})$ is classical (section 2.1).

We show exactness at $I\!K_0(\mathcal{C})$. The composition $I\!K_0(\mathcal{B}) \to I\!K_0(\mathcal{C}) \to I\!K_0(\mathcal{SA})$ is zero because $p_{\mathcal{B}} i_{\mathcal{B}}(b) = 0$ for $b \in \widetilde{D\mathcal{B}}$. Let $c \in \widetilde{D\mathcal{C}}$ with $\delta(c) = [p_{\mathcal{B}}(B)] = 0$ in $I\!K_0(\mathcal{SA})$ for some object B of $D\mathcal{FB}$ with $p_{\mathcal{F}}(B) \cong i_{\mathcal{C}}(c)$. Since the image of $p_{\mathcal{A}}$ is a triangulated category equivalent to the full triangulated subcategory of $\widetilde{D\mathcal{SA}}$ on objects with trivial class in $I\!K_0(\mathcal{SA})$ (section 2.1), there is an object A of $D\mathcal{FA}$ such that $p_{\mathcal{A}}(A) \cong p_{\mathcal{B}}(B)$. Then there are $D\mathcal{FB}$-maps $B \to B'$, $A \to B'$ with cones b, b' in $\widetilde{D\mathcal{B}}$. It follows that $p_{\mathcal{F}}B' \cong p_{\mathcal{F}}(b')$, and so there is a distinguished triangle $c \to p(b') \to p(b) \to \Sigma c$ in $\widetilde{D\mathcal{C}}$. Hence $[c] = p([b'] - [b])$ in $I\!K_0(\mathcal{C})$, so $[c]$ is in the image of $I\!K_0(\mathcal{B}) \to I\!K_0(\mathcal{C})$.

We now show exactness at $I\!K_{-1}(\mathcal{A}) = I\!K_0(\mathcal{SA})$. The composition $I\!K_0(\mathcal{C}) \to I\!K_0(\mathcal{SA}) \to I\!K_0(\mathcal{SB})$ is zero, since it factors through $I\!K_0(\mathcal{FB})$, and $I\!K_0(\mathcal{FB}) = 0$. Let A be an object of $\widetilde{D\mathcal{SA}}$ whose class in $I\!K_0(\mathcal{SB})$ is trivial. It follows that there is an object B in $D\mathcal{FB}$ such that $p_{\mathcal{B}}(B) \cong A$. As we have $p_{\mathcal{C}} p_{\mathcal{F}}(B) \cong p_{\mathcal{S}}(A) = 0$, the object $p_{\mathcal{F}}(B)$ lies in $\widetilde{D\mathcal{C}}$, and by definition we have $\delta(p_{\mathcal{F}}(B)) = [A]$. □

Corollary 1 *Let $f : A \to B$ be a map in $\mathfrak{M}$ such that $D(f)$ is cofinal, e.g. an equivalence of categories. Then $I\!K_i(f) : I\!K_i(A) \to I\!K_i(B)$ is an isomorphism for $i \leq 0$.*

Proof This follows from theorem 1 applied to the exact sequence $0 \to A \to B$. □

Remark 1 ($I\!K_{-1}$ as obstruction group) Let M be an object of $\mathfrak{M}$. Then $I\!K_{-1}(M) = 0$ if and only if for all exact sequences $M \to N \to P$ in $\mathfrak{M}$ the Verdier quotient $\widetilde{DN}/\widetilde{DM}$ is idempotent complete.

If $I\!K_{-1}(M) = 0$, then $I\!K_0(N) \to I\!K_0(P)$ is surjective which implies that $\widetilde{DN}/\widetilde{DM} \to \widetilde{DP}$ is an equivalence by Thomason's classification of dense subcategories. Thus $\widetilde{DN}/\widetilde{DM}$ is idempotent complete. For the other direction, we have in particular $DSM = \widetilde{D\mathcal{F}M}/\widetilde{DM}$ idempotent complete. Thus $0 = I\!K_0(\mathcal{F}M) \to I\!K_0(\mathcal{S}M) = I\!K_{-1}(M)$ is surjective.

3 C-compactly generated triangulated categories

Definition 3 (compare [Nee92]) Let $\mathcal{T}$ be a triangulated category with countable coproducts. We call an object X of $\mathcal{T}$ *c-compact* (short for "countably compact") if the functor $\hom(X,\quad) : \mathcal{T} \to \mathbb{Z}{-}\mathrm{Mod}$ commutes with countable coproducts. We denote by $\mathcal{T}^c$ the full subcategory of c-compact objects.

Clearly, $\mathcal{T}^c$ it is a triangulated category. It is idempotent complete because a direct factor of a c-compact object is c-compact and $\mathcal{T}$ is idempotent complete as it has countable coproducts [BN93, 3.2].

3.1 Homotopy colimits

Recall from [Nee92] the definition of homotopy colimits in triangulated categories. Let $\mathcal{T}$ be a triangulated category with countable coproducts. Given a sequence $A_0 \stackrel{f_0}{\to} A_1 \stackrel{f_1}{\to} \cdots$ of objects and maps in $\mathcal{T}$ indexed over the positive integers, the *homotopy colimit* of this sequence is by definition the third object in the distinguished triangle

$$\bigoplus_i A_i \xrightarrow{1-\mathrm{shift}} \bigoplus_i A_i \longrightarrow \mathrm{hocolim}_i A_i \longrightarrow \bigoplus_i A_i[1] \tag{3}$$

where the map shift, restricted to A_i, is $A_i \stackrel{f_i}{\to} A_{i+1} \to \bigoplus_j A_j$. The functor $\hom_{\mathcal{T}}(A,\quad)$ applied to the distinguished triangle yields a long exact sequence of abelian groups. If A is a c-compact object, then this sequence decomposes into short exact sequences giving rise to the isomorphism

$$\mathrm{colim}_i \hom_{\mathcal{T}}(A, A_i) \xrightarrow{\sim} \hom_{\mathcal{T}}(A, \mathrm{hocolim}_i A_i). \tag{4}$$

Definition 4 A triangulated category $\mathcal{T}$ with countable coproducts is called *c-compactly generated* if there is a set S of c-compact objects such that every object in $\mathcal{T}$ is the homotopy colimit of a (countable) sequence of objects in $< S >$. Here $< S >$ denotes the smallest triangulated subcategory of $\mathcal{T}$ which is closed under isomorphisms in $\mathcal{T}$ and which contains S. We call S a generating set (of c-compact objects) for $\mathcal{T}$.

Note that the notion "c-compactly generated" is not the same as Neeman's notion "compactly generated" as Neeman requires all set indexed direct sums to exist whereas we only require countable direct sums to exist.

Lemma 2 *Let $\mathcal{T}$ be a c-compactly generated triangulated category with generating set S of c-compact objects, then the category $\mathcal{T}^c$ is the smallest idempotent complete triangulated subcategory of $\mathcal{T}$ which is closed under isomorphisms and which contains S.*

Proof We know that $\mathcal{T}^c$ is idempotent complete (definition 3). Let $X \in \mathcal{T}$ be c-compact. Write X as $\mathrm{hocolim}_i S_i$ with $S_i \in < S >$. It follows from the isomorphism (4) that the identity map on X factors through some S_i. Thus X is a direct factor of some S_i. □

Corollary 2 *Let $\mathcal{R} \to \mathcal{T}$ be a triangle functor between c-compactly generated triangulated categories which preserves countable coproducts and c-compact objects. Then $\mathcal{R} \to \mathcal{T}$ is fully faithful if and only if $\mathcal{R}^c \to \mathcal{T}^c$ is. It is an equivalence if and only if $\mathcal{R}^c \to \mathcal{T}^c$ is. It is trivial, i.e., every object is sent to the zero object, if and only if $\mathcal{R}^c \to \mathcal{T}^c$ is trivial.*

Proof The triangle functor $\mathcal{R} \to \mathcal{T}$ preserves countable coproducts, and hence homotopy colimits. Given two objects X, Y of $\mathcal{R}$, write them as homotopy colimits of c-compact objects $X = \operatorname{hocolim}_i A_i$, $Y = \operatorname{hocolim}_j B_j$. Applying $\hom(\ ,Y)$ to (3), and using (4), we get long exact sequences of homomorphism groups

$$\begin{array}{l} \cdots \longrightarrow \Pi_i \operatorname{colim}_j \hom(A_i[1], B_j) \longrightarrow \Pi_i \operatorname{colim}_j \hom(A_i[1], B_j) \longrightarrow \hom(X,Y) \\ \qquad \longrightarrow \Pi_i \operatorname{colim}_j \hom(A_i, B_j) \longrightarrow \Pi_i \operatorname{colim}_j \hom(A_i, B_j) \longrightarrow \cdots \end{array} \tag{5}$$

The functor $\mathcal{R} \to \mathcal{T}$ induces a map from the long exact sequence (5) for homomorphism groups in $\mathcal{R}$ to the long exact sequence (5) for homomorphism groups in $\mathcal{T}$. By the 5-lemma, if $\mathcal{R}^c \to \mathcal{T}^c$ is fully faithful, then so is $\mathcal{R} \to \mathcal{T}$.

If $\mathcal{R}^c \to \mathcal{T}^c$ is an equivalence, it is fully faithful, so by the above argument, $\mathcal{R} \to \mathcal{T}$ is fully faithful. It is essentially surjective because every object of $\mathcal{T}$ is a homotopy colimit of c-compact objects.

If $\mathcal{R}^c \to \mathcal{T}^c$ is trivial, it sends all maps to zero. Let $X = \operatorname{hocolim}_i A_i$ be an object of $\mathcal{R}$ written as a homotopy colimit of objects in $\mathcal{R}^c$. The exact sequence (5) for homomorphism groups in $\mathcal{T}$ and $Y = X$ shows that $\hom_{\mathcal{T}}(X,X) = 0$. Thus $X = 0$ in $\mathcal{T}$.

The other implications are trivial. □

The following is a variant, suitable for our applications, of a theorem of Neeman [Nee92].

Theorem 2 *Let $\mathcal{T}$ be a c-compactly generated triangulated category. Let $\mathcal{R} \subset \mathcal{T}$ be a c-compactly generated full triangulated subcategory closed under countable coproducts such that $\mathcal{R}^c \subset \mathcal{T}^c$. Then $\mathcal{T} \to \mathcal{T}/\mathcal{R}$ preserves countable coproducts, and $\mathcal{T}/\mathcal{R}$ is c-compactly generated by the image of $\mathcal{T}^c$ in $\mathcal{T}/\mathcal{R}$. Moreover,*

$$\mathcal{R}^c \to \mathcal{T}^c \to (\mathcal{T}/\mathcal{R})^c$$

is an exact sequence of triangulated categories.

Proof The functor $\mathcal{T} \to \mathcal{T}/\mathcal{R}$ preserves countable coproducts (and thus homotopy colimits) because $\mathcal{R} \subset \mathcal{T}$ is closed under countable coproducts. In detail, the set of maps in $\mathcal{T}$ with cone in $\mathcal{R}$ satisfies a calculus of fractions. This implies that for any map $T \to \bigoplus_i X_i$ in $\mathcal{T}$ with cone in $\mathcal{R}$, there are maps $t_i : T_i \to X_i$ with cone in $\mathcal{R}$ and maps $T_i \to T$ such that the composition $\bigoplus_i T \to T \to \bigoplus_i X_i$ equals $\bigoplus_i t_i$. Since $\mathcal{R}$ is closed under countable coproducts, and countable coproducts of distinguished triangles are distinguished [Kel96, 8.4], the cone of $\bigoplus_i t_i$ is in $\mathcal{R}$. By the calculus of fractions, this proves the preservation of countable coproducts.

Next, we show that the canonical functor $\mathcal{T}^c/\mathcal{R}^c \to \mathcal{T}/\mathcal{R}$ is fully faithful. Let $X \to A$ be a map in $\mathcal{T}$ with cone, say R, in $\mathcal{R}$. Suppose that A is in $\mathcal{T}^c$. As $R = \text{hocolim}_i R_i$ with $R_i \in \mathcal{R}^c$ (definition 4) and A is c-compact, the map $A \to R$ factors over some $R_i \to R$. Choose a distinguished triangle $B \to A \to R_i \to B[1]$ in $\mathcal{T}^c$. The map $B \to A$ factors through $X \to A$ and has cone in $\mathcal{R}^c$. This implies full faithfulness.

We show that $\mathcal{T} \to \mathcal{T}/\mathcal{R}$ preserves c-compact objects. Let $A \to \bigoplus_i X_i$ be a map in $\mathcal{T}/\mathcal{R}$ with $A \in \mathcal{T}^c$. Represent it by a fraction given by the $\mathcal{T}$-maps $X \to A$ with cone in $\mathcal{R}$ and $X \to \bigoplus_i X_i$. By the previous paragraph, there is a $\mathcal{T}$-map $B \to X$ with cone in $\mathcal{R}$ and B in $\mathcal{T}^c$. The $\mathcal{T}$-map $B \to \bigoplus_i X_i$ factors through a finite sum and so the $\mathcal{T}/\mathcal{R}$-map $A \to \bigoplus_i X_i$ factors through this finite sum.

The functor $\mathcal{T} \to \mathcal{T}/\mathcal{R}$ preserves homotopy colimits. Since $\mathcal{T}^c$ generates $\mathcal{T}$, its image in $\mathcal{T}/\mathcal{R}$ generates $\mathcal{T}/\mathcal{R}$. The exactness of the sequence of c-compact objects follows from full faithfulness proved in the second paragraph and from lemma 2. □

Corollary 3 *Let $\mathcal{R} \to \mathcal{S} \to \mathcal{T}$ be a sequence of c-compactly generated triangulated categories, preserving countable coproducts and c-compact objects. Then $\mathcal{R} \to \mathcal{S} \to \mathcal{T}$ is an exact sequence if and only if $\mathcal{R}^c \to \mathcal{S}^c \to \mathcal{T}^c$ is.*

Proof By theorem 2, exactness of $\mathcal{R} \to \mathcal{S} \to \mathcal{T}$ implies exactness f $\mathcal{R}^c \to \mathcal{S}^c \to \mathcal{T}^c$.

If $\mathcal{R}^c \to \mathcal{S}^c \to \mathcal{T}^c$ is exact, then $\mathcal{R} \to \mathcal{S}$ is fully faithful, and the composition $\mathcal{R} \to \mathcal{S} \to \mathcal{T}$ is trivial (corollary 2). The Verdier quotient $\mathcal{S}/\mathcal{R}$ has countable coproducts and is c-compactly generated by $\mathcal{S}^c$ (theorem 2). It follows that $\mathcal{S}/\mathcal{R} \to \mathcal{T}$ preserves countable coproducts and c-compact objects. By the exactness assumption and lemma 2, the functor $(\mathcal{S}/\mathcal{R})^c \to \mathcal{T}^c$ is an equivalence. Thus $\mathcal{S}/\mathcal{R} \to \mathcal{T}$ is an equivalence (corollary 2). □

Lemma 3 *In the situation of Theorem 2, we have an exact sequence of triangulated categories*

$$\mathcal{R}/\mathcal{R}^c \to \mathcal{T}/\mathcal{T}^c \to (\mathcal{T}/\mathcal{R})/(\mathcal{T}/\mathcal{R})^c.$$

Proof By the universal property of Verdier quotients it is clear that the last category is the Verdier quotient of $\mathcal{T}/\mathcal{T}^c$ by the full triangulated subcategory generated by the image of $\mathcal{R}/\mathcal{R}^c \to \mathcal{T}/\mathcal{T}^c$. So we only have to show full faithfulness of the first functor. Given a $\mathcal{T}$-map $R \to T$ with cone in $\mathcal{T}^c$ and R in $\mathcal{R}$, choose a distinguished triangle $A \to R \to T \to A[1]$ in $\mathcal{T}$. Then A is in $\mathcal{T}^c$. Write R as $hocolim_i R_i$ with R_i in $\mathcal{R}^c$ (definition 4). Since $\mathcal{R} \to \mathcal{T}$ preserves countable coproducts, and hence homotopy colimits, and since A is c-compact, the map $A \to R$ factors through some $R_i \to R$. Choose a distinguished triangle $R_i \to R \to X \to R_i[1]$ in $\mathcal{R}$. Then the map $R \to X$ factors through $T \to X$ and $R \to X$ has cone in $\mathcal{R}^c$. This is enough to prove full faithfulness. □

Question 1 Given a small triangulated category $\mathcal{A}$, is there a fully faithful triangle embedding of $\mathcal{A}$ into a c-compactly generated triangulated category $\mathcal{T}$ such that $\mathcal{A}$ constitutes a set of c-compact generators for $\mathcal{T}$?

4 Frobenius pairs

4.1 Exact categories

Recall that an *exact category* is an additive category equipped with a class of short exact sequences satisfying the axioms Ex0-Ex2$^{\text{op}}$ of [Kel96, 4]. These axioms are equivalent to Quillen's original axioms in [Qui73] (cf. [Kel90, appendix]). In [Kel96, 4] admissible monomorphisms are called inflations, admissible epimorphisms deflations and short exact sequences conflations. We will adopt this terminology.

4.2 The embedding $\mathcal{E} \subset \mathrm{Lex}\mathcal{E}$

Let $\mathcal{E}$ be a small exact category and let $\mathrm{Lex}\mathcal{E}$ be the category of left exact additive functors from $\mathcal{E}^{op}$ to $\langle ab \rangle$, the category of abelian groups. We identify $\mathcal{E}$ with the representable functors via the Yoneda embedding. The category $\mathrm{Lex}\mathcal{E}$ is a Grothendieck abelian category with generating set $\mathcal{E}$. The Yoneda embedding $\mathcal{E} \to \mathrm{Lex}\mathcal{E}$ is exact, reflects conflations, and is closed under extensions.

If $\mathcal{E}$ is idempotent complete, then the inclusion $\mathcal{E} \subset \mathrm{Lex}\mathcal{E}$ is also closed under kernels of epimorphisms. In particular, if $f \circ g$ is a deflation in $\mathcal{E}$ then so is f.

For details we refer the reader to [Kel90, Appendix A], [TT90, Appendix A].

4.3 Frobenius categories

A *Frobenius category* [Kel96], [Hap87] is an exact category $\mathcal{E}$ which has enough projective and injective objects, and in which projectives and injectives coincide. We write $\mathcal{E}$-prinj for the full subcategory of projective-injective objects of a Frobenius category $\mathcal{E}$. A *map of Frobenius categories* is an exact functor preserving projective-injective objects. Given a Frobenius category $\mathcal{E}$, its stable category $\underline{\mathcal{E}}$ has the same objects as $\mathcal{E}$. Morphisms are morphisms in $\mathcal{E}$ modulo those which factor through an injective-projective object. The stable category is a triangulated category [Hap87]. Its construction is functorial for maps of Frobenius categories.

Definition 5 A Frobenius pair $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0)$ is a fully faithful inclusion $\mathcal{A}_0 \to \mathcal{A}$ of small Frobenius categories. Recall (section 4.3) that by definition $\mathcal{A}_0-\mathrm{prinj} \subset \mathcal{A}-\mathrm{prinj}$. A map of Frobenius pairs $(\mathcal{A}, \mathcal{A}_0) \to (\mathcal{B}, \mathcal{B}_0)$ is a map of Frobenius categories $\mathcal{A} \to \mathcal{B}$ such that $\mathcal{A}_0$ is mapped into $\mathcal{B}_0$.

Given a small Frobenius category $\mathcal{A}$, we write $\mathcal{A}$ for the Frobenius pair $(\mathcal{A}, \mathcal{A}-\mathrm{prinj})$.

Definition 6 If $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0)$ is a Frobenius pair, then the map $\underline{\mathcal{A}_0} \to \underline{\mathcal{A}}$ of small triangulated categories is fully faithful. This is because any map to an injective object factors over any chosen inflation into an injective object. We define the *derived category* $D\boldsymbol{\mathcal{A}} = D(\mathcal{A}, \mathcal{A}_0)$ of the Frobenius pair $\boldsymbol{\mathcal{A}}$ to be the the Verdier quotient [Ver96, II 2]

$$D\boldsymbol{\mathcal{A}} = \underline{\mathcal{A}}/\underline{\mathcal{A}_0}.$$

A map of Frobenius pairs induces a map of derived categories by passing to stable categories and Verdier quotients.

We refer the reader to section 6 for examples of Frobenius pairs and their derived categories.

5 The functors $\mathcal{F}$ and $\mathcal{S}$

5.1 Countable envelopes

Let $\mathcal{E}$ be a small exact category. We write $\mathcal{F}\mathcal{E}$ for the countable envelope of $\mathcal{E}$ [Kel90, Appendix B] (denoted by $\mathcal{E}^{\sim}$ in *loc.cit*). We review definitions and basic properties from *loc.cit*. The category $\mathcal{F}\mathcal{E}$ is an exact category whose objects are sequences $A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \ldots$ of inflations in $\mathcal{E}$. The morphism set from a sequence A to B is $\lim_i \operatorname{colim}_j \hom_{\mathcal{E}}(A_i, B_j)$. The functor colim : $\mathcal{F}\mathcal{E} \to \mathrm{Lex}\mathcal{E}$ which sends a sequence A to $\operatorname{colim}_i A_i$ is fully faithful and extension closed and thus induces an exact structure on $\mathcal{F}\mathcal{E}$ [Kel96, 11.7, 12.1]. A sequence in $\mathcal{F}\mathcal{E}$ is a conflation if and only if it is isomorphic to maps of sequences $A \to B \to C$ with $A_i \to B_i \to C_i$ a conflation in $\mathcal{E}$. Therefore, the exact structure does not depend on the embedding $\mathcal{E} \to \mathrm{Lex}\mathcal{E}$ and $\mathcal{F}$ defines a functor from exact categories to exact categories.

Colimits of sequences of inflations in $\mathcal{F}\mathcal{E}$ exist in $\mathcal{F}\mathcal{E}$ and are exact. In particular, $\mathcal{F}\mathcal{E}$ has exact, countable coproducts.

The Yoneda embedding $\mathcal{E} \to \mathrm{Lex}\mathcal{E}$ factors through $\mathcal{F}\mathcal{E}$ and defines an exact functor $\mathcal{E} \to \mathcal{F}\mathcal{E}$ which sends an object $E \in \mathcal{E}$ to the constant sequence $E \xrightarrow{id} E \xrightarrow{id} E \xrightarrow{id} \cdots$.

Lemma 4 *The countable envelope $\mathcal{F}\mathcal{E}$ of an exact category $\mathcal{E}$ is flasque, i.e., there is an exact functor $T : \mathcal{F}\mathcal{E} \to \mathcal{F}\mathcal{E}$ and a natural equivalence $T \oplus id \cong T$ of functors.*

Proof Countable direct sums exist in $\mathcal{F}\mathcal{E}$ and are exact, so the functor $\mathcal{F}\mathcal{E} \to \mathcal{F}\mathcal{E}$: $E \mapsto \bigoplus_{\mathbb{N}} E$ and the natural equivalence $E \oplus \bigoplus_{\mathbb{N}} E \cong \bigoplus_{\mathbb{N}} E$ make $\mathcal{F}\mathcal{E}$ into a flasque exact category.

More precisely, we define the functor T as follows. For $A = (A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \cdots)$ an object of $\mathcal{F}\mathcal{E}$, we let tA denote the object $(0 \hookrightarrow A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \cdots)$. Clearly $A \mapsto tA$ is a functor, and we have a natural isomorphism $A \to tA$ induced by the maps $A_i \to A_{i+1}$. Now the functor $A \mapsto TA = \bigoplus_{i \geq 0} t^i A$ makes sense and is exact. We have natural isomorphisms $A \oplus TA \cong A \oplus tTA = TA$. □

5.2 Definition of the functor $\mathcal{F}$

Let $\mathcal{A}$ be a Frobenius category, then its countable envelope $\mathcal{F}\mathcal{A}$ is also a Frobenius category [Kel90, appendix B]. The projective-injective objects are the direct factors of objects of $\mathcal{F}(\mathcal{A}-\mathrm{prinj})$. Clearly, $\mathcal{F}$ defines a functor from Frobenius categories to Frobenius categories.

Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0)$ be a Frobenius pair. Then $\mathcal{F}\mathcal{A}_0 \to \mathcal{F}\mathcal{A}$ is an exact, fully faithful map of small Frobenius categories. Thus the following pair defines a Frobenius pair

$$\mathcal{F}\boldsymbol{\mathcal{A}} = (\mathcal{F}\mathcal{A}, \mathcal{F}\mathcal{A}_0).$$

Clearly, $\mathcal{F}$ defines a functor from Frobenius pairs to Frobenius pairs.

The functor $\mathcal{A} \to \mathcal{F}\mathcal{A}$ (section 5.1) induces a map of Frobenius pairs $\boldsymbol{\mathcal{A}} \to \mathcal{F}\boldsymbol{\mathcal{A}}$ natural in $\boldsymbol{\mathcal{A}}$.

Proposition 1 *Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0)$ be a Frobenius pair. Then the map $\boldsymbol{\mathcal{A}} \to \mathcal{F}\boldsymbol{\mathcal{A}}$ induces a fully faithful map $D\boldsymbol{\mathcal{A}} \hookrightarrow D\mathcal{F}\boldsymbol{\mathcal{A}}$ of triangulated categories. Moreover, $D\mathcal{F}\boldsymbol{\mathcal{A}}$ has countable coproducts, and it is c-compactly generated by $D\boldsymbol{\mathcal{A}}$. In particular, the inclusion induces a triangle equivalence (lemma 2)*

$$\widetilde{D\boldsymbol{\mathcal{A}}} \xrightarrow{\sim} D^c\mathcal{F}\boldsymbol{\mathcal{A}}.$$

Proof We first prove the proposition for $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}\text{−prinj})$ a small Frobenius category. The triangle functor $\underline{\mathcal{A}} \to \underline{\mathcal{F}\mathcal{A}}$ is fully faithful because the fully faithful map $\mathcal{A} \to \mathcal{F}\mathcal{A}$ preserves projective-injective objects. Since $\mathcal{F}\mathcal{A}$ has countable coproducts, and projective-injective objects are closed under countable coproducts, $\underline{\mathcal{F}\mathcal{A}}$ has countable coproducts as well. Any object A of $\underline{\mathcal{A}}$ is c-compact in $\underline{\mathcal{F}\mathcal{A}}$ because any object in $\mathcal{A}$ is compact in $\mathcal{F}\mathcal{A} \subset \mathrm{Lex}\mathcal{A}$.

Let $\mathrm{colim}_i A_i = (A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \cdots)$ be an object of $\underline{\mathcal{F}\mathcal{A}}$. The short exact sequence

$$0 \to \bigoplus_i A_i \xrightarrow{1-\text{shift}} \bigoplus_i A_i \longrightarrow \mathrm{colim}_i A_i \to 0 \tag{6}$$

in $\mathcal{F}\mathcal{A}$ gives rise to a triangle in $\underline{\mathcal{F}\mathcal{A}}$. This identifies $\mathrm{colim}_i A_i$ with $\mathrm{hocolim}_i A_i$ in $\underline{\mathcal{F}\mathcal{A}}$. Thus $\underline{\mathcal{F}\mathcal{A}}$ is c-compactly generated by $\underline{\mathcal{A}}$.

Now we prove the general case. Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0)$ be a Frobenius pair. The functor $\underline{\mathcal{F}\mathcal{A}_0} \to \underline{\mathcal{F}\mathcal{A}}$ is fully faithful and preserves countable coproducts because the map $\mathcal{F}\mathcal{A}_0 \to \mathcal{F}\mathcal{A}$ of Frobenius categories is fully faithful, preserves projective-injective objects and countable coproducts. It preserves c-compact objects as it sends $\underline{\mathcal{A}_0}$ to $\underline{\mathcal{A}}$. The rest follows from Theorem 2. □

Remark 2 Let $\boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ be a map of Frobenius pairs. Then $D\mathcal{F}\boldsymbol{\mathcal{A}} \to D\mathcal{F}\boldsymbol{\mathcal{B}}$ preserves countable coproducts since $\mathcal{F}\mathcal{A} \to \mathcal{F}\mathcal{B}$ does. It preserves c-compact objects because it sends $D\boldsymbol{\mathcal{A}}$ into $D\boldsymbol{\mathcal{B}}$.

5.3 Definition of the functor $\mathcal{S}$

We define a functor $\mathcal{S}$, called *suspension*, from the category of Frobenius pairs into itself as follows. Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0)$ be a Frobenius pair. By proposition 1, the functor $D\boldsymbol{\mathcal{A}} \to D\mathcal{F}\boldsymbol{\mathcal{A}}$ is fully faithful. Let $\mathcal{S}\boldsymbol{\mathcal{A}} = (\mathcal{F}\mathcal{A}, \mathcal{S}_0\boldsymbol{\mathcal{A}})$ be the Frobenius pair with $\mathcal{S}_0\boldsymbol{\mathcal{A}}$ the full subcategory of $\mathcal{F}\mathcal{A}$ of objects sent to zero in $D\mathcal{F}\boldsymbol{\mathcal{A}}/D\boldsymbol{\mathcal{A}}$. The natural transformation $id \to \mathcal{F}$ makes $\mathcal{S}$ into a functor from Frobenius pairs to itself. The identity functor $id : \mathcal{F}\mathcal{A} \to \mathcal{F}\mathcal{A}$ defines a map $\mathcal{F}\boldsymbol{\mathcal{A}} \to \mathcal{S}\boldsymbol{\mathcal{A}}$ of Frobenius pairs which on derived categories is the localization functor $D\mathcal{F}\boldsymbol{\mathcal{A}} \to D\mathcal{F}\boldsymbol{\mathcal{A}}/D\boldsymbol{\mathcal{A}} = D\mathcal{S}\boldsymbol{\mathcal{A}}$. In other words, the sequence of functors $id \to \mathcal{F} \to \mathcal{S}$ is exact.

We will show that the functors $\mathcal{F}$ and $\mathcal{S}$ preserve exact sequences.

Theorem 3 *If we take $\mathfrak{M}$ to be the category of Frobenius pairs, then the sequence $id \to \mathcal{F} \to \mathcal{S}$ of functors from Frobenius pairs to Frobenius pairs satisfies the hypothesis of the set-up (section 2.2).*

Proof For $\mathcal{A}$ a Frobenius pair, the sequence $\mathcal{A} \to \mathcal{F}\mathcal{A} \to \mathcal{S}\mathcal{A}$ is exact by the definition of the functors $\mathcal{F}$ and $\mathcal{S}$ (definition 5.3). We have $I\!K_0(\mathcal{F}\mathcal{A}) = 0$ because $D\mathcal{F}\mathcal{A}$ has countable coproducts (proposition 1). The functor $\mathcal{F}$ preserves exact sequences in $\mathfrak{M}$ because of proposition 1, 2 and corollary 3. It follows from lemma 3 that the functor $\mathcal{S}$ also preserves exact sequences in $\mathfrak{M}$. □

6 Examples of Frobenius pairs

Definition 7 Let $\mathcal{D}$ be a small triangulated category. We say that $\mathcal{D}$ admits a *Frobenius model* if there is a Frobenius pair $\mathcal{A}$ and a triangle equivalence $D\mathcal{A} \cong \mathcal{D}$.

6.1 Frobenius models for $\mathcal{D} \subset D\mathcal{A}$ and $D\mathcal{A}/\mathcal{D}$

Let $\mathcal{A} = (\mathcal{A}, \mathcal{A}_0)$ be a Frobenius pair and $\mathcal{D} \subset D\mathcal{A}$ a full triangulated subcategory. Let $\mathcal{B} \subset \mathcal{A}$ be the full subcategory of objects which are isomorphic in $D\mathcal{A}$ to an object of $\mathcal{D}$. Then $\mathcal{B} \subset \mathcal{A}$ is closed under extensions. Declaring a sequence in $\mathcal{B}$ to be exact if it is exact in $\mathcal{A}$, gives $\mathcal{B}$ the structure of an exact category. Since $\mathcal{B}$ contains all projective-injective objects of $\mathcal{A}$, and since $\mathcal{B} \subset \mathcal{A}$ is moreover closed under kernels of deflations and cokernels of inflations, $\mathcal{B}$ is a Frobenius category with $\mathcal{B}-$prinj $= \mathcal{A}-$prinj. Therefore, $\mathcal{B} = (\mathcal{B}, \mathcal{A}_0)$ and $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ are Frobenius pairs. The inclusions $\mathcal{B} \to \mathcal{A} \to \mathcal{C}$ yield a short exact sequence of Frobenius pairs whose sequence of derived categories is equivalent to $\mathcal{D} \to D\mathcal{A} \to D\mathcal{A}/\mathcal{D}$. In particular, any full triangulated subcategory and any Verdier quotient of $D\mathcal{A}$ admits a Frobenius model.

6.2 Frobenius models for $D(\mathcal{E})$, $\mathcal{E}$ an exact category

Let $\mathcal{E}$ be an exact category. Let $\mathrm{Ch}^b(\mathcal{E})$, $\mathrm{Ch}^+(\mathcal{E})$, $\mathrm{Ch}^-(\mathcal{E})$, $\mathrm{Ch}(\mathcal{E})$ be the category of bounded, bounded below, bounded above, unbounded chain complexes in $\mathcal{E}$. Declare a sequence $A^* \to B^* \to C^*$ in $\mathrm{Ch}^\sharp\,\mathcal{E}$ ($\sharp \in \{b, +, -, \emptyset\}$) to be a conflation if $A^i \to B^i \to C^i$ is isomorphic to the split conflation $A^i \to A^i \oplus C^i \to C^i$ for all $i \in \mathbb{Z}$. This makes $\mathrm{Ch}^\sharp\,\mathcal{E}$ into an exact category. One checks that $\mathrm{Ch}^\sharp\,\mathcal{E}$ is a Frobenius category whose projective-injective objects are the contractible chain complexes in $\mathrm{Ch}^\sharp\,\mathcal{E}$. Its stable category is the usual homotopy category $\mathcal{K}^\sharp(\mathcal{E})$ of chain complexes where maps are chain maps up to chain homotopy.

Let $\mathrm{Ac}^\sharp(\mathcal{E}) \subset \mathrm{Ch}^\sharp(\mathcal{E})$ be the full subcategory of chain complexes which are homotopy equivalent to acyclic chain complexes in $\mathcal{E}$. Recall that a complex (E^*, d^*) is acyclic if the differentials d^i admit factorizations $E^i \to Z^{i+1} \to E^{i+1}$ such that $Z^i \to E^i \to Z^{i+1}$ is a conflation in $\mathcal{E}$ for all $i \in \mathbb{Z}$. The inclusion $\mathrm{Ac}^\sharp(\mathcal{E}) \subset \mathrm{Ch}^\sharp(\mathcal{E})$ is closed under extensions, kernels of deflations and cokernels of inflations, and $\mathrm{Ac}^\sharp(\mathcal{E})$ contains all projective-injective objects. It follows that $\mathbf{Ch}^\sharp\mathcal{E} = (\mathrm{Ch}^\sharp\,\mathcal{E}, \mathrm{Ac}^\sharp\,\mathcal{E})$ is a Frobenius pair. Its derived category $D(\mathrm{Ch}^\sharp\,\mathcal{E}, \mathrm{Ac}^\sharp\,\mathcal{E})$ is the bounded (bounded below, bounded above, unbounded) derived category $D^\sharp(\mathcal{E})$ of $\mathcal{E}$ as defined in [Kel96].

Definition 8 We define the negative K-groups of an exact category $\mathcal{E}$ by

$$I\!K_i(\mathcal{E}) = I\!K_i(\mathrm{Ch}^b\,\mathcal{E}, \mathrm{Ac}^b\,\mathcal{E}).$$

We write $I\!K_i(R) = I\!K_i(R-\mathrm{proj})$ for R a ring and $R-$proj the category of finitely generated projective R-modules.

Remark 3 Recall [BS01] that if $\mathcal{E}$ is idempotent complete, then so is $D^b(\mathcal{E})$. In this case $I\!K_0(\mathcal{E})$ is the usual $K_0(\mathcal{E})$. If $\mathcal{E}$ is not idempotent complete, then $I\!K_0(\mathcal{E}) = K_0(\tilde{\mathcal{E}})$.

Given a sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ of exact categories such that $D^b\mathcal{A} \to D^b\mathcal{B} \to D^b\mathcal{C}$ is exact, Theorem 1 yields a long exact sequence

$$I\!K_0(\mathcal{A}) \to I\!K_0(\mathcal{B}) \to I\!K_0(\mathcal{C}) \to I\!K_{-1}(\mathcal{A}) \to I\!K_{-1}(\mathcal{B}) \cdots$$

For example, let R be a ring, and let $S \subset R$ be a multiplicative set of central non-zero-divisors. Then one can take $\mathcal{B} = \mathcal{P}^1(R)$ to be the category of R-modules M of projective dimension at most 1 with $S^{-1}M$ a projective $S^{-1}R$-module, $\mathcal{B} \to \mathcal{C} = S^{-1}R-\mathrm{proj}$ to be the localization map, and $\mathcal{A}$ to be the full subcategory of $\mathcal{P}^1(R)$ of S-torsion modules. Moreover, $R-\mathrm{proj} \subset \mathcal{P}^1(R)$ is a derived equivalence by resolution. The resulting long exact sequence

$$I\!K_0(\mathcal{A}) \to I\!K_0(R) \to I\!K_0(S^{-1}R) \to I\!K_{-1}(\mathcal{A}) \to I\!K_{-1}(R) \cdots$$

is classical [Car80].

6.3 Frobenius models for $\mathcal{D} \subset D\mathcal{A}$, $\mathcal{A}$ a Grothendieck abelian category

Let $\mathcal{A}$ be a Grothendieck abelian category. We claim that any small triangulated subcategory $\mathcal{D}$ of the derived category $D(\mathcal{A})$ of $\mathcal{A}$ admits a Frobenius model. But first, a certainly well-known lemma.

Lemma 5 *Let $\mathcal{U}$ be an abelian category in which countable filtered colimits exist and are exact. Let $\mathcal{A} \subset \mathcal{U}$ be a Serre subcategory closed under the formation of countable filtered colimits. Suppose that for any epimorphism $X \to A$ from an object X of $\mathcal{U}$ to an object A of $\mathcal{A}$ there is a subobject $B \subset X$ with B an object of $\mathcal{A}$ such that the composition $B \to A$ is an epimorphism. Then the following triangle functor is fully faithful*

$$D(\mathcal{A}) \to D(\mathcal{U}).$$

*Proof (*sketch) One shows that for any chain complex X in $\mathcal{U}$ with $H^*(X) \in \mathcal{A}$ there is a quasi-isomorphism $A \to X$ with A a chain complex in $\mathcal{A}$. To see this, one constructs a sequence $A_k \to A_{k+1} \to \cdots \to X$, $k \in \mathbb{N}$, with A_k chain complexes in $\mathcal{A}$ and $H^*(A_k) \to H^*(X)$ surjective such that the kernel of $H^*(A_k) \to H^*(X)$ maps to zero in $H^*(A_{k+1})$. Then $\mathrm{colim}_k A_k \to X$ is the desired quasi-isomorphism. □

Let $\mathcal{A}$ be a Grothendieck abelian category. It is well known that $D(\mathcal{A})$ has small hom-sets [Fra01]. We review a short proof of this fact which will provide small full triangulated subcategories $\mathcal{D}$ of $\mathcal{D}(A)$ with Frobenius models.

Let G be a generator for $\mathcal{A}$. Define the size size(X) of an object X of $\mathcal{A}$ to be the cardinality of $\hom_{\mathcal{A}}(G, X)$. For a cardinal κ, let $\mathcal{A}_\kappa$ be the full subcategory of $\mathcal{A}$ of objects with size $\leq \kappa$. The categories $\mathcal{A}_\kappa$ are (essentially) small as for every object X we have an epimorphism $\bigoplus G \to X$ where the direct sum is indexed over $\hom_{\mathcal{A}}(G, X)$, and $\bigoplus_\kappa G$ has only set-many quotients (up to isomorphism). For any cardinal κ there is a cardinal $\kappa_0 \geq \kappa$ such that $\mathcal{A}_{\kappa_0} \subset \mathcal{A}$ is a Serre subcategory closed under colimits over ordinal numbers $\leq \kappa_0$ [Fra01, 3.7]. Moreover, the inclusion $\mathcal{A}_{\kappa_0} \subset \mathcal{A}$ satisfies the hypothesis of lemma 5 [Fra01, 3.8]. It follows that $D(\mathcal{A})$ has small hom-sets and maps $A \to B$ in $D(\mathcal{A})$ can be calculated in $D(\mathcal{A}_{\kappa_0})$ whenever $A, B \in \mathcal{A}_{\kappa_0}$.

Any small full triangulated subcategory $\mathcal{D}$ of $D\mathcal{A}$ lies in $D\mathcal{A}_{\kappa_0}$ for some κ_0. Since $D\mathcal{A}_{\kappa_0}$ has a Frobenius model (section 6.2), $\mathcal{D}$ has also a Frobenius model (section 6.1).

6.4 Frobenius models for $\mathcal{D} \subset D(A)$, A a dga

Let A be a differential graded algebra (dga). Its derived category $D(A)$ is obtained from the category of differential graded left A-modules (short: dg A-module) by formally inverting quasi-isomorphisms, *i.e.,* those morphisms $M \to N$ such that $H^*(M) \to H^*(N)$ is an isomorphism. We claim that any small full triangulated subcategory $\mathcal{D} \subset D(A)$ has a Frobenius model.

We first review from [KM95, part III] an explicit construction of $D(A)$. A cell A-module is a dg A-module M which admits a filtration $0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M = \bigcup_{i \in \mathbb{N}} M_i$ by dg A-submodules such that M_{i+1}/M_i is a free dg A-module, *i.e.,* a direct sum of the dg A-module A with generators placed in various degrees. In particular, any cell A-module is a free A-module (forgetting differentials). Write $\mathcal{C}(A)$ for the full subcategory of dg A-modules which consists of cell A-modules. Given a dg A-module M, its cone CM is the dg A-module with underlying A-module $M \oplus M[1]$, *i.e.,* $(CM)^i = M^i \oplus M^{i+1}$ and with differential $d : (x, y) \mapsto (dx + (-1)^i y, dy)$. If M is a cell A-module, then so is CM. A map $f : M \to N$ is homotopic to zero if it factors through CM.

Denote by $\mathcal{H}(A)$ the homotopy category of cell A-modules. Its objects are cell A-modules and its maps are dg A-module maps modulo those which are homotopic to zero. The inclusion of $\mathcal{C}(A)$ into all dg A-modules induces an equivalence of triangulated categories $\mathcal{H}(A) \xrightarrow{\sim} D(A)$ [KM95, part III, 2.7].

The category $\mathcal{C}(A)$ of cell A-modules is a (large) Frobenius category when we declare a sequence to be a conflation if it is a (split) exact sequence of A-modules (forgetting differentials). Every dg A-module of the form CM is projective and injective for this exact structure. This is because the A-module maps $(0 \ \ 1) : M[1] \to CM$ and $(1 \ \ 0) : CM \to M$ induce isomorphisms $\hom_{\mathrm{dg}\ A-\mathrm{mod}}(CM, N) \to \hom_{A-\mathrm{mod}}(M[1], N)$ and $\hom_{\mathrm{dg}\ A-\mathrm{mod}}(N, CM) \to \hom_{A-\mathrm{mod}}(N, M)$ for M, N dg A-modules. Moreover, we have inclusions $(1\ 0) : M \to CM$ and epimorphisms $CM[-1] \to M$ of dg A-modules. Thus the category of cell A-modules has enough injectives and projectives and is a Frobenius category. It is clear that its stable category is $\mathcal{H}(A)$ and thus is equivalent to $D(A)$.

For an infinite cardinal κ, let $\mathcal{C}_\kappa(A)$ be the full subcategory of $\mathcal{C}(A)$ consisting of cell A-modules of size (cardinality of a basis as an A-module) $< \kappa$. Then $\mathcal{C}_\kappa(A) \subset \mathcal{C}(A)$ is a (essentially) small extension closed Frobenius subcategory. Its stable category $\mathcal{H}_\kappa(A)$ is a (essentially) small full triangulated subcategory of $\mathcal{H}(A)$ which admits a Frobenius model, namely $(\mathcal{C}_\kappa(A), \mathcal{C}_\kappa(A)\text{-prinj})$. Since any small triangulated subcategory $\mathcal{D}$ of $D(A)$ is contained in some $\mathcal{H}_\kappa(A)$, $\mathcal{D}$ has a Frobenius model by section 6.1.

Definition 9 Let A be a dga. We define

$$\mathbb{K}_i(A) = \mathbb{K}_i(\text{cell } A\text{-mod}),$$

where cell A−mod is the Frobenius category of finite cell A-modules.

Remark 4 A map $A \to B$ of dg-algebras induces a map $\otimes_A B$: cell A−mod $\to$ cell B−mod of Frobenius pairs, and thus a map $\mathbb{K}_i(A) \to \mathbb{K}_i(B)$. If $A \to B$ is a quasi-isomorphism, then $\otimes_A B$ induces an equivalence of derived categories and thus isomorphisms $\mathbb{K}_i(A) \xrightarrow{\sim} \mathbb{K}_i(B)$ (corollary 1).

6.5 Complicial BiWaldhausen categories

A complicial BiWaldhausen category [TT90, 1.2.11] is a full subcategory $\mathcal{C}$ of the category $\mathrm{Ch}(\mathcal{A})$ of complexes in some abelian category $\mathcal{A}$. It comes equipped with a set ω of maps in $\mathcal{C}$ called weak equivalences which we suppose to be closed under retracts. These data are subject to certain conditions specified in [TT90, 1.2.11]. Suppose, as in [TT90, 1.9.6], that $\mathcal{C} \subset \mathrm{Ch}(\mathcal{A})$ is closed under canonical homotopy push-outs and canonical homotopy pull-backs.

Declare a sequence in $\mathcal{C}$ to be a conflation if it is degree-wise split exact. Then the axioms imposed in [TT90, 1.2.11] and [TT90, 1.9.6] imply that $\mathcal{C}$ is a Frobenius category, with projective-injective objects being the contractible chain complexes in $\mathcal{C}$. Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ of objects X for which $0 \to X$ is a weak equivalence. Then $\mathcal{C}_0 \subset \mathcal{C}$ is closed under extensions, kernels of deflations, cokernels of inflations, direct factors, and it contains all projective-injective objects in $\mathcal{C}$. It follows that $(\mathcal{C}, \mathcal{C}_0)$ is a Frobenius pair. Its derived category $D(\mathcal{C}, \mathcal{C}_0)$ is isomorphic to the derived category $\omega^{-1}\mathcal{C}$ as constructed in [TT90, 1.9.6].

For examples of complicial BiWaldhausen categories we refer the reader to [TT90, 3.1, 3.2, 3.3]. Let X be a quasi-compact and quasi-separated scheme, we define $\mathbb{K}_i(X)$, $i \le 0$, to be the negative $\mathbb{K}$-groups of the Frobenius pair associated with the complicial BiWaldhausen category of [TT90, 3.1].

Question 2 Given a stable model category, its homotopy category is a triangulated category [Hov99, chapter 7]. Suppose that the model category is an additive category. Does every small triangulated subcategory of the homotopy category admit a Frobenius model?

7 Additivity and Colimits

Theorem 4 (Additivity) *Let $F \to G : \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ be a natural transformation of maps between Frobenius pairs. If $F(A) \to G(A)$ is an inflation for all objects A*

of $\mathcal{A}$, then $G/F : \mathcal{A} \to \mathcal{B}$ is a map of Frobenius pairs and $\mathbb{K}_i(F) + \mathbb{K}_i(G/F) = \mathbb{K}_i(G) : \mathbb{K}_i(\mathcal{A}) \to \mathbb{K}_i(\mathcal{B})$ for all $i \leq 0$.

Proof It is clear that G/F is a map of Frobenius pairs.

The statement of the theorem is true for $\mathbb{K}_0$. Since $\mathcal{S}^i F \to \mathcal{S}^i G : \mathcal{S}^i \mathcal{A} \to \mathcal{S}^i \mathcal{B}$ also satisfy the hypothesis of the theorem, we are done. □

Corollary 4 *Let $0 \to F \to G \to H \to 0 : \mathcal{A} \to \mathcal{B}$ be an exact sequence of exact functors between exact categories $\mathcal{A}$ and $\mathcal{B}$. Then $\mathbb{K}_i(F) + \mathbb{K}_i(H) = \mathbb{K}_i(G) : \mathbb{K}_i(\mathcal{A}) \to \mathbb{K}_i(\mathcal{B})$ for all $i \leq 0$.*

Proof A map $\mathcal{A} \to \mathcal{B}$ of Frobenius pairs which factors over $\mathcal{B}_0$ induces the 0 map in K-theory. Thus theorem 4 implies that $\mathbb{K}_i(F) \oplus \mathbb{K}_i(\text{cone}(F \to G)) = \mathbb{K}_i(G)$. Let $j : \text{cone}(F \to G) \to H$ be the canonical map. Additivity (theorem 4) yields $\mathbb{K}_i(\text{cone}(F \to G)) \oplus \mathbb{K}_i(\text{cone}(j)) = \mathbb{K}_i(H)$. But $\text{cone}(j)$ is acyclic, thus $\mathbb{K}_i(\text{cone}(j)) = 0$. □

We conclude the section by showing that the functors $\mathbb{K}_i$ commute with filtered colimits.

Lemma 6 *Let $i \mapsto \mathcal{A}_i$ be a functor from a small, filtered index category I to the category of Frobenius pairs. Then $\text{colim}\mathcal{A}_i$ is a Frobenius pair and the natural map*

$$\text{colim}\,\mathbb{K}_n(\mathcal{A}_i) \xrightarrow{\sim} \mathbb{K}_n(\text{colim}_i \mathcal{A}_i)$$

is an isomorphism for $n \leq 0$.

Proof Clearly, $\text{colim}_i \mathcal{A}_i$ is a Frobenius pair. The natural map $\text{colim}\, D\mathcal{A}_i \to D\text{colim}_i \mathcal{A}_i$ is an equivalence. Since K_0 commutes with filtered colimits of triangulated categories, the case $n = 0$ follows.

Filtered colimits preserve exact sequences of triangulated categories, so they preserve exact sequences of Frobenius pairs. Moreover, $\text{colim}_i \mathcal{F}\mathcal{A}_i$ is flasque, as the functor T and the natural equivalence $id \oplus T \cong T$ of lemma 4 extend to $\text{colim}_i \mathcal{F}\mathcal{A}_i$. By additivity (theorem 4), we have $\mathbb{K}_n(\text{colim}\mathcal{F}\mathcal{A}) = 0$, $n \leq 0$. The long exact sequences 1 associated to the diagram

$$\begin{tikzcd}
\text{colim}_i \mathcal{A}_i \arrow[r] \arrow[d, equal] & \text{colim}_i \mathcal{F}\mathcal{A}_i \arrow[r] \arrow[d] & \text{colim}_i \mathcal{S}\mathcal{A}_i \arrow[d] \\
\text{colim}_i \mathcal{A}_i \arrow[r] & \mathcal{F}\text{colim}_i \mathcal{A}_i \arrow[r] & \mathcal{S}\text{colim}_i \mathcal{A}_i
\end{tikzcd}$$

of exact sequences of Frobenius pairs, together with the vanishing of $\mathbb{K}_i$ of the middle terms yield isomorphisms $\mathbb{K}_n(\text{colim}\mathcal{S}\mathcal{A}_i) \xrightarrow{\sim} \mathbb{K}_n(\mathcal{S}\text{colim}\mathcal{A}_i)$, as both are isomorphic, via the boundary map, to $\mathbb{K}_{n-1}(\text{colim}\mathcal{A}_i)$, $n \leq 0$. Iterating, we see that the maps $\text{colim}\,\mathbb{K}_0(\mathcal{S}^n\mathcal{A}_i) \xrightarrow{\sim} \mathbb{K}_0(\text{colim}\mathcal{S}^n\mathcal{A}_i) \xrightarrow{\sim} \mathbb{K}_0(\mathcal{S}^n\text{colim}\mathcal{A}_i)$ are isomorphisms whose composition is $\text{colim}\,\mathbb{K}_{-n}(\mathcal{A}_i) \to \mathbb{K}_{-n}(\text{colim}\mathcal{A}_i)$, $n \geq 0$. □

Corollary 5 *Let $\mathcal{E}_i$, $i \in I$, be a diagram of exact categories and exact functors indexed over a filtered category I. Then the natural map $\text{colim}_i \mathbb{K}_n(\mathcal{E}_i) \to \mathbb{K}_n(\text{colim}\mathcal{E}_i)$ is an isomorphism for $n \leq 0$.*

Proof This is because $\operatorname{colim}(\mathrm{Ch}^b\,\mathcal{E}_i, \mathrm{Ac}^b\,\mathcal{E}_i) \to (\mathrm{Ch}^b\,\operatorname{colim}\mathcal{E}_i, \mathrm{Ac}^b\,\operatorname{colim}\mathcal{E}_i)$ is an equivalence of Frobenius pairs. Now apply lemma 6. □

8 Agreement

In this section we show that our definition of negative K-groups extends the definitions of Bass [Bas68], Karoubi [Kar70], Pedersen [Ped84], Pedersen-Weibel [PW89] and Thomason [TT90]. Agreement with Carter's [Car80] and Yao's [Yao92] definitions is outlined in remark 5.

Theorem 5 *Let R be a ring. Then there are natural isomorphisms between Bass' and Pedersen's groups $K_i(R)$ and the groups $\mathbb{K}_i(R)$ defined in definition 8 for $i \leq 0$.*

Let X be a quasi-compact and quasi-separated scheme. Then there are natural isomorphisms between Thomason's groups $K_i^B(X)$ and the groups $\mathbb{K}_i(X)$ defined in section 6.5 for $i \leq 0$.

Let $\mathcal{A}$ be an additive category. Then there are natural isomorphisms between Karoubi's and Pedersen-Weibel's groups $K_i(\mathcal{A})$ and the groups $\mathbb{K}_i(\mathcal{A})$ defined in definition 8 for $i \leq 0$.

Proof Thomason's proof of the projective space bundle theorem [Tho93] only uses exact sequences of derived categories and carries over to our framework. More precisely, let X be a quasi-compact and quasi-separated scheme, and let $p : \mathbb{P}^1_X \to X$ be the projection from the projective line over X to X. Then the triangle maps $Lp^* : D_{\mathrm{parf}}(X) \to D_{\mathrm{parf}}(\mathbb{P}^1_X)$ and $O(-1) \otimes Lp^* : D_{\mathrm{parf}}(X) \to D_{\mathrm{parf}}(\mathbb{P}^1_X)$, which are induced by maps of their Frobenius models, induce isomorphisms

$$(Lp^*, O(-1) \otimes Lp^*) : \mathbb{K}_i(X) \oplus \mathbb{K}_i(X) \xrightarrow{\sim} \mathbb{K}_i(\mathbb{P}^1_X), \quad i \leq 0.$$

The proof is the same as in [Tho93]. It follows from the proof of Bass' fundamental theorem given in [TT90, 6.6 (b)] that there is an exact sequence of abelian groups

$$\begin{aligned} 0 \to \mathbb{K}_i(X) \to \mathbb{K}_i(X[T]) \oplus \mathbb{K}_i(X[T^{-1}]) \\ \to \mathbb{K}_i(X[T, T^{-1}]) \to \mathbb{K}_{i-1}(X) \to 0 \end{aligned}$$

for $i \leq 0$. Since $\mathbb{K}_0(X) = K_0^B(X)$ for any quasi-compact and quasi-separated scheme X, our negative K-groups coincide with Thomason's negative K-groups, and hence with Bass' groups in the commutative case.

We show agreement with Karoubi's and Pedersen-Weibel's negative K-groups. For $\mathcal{A}$ an additive category, let $\mathcal{CA}$ be the cone category [Kar70], [PW89], and let $\mathcal{SA} = \mathcal{CA}/\mathcal{A}$ be the suspension category. It follows from [CP97] that the sequence of additive categories $\mathcal{A} \to \mathcal{CA} \to \mathcal{SA}$ induces an exact sequence of triangulated categories $D^b\mathcal{A} \to D^b\mathcal{CA} \to D^b\mathcal{SA}$. A more explicit proof and a generalization to exact categories can be found in [Sch04, Proposition 2.6]. Since $\mathcal{CA}$ is flasque, it follows that $\mathbb{K}_i(\mathcal{A}) = \mathbb{K}_{i+n}(S^n\mathcal{A})$ for $i + n \leq 0$ (theorem 1). In particular, $\mathbb{K}_{-i}(\mathcal{A}) = \mathbb{K}_0(S^i\mathcal{A}) = K_0((S^i\mathcal{A})^\sim)$ for $i \geq 0$. But the last group is Karoubi's and Pedersen-Weibel's $-i$-th K-group of $\mathcal{A}$ [Kar70], [PW89].

In particular, negative K-groups as defined by Bass and Pedersen are isomorphic to our negative $\mathbb{K}$-groups for a (not necessarily commutative) ring. Karoubi

[Kar71] (Pedersen [Ped84]) showed that his groups coincide with Bass groups. Pedersen-Weibel's definition is a generalization of Pedersen's definition. □

Remark 5 Alternatively, one can prove a projective space bundle theorem for the non-commutative projective line over a non-commutative ring R or an "admissible abelian category" [Yao92] following Thomason's arguments for the commutative case [Tho93]. This would lead to an alternative proof of agreement for non-commutative rings and for Yao's "admissible abelian categories". In particular, this would show agreement with Carter's negative K-groups [Car80]. However, details remain to be written down.

Remark 6 If X is a quasi-compact and quasi-separated scheme which admits an ample family of line bundles, then the inclusion of bounded chain complexes of vector bundles of finite rank on X into perfect complexes induces an equivalence of triangulated categories $D^b(\mathrm{Vect}(X)) \xrightarrow{\sim} D_{\mathrm{parf}}(X)$ [TT90]. Thus $\mathbb{K}_i(\mathrm{Vect}(X)) = \mathbb{K}_i(X) = K_i^B(X)$ (corollary 1, theorem 5).

9 Generators and relations

Lemma 7 *Let $\mathcal{E}$ be an exact category and $D(\mathcal{E}) = D(\mathrm{Ch}\,\mathcal{E}, \mathrm{Ac}\,\mathcal{E})$ its unbounded derived category [Nee90], [Kel96]. Then $\mathbb{K}_{i-1}(\mathcal{E}) = \mathbb{K}_i(\mathrm{Ch}\,\mathcal{E}, \mathrm{Ac}\,\mathcal{E})$, $i \leq 0$. In particular, $\mathbb{K}_{-1}(\mathcal{E}) = K_0(\widetilde{D(\mathcal{E})})$.*

Proof Let $D^+(\mathcal{E})$ (resp. $D^-(\mathcal{E})$) be the derived category of bounded below (resp. bounded above) complexes in $\mathcal{E}$. These are the derived categories of section 6.2. In the diagram of derived categories

$$\begin{tikzcd} D^b(\mathcal{E}) \arrow[r] \arrow[d] & D^+(\mathcal{E}) \arrow[d] \\ D^-(\mathcal{E}) \arrow[r] & D(\mathcal{E}) \end{tikzcd}$$

all functors are fully faithful, and the induced functors on quotients are equivalences. Since the diagram is induced by a commutative diagram of the corresponding Frobenius pairs, Theorem 1 yields isomorphisms $\mathbb{K}_{n-1}(\mathcal{E}) = \mathbb{K}_n(\mathrm{Ch}\,\mathcal{E}, \mathrm{Ac}\,\mathcal{E})$, $n \leq 0$, once we know that

$$\mathbb{K}_n(\mathrm{Ch}^-\,\mathcal{E}, \mathrm{Ac}^-\,\mathcal{E}) = \mathbb{K}_n(\mathrm{Ch}^+\,\mathcal{E}, \mathrm{Ac}^+\,\mathcal{E}) = 0, \quad n \leq 0.$$

For bounded above complexes this follows from the usual "Eilenberg swindle": the functor $T = \bigoplus_{n\in\mathbb{N}}[2n]$ satisfies $T[2] \oplus id = T$, hence $\mathbb{K}_n(id) = 0$ as $\mathbb{K}_n(T[2]) = \mathbb{K}_n(T)$ by additivity. The argument for bounded below complexes is similar. □

Corollary 6 *The group $\mathbb{K}_{-1}(\mathcal{E})$ is the quotient of the abelian monoid of isomorphism classes of idempotents in $D(\mathcal{E})$, under direct sum operation modulo the submonoid of those idempotents which split in $D(\mathcal{E})$. In particular, $\mathbb{K}_{-1}(\mathcal{E}) = 0$ if and only if $D(\mathcal{E})$ is idempotent complete.*

Proof Let $\mathcal{T}$ be a triangulated category, the cokernel of $K_0(\mathcal{T}) \to K_0(\tilde{\mathcal{T}})$ is isomorphic to the quotient of the abelian monoid of isomorphism classes of objects in $\tilde{\mathcal{T}}$, under direct sum operation, modulo the submonoid of isomorphism classes of objects in $\mathcal{T}$ [Tho97, proof of 2.2]. Since we have a surjection $0 = K_0(D^+\mathcal{E}) \oplus K_0(D^-\mathcal{E}) \to K_0(D\mathcal{E})$, we have $K_0(D\mathcal{E}) = 0$, and the claim follows from lemma 7. □

9.1 The map $K_{-1}(R) \to I\!K_{-1}(R\text{−proj}) = K_0(D(R\text{−proj})^{\sim})$

In case $\mathcal{E} = R$−proj, the category of finitely generated projective R-modules, we construct a map from Bass' group $K_{-1}(R)$ to $K_0(D(R\text{−proj})^{\sim})$. By theorem 5, we know that these two groups are isomorphic. We leave it to the reader to check that the map below realizes the isomorphism.

Write $D(R)$ for $D(R$−proj$)$. The map $\rho : K_0(R[t, t^{-1}]) \to K_0(D(R)^{\sim})$ is defined as follows. Let $q = \sum_{i=-k}^{k} a_i t^i : R[t, t^{-1}]^n \to R[t, t^{-1}]^n$ be an idempotent, *i.e.*, $q^2 = q$, representing a finitely generated projective $R[t, t^{-1}]$-module. The a_i's are R-linear maps $a_i : V \to V$ with $V = R^n$. Let $p = (a_{j-i})_{i,j=0,\ldots k} : V^{k+1} \to V^{k+1}$, $d = (a_{k+j-i})_{i,j=0,\ldots k} : V^{k+1} \to V^{k+1}$, and $h = (a_{-k+j-i})_{i,j=0,\ldots k} : V^{k+1} \to V^{k+1}$. The equation $q = q^2$ implies 1) $p = p^2 + dh + hd$, 2) $d = pd + dp$ and 3) $0 = d^2$. Hence we get an endomorphism of a chain complex

$$\begin{tikzcd}
\cdots \arrow[r] & V^{k+1} \arrow[r, "d"] \arrow[d, "p"'] & V^{k+1} \arrow[r, "d"] \arrow[d, "1-p"'] & V^{k+1} \arrow[r, "d"] \arrow[d, "p"'] & V^{k+1} \arrow[r, "d"] \arrow[d, "1-p"'] & V^{k+1} \arrow[r, "d"] \arrow[d, "p"'] & \cdots \\
\cdots \arrow[r] & V^{k+1} \arrow[r, "d"] & V^{k+1} \arrow[r, "d"] & V^{k+1} \arrow[r, "d"] & V^{k+1} \arrow[r, "d"] & V^{k+1} \arrow[r, "d"] & \cdots .
\end{tikzcd}$$

3) ensures that the rows are really complexes, 2) that it is a map of chain complexes and 1) that the endomorphism is an idempotent up to homotopy. This defines an element of $K_0(D(R)^{\sim})$, the image under ρ of the element we started with. If a module comes from $K_0(R[t])$ or $K_0(R[t^{-1}])$, then h or d is zero and therefore $p = p^2$, not only up to homotopy, and we can take the image degree-wise. This means the idempotent has image in $D(R)$ and is thus zero in $K_0(D(R)^{\sim})$. It follows that ρ defines a map $\rho : K_{-1}(R) \to K_0(D(R)^{\sim}) = I\!K_{-1}(R)$.

10 Negative K-groups of abelian categories

Theorem 6 *Let $\mathcal{A}$ be a small abelian category. Then $I\!K_{-1}(\mathcal{A}) = 0$.*

Proof The t-structure on $D^b(\mathcal{A})$ of [BBD82] extends to a t-structure on $D(\mathcal{A})$. The truncations $\tau_{\leq 0}$ and $\tau_{\geq 1}$ have images in $D^-(\mathcal{A})$ and $D^+(\mathcal{A})$, respectively. Recall that there is a distinguished triangle

$$\tau_{\leq 0}X \to X \to \tau_{\geq 1}X \to (\tau_{\leq 0}X)[1]$$

which is functorial in X. The t-structure extends to a t-structure on the idempotent completion $D(\mathcal{A})^{\sim}$.

Suppose that X is an object of $D(\mathcal{A})^{\sim}$. The objects $\tau_{\leq 0}X$ and $\tau_{\geq 1}X$ are objects of $D^{-}(\mathcal{A})^{\sim}$ and $D^{+}(\mathcal{A})^{\sim}$, respectively. The two categories $D^{-}(\mathcal{A})$ and $D^{+}(\mathcal{A})$ are idempotent complete, by the Eilenberg swindle, which implies $\mathbb{K}_0(D^{-}(\mathcal{A})) = \mathbb{K}_0(D^{+}(\mathcal{A})) = 0$, and by fact 2 of section 2.1. Thus $\tau_{\leq 0}X$ and $\tau_{\geq 1}X$ are objects of $D^{-}(\mathcal{A})$ and $D^{+}(\mathcal{A})$, respectively. As a triangulated category, $D(\mathcal{A})$ is extension closed in its idempotent completion [BS01]. Therefore, X is an object of $D(\mathcal{A})$. It follows that $D(\mathcal{A})$ is idempotent complete, hence $\mathbb{K}_{-1}(\mathcal{A}) = 0$ (corollary 6). □

10.1 Noetherian abelian categories

Recall [Pop73, 5.7] that an object A of an abelian category $\mathcal{A}$ is called noetherian if any ascending chain $A_0 \subset A_1 \subset A_2 \subset \ldots \subset A$ of subobjects of A eventually stops. A small abelian category is called noetherian if all of its objects are noetherian. Any Serre subcategory of a noetherian abelian category is noetherian. Any quotient of a noetherian abelian category by a Serre subcategory is noetherian (exercise!). Examples of noetherian abelian categories are the abelian categories of finitely generated R-modules for R a noetherian ring, and the category of coherent O_X-modules for X a noetherian scheme.

Theorem 7 *Let $\mathcal{A}$ be a small noetherian abelian category. Then $\mathbb{K}_i\mathcal{A} = 0$ for $i < 0$.*

Proof As in section 4.2, let Lex$\mathcal{A}$ be the Grothendieck abelian category of left exact additive functors $\mathcal{A}^{op} \to \langle ab \rangle$. The Yoneda embedding $\mathcal{A} \to \mathrm{Lex}(\mathcal{A})$ identifies $\mathcal{A}$ (up to equivalence of categories) with the Serre subcategory of noetherian objects of Lex$(\mathcal{A})$ [Pop73, 5.8.8], [Pop73, 5.8.9], [Gab62, Théorème 1, p. 356].

For an abelian category $\mathcal{B}$, let End$\mathcal{B}$ be the category of endomorphisms of $\mathcal{B}$. Objects are endomorphisms and morphisms are maps of objects in $\mathcal{B}$ commuting with the respective endomorphisms. Obviously, End$\mathcal{B}$ is an abelian category. For any $A \in \mathcal{A} \subset \mathrm{Lex}\mathcal{A}$ there is an object $A[t] \in \mathrm{End}(\mathrm{Lex}\mathcal{A})$ constructed as follows. The underlying Lex$\mathcal{A}$-object is $A[t] = A \oplus At \oplus At^2 \ldots$, where At^i stands for a copy of A. The endomorphism of $A[t]$ is "multiplication by t", *i.e.,* the map which sends At^i to At^{i+1} identifying them. This construction is obviously functorial in A. Let $\mathcal{A}[t]$ be the full subcategory of End(Lex$\mathcal{A}$) of objects X for which there is an End(Lex$\mathcal{A}$)-epimorphism $A[t] \to X$ for some object $A \in \mathcal{A}$. The category $\mathcal{A}[t]$ is a noetherian abelian category. The proof of this fact is a simple adaptation of the proof that if R is a noetherian ring then so is $R[t]$.

Let $Nil \subset \mathcal{A}[t]$ be the full subcategory of nilpotent endomorphisms. It is a Serre subcategory, and we write $\mathcal{A}[t, t^{-1}]$ for the quotient of $\mathcal{A}[t]$ by the Serre subcategory Nil. The category $\mathcal{A}[t, t^{-1}]$ is again an (essentially) small noetherian abelian category. By lemma 8 below, the sequence of noetherian abelian categories $Nil \to \mathcal{A}[t] \to \mathcal{A}[t, t^{-1}]$ induces a short exact sequence of triangulated categories $D^b(Nil) \to D^b(\mathcal{A}[t]) \to D^b(\mathcal{A}[t, t^{-1}])$. Thus it induces a long exact sequence of negative K-groups (theorem 1, theorem 3).

There is an exact functor $j : \mathcal{A} \to Nil$ which sends the object A to the object A equipped with the zero endomorphism. We write i for the composition of j with the inclusion $Nil \subset \mathcal{A}[t]$. There is an exact functor $f : \mathcal{A} \to \mathcal{A}[t]$ sending A to $A[t]$. We have a functorial exact sequence $0 \to f \to f \to i \to 0$ which is $0 \to A[t] \stackrel{\times t}{\to} A[t] \to A \to 0$ for $A \in \mathcal{A}$. By additivity (corollary 4), it follows that $I\!K_n(f) + I\!K_n(i) = I\!K_n(f) : I\!K_n(\mathcal{A}) \to I\!K_n(\mathcal{A}[t])$, $n \leq 0$. Hence $0 = I\!K_n(i) : I\!K_n(\mathcal{A}) \to I\!K_n(\mathcal{A}[t])$, $n \leq 0$.

Any object $e : B \to B$ of $\mathcal{A}[t]$ is noetherian in $\mathcal{A}[t]$. If $e = 0$, then the object B of $\mathcal{A}$ is noetherian in $\mathcal{A}$, since e in $\mathcal{A}[t]$, and B in $\mathcal{A}$ have isomorphic posets of subobjects. Since an object $e : B \to B$ of Nil is a finite extension of objects equipped with the trivial endomorphism, B is an object of $\mathcal{A}$. Thus we have an exact forgetful functor $Nil \to \mathcal{A} : (e : B \to B) \mapsto B$ which is a retraction of j. It follows that $I\!K_n(j)$ is injective. Since $I\!K_n(i) = 0$, $I\!K_n(\mathcal{A})$ is a subquotient of $I\!K_{n+1}(\mathcal{A}[t, t^{-1}])$, $n < 0$, by the long exact sequence of negative K-groups associated to $Nil \to \mathcal{A}[t] \to \mathcal{A}[t, t^{-1}]$.

Descending induction on n starting with $n = -1$ (theorem 6) shows that $I\!K_n(\mathcal{A}) = 0$, $n < 0$, for any noetherian abelian category $\mathcal{A}$. □

Lemma 8 *Let $\mathcal{A}$ be a noetherian abelian category. The sequence of noetherian abelian categories $Nil \subset \mathcal{A}[t] \to \mathcal{A}[t, t^{-1}]$ induces an exact sequence of triangulated categories*

$$D^b(Nil) \longrightarrow D^b(\mathcal{A}[t]) \longrightarrow D^b(\mathcal{A}[t, t^{-1}]).$$

Proof By [Kel99, 1.15 Lemma b)], we only have to show that $D^b(Nil) \to D^b(\mathcal{A}[t])$ is fully faithful. We will verify Keller's criterion [Kel96, 12 C2] for the inclusion $Nil \subset \mathcal{A}[t]$.

The category $\mathrm{End}(\mathrm{Lex}\mathcal{A})$ is locally noetherian, and its category of noetherian objects is $\mathcal{A}[t]$. Let I be an indecomposable injective object in $\mathrm{End}(\mathrm{Lex}\mathcal{A})$. Then the endomorphism $\times t : I \to I$ is an isomorphism or every noetherian subobject of I is nilpotent. The proof is the same as in [Pop73, lemma 5.9.10].

Given an injection $N \subset X$ in $\mathcal{A}[t]$ with N nilpotent, let $X \subset E(X)$ be an injective envelope of X in $\mathrm{Lex}\mathcal{A}[t]$. Since X is noetherian, $E(X)$ is a finite direct sum of indecomposable injectives. So we can write $E(X) = I \oplus J$ with $\times t : J \to J$ an isomorphism and every noetherian subobject of I being nilpotent. The map $N \to J$ is trivial as N is nilpotent, so $N \to I$ is injective. Let M be the image of $X \to I$. Since X is noetherian, M is noetherian, thus nilpotent, and the map $N \to M$ is injective. □

*Remark 7 (*Regular rings) Let R be a regular noetherian ring. Then the inclusion of the category of finitely generated projective R-modules into the category of all finitely generated R-modules induces an equivalence of bounded derived categories. As the latter category is noetherian abelian, it follows that $K_i(R) = 0$ for $i < 0$ (theorem 5, theorem 7, corollary 1). This is a well-known theorem of Bass.

Remark 8 (G-theory) Let X be a noetherian scheme. Its G-theory is the K-theory associated to the noetherian abelian category of coherent O_X-modules. By Theorem 7, negative G-theory is trivial.

Conjecture 1 Let $\mathcal{A}$ be a small abelian category. We conjecture that

$$I\!K_i(\mathcal{A}) = 0, \quad \text{for all } i < 0.$$

Evidence is given in theorem 6 and theorem 7.

Remark 9 (Gabber) Even though $I\!K_i$ commutes with filtered colimits (corollary 5), vanishing for noetherian abelian categories does not imply the conjecture.

Not every small abelian category is the filtered colimit of small noetherian abelian categories. Take an abelian category $\mathcal{A}$ containing a morphism $f : A \to B$ such that $\ker f \subset \ker f^2 \subset \ker f^3 \subset \ldots \subset A$ is a strictly increasing sequence of subobjects. If $\mathcal{A}$ was a filtered colimit of noetherian categories, then the map f (and thus the whole sequence of kernels) would have to lie in one of the noetherian abelian categories, and the sequence of subobjects would have to stop.

11 Exact versus additive K-theory

Let $\mathcal{E}$ be an exact category. We write $\mathcal{E}^{\oplus}$ for the split exact category which, as an additive category, is $\mathcal{E}$. The identity functor $\mathcal{E}^{\oplus} \to \mathcal{E}$ is an exact functor.

If conjecture 1 is true, then the following proposition implies that

$$I\!K_i(\mathcal{E}^{\oplus}) \to I\!K_i(\mathcal{E})$$

is an isomorphism for $i < 0$. This is plainly false for $i \geq 0$. It is an open question for $i < 0$.

Proposition 2 *Let $\mathcal{E}$ be an exact category. There is an abelian category $\mathcal{A}$ and a zigzag of Frobenius pairs between* $\mathrm{Ac}^b(\mathcal{E})$ *and* $(\mathrm{Ch}^b(\mathcal{A}), \mathrm{Ac}^b(\mathcal{A}))$ *inducing equivalences of derived categories. In particular, there is a long exact sequence for* $i \leq 0$

$$I\!K_i(\mathcal{A}) \to I\!K_i(\mathcal{E}^{\oplus}) \to I\!K_i(\mathcal{E}) \to I\!K_{i-1}(\mathcal{A}) \to I\!K_{i-1}(\mathcal{E}^{\oplus}) \to \cdots$$

11.1 Effaceable functors in Lex$\mathcal{E}$

In the proof of proposition 2, we can assume $\mathcal{E}$ to be idempotent complete as $\mathcal{E} \to \tilde{\mathcal{E}}$ and $\mathcal{E}^{\oplus} \to \tilde{\mathcal{E}^{\oplus}}$ are $I\!K$-theory equivalences. Before proving the proposition we recall from [Kel90, appendix A] a more precise description of the category Lex$\mathcal{E}$ (section 4.2). Let Mod$\mathcal{E}$ be the category of additive functors from $\mathcal{E}^{op}$ to the category $\langle ab \rangle$ of abelian groups. It is a Grothendieck abelian category with $\mathcal{E}$ a set of small projective generators, where, as usual, $\mathcal{E}$ is identified with its image in Mod$\mathcal{E}$ via the Yoneda embedding. The full subcategory of finitely generated projective objects in Mod$\mathcal{E}$ is then equivalent to $\mathcal{E}$, via this identification, since we assume $\mathcal{E}$ to be idempotent complete.

Let $\mathcal{C} \subset$ Mod$\mathcal{E}$ be the full subcategory of effaceable functors, *i.e.,* those functors F such that for every $A \in \mathcal{E}$ and every map $A \to F$ in Mod$\mathcal{E}$ there is a deflation $B \to A$ in $\mathcal{E}$ such that the composition $B \to F$ is zero. The category $\mathcal{C}$ is a localizing subcategory of Mod$\mathcal{E}$. Moreover, Lex$\mathcal{E}$ can be identified with the quotient abelian category of Mod$\mathcal{E}$ by $\mathcal{C}$ such that the natural embedding $i_{\mathcal{E}}$: Lex$\mathcal{E} \subset$ Mod$\mathcal{E}$ is the section functor to the localization functor $a_{\mathcal{E}}$: Mod$\mathcal{E} \to$ Lex$\mathcal{E}$.

Let $fp\mathcal{C}$ be the full subcategory of finitely presented functors which are effaceable, *i.e.,* those functors $F \in \mathcal{C}$ for which there is an exact sequence $A \to B \to F \to 0$ in Mod$\mathcal{E}$ with A and B in $\mathcal{E}$.

Lemma 9 *Let $\mathcal{E}$ be an idempotent complete exact category. Then the category $fp\mathcal{C}$ of effaceable, finitely presented functors is an abelian, extension closed subcategory of the category $\mathcal{C}$ of effaceable functors in* $\mathrm{Mod}\mathcal{E}$.

Proof First we show that for a functor $F : \mathcal{E}^{op} \to \langle ab \rangle$ we have $F \in fp\mathcal{C}$ if and only if there is an exact sequence

$$0 \to A \to B \to C \to F \to 0 \tag{7}$$

in $\mathrm{Mod}\mathcal{E}$ with $A \to B \to \mathcal{C}$ a conflation in $\mathcal{E}$. Let F be in $fp\mathcal{C}$. Choose an exact sequence $X \stackrel{x}{\to} C \to F \to 0$ in $\mathrm{Mod}\mathcal{E}$ with $X, C \in \mathcal{E}$. By the effaceability of F, there is a deflation $y : Y \to C$ in $\mathcal{E}$ such that the composition $Y \to F$ is zero. Then $(x, y) : B := X \oplus Y \to C$ is a deflation in $\mathcal{E}$ (section 4.2) and we denote by A its kernel. The sequence (7) is exact by construction and the left exactness of the Yoneda embedding $\mathcal{E} \to \mathrm{Mod}\mathcal{E}$.

Given an exact sequence (7) in $\mathrm{Mod}\mathcal{E}$ with $A \to B \to C$ a conflation in $\mathcal{E}$, the object F is easily seen to be effaceable, and thus is in $fp\mathcal{C}$.

As usual, finitely presented functors are closed under cokernels and extensions. Since this is also true for $\mathcal{C} \subset \mathrm{Mod}\mathcal{E}$, it is true for $fp\mathcal{C}$ as well.

The category $fp\mathcal{C}$ is also closed under kernels of epimorphisms in $\mathrm{Mod}\mathcal{E}$. Given an epimorphism $F \to G$ of objects in $fp\mathcal{C}$, choose projective resolutions for F and G as in (7). The map $F \to G$ extends to a map of resolutions and we denote by

$$0 \to A \to B \to C \to D \to 0 \tag{8}$$

its cone. Note that $A, B, C, D \in \mathcal{E}$. The surjectivity of $F \to G$ implies that (8) is exact except, possibly, at C. Since D is projective in $\mathrm{Mod}\mathcal{E}$, the map $C \to D$ splits and therefore has a kernel E in $\mathcal{E}$. The sequence $0 \to A \to B \to E$ is a resolution for the kernel of $F \to G$.

It follows that $fp\mathcal{C} \subset \mathrm{Mod}\mathcal{E}$ is closed under kernels as well. Hence, $fp\mathcal{C}$ is an abelian category. □

Proof (of proposition 2) Recall that we can assume $\mathcal{E}$ to be idempotent complete. We will show that we can take $\mathcal{A} = fp\mathcal{C}$.

Let $\hat{\mathcal{E}}$ be the full subcategory of $\mathrm{Mod}\mathcal{E}$ consisting of those functors $F : \mathcal{E}^{op} \to \langle ab \rangle$ for which there is an exact sequence $0 \to A \to F \to G \to 0$ in $\mathrm{Mod}\mathcal{E}$ with A in $\mathcal{E}$ and G in $fp\mathcal{C}$. Then $\hat{\mathcal{E}}$ is extension closed in $\mathrm{Mod}\mathcal{E}$.

To see this, let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence in $\mathrm{Mod}\mathcal{E}$ with F_1, F_3 in $\hat{\mathcal{E}}$. Choose exact sequences $0 \to A_i \to F_i \to G_i \to 0$, with A_i in $\mathcal{E}$ and G_i in $fp\mathcal{C}$, $i = 1, 3$. As A_3 is projective, the injection $A_3 \to F_3$ lifts to a map $A_3 \to F_2$ which is an injection as well. Let G_2 be the pushout of the epimorphism $F_1 \to G_1$ along the injection $F_1 \to F_2/A_3$. It is an extension of G_1 and G_3 and thus in $fp\mathcal{C}$. The kernel of the epimorphism $F_2 \to G_2$ is an extension of A_1 and A_3 and thus in $\mathcal{E}$. Hence, F_2 is in $\hat{\mathcal{E}}$.

We make $\hat{\mathcal{E}}$ into an exact category by declaring those sequences to be conflations which are also conflations in $\mathrm{Mod}\mathcal{E}$.

Standard arguments in homological algebra show that any F in $\hat{\mathcal{E}}$ has a resolution as in (7) with A, B, C in $\mathcal{E}$. Thus, the inclusion $\mathcal{E}^{\oplus} \to \hat{\mathcal{E}}$ induces an essentially surjective functor of bounded derived categories $D^b(\mathcal{E}^{\oplus}) \to D^b(\hat{\mathcal{E}})$. The triangle functor is fully faithful by resolution or Keller's criterion [Kel96, 11.7, 12.1]. Hence, we have an equivalence $D^b(\mathcal{E}^{\oplus}) \xrightarrow{\sim} D^b(\hat{\mathcal{E}})$.

Next the inclusion $fp\mathcal{C} \to \hat{\mathcal{E}}$ satisfies the dual of Keller's criterion [Kel96, 11.7, 12.1]. To see this, let $F \to G \to H$ be a conflation in $\hat{\mathcal{E}}$ with F in $fp\mathcal{C}$. Choose a conflation $A \to H \to K$ in $\hat{\mathcal{E}}$ with K in $fp\mathcal{C}$ and $A \in \mathcal{E} = \mathrm{Mod}\mathcal{E}{-}\mathrm{proj}$. The inflation $A \to H$ lifts to an inclusion $A \to G$ whose cokernel, say L, is an extension of F and K. It follows that L is in $fp\mathcal{C}$ and that $F \to L$ is an inflation in $fp\mathcal{C}$.

Summarizing, we have a fully faithful triangle functor $D^b(fp\mathcal{C}) \hookrightarrow D^b(\hat{\mathcal{E}}) \cong D^b(\mathcal{E}^{\oplus})$. Let F be an object of $fp\mathcal{C}$. Choose an exact sequence in $\mathrm{Mod}\mathcal{E}$ as in (7) with $A \to B \to C$ a conflation in $\mathcal{E}$. Then the object F, considered as a chain complex concentrated in degree zero, is sent to the acyclic chain complex $0 \to A \to B \to C \to 0$ in $D^b(\mathcal{E}^{\oplus})$. Such acyclic chain complexes generate the triangulated subcategory of all acyclic chain complexes in $D^b(\mathcal{E}^{\oplus})$. Thus, the inclusion $D^b(fp\mathcal{C}) \hookrightarrow D^b(\hat{\mathcal{E}}) \cong D^b(\mathcal{E}^{\oplus})$ identifies $D^b(fp\mathcal{C})$ with the triangulated subcategory of $D^b(\mathcal{E}^{\oplus})$ consisting of the acyclic chain complexes.

On the level of Frobenius pairs, the map $\mathrm{Ch}^b\,\mathcal{E}^{\oplus} \to (\mathrm{Ch}^b\,\hat{\mathcal{E}}, \mathrm{Ac}^b\,\hat{\mathcal{E}})$ induces an equivalence of derived categories. Let $\mathcal{B}$ be the full subcategory of $\mathrm{Ch}^b\,\hat{\mathcal{E}}$ of objects which are isomorphic in $D^b\hat{\mathcal{E}}$ to an object of $\mathrm{Ac}^b\,\mathcal{E}$. By the arguments above, we have maps of Frobenius pairs

$$(\mathrm{Ch}^b\, fp\mathcal{C}, \mathrm{Ac}^b\, fp\mathcal{C}) \to (\mathcal{B}, \mathrm{Ac}^b\,\hat{\mathcal{E}}) \leftarrow \mathrm{Ac}^b\,\mathcal{E}$$

which induce equivalences of derived categories. □

12 Higher algebraic K-theory

In this section we will construct a functor $\mathbb{K}$ from Frobenius pairs to spectra whose negative homotopy groups are the negative $\mathbb{K}$-groups introduced in sections 2 and 5. Short exact sequences of Frobenius pairs will give rise to homotopy fibrations of $\mathbb{K}$-theory spectra. This extends the results of the previous sections to higher algebraic K-theory. In particular, theorem 1, remark 3, remark 4, lemma 7, and proposition 2 also hold for higher algebraic K-groups, that is for $i \in \mathbb{Z}$ instead of just $i \leq 0$.

Without loss of generality, we will assume all additive categories (in particular, all Frobenius pairs) in this section to have a unital and associative direct sum operation $\oplus$. Moreover, functors of additive categories are to preserve the unit and the direct sum operation. We can do so because there is a functorial strictification which embeds an additive category into an equivalent additive category which has a unital and associative coproduct [May74].

Our reference for spectra is [BF78].

Definition 10 Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0)$ be a Frobenius pair. Its associated category with cofibrations and weak equivalences (also called Waldhausen category) [Wal85] is the category $\mathcal{A}$ with cofibrations $\mathrm{cof}\boldsymbol{\mathcal{A}}$ the inflations in $\mathcal{A}$ and weak equivalences $w\boldsymbol{\mathcal{A}}$ the maps in $\mathcal{A}$ which are isomorphisms in $D\boldsymbol{\mathcal{A}}$. By abuse of notation we still write $\boldsymbol{\mathcal{A}}$ for this category with cofibrations and weak equivalences. The *K-theory space of* $\boldsymbol{\mathcal{A}}$ is defined, according to [Wal85], as

$$K(\boldsymbol{\mathcal{A}}) = \Omega|wS.\boldsymbol{\mathcal{A}}|.$$

The association $\boldsymbol{\mathcal{A}} \mapsto K(\boldsymbol{\mathcal{A}})$ defines a functor from Frobenius pairs to spaces.

Remark 10 The category with cofibrations and weak equivalences $\boldsymbol{\mathcal{A}}$ is a Waldhausen category with factorizations (definition 11): Every map $f : A \to B$ is the composition of an inflation $(f, i) : A \to B \oplus I$ and a weak equivalence $(1, 0) : B \oplus I \to B$, where $A \hookrightarrow I$ is an inflation into an injective object. Moreover, $K(\mathcal{A})$ and $K(\mathcal{A}^{op})$ are isomorphic.

Lemma 10 *There is a contraction $H_{\boldsymbol{\mathcal{A}}} : I \wedge K(\mathcal{F}\boldsymbol{\mathcal{A}}) \to K(\mathcal{F}\boldsymbol{\mathcal{A}})$, i.e., a map with $H_{\boldsymbol{\mathcal{A}}}(0,\ ) = 0$ and $H_{\boldsymbol{\mathcal{A}}}(1,\ ) = id$, which is functorial in the Frobenius pair $\boldsymbol{\mathcal{A}}$.*

Proof Let X be a pointed space. The co-H space structure $S^1 \to S^1 \vee S^1$ induces an H-space structure $\nu : \Omega X \times \Omega X \to \Omega X$.

If $\mu : X \times X \to X$ is a unital and associative H-space, then the H-space structure $\Omega\mu : \Omega X \times \Omega X \to \Omega X$ is homotopic to ν via a homotopy which is functorial for unital and associative H-spaces by the following argument. The diagram

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[dr, "id \times id"'] & S^1 \vee S^1 \arrow[r, "f_1 \vee f_2"] \arrow[d] & X \vee X \arrow[d] \arrow[dr, "id \vee id"] & \\
& S^1 \times S^1 \arrow[r, "f_1 \times f_2"'] & X \times X \arrow[r, "\mu"'] & X
\end{tikzcd}$$

commutes, except for the left hand triangle which commutes up to homotopy. The upper composition yields ν, the lower $\Omega\mu$. A choice of homotopy for the left hand triangle defines a homotopy $\nu \simeq \Omega\mu$ which is functorial for unital and associative H-spaces.

Now we prove the claim of the lemma. The associative and unital direct sum operation $\oplus$ on $\boldsymbol{\mathcal{A}}$ induces an associative and unital H-space structure on $X_{\boldsymbol{\mathcal{A}}} = |wS.\boldsymbol{\mathcal{A}}|$, thus on $K(\boldsymbol{\mathcal{A}}) = \Omega X_{\boldsymbol{\mathcal{A}}}$, which is functorial in $\mathcal{A}$. By the above, its H-space multiplication is homotopic to ν, functorially in $\mathcal{A}$. The map $T : \mathcal{F}\mathcal{A} \to \mathcal{F}\mathcal{A}$ of lemma 4 induces a map $T : \mathcal{F}\boldsymbol{\mathcal{A}} \to \mathcal{F}\boldsymbol{\mathcal{A}}$ of Frobenius pairs. The natural equivalence $id \oplus T \cong T$ of lemma 4 is functorial in $\mathcal{A}$ (for maps preserving the unital and associative coproduct). By the argument above, we have a functorial homotopy $K(T) \simeq \Omega\mu(K(T), id) \simeq \eta(K(T, id))$. Since ν has an H-space inverse which is functorial in X, there is a homotopy $* \simeq id_{K(\boldsymbol{\mathcal{A}})}$ functorial in $\boldsymbol{\mathcal{A}}$. □

12.1 Definition of the $I\!\!K$-theory spectrum

Let $\boldsymbol{\mathcal{A}}$ be a Frobenius pair. The map $\mathcal{A} \to \mathcal{F}\mathcal{A}$ of Frobenius categories induces maps $\boldsymbol{\mathcal{A}} \to \mathcal{F}\boldsymbol{\mathcal{A}} \to \mathcal{S}\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}} \to (\mathcal{A}, \mathcal{A}) \to \mathcal{S}\boldsymbol{\mathcal{A}}$ of Frobenius pairs whose compositions coincide. Thus we obtain a commutative diagram of K-theory spaces

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r] \arrow[d] & K(\mathcal{A}, \mathcal{A}) \simeq * \arrow[d] \\ * \simeq K(\mathcal{F}\mathcal{A}) \arrow[r] & K(\mathcal{S}\mathcal{A}) \end{tikzcd} \tag{9}$$

functorial in $\mathcal{A}$. By lemma 10, the space $K(\mathcal{F}\mathcal{A})$ is contractible, functorially in $\mathcal{A}$. The space $K(\mathcal{A}, \mathcal{A})$ is contractible, functorially in $\mathcal{A}$, because $wS_n(\mathcal{A}, \mathcal{A})$ has an initial object, namely the unit 0, as $wS_n(\mathcal{A}, \mathcal{A}) = S_n\mathcal{A}$ is an additive category.

The square and the two contractions yield a map

$$K(\mathcal{A}) \to \Omega K(\mathcal{S}\mathcal{A}), \tag{10}$$

and we define the spectrum $\mathbb{K}(\mathcal{A})$, associated with the Frobenius pair $\mathcal{A}$, to be the sequence of spaces $K(\mathcal{A})$, $K(\mathcal{S}\mathcal{A})$, $K(\mathcal{S}^2\mathcal{A})$, ... together with the structure maps given by (10).

By the functoriality of the square (9) and the functorial contractions, $\mathbb{K}$ defines a functor from Frobenius pairs to spectra.

Theorem 8 *Let $\mathcal{A}$ be a Frobenius pair. Then the spectrum $\Omega\mathbb{K}(\mathcal{A})$ is an Ω-spectrum. The homotopy groups of $\mathbb{K}(\mathcal{A})$ are given by*

$$\pi_i\mathbb{K}(\mathcal{A}) = \begin{cases} \pi_i K(\mathcal{A}) & i > 0 \text{ as defined in definition 10} \\ \mathbb{K}_0(\mathcal{A}) = K_0(D(\mathcal{A})^{\sim}) & i = 0 \\ \mathbb{K}_i(\mathcal{A}) & i < 0 \text{ as defined in sections 2 and 5.} \end{cases}$$

Proof Recall that $D\mathcal{A} \subset D\mathcal{F}\mathcal{A} \to D\mathcal{S}\mathcal{A}$ is an exact sequence of triangulated categories (theorem 3). Let $\mathcal{B}$ be the full subcategory of $\mathcal{F}\mathcal{A}$ whose objects are zero in $D\mathcal{S}\mathcal{A}$. Then $\mathcal{B}$ inherits an exact structure from $\mathcal{F}\mathcal{A}$ which makes it into Frobenius category. Write $\mathcal{A}\hat{}$ for the Frobenius pair $(\mathcal{B}, \mathcal{F}\mathcal{A}_0)$. The map of Frobenius pairs $\mathcal{A} \to \mathcal{A}\hat{}$ induces an idempotent completion $D\mathcal{A} \to D\mathcal{A}\hat{}$, because $D\mathcal{F}\mathcal{A}$ is idempotent complete, as it has countable coproducts. By cofinality (proposition 4), the map $K(\mathcal{A}) \to K(\mathcal{A}\hat{})$ is an isomorphism on π_i, $i > 0$, and a monomorphism on π_0. By proposition 5, we have a homotopy cartesian square

$$\begin{tikzcd} K(\mathcal{A}\hat{}) \arrow[r] \arrow[d] & K(\mathcal{F}\mathcal{A}, \mathcal{F}\mathcal{A}) \simeq * \arrow[d] \\ * \simeq K(\mathcal{F}\mathcal{A}) \arrow[r] & K(\mathcal{S}\mathcal{A}). \end{tikzcd} \tag{11}$$

Thus, the induced maps

$$K(\mathcal{A}\hat{}) \to \Omega K(\mathcal{S}\mathcal{A}) \to \Omega K((\mathcal{S}\mathcal{A})\hat{}) \tag{12}$$

are homotopy equivalences. Since $\Omega K(\mathcal{A}) \to \Omega K(\mathcal{A}\hat{})$ is a homotopy equivalence as well, the spectrum $\Omega\mathbb{K}(\mathcal{A})$ is an Ω-spectrum.

The above argument actually shows that the spectrum

$$\hat{\mathbb{K}}(\mathcal{A}) = \{K(\mathcal{A}\hat{}), K((\mathcal{S}\mathcal{A})\hat{})K((\mathcal{S}^2\mathcal{A})\hat{}), ...\}$$

with structure maps given by (12) is an Ω-spectrum, and the map $\mathbb{K} \to \hat{\mathbb{K}}$, induced by $\mathcal{A} \to \mathcal{A}\hat{}$, is a weak equivalence of spectra. Thus, for $i > 0$ we have $\pi_i \mathbb{K}(\mathcal{A}) = \pi_i \hat{\mathbb{K}}(\mathcal{A}) = K_i(\mathcal{A}\hat{}) = K_i(\mathcal{A})$. The last equality holds by cofinality. For $i \leq 0$, we have $\pi_i \mathbb{K}(\mathcal{A}) = \pi_i \hat{\mathbb{K}}(\mathcal{A}) = K_0((\mathcal{S}^{-i}\mathcal{A})\hat{}) = K_0((D\mathcal{S}^{-i}\mathcal{A})^{\sim}) = \mathbb{K}_i(\mathcal{A})$. $\square$

Theorem 9 *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be an exact sequence of Frobenius pairs. Then, applying the $\mathbb{K}$-theory functor yields a homotopy cartesian square*

$$\begin{tikzcd} \mathbb{K}(\mathcal{A}) \arrow[r] \arrow[d] & \mathbb{K}(\mathcal{B}) \arrow[d] \\ * \simeq \mathbb{K}(\mathcal{A}, \mathcal{A}) \arrow[r] & \mathbb{K}(\mathcal{C}) \end{tikzcd} \tag{13}$$

of $\mathbb{K}$-theory spectra in which the lower left corner is contractible.

In particular, there is a long exact sequence of abelian groups, $i \in \mathbb{Z}$,

$$\cdots \to \mathbb{K}_i(\mathcal{A}) \to \mathbb{K}_i(\mathcal{B}) \to \mathbb{K}_i(\mathcal{C})$$
$$\xrightarrow{\delta} \mathbb{K}_{i-1}(\mathcal{A}) \to \mathbb{K}_{i-1}(\mathcal{B}) \to \mathbb{K}_{i-1}(\mathcal{C}) \to \cdots$$

The square and the long exact sequence are functorial for maps of exact sequences of Frobenius pairs.

Proof By theorem 3, $\mathcal{S}^n\mathcal{A} \to \mathcal{S}^n\mathcal{B} \to \mathcal{S}^n\mathcal{C}$ is also an exact sequence of Frobenius pairs. Since $\Omega\mathbb{K}$ is an Ω-spectrum (theorem 8), it suffices to show that

$$\begin{tikzcd} \Omega K(\mathcal{A}) \arrow[r] \arrow[d] & \Omega K(\mathcal{B}) \arrow[d] \\ * \simeq \Omega K(\mathcal{A}, \mathcal{A}) \arrow[r] & \Omega K(\mathcal{C}) \end{tikzcd} \tag{14}$$

is a homotopy cartesian square of spaces. Let $\mathcal{B}_1$ be the full subcategory of $\mathcal{B}$ of objects which are zero in $D\mathcal{C}$. As usual, the exact structure on $\mathcal{B}$ induces an exact structure on $\mathcal{B}_1$ which makes $\mathcal{B}_1$ into a Frobenius category with $\mathcal{B}-\text{prinj} = \mathcal{B}_1-\text{prinj}$. The claim now follows from proposition 5 and cofinality (proposition 4) applied to $\mathcal{A} \to (\mathcal{B}_1, \mathcal{B}_0)$ and $(\mathcal{B}, \mathcal{B}_1) \to \mathcal{C}$. The last map may only exist when $\mathcal{C}$ is saturated (*i.e.,* if $\mathcal{C}_0$ consists of all the objects of $\mathcal{C}$ which are zero in $D\mathcal{C}$), but, replacing $\mathcal{C}$ with its saturation doesn't change K-theory as both have isomorphic associated Waldhausen categories. $\square$

12.2 The spectrum $\mathbb{K}(\mathcal{E})$, $\mathcal{E}$ an exact category

As in definition 8 we define the $\mathbb{K}$-theory spectrum of an exact category $\mathcal{E}$ by

$$\mathbb{K}(\mathcal{E}) = \mathbb{K}(\text{Ch}^b\,\mathcal{E}, \text{Ac}^b\,\mathcal{E}).$$

We write $\mathbb{K}(R) = \mathbb{K}(R-\text{proj})$ for R a ring. By [TT90, 1.11.7] and theorem 8, we have $\mathbb{K}_i(\mathcal{E}) = \mathbb{K}_i^Q(\mathcal{E})$, Quillen's K-groups of $\mathcal{E}$, $i > 0$. By theorem 5, negative

$I\!K$-groups as defined here coincide with negative K-groups of Bass and Thomason whenever they have defined them.

By theorem 9, a sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ of exact categories such that $D^b(\mathcal{A}) \to D^b(\mathcal{B}) \to D^b(\mathcal{C})$ is exact, induces a long exact sequence of K-groups including the negative K-groups. For situations in which the theorem can be applied, we refer the reader to [TT90], [Kel99], [Nee92].

12.3 The Spectrum $I\!K(A)$, A a dga

Let A be a dga. As in definition 9 we define

$$I\!K(A) = I\!K(\text{cell } A\text{−mod}).$$

If $A \to B$ is a quasi-isomorphism, then $\otimes_A B$ induces an equivalence of derived categories and thus isomorphisms (theorem 9) $I\!K_i(A) \xrightarrow{\sim} I\!K_i(B)$, $i \in \mathbb{Z}$.

We conclude the section with 3 propositions which we needed in the proofs of theorem 8 and theorem 9.

Proposition 3 *Let $F : \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ be a map of Frobenius pairs such that $D\boldsymbol{\mathcal{A}} \to D\boldsymbol{\mathcal{B}}$ is a an equivalence. Then $K(\boldsymbol{\mathcal{A}}) \to K(\boldsymbol{\mathcal{B}})$ is a homotopy equivalence.*

Proof We translate Thomason's proof [TT90, 1.9.8.] with minor changes into the language of Frobenius pairs in order to see that no functorial (co-) cylinders are required.

Given an exact functor $F : \mathcal{A} \to \mathcal{B}$ between exact categories, let $\mathcal{C}_F$ be the category whose objects are data $(A, i : F(A) \hookrightarrow B)$ with A an object of $\mathcal{A}$ and $i : F(A) \hookrightarrow B$ an inflation of $\mathcal{B}$. A map from $(A, i : F(A) \to B)$ to $(A', i' : F(A') \to B')$ is a pair of maps $a : A \to A'$ in $\mathcal{A}$ and $b : B \to B'$ in $\mathcal{B}$ such that $i'F(a) = bi$. Declare a sequence in $\mathcal{C}_F$ to be a conflation if it is a conflation when evaluated at A, B and $B/F(A)$. This makes $\mathcal{C}_F$ into an exact category.

If $F : \mathcal{A} \to \mathcal{B}$ is a map of Frobenius categories, then $\mathcal{C}_F$ is a Frobenius category as well. Its projective-injective objects are those $(A, F(A) \hookrightarrow B)$ with A and B projective-injective in $\mathcal{A}$ and $\mathcal{B}$ respectively.

Let $(F, F_0) : \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ a map of Frobenius pairs with $D\boldsymbol{\mathcal{A}} \to D\boldsymbol{\mathcal{B}}$ an equivalence. We can assume $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ to be saturated, *i.e.,* the objects in $\mathcal{A}_0$, $\mathcal{B}_0$ are exactly those which are zero in the derived categories. This is because the Waldhausen categories of a Frobenius pair and its saturation are the same.

We define a Frobenius pair $\boldsymbol{\mathcal{C}} = (\mathcal{C}, \mathcal{C}_0)$ as follows. The category $\mathcal{C}$ is the full subcategory of $\mathcal{C}_F$ of objects $(A, F(A) \hookrightarrow B)$ with $B/F(A)$ zero in $D\mathcal{B}$, or equivalently, with $F(A) \hookrightarrow B$ a weak equivalence in $\boldsymbol{\mathcal{B}}$. The inclusion $\mathcal{C} \subset \mathcal{C}_F$ is closed under extensions, kernels of deflations and cokernels of inflations. It contains all projective-injective objects of $\mathcal{C}_F$. Thus, declaring a sequence in $\mathcal{C}$ to be a deflation if it is a deflation in $\mathcal{C}_F$, makes $\mathcal{C}$ into a Frobenius category. We let $\mathcal{C}_0 = \mathcal{C}_{F_0}$. Since $\mathcal{C}_0 \subset \mathcal{C}$ preserves projective-injective objects, $\boldsymbol{\mathcal{C}}$ is a Frobenius pair. In terms of Waldhausen categories, weak equivalences in $\boldsymbol{\mathcal{C}}$ are exactly point-wise weak equivalences, *i.e.,* those maps which are weak equivalences when evaluated at A and at B.

The map $\mathcal{A} \to \mathcal{C} : A \mapsto (A, id : F(A) \to F(A))$ has a retraction $\mathcal{C} \to \mathcal{A} : (A, F(A) \to B) \mapsto A$. It is a K-theory equivalence because the composition $\mathcal{C} \to \mathcal{A} \to \mathcal{C}$ is naturally weakly equivalent to the identity functor: let $(A, i : F(A) \hookrightarrow B)$ be an object of $\mathcal{C}$, the natural weak equivalence is $(id, i) : (A, id : F(A) \to F(A)) \to (A, i : F(A) \to B)$.

The functor $\mathcal{C} \to \mathcal{B} : (A, F(A) \to B) \to B$ will be a K-theory equivalence by the dual of the approximation theorem 10. Since any Frobenius pair is a Waldhausen category with factorization (remark 10) we only have to check definition A.1 $\mathrm{appr}_{F^{op}}$.

Call two maps $f, g : X \to Y$ in a Frobenius category homotopic if their difference factors through a projective-injective object. Given maps $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ such that gf is homotopic to h, if f is an inflation, then there is a g', homotopic to g, such that $g'f = h$.

We check $\mathrm{appr}_{F^{op}}$. Given a diagram $F(A) \overset{\sim}{\hookrightarrow} B \overset{b'}{\leftarrow} B'$ in $\mathcal{B}$, construct a $\mathcal{B}$-diagram as in [TT90, 1.9.8.3]

$$\begin{tikzcd}
FA \arrow[r, tail, "\sim"] \arrow[d, "F( )"', "\sim"] & B \arrow[d, "\sim"] & \\
FA_2 \arrow[r, "\sim"] & B_2 & \\
FA_1 \arrow[r, "\sim"] \arrow[u, "F( )"] & B_1 \arrow[u] & B' \arrow[l, "\sim"'] \arrow[uul, bend right, "b'"']
\end{tikzcd} \tag{15}$$

which is commutative up to homotopy. By remark 10, we can assume $FA_2 \to B_2$ to be an inflation. Since $FA \to B$ is an inflation, we can change $B \to B_2$, up to homotopy, such that the upper square commutes. Next, choose inflations $FA_1 \hookrightarrow I$ and $B' \hookrightarrow J$ into injective objects. Since the lower square commutes up to homotopy, the difference of the two compositions factor through I. We can make the lower square commutative by replacing B_1 with $B_1 \oplus I$. Similarly, we can make the right hand square commutative by replacing the new B_1 with $B_1 \oplus J$ without destroying the commutativity of the lower square. So we can assume (15) to commute, and the horizontal maps to be inflations. Choose a $\mathcal{C}$-deflation $(P, F(P) \hookrightarrow Q) \to (A_2, F(A_2) \hookrightarrow B_2)$ with P, Q projective. Replacing the lower vertical maps with the $\mathcal{C}$-deflation $(A_1 \oplus P, F(A_1 \oplus P) \hookrightarrow B_1 \oplus Q) \longrightarrow (A_2, FA_2 \hookrightarrow B_2)$, we can assume (15) to be commutative, the horizontal maps to be inflations and the lower vertical maps to be deflations. Let A_3 be the pull-back of $A \to A_2$ along the $\mathcal{A}$-deflation $A_1 \to A_2$, let B_3 be the pull-back of $B \to B_2$ along the $\mathcal{B}$-deflation $B_1 \to B_2$. The map $FA_3 \to B_3$ is an inflation and a weak equivalence as it is a pull back in $\mathcal{C}$. The map $(A_3, FA_3 \hookrightarrow B_3) \longrightarrow (A, FA \hookrightarrow B)$ is a $\mathcal{C}$ deflation as it is the pull back of a $\mathcal{C}$-deflation. The universal properties of pull-backs yield a commutative diagram

$$\begin{tikzcd}
FA \arrow[r, tail, "\sim"] & B & \\
FA_3 \arrow[r, tail, "\sim"] \arrow[u, "F( )"] & B_3 \arrow[u] & B' . \arrow[l, "\sim"'] \arrow[ul, bend right, "b'"']
\end{tikzcd}$$

This verifies the approximation property. □

Proposition 4 (Cofinality) *Let $\boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ be a map of Frobenius pairs such that $D\boldsymbol{\mathcal{A}} \to D\boldsymbol{\mathcal{B}}$ is cofinal (section 2.1). Then $K(\boldsymbol{\mathcal{A}}) \to K(\boldsymbol{\mathcal{B}})$ induces isomorphisms on π_i, $i > 0$ and a monomorphism on π_0.*

Proof Let $\mathcal{B}_1 \subset \mathcal{B}$ be the full subcategory of objects which are isomorphic in $D\mathcal{B}$ to an object of $D\boldsymbol{\mathcal{A}}$. Then $D\boldsymbol{\mathcal{A}} \to D(\mathcal{B}_1, \mathcal{B}_0)$ is an equivalence. So $K(\boldsymbol{\mathcal{A}}) \to K(\mathcal{B}_1, \mathcal{B}_0)$ is a homotopy equivalence (proposition 3). Moreover, $K(\mathcal{B}_1, \mathcal{B}_0) \to K(\boldsymbol{\mathcal{B}})$ induces isomorphisms on π_i, $i > 0$ and a monomorphism on π_0 by cofinality (theorem 12). □

Proposition 5 *Let $\mathcal{B}$ be a small Frobenius category and let $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \underline{\mathcal{B}}$ full triangulated subcategories which are closed under direct factors in $\underline{\mathcal{B}}$. Let $\mathcal{B}_i \subset \mathcal{B}$ be the full subcategory of those objects which lie in $\mathcal{D}_i$, $i = 0, 1$. It is closed under kernels of deflations, cokernels of inflations, extensions, and contains $\mathcal{B}-$*prinj. *Thus $\mathcal{B}_i$ inherits an exact structure from $\mathcal{B}$ that makes $\mathcal{B}_i$ into a Frobenius category. Then the sequence of Frobenius pairs $(\mathcal{B}_1, \mathcal{B}_0) \to (\mathcal{B}, \mathcal{B}_0) \to (\mathcal{B}, \mathcal{B}_1)$ induces a homotopy cartesian square of K-theory spaces*

$$\begin{tikzcd} K(\mathcal{B}_1, \mathcal{B}_0) \arrow[r] \arrow[d] & K(\mathcal{B}, \mathcal{B}) \simeq * \arrow[d] \\ K(\mathcal{B}, \mathcal{B}_0) \arrow[r] & K(\mathcal{B}, \mathcal{B}_1). \end{tikzcd} \tag{16}$$

Proof The homotopy cartesian square is Waldhausen's fibrations theorem for non-functorial cylinders (theorem 11) applied to the change of weak equivalences from $(\mathcal{B}, \mathcal{B}_0)$ to $(\mathcal{B}, \mathcal{B}_1)$. Recall (remark 10), that Frobenius pairs are Waldhausen categories with factorizations,*i.e.,* every map is a cofibration followed by a weak equivalence. □

A Appendix: Getting rid of cylinder functors

We give a proof of Waldhausen's Approximation [Wal85, 1.6.7] and Fibration theorem [Wal85, 1.6.4] and of Thomason's Cofinality [TT90, 1.10.1] result using slightly weaker hypotheses, replacing the existence of a cylinder functor and the cylinder axiom by the existence of the factorization of every map into a cofibration followed by a weak equivalence. No functoriality is required. Some of the results in this appendix have been found independently by Denis-Charles Cisinsk.

A.1 The approximation property

Let $F : \mathcal{A} \to \mathcal{B}$ be a map between categories with cofibration and weak equivalences, *i.e.,* F is a functor preserving cofibrations, push-out diagrams along cofibrations and weak equivalences. We say that F satisfies the approximation property if the following is true.

appr$_F$: For every map $b : F(A) \to B$ in $\mathcal{B}$ there are an object A' of $\mathcal{A}$, a cofibration $a : A \rightarrowtail A'$ in $\mathcal{A}$ and a weak equivalence $\beta : F(A') \xrightarrow{\sim} B$ in $\mathcal{B}$ such that $b = \beta \circ F(a)$. Moreover, a map α in $\mathcal{A}$ is a weak equivalence if and only if $F(\alpha)$ is.

Remark that this is exactly Waldhausen's approximation property on page 352 of [Wal85]. Remark also that appr$_{id_\mathcal{A}}$ means nothing else than that every morphism in $\mathcal{A}$ is the composition of a cofibration followed by a weak equivalence. In this appendix we will prove the following versions of Waldhausen's approximation and fibration theorems.

Theorem 10 (Approximation, compare [Wal85, 1.6.7.]) *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with cofibrations and weak equivalences. Suppose the weak equivalences in $\mathcal{A}$ and $\mathcal{B}$ satisfy the saturation axiom. Suppose further that the identity functor on $\mathcal{A}$ satisfies approximation, i.e., every morphism in $\mathcal{A}$ is the composition of a cofibration followed by a weak equivalence. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor satisfying the approximation property. Then the induced maps $w\mathcal{A} \to w\mathcal{B}$ and $wS.\mathcal{A} \to wS.\mathcal{B}$ are homotopy equivalences.*

The situation of the next theorem is the following (taken from [Wal85]). Suppose that $\mathcal{C}$ is a category with cofibrations and that $\mathcal{C}$ is equipped with two categories of weak equivalences, one finer than the other, $v\mathcal{C} \subset w\mathcal{C}$. Let $\mathcal{C}^w$ denote the subcategory with cofibrations of $\mathcal{C}$ given by the objects A in $\mathcal{C}$ having the property that the map $* \to A$ is in $w\mathcal{C}$. It inherits categories of weak equivalences $v\mathcal{C}^w = \mathcal{C}^w \cap v\mathcal{C}$ and $w\mathcal{C}^w = \mathcal{C}^w \cap w\mathcal{C}$.

Theorem 11 (Fibration, compare [Wal85, 1.6.4.]) *If every map in $\mathcal{C}$ is the composition of a cofibration followed by a map in $w\mathcal{C}$ and if $w\mathcal{C}$ satisfies the saturation and extension axiom, then the square*

$$\begin{tikzcd} vS.\mathcal{C}^w \arrow[r] \arrow[d] & wS.\mathcal{C}^w \simeq * \arrow[d] \\ vS.\mathcal{C} \arrow[r] & wS.\mathcal{C} \end{tikzcd}$$

is homotopy cartesian, and the upper right term is contractible.

Theorem 12 (Cofinality, compare [TT90, 1.10.1]) *Let* $(\mathcal{C}, \mathrm{cof}\mathcal{C}, w\mathcal{C})$ *be a category with cofibrations and weak equivalences such that every morphism in $\mathcal{C}$ is a cofibration followed by a weak equivalence. Suppose that the initial object $\emptyset \in \mathcal{C}$ is also a terminal object* 0. *Let $p : K_0(\mathcal{C}) \to G$ be a surjective group homomorphism, and let $\hat{\mathcal{C}}$ be the full subcategory of $\mathcal{C}$ of objects C whose class in $K_0(\mathcal{C})$ is trivial in G. Then* $(\hat{\mathcal{C}}, \hat{\mathcal{C}} \cap \mathrm{cof}\mathcal{C}, \hat{\mathcal{C}} \cap w\mathcal{C})$ *is a category with cofibrations and weak equivalences and the map $K_i(\hat{\mathcal{C}}) \to K_i(\mathcal{C})$ is an isomorphism for $i > 0$ and the monomorphism* $\ker p \subset K_0(\mathcal{C})$ *for $i = 0$.*

Definition 11 A *Waldhausen category with factorization* is a category with cofibrations and weak equivalences $\mathcal{C}$ satisfying appr$_{id_\mathcal{C}}$, *i.e.,* every map is a composition of a cofibration followed by a weak equivalence.

A.2 Diagram categories

Let $\mathcal{P}$ be a finite poset considered as a category in the usual way. The fundamental example is the poset $[1] = \{0 < 1\}$. A subposet $\mathcal{S} \subset \mathcal{P}$ is called saturated if for every element $S \in \mathcal{S}$ every predecessor of S in $\mathcal{P}$ also belongs to $\mathcal{S}$.

Let $\mathcal{C}$ be a category with cofibrations [Wal85, 1.1]. Write $\mathcal{C}^{\mathcal{P}}$ for the category of $\mathcal{P}$ diagrams in $\mathcal{C}$, *i.e.,* of functors $\mathcal{P} \to \mathcal{C}$. A map of $\mathcal{P}$-diagrams is a natural transformation $\phi : X_0 \to X_1$, which is the same as a functor $X : [1] \times \mathcal{P} \to \mathcal{C}$ whose restriction to $\{i\} \times \mathcal{P}$ is X_i, $i = 0, 1$. Let $P \in \mathcal{P}$ be an element and $\mathcal{S} \subset \mathcal{P}$ a saturated subposet such that $Q < P$ for all $Q \in \mathcal{S}$. For example one could think of $\mathcal{S} = \mathcal{S}_{<P} := \{Q \in \mathcal{P} \mid Q < P\}$. Write $I(S, P)$ for the full subcategory of $[1] \times \mathcal{P}$ which contains precisely all objects of $[1] \times \mathcal{S}$ and $(0, P)$. A map of $\mathcal{P}$ diagrams $X : [1] \times \mathcal{P} \to \mathcal{C}$ yields a functor $I(S, P) \to \mathcal{C}$ by restriction. The map is called a *cofibration* in $\mathcal{C}^{\mathcal{P}}$ if (*) is satisfied for all $P \in \mathcal{P}$:

(*) The colimit $\mathrm{colim}_{I(\mathcal{S}_{<P}, P)} X$ exists in $\mathcal{C}$ and the canonical map

$$\mathrm{colim}_{I(\mathcal{S}_{<P}, P)} X \to X_1(P)$$

is a cofibration.

As usual, an object $X \in \mathcal{C}^{\mathcal{P}}$ is called *cofibrant* if $\emptyset \to X$ is a cofibration.

Lemma 11 *Let $X : [1] \times \mathcal{P} \to \mathcal{C}$ be a map in $\mathcal{C}^{\mathcal{P}}$. Let $P \in \mathcal{P}$ be an element and $\mathcal{S}' \subset \mathcal{S} \subset \mathcal{S}_{<P}$ saturated subposets of $\mathcal{P}$. If (*) is satisfied for all $Q < P$, then* $\mathrm{colim}_{I(S,P)} X$ *exists and the canonical map* $\mathrm{colim}_{I(S',P)} X \to \mathrm{colim}_{I(S,P)} X$ *is a cofibration in $\mathcal{C}$. In particular, if X_1 is cofibrant in $\mathcal{C}^{\mathcal{P}}$, then* $\mathrm{colim}_{\mathcal{P}} X_1$ *exists.*

Proof We proceed by induction on the size of $\mathcal{S}$. For $\mathcal{S} = \emptyset$ the colimit exists as it is $X_0(P)$ and the canonical map is a cofibration as it is necessarily the identity map ($\mathcal{S}' = \emptyset$). Suppose the lemma is true for all $\mathcal{S}$ of cardinality $< n$.

Let $\mathcal{S}$ have cardinality n. If $\mathcal{S}' = \mathcal{S}$, then the canonical map (once it exists) is obviously a cofibration and we choose $Q \in S$ maximal. Otherwise we choose $Q \in \mathcal{S}$ maximal such that $Q \notin \mathcal{S}'$. In the diagram

$$\mathrm{colim}_{I(S\setminus\{Q\},P)} X \leftarrow \mathrm{colim}_{I(S_{<Q},Q)} X \hookrightarrow X_1(Q)$$

the left hand term exists by induction hypothesis, the middle term exists and the right hand arrow is a cofibration by the assumptions of the lemma. In particular, the pushout of the diagram exists in $\mathcal{C}$. Since $\mathrm{colim}_{I(S,P)} X$ has the universal property of this pushout, it exists, and the canonical map $\mathrm{colim}_{I(S\setminus\{Q\},P)} X \to \mathrm{colim}_{I(S,P)} X$ is a cofibration. By induction hypothesis, $\mathrm{colim}_{I(S',P)} X \to \mathrm{colim}_{I(S\setminus\{Q\},P)} X$ is a cofibration. Composition with the previous map yields a cofibration.

Let $X_1 \in \mathcal{C}^{\mathcal{P}}$ be cofibrant, *i.e.,* $X_0 = * \to X_1$ is a cofibration. Let $\mathcal{P}^+$ be the poset consisting of $\mathcal{P}$ and an additional maximal element $+$. Define $X_0(+) = *$. Then according to what was shown above, $\mathrm{colim}_{I(\mathcal{P},+)} X$ exists. But $\mathrm{colim}_{I(\mathcal{P},+)} X = \mathrm{colim}_{\mathcal{P}} X_1$. □

Lemma 12 *Let $X : [1] \times \mathcal{P} \to \mathcal{C}$ be a cofibration in $\mathcal{C}^{\mathcal{P}}$. If X_0 takes values in* $(\mathrm{cof}\mathcal{C})^{\mathcal{P}}$*, then so does X_1 and the canonical map $X_1(Q) \to$* $\mathrm{colim}_{I(S,P)} X$ *is a cofibration for all saturated $\mathcal{S} \subset S_{<P}$, $Q \in \mathcal{S}$ maximal and $P \in \mathcal{P}$.*

Proof The pushout of the diagram

$$\mathrm{colim}_{I(S_{<Q},Q)}X \leftarrow X_0(Q) \rightarrow X_0(P)$$

is $\mathrm{colim}_{I(S_{<Q},P)}X$. Since $X_0(Q) \rightarrow X_0(P)$ is a cofibration by assumption, the map $\mathrm{colim}_{I(S_{<Q},Q)}X \rightarrow \mathrm{colim}_{I(S_{<Q},P)}X$ is a cofibration. Composing with the cofibration $\mathrm{colim}_{I(S_{<Q},P)}X \hookrightarrow \mathrm{colim}_{I(S\setminus\{Q\},P)}X$ of lemma 11 yields a cofibration. Thus $X_1(Q) \rightarrow \mathrm{colim}_{I(S,P)}X$ is a cofibration as it is the pushout of the previous composition along $\mathrm{colim}_{I(S_{<Q},Q)}X \rightarrow X_1(Q)$. In particular, for $Q \in \mathcal{S}_{<P}$ maximal, we have a cofibration $X_1(Q) \rightarrow \mathrm{colim}_{I(S_{<P},P)}X$. Composed with the cofibration $\mathrm{colim}_{I(S_{<P},P)}X \rightarrow X_1(P)$ yields a cofibration $X_1(Q) \rightarrow X_1(P)$. It follows that X_1 takes values in $(\mathrm{cof}\mathcal{C})^{\mathcal{P}}$. □

Lemma 13 *Let $\mathcal{C}$ be a Waldhausen category with factorization. Call a map $X_0 \rightarrow X_1$ in $\mathcal{C}^{\mathcal{P}}$ a weak equivalence if for all $P \in \mathcal{P}$ the map $X_0(P) \rightarrow X_1(P)$ is a weak equivalence in $\mathcal{C}$. Then every map $X \rightarrow Y$ in $\mathcal{C}^{\mathcal{P}}$ factors as $X \hookrightarrow Z \xrightarrow{\sim} Y$ with $X \hookrightarrow Z$ a cofibration and $Z \xrightarrow{\sim} Y$ a weak equivalence.*

Proof The proof proceeds by induction on the size of $\mathcal{P}$. If the cardinality of $\mathcal{P}$ is 1, then the claim is the factorization property in $\mathcal{C}$. Let $\mathcal{P}$ be a finite poset. Choose a maximal element $P \in \mathcal{P}$. By induction hypothesis there is a factorization $X_{|\mathcal{P}\setminus\{P\}} \hookrightarrow Z_{|\mathcal{P}\setminus\{P\}} \xrightarrow{\sim} Y_{|\mathcal{P}\setminus\{P\}}$ with the first map a cofibration and the last map a weak equivalence. By lemma 11, the colimit $\mathrm{colim}_{I(\mathcal{S}_{<P},P)}(X \rightarrow Z)$ exists. By the factorization property in $\mathcal{C}$, the canonical map $\mathrm{colim}_{I(\mathcal{S}_{<P},P)}(X \rightarrow Z) \rightarrow Y(P)$ can be factored as $\mathrm{colim}_{I(\mathcal{S}_{<P},P)}(X \rightarrow Z) \hookrightarrow Z(P) \xrightarrow{\sim} Y(P)$ into a cofibration followed by a weak equivalence. This defines $Z : \mathcal{P} \rightarrow \mathcal{C}$. By construction, we have a factorization $X \hookrightarrow Z \xrightarrow{\sim} Y$ of $X \rightarrow Y$ into a cofibration followed by a weak equivalence. □

Lemma 14 (extracted from [Wal85]) *Let $\mathcal{C}$ be a non-empty category such that every functor $\mathcal{P} \rightarrow \mathcal{C}$, with $\mathcal{P}$ a finite poset, is homotopic to zero. Then $\mathcal{C}$ is contractible.*

Proof Choose for every connected component of $\mathcal{C}$ a vertex lying in that component. The choice yields a functor from the discrete category $\pi_0\mathcal{C}$ to $\mathcal{C}$ which, by hypothesis, is contractible. So $\mathcal{C}$ is connected.

Pick a zero simplex of $\mathcal{C}$ and call it base point. We have to show that $\pi_n\mathcal{C} = 0$, $n > 0$. Every element $[\alpha] \in \pi_n\mathcal{C}$ is represented by a pointed map $\alpha : Sd^k S^n \rightarrow N_*\mathcal{C}$ for some $k \in \mathbb{N}$. Here $S^n = (\Delta^1/\partial\Delta^1)^{\wedge n}$ is the simplicial n-sphere, $N_*\mathcal{C}$ is the nerve of $\mathcal{C}$, and $Sd^k S^n$ is the k-th normal subdivision of S^n [FP90, 4.6]. We can assume $k \geq 2$. For any simplicial set Z, the simplicial set $Sd^2 Z$ is the nerve of a poset [Tho80]. Since S^n, hence $Sd^k S^n = N_*\mathcal{P}_k$, is a finite simplicial set, $\mathcal{P}_k$ is a finite poset. Moreover, $N_* : cat \rightarrow \Delta^{op}Sets$ is fully faithful, so α is the nerve of a map $\mathcal{P}_k \rightarrow \mathcal{C}$. By hypothesis, this map is contractible, thus α is contractible. Every pointed map from S^n to a topological space X, which is homotopic to zero, is also homotopic to zero via a base point preserving homotopy. Thus α is contractible via a base point preserving homotopy, so $[\alpha] = 0$. □

*Proof (*of Approximation, theorem 10) By [Wal85, 1.6.6.], the map of Waldhausen categories $S_n\mathcal{A} \to S_n\mathcal{B}$ satisfies the hypothesis of theorem 10 as well. We show that $F : w\mathcal{A} \to w\mathcal{B}$ is a homotopy equivalence. The same argument applied to $S_n\mathcal{A} \to S_n\mathcal{B}$ yields homotopy equivalences $wS_n\mathcal{A} \to wS_n\mathcal{B}$ for all $n \in \mathbb{N}$ and thus a homotopy equivalence after realization $wS_*\mathcal{A} \to wS_*\mathcal{B}$.

In order to show that F is a homotopy equivalence, it suffices to show that for every object B of $w\mathcal{B}$ the category $(F \downarrow B)$ is contractible [Qui73, Theorem A]. The categories $(F \downarrow B)$ are non-empty since the map $\emptyset = F(\emptyset) \to B$ admits a factorization $\emptyset \to F(A) \xrightarrow{\sim} B$ where the latter map is a weak equivalence. Let $\mathcal{P}$ be a finite poset. A functor $\mathcal{P} \to (F \downarrow B)$ is a $\mathcal{P}$-diagram X in $w\mathcal{A}$ together with a map $F(X) \to cB$ in $(w\mathcal{B})^{\mathcal{P}}$, where cB stands for the constant diagram $cB(Q \leq P) = id_B$. Factor $\emptyset \to X$ into $\emptyset \hookrightarrow Y \xrightarrow{\sim} X$ (lemma 13). Since Y is cofibrant, the colimit $\mathrm{colim}_{\mathcal{P}}Y$ exists (lemma 11). We have $F(\mathrm{colim}_{\mathcal{P}}Y) = \mathrm{colim}_{\mathcal{P}}F(Y)$ as the colimit is a successive pushout along cofibrations and F preserves them. Thus there is an induced map $\mathrm{colim}_{\mathcal{P}}F(Y) \to B$ which we can factor as $F(\mathrm{colim}_{\mathcal{P}}Y) \to F(Z) \xrightarrow{\sim} B$ by appr_F. Saturation and the condition appr_F ensure that $Y \to cZ$ is a point-wise weak equivalence. The sequence of maps $X \xleftarrow{\sim} Y \xrightarrow{\sim} cZ$ in $(F \downarrow B)^{\mathcal{P}}$ define a null homotopy. Thus any functor $\mathcal{P} \to (F \downarrow B)^{\mathcal{P}}$ is homotopic to zero. By lemma 14, $(F \downarrow B)$ is contractible. □

Lemma 15 *Let $\mathcal{C}$ be a Waldhausen category with factorization (definition 11) satisfying the saturation axiom. Let $\bar{w}\mathcal{C} = w\mathcal{C} \cap \mathrm{cof}\mathcal{C}$. Then the inclusion $F : \bar{w}\mathcal{C} \to w\mathcal{C}$ is a homotopy equivalence.*

Proof By Quillen's Theorem A, it suffices to show that $(F \downarrow B)$ is contractible for every object B of $w\mathcal{C}$. It is non-empty since id_B is an object of $(F \downarrow B)$.

Let $\mathcal{P} \to (F \downarrow B)$ be a functor with $\mathcal{P}$ a finite poset. It is given by a $\mathcal{P}$ diagram X in $\bar{w}\mathcal{C}$ together with a map $X \to cB$ in $(w\mathcal{C})^{\mathcal{P}}$. Here cB is the constant $\mathcal{P}$-diagram associated with B. Factor $X \vee cB \to cB$ as a cofibration followed by a weak equivalence in $\mathcal{C}^{\mathcal{P}}$ (lemma 11): $X \vee cB \hookrightarrow Y \xrightarrow{\sim} cB$. By lemma 12, Y is an object of $(\mathrm{cof}\mathcal{C})^{\mathcal{P}}$. By the saturation axiom applied to the maps occurring in $Y \xrightarrow{\sim} cB$, Y is in $(w\mathcal{C})^{\mathcal{P}}$ and thus in $(\bar{w}\mathcal{C})^{\mathcal{P}}$. Since $X \to X \vee cB$, $cB \to X \vee cB$ and $X \vee cB \to Y$ are point-wise cofibrations, their compositions $X \to Y$ and $cB \to Y$ are point-wise cofibrations. By the saturation axiom, both maps are also point-wise weak equivalences. Thus the sequence of maps $X \to Y \leftarrow cB$ over cB defines a null-homotopy of $\mathcal{P} \to (F \downarrow B)$. By lemma 14, we are done. □

*Proof (*of Fibration, theorem 11) Replacing [Wal85, 1.6.3.] with lemma 15 in Waldhausen's proof of [Wal85, 1.6.4.] yields a proof of theorem 11. □

*Proof (*of Cofinality, theorem 12) The proof is the same as the proof of [TT90, 1.10.1.] replacing [TT90, 1.8.2.] by theorem 11 and [TT90, 1.5.7.] by the (non-functorial) factorization of $A \to 0$ as a cofibration followed by a weak equivalence $A \hookrightarrow I \xrightarrow{\sim} 0$ and setting $\Sigma A = I/A$. The non-functoriality of ΣA is irrelevant in the proof of [TT90, 1.10.1.]. □

Acknowledgements I would like to thank Bernhard Keller for his support. The concept of Frobenius pairs is due to him. Also, I would like to thank Max Karoubi for his support and his comment that, up to now, there is no theory of negative K-groups for exact categories which triggered our interest in this problem. His [Kar70] has much influenced this work. Thanks go to Chuck Weibel for his motivating comments while the author was a PhD student. I am grateful to Marc Levine for inviting me to participate in the Wolfgang Paul Program and to the University of Essen with its DFG-Schwerpunkt "Globale Methoden in der Komplexen Geometrie" where the final version of this work was written up. Finally, I'd like to thank the referee for his careful reading of the manuscript.

References

[BFJR04] Bartels, A., Farrell, T., Jones, L., Reich, H.: On the isomorphism conjecture in algebraic K-theory. Topology. **43**(1), 157–213 (2004)
[Bas68] Bass, H.: Algebraic K-theory. New York-Amsterdam: W. A. Benjamin, Inc., 1968
[BBD82] Beĭlinson, A.A., Bernstein, J., Deligne, P.: Faisceaux pervers. In: Analysis and topology on singular spaces, I (Luminy, 1981), pp 5–171. Soc. Math. France, Paris, 1982
[BF78] Bousfield, A.K., Friedlander, E.M.: Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In: Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, vol. 658 of Lecture Notes in Math. pp 80–130. Springer, Berlin, 1978
[BN93] Bökstedt, M., Neeman, A.: Homotopy limits in triangulated categories. Compositio Math. **86**(2), 209–234, (1993)
[BS01] Balmer, P., Schlichting, M.: Idempotent completion of triangulated categories. J. Algebra. **236**, 819–834 (2001)
[Car80] Carter, D.W.: Localization in lower algebraic K-theory. Comm. Algebra. **8**(7), 603–622 (1980)
[CP97] Cárdenas, M., Pedersen, E.K.: On the Karoubi filtration of a category. K-Theory. **12**(2), 165–191 (1997)
[FP90] Fritsch, R., Piccinini, R.A.: Cellular structures in topology, vol. 19 of Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1990
[Fra01] Franke, J.: On the Brown representability theorem for triangulated categories. Topology. **40**(4), 667–680 (2001)
[Gab62] Gabriel, P.: Des catégories abéliennes. Bull. Soc. Math. France. **90**, 323–448 (1962)
[HS04] Hornbostel, J., Schlichting, M.: Localization in Hermitian K-theory of rings. J. London Math. Soc. (2). **70**(1), 77–124 (2004)
[Hap87] Happel, D.: On the derived category of a finite-dimensional algebra. Comment. Math. Helv. **62**(3), 339–389 (1987)
[Hov99] Hovey, M.: Model categories, vol. 63 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999
[Hsi84] Hsiang, WC.: Geometric applications of algebraic K-theory. In: Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), pp 99–118, Warsaw, PWN, 1984
[Kar70] Karoubi, M.: Foncteurs dérivés et K-théorie. In: Séminaire Heidelberg-Saarbrücken-Strasbourg sur la K-théorie (1967/68), vol. 136 of Lecture Notes in Mathematics, pp 107–186, Springer, Berlin, 1970
[Kar71] Karoubi, M.: La périodicité de Bott en K-théorie générale. Ann. Sci. École Norm. Sup. (4). **4**, 63–95 (1971)
[Kel90] Keller, B.: Chain complexes and stable categories. Manuscripta Math. **67**(4), 379–417 (1990)
[Kel96] Keller, B.: Derived categories and their uses. In: Handbook of algebra, Vol. 1, pp. 671–701. North-Holland, Amsterdam, 1996
[Kel99] Keller, B.: On the cyclic homology of exact categories. J. Pure Appl. Algebra. **136**(1), 1–56, (1999)
[KM95] Kříž I., May, J.P.: Operads, algebras, modules and motives. Astérisque. (233), iv+145pp (1995)
[May74] May, J.P.: E_∞ spaces, group completions, and permutative categories. pp 61–93. London Math. Soc. Lecture Note Ser. **11** (1974)
[Nee90] Neeman, A.: The derived category of an exact category. J. Algebra. **135**(2), 388–394 (1990)

[Nee92] Neeman, A.: The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. Ann. Sci. École Norm. Sup. (4). **25**(5), 547–566 (1992)

[Nee01] Neeman, A.: Triangulated categories, vol. 148 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2001

[Ped84] Pedersen, E.: On the K_{-i}-functors. J. Algebra. **90**(2), 461–475 (1984)

[Pop73] Popescu, N.: Abelian categories with applications to rings and modules. Academic Press, London, 1973. London Mathematical Society Monographs, No. 3

[PW89] Pedersen, E.K., Weibel, C.A.: K-theory homology of spaces. In: Algebraic topology (Arcata, CA, 1986), volume 1370 of Lecture Notes in Math. pp 346–361. Springer, Berlin, 1989

[Qui73] Quillen, D.: Higher algebraic K-theory. I. pp 85–147. Lecture Notes in Math. **341**, 1973

[Qui75] Quillen, D.: Higher algebraic K-theory. In: Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, pp 171–176. Canad. Math. Congress, Montreal, Que. 1975

[Sch02] Schlichting, M.: A note on K-theory and triangulated categories. Invent. Math. **150**(1), 111–116 (2002)

[Sch04] Schlichting, M.: Delooping the K-theory of exact categories. Topology. **43**(5), 1089–1103 (2004)

[SGA5] Cohomologie l-adique et fonctions L. Séminaire de Géometrie Algébrique du Bois-Marie 1965–1966 (SGA 5), Edité par Luc Illusie, Lecture Notes in Mathematics, Vol. 589. Springer, Berlin, 1977

[She89] Sherman, C.: On the homotopy fiber of the map $BQ\mathcal{A}^{\oplus} \to BQ\mathcal{A}$ (after M. Auslander). In: Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), vol. 83 of Contemp. Math. pp 343–348. Amer. Math. Soc. Providence, RI, 1989

[Tho80] Thomason, R.W.: Cat as a closed model category. Cahiers Topologie Géom. Différentielle. **21**(3), 305–324 (1980)

[Tho93] Thomason, R.W.: Les K-groupes d'un fibré projectif. In: Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991), volume 407 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. pp 243–248. Kluwer Acad. Publ., Dordrecht, 1993

[Tho97] Thomason, R.W.: The classification of triangulated subcategories. Compositio Math. **105**(1), 1–27 (1997)

[TT90] Thomason, R.W., Trobaugh, T.: Higher algebraic K-theory of schemes and of derived categories. In: The Grothendieck Festschrift, Vol. III, pp 247–435. Birkhäuser Boston, Boston, MA, 1990

[Ver96] Verdier, J-L.: Des catégories dérivées des catégories abéliennes. Astérisque, (239):xii+253 pp. (1997), 1996. With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis

[Wag72] Wagoner, J.B.: Delooping classifying spaces in algebraic K-theory. Topology. **11**, 349–370 (1972)

[Wal85] Waldhausen, F.: Algebraic K-theory of spaces. In: Algebraic and geometric topology (New Brunswick, N.J., 1983), pp 318–419. Springer, Berlin, 1985

[Wei80] Weibel, C.A.: K-theory and analytic isomorphisms. Invent. Math. **61**(2), 177–197 (1980)

[Wei01] Weibel, C.: The negative K-theory of normal surfaces. Duke Math. J. **108**(1), 1–35 (2001)

[Yao92] Yao, D.: Higher algebraic K-theory of admissible abelian categories and localization theorems. J. Pure Appl. Algebra. **77**(3), 263–339 (1992)