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Complete toric varieties with reductive automorphism group

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Abstract We give equivalent and sufficient criteria for the automorphism group of a complete toric variety, respectively a Gorenstein toric Fano variety, to be reductive. In particular we show that the automorphism group of a Gorenstein toric Fano variety is reductive, if the barycenter of the associated reflexive polytope is zero. Furthermore a sharp bound on the dimension of the reductive automorphism group of a complete toric variety is proven by studying the set of Demazure roots.

Keywords Toric varieties · Algebraic groups · Fano varieties · Lattice polytopes

1 Introduction

In this paper we study criteria for the automorphism group of a complete toric variety to be reductive. Here one source of motivation comes from the following result:

Theorem 1.1 **(Matsushima 1957)** *If a nonsingular Fano variety X admits an Einstein-Kähler metric, then* $\operatorname{Aut}(X)$ *is a reductive algebraic group.*

In 1983 Futaki introduced the so called *Futaki character*, whose vanishing is another important obstruction for the existence of an Einstein-Kähler metric. For a nonsingular toric Fano variety with reductive automorphism group there is an explicit criterion (see [12, Cor. 5.5]):

Theorem 1.2 **(Mabuchi 1987)** *Let X be a nonsingular toric Fano variety with* $\operatorname{Aut}(X)$ *reductive.*

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The Futaki character of X vanishes if and only if the barycenter of P is zero, where P is the associated reflexive polytope, i.e., the fan of normals of P is associated to X.

In [2, Thm. 1.1] Batyrev and Selivanova were able to give a sufficient criterion for the existence of an Einstein-Kähler metric:

Theorem 1.3 (Batyrev/Selivanova 1999) *Let X be a nonsingular toric Fano variety. We denote by P the associated reflexive polytope.*

If X is symmetric, i.e., the group of lattice automorphisms leaving P invariant has no non-zero fixpoints, then X admits an Einstein-Kähler metric.

In particular they got as a corollary [2, Cor. 1.2] that the automorphism group of such a symmetric X is reductive. Expressed in combinatorial terms this just means that the set of lattice points in the relative interiors of facets of P is centrally symmetric. So they asked whether a direct proof of this result exists. Indeed there is the following generalization with a simple combinatorial proof (see Theorem 5.2(1) and Remark 5.5):

Theorem 1.4 *Let X be a complete toric variety.*

If the group of automorphisms of the associated fan has no non-zero fixpoints, then $\mathrm{Aut}(X)$ *is reductive.*

Motivated by above results it was conjectured by Batyrev that in the case of a nonsingular toric Fano variety already the vanishing of the barycenter of the associated reflexive polytope were sufficient for the automorphism group to be reductive. Indeed there is even the following more general result that has a purely convex-geometrical proof (see Theorem 5.2(2i)):

Theorem 1.5 *Let X be a Gorenstein toric Fano variety.*

If the barycenter of the associated reflexive polytope is zero, then $\mathrm{Aut}(X)$ *is reductive.*

Only very recently Xu-Jia Wang and Xiaohua Zhu could prove that the vanishing of the Futaki character is even sufficient for the existence of an Einstein-Kähler metric in the toric case (see [16, Cor. 1.3]):

Theorem 1.6 (Wang/Zhu 2004) *Let X be a nonsingular toric Fano variety.*

Then X admits an Einstein-Kähler metric if and only the Futaki character of X vanishes.

Combining the previous results this yields a generalization of the above theorem of Mabuchi that is also implicit in [16, Lemma 2.2]:

Corollary 1.7 *Let X be a nonsingular toric Fano variety.*

Then X admits an Einstein-Kähler metric if and only if the barycenter of P is zero, where P is the associated reflexive polytope.

It is now conjectured by Batyrev that this result may also hold in the singular case of a Gorenstein toric Fano variety.

Another source of motivation that orginated this research was the aim to give mathematical explanations for observations made by Batyrev, Kreuzer and the author in the computer database [11] of 3- and 4-dimensional reflexive polytopes. Here one of the main results is a necessary condition for the automorphism group of a complete toric variety to be reductive that is given by the following sharp upper bound on the dimension (see Theorem 3.23):

Theorem 1.8 *Let X be a d-dimensional complete toric variety that is not a product of projective spaces.*

If $\mathrm{Aut}(X)$ *is reductive, then* $\dim \mathrm{Aut}(X) \begin{cases} = 2, & \text{for } d = 2 \\ \leq d^2 - 2d + 4\ , & \text{for } d \geq 3 \end{cases}$

The paper is organized as follows:

In section 2 the notation is fixed and basic definitions are given.

Section 3 deals with the automorphism group $\mathrm{Aut}(X)$ of a d-dimensional complete toric variety X. Here the set of roots $\mathcal{R}$ plays a crucial part in determining the dimension and whether the group is reductive (see Prop. 3.2). Using results of Cox in [7] we construct families of roots that parametrize the set of semisimple roots $\mathcal{S} := \mathcal{R} \cap -\mathcal{R}$ in a geometrically convenient way, these are called $\mathcal{S}$-root bases. As an application we show in Prop. 3.18 that X is isomorphic to a product of projective spaces if and only if there are d linearly independent semisimple roots. When $\mathrm{Aut}(X)$ is reductive, we obtain the bound $\dim \mathrm{Aut}(X) \leq d^2 + 2d$, with equality if and only if $X \cong \mathbb{P}^d$ (see 3.19). Moreover studying this approach in more detail we get in Prop. 3.20 the existence of some special families of roots that yields several restrictions on the set $\mathcal{R}$ (see 3.21 and 3.22). From this we can derive the above bound on $\dim \mathrm{Aut}(X)$ in Theorem 3.23.

In section 4 we more closely examine the case of a d-dimensional Gorenstein toric Fano variety X associated to a reflexive polytope P (see [13]). Here a root of X is just a lattice point in the relative interior of a facet of P, so the results of the previous section have a direct geometric interpretation. For instance we obtain that P has at most $2d$ facets containing roots of P, with equality if and only if X is the product of d projective lines (see Corollary 4.4). Furthermore the intersection of P with the space spanned by all semisimple roots is a reflexive polytope associated to a product of projective spaces (see Theorem 4.10).

In section 5 we present and discuss several combinatorial equivalent and sufficient criteria for the automorphism group of a complete toric variety, respectively a Gorenstein toric Fano variety, to be reductive (see Theorem 5.2).

In section 6 we investigate the roots of d-dimensional centrally symmetric reflexive polytopes. As an application we finish in Theorem 6.3(3) the proof of [13, Thm. 6.4] saying that such a lattice polytope has at most 3^d lattice points, with equality if and only if it is isomorphic to $[-1, 1]^d$.

2 Notation and basic definitions

In this section we shortly repeat the standard notation for polytopes and toric varieties, as it can be found in [8], [9] or [15]. In [1] reflexive polytopes were introduced.

Let $N \cong \mathbb{Z}^d$ be a d-dimensional lattice and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^d$ the dual lattice with $\langle \cdot, \cdot \rangle$ the nondegenerate symmetric pairing. As usual, $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^d$ and $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^d$ (respectively $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$) will denote the rational (respectively real) scalar extensions.

For a subset S of a real vector space let $\mathrm{lin}(S)$ (respectively $\mathrm{aff}(S)$, $\mathrm{conv}(S)$, $\mathrm{pos}(S)$) be the linear (respectively affine, convex, positive) hull of S. A subset $P \subseteq M_{\mathbb{R}}$ is called a polytope, if it is the convex hull of finitely many points in $M_{\mathbb{R}}$. The boundary of P is denoted by ∂P, the relative interior of P by $\mathrm{relint} P$. When P is full-dimensional, its relative interior is just the interior $\mathrm{int} P$.

A face F of P is denoted by $F \leq P$. The vertices of P form the set $\mathcal{V}(P)$, the facets of P the set $\mathcal{F}(P)$. P is called a lattice polytope, respectively rational polytope, if $\mathcal{V}(P) \subseteq M$, respectively $\mathcal{V}(P) \subseteq M_{\mathbb{Q}}$. An isomorphism of lattice polytopes is an isomorphism of the associated lattices such that the induced real linear isomorphism maps the polytopes onto each other.

We usually denote by $\triangle$ a complete fan in $N_{\mathbb{R}}$. The k-dimensional cones of $\triangle$ form a set $\triangle(k)$. The elements in $\triangle(1)$ are called rays, and given $\tau \in \triangle(1)$, we let v_τ denote the unique generator of $N \cap \tau$.

Let $P \subseteq M_{\mathbb{R}}$ be a rational d-dimensional polytope with $0 \in \mathrm{int} P$. Then we have the important notion of the dual polytope

$$P^* := \{y \in N_{\mathbb{R}} \ : \ \langle x, y \rangle \geq -1 \ \forall x \in P\},$$

that is also a rational d-dimensional polytope with $0 \in \mathrm{int} P^*$. Duality means that $(P^*)^* = P$. The fan $\mathcal{N}_P := \{\mathrm{pos}(F) \ : \ F \leq P^*\}$ is called the normal fan of P.

There is a correspondence between i-dimensional faces of P and $(d-1-i)$-dimensional faces of P^* that reverses inclusion. For a facet $F \leq P$ we let $\eta_F \in N_{\mathbb{Q}}$ denote the uniquely determined inner normal satisfying $\langle \eta_F, F \rangle = -1$. We have

$$\mathcal{V}(P^*) = \{\eta_F \ : \ F \in \mathcal{F}(P)\}.$$

The dual of the product of d_i-dimensional polytopes $P_i \subseteq \mathbb{R}^{d_i}$ with $0 \in \mathrm{int} P_i$ for $i = 1, 2$ is given by

$$(P_1 \times P_2)^* = \mathrm{conv}(P_1^* \times \{0\}, \{0\} \times P_2^*) \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}. \tag{2.1}$$

By a well-known construction a fan $\triangle$ in $N_{\mathbb{R}}$ defines a toric variety $X := X(N, \triangle)$, i.e., a normal irreducible algebraic variety over $\mathbb{C}$ such that an open embedded algebraic torus $(\mathbb{C}^*)^d$ acts on X in extension of its own action.

Let $P \subseteq M_{\mathbb{R}}$ be a rational polytope. We define the associated toric variety

$$X_P := X(N, \mathcal{N}_P).$$

For d-dimensional rational polytopes P_1, P_2 equation (2.1) implies

$$X_{P_1} \times X_{P_2} \cong X_{P_1 \times P_2}.$$

Definition 2.1 A complex variety X is called *Gorenstein Fano variety*, if X is projective, normal and its anticanonical divisor is an ample Cartier divisor.

In the toric case the following definition was introduced by Batyrev [1]:

Definition 2.2 A d-dimensional polytope $P \subseteq M_{\mathbb{R}}$ with $0 \in \operatorname{int} P$ is called *reflexive polytope*, if P is a lattice polytope and P^* is a lattice polytope.

Especially reflexive polytopes always appear in dual pairs. There is the following fundamental result (see [1] or [13]):

Theorem 2.3 *Under the map $P \mapsto X_P$ reflexive polytopes correspond uniquely up to isomorphism to Gorenstein toric Fano varieties. There are only finitely many isomorphism types of d-dimensional reflexive polytopes.*

Here X_P is a nonsingular toric Fano variety if and only if the vertices of any facet of P^* form a $\mathbb{Z}$-basis of the lattice M.

The following property [13, Lemma 1.13] characterizes reflexive polytopes.

Lemma 2.4 *Let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope.*

For any $F \in \mathcal{F}(P)$ and $m \in F \cap M$ there is a $\mathbb{Z}$-basis $e_1, \ldots, e_{d-1}, e_d$ of M such that $e_d = m$ and $F \subseteq \{x \in M_{\mathbb{R}} : x_d = 1\}$; in particular $\eta_F = -e_d^*$ in the dual basis $e_1^*, \ldots, e_d^*$ of N.

Furthermore $\operatorname{int} P \cap M = \{0\}$.

3 The set of roots of a complete toric variety

In this section the set of roots of a complete toric variety is investigated, and some classification results and bounds on the dimension of the automorphism group are achieved.

Throughout the section let $\triangle$ be a complete fan in $N_{\mathbb{R}}$, i.e., $\cup_{\sigma \in \triangle} \sigma = N_{\mathbb{R}}$, with associated complete toric variety $X = X(N, \triangle)$.

Definition 3.1 Let $\mathcal{R}$ be the set of *Demazure roots* of $\triangle$, i.e.,

$$\mathcal{R} := \{m \in M \mid \exists\, \tau \in \triangle(1) : \langle v_\tau, m\rangle = -1,\ \ \langle v_{\tau'}, m\rangle \geq 0\ \ \forall\, \tau' \in \triangle(1)\backslash\{\tau\}\}.$$

For $m \in \mathcal{R}$ we denote by η_m the unique primitive generator v_τ of the unique ray τ with $\langle v_\tau, m\rangle = -1$. For a subset $A \subseteq \mathcal{R}$ we define $\eta(A) := \{\eta_m : m \in A\}$.

Let $\mathcal{S} := \mathcal{R} \cap (-\mathcal{R}) = \{m \in \mathcal{R} : -m \in \mathcal{R}\}$ be the set of *semisimple* roots and $\mathcal{U} := \mathcal{R}\backslash\mathcal{S} = \{m \in \mathcal{R} : -m \notin \mathcal{R}\}$ the set of *unipotent* roots. We say that $\triangle$ is *semisimple*, if $\mathcal{R} = \mathcal{S}$, or equivalently $\mathcal{U} = \emptyset$.

Furthermore we define $\mathcal{S}_1 := \{x \in \mathcal{S} : \eta_x \notin \eta(\mathcal{U})\}$ and $\mathcal{S}_2 := \mathcal{S}\backslash\mathcal{S}_1$, analogously $\mathcal{U}_1 := \{x \in \mathcal{U} : \eta_x \notin \eta(\mathcal{S})\}$ and $\mathcal{U}_2 := \mathcal{U}\backslash\mathcal{U}_1$. In particular we have $\eta(\mathcal{S}_1) \cap \eta(\mathcal{S}_2) = \emptyset$ and $\eta(\mathcal{S}_2) = \eta(\mathcal{U}_2)$.

Usually the set $-\mathcal{R}$ is called the set of Demazure roots (see [15, Prop. 3.13]), however the sign convention here will turn out to be more convenient when considering normal fans of polytopes. Note that $\mathcal{R}$ only depends on the set of rays $\triangle(1)$.

For a root $m \in \mathcal{R}$ there is a one-parameter subgroup $x_m : \mathbb{C} \to \operatorname{Aut}(X)$. The identity component $\operatorname{Aut}^\circ(X)$ is a semidirect product of a reductive algebraic subgroup containing the big torus $(\mathbb{C}^*)^d$ and having $\mathcal{S}$ as a root system and the unipotent radical that is generated by $\{x_m(\mathbb{C}) : m \in \mathcal{U}\}$. Futhermore $\mathrm{Aut}(X)$ is generated by $\mathrm{Aut}^\circ(X)$ and the finite number of automorphisms that are induced by lattice automorphisms of the fan Δ. These results are due to Demazure (see [15, p. 140]) in the nonsingular complete case, and were generalized by Cox [7, Cor. 4.7] and Bühler [6]. Bruns and Gubeladze considered the case of a projective toric variety in [4, Thm. 5.4]. In particular there is the following result (recall that an algebraic group is *reductive*, if the unipotent radical is trivial):

Proposition 3.2 $\mathrm{Aut}(X)$ *has dimension* $|\mathcal{R}| + d$. *It is reductive if and only if* Δ *is semisimple.*

Moreover each irreducible component of the root system $\mathcal{S}$ is of type **A** (in the nonsingular case see [15, p. 140]). Here we will give an explicit description of $\mathcal{S}$ and $\eta(\mathcal{R})$ by special families of roots that will turn out to be useful for geometric applications.

Definition 3.3 A pair of roots $v, w \in \mathcal{R}$ is called *orthogonal*, in symbols $v \perp w$, if $\langle \eta_v, w\rangle = 0 = \langle \eta_w, v\rangle$. In particular $\eta_{-v} \neq \eta_w \neq \eta_v \neq \eta_{-w}$.

We remark that the term ‘orthogonal’ may be misleading, because most standard properties do not hold, e.g., $v \perp w$ does not necessarily imply $(-v) \perp w$.

Lemma 3.4 *Let* $B = \{b_1, \dots, b_l\}$ *be a non-empty set of roots such that* $\langle \eta_{b_i}, b_j\rangle = 0$ *for* $1 \leq j < i \leq l$. *Then* B *is a* $\mathbb{Z}$*-basis of* $\mathrm{lin}_{\mathbb{R}}(B) \cap M$.

Proof We prove the base property by induction on l. Let $x := \lambda_1 b_1 + \dots + \lambda_l b_l \in M$ with $\lambda_1, \dots, \lambda_l \in \mathbb{R}$. Then $\lambda_l = -\langle \eta_{b_l}, x\rangle \in \mathbb{Z}$. So $x - \lambda_l b_l = \lambda_1 b_1 + \dots + \lambda_{l-1} b_{l-1} \in M$. Now proceed by induction. □

Definition 3.5 Let $A \subseteq \mathcal{R}$. A pairwise orthogonal family $B \subseteq A$ is called

- *A-facet basis*, if $\eta(A) = \{\eta_b : b \in B\} \cup \{\eta_{-b} : b \in B, -b \in A\}$.
- *A-root basis*, if $A = \mathcal{R} \cap \mathrm{lin}(B)$.

Remark 3.6 When B is an A-root basis, we have $\mathrm{lin}(A) = \mathrm{lin}(B)$, so $\dim_{\mathbb{R}} \mathrm{lin}(A) = |B|$ by 3.4. If furthermore B is a subset of $\mathcal{S}$, then Prop. 3.11 below implies that also A is contained in $\mathcal{S}$ and can be easily described by B, moreover B is also an A-facet basis. Note however that in general even an $\mathcal{S}$-root basis is *not* a fundamental system for the root system $\mathcal{S}$ in the usual sense.

For arbitrary $A \subseteq \mathcal{R}$ we can not expect the existence of an A-root basis. However it is one of the goals of this section to show that there are always $\mathcal{R}$-facet bases (3.20(2)) and $\mathcal{S}$-root bases (3.15). To explicitly construct these families an algebraic approach due to Cox shall now be discussed:

In [7] Cox described $\mathcal{R}$ as a set of *ordered* pairs of monomials in the homogeneous coordinate ring of the toric variety. For this we denote by $S := \mathbb{C}[x_\tau : \tau \in \Delta(1)]$ the homogeneous coordinate ring of X, i.e., S is just a polynomial ring where any monomial in S is naturally graded by the class group $\mathrm{Cl}(X)$, i.e., the degree of a monomial $\prod_\tau x_\tau^{k_\tau}$ is the class of the Weil divisor $\sum_\tau k_\tau \mathcal{V}_\tau$, where $\mathcal{V}_\tau$ is the torus-invariant prime divisor corresponding to the ray τ. Recall that $\tau \cap N$ is generated by v_τ.

We let Y denote the set of indeterminates $\{x_\tau : \tau \in \triangle(1)\}$ and $\mathcal{M}$ the set of monomials in S. For any root $m \in \mathcal{R}$ we define $\tau_m := \operatorname{pos}(\eta_m) \in \triangle(1)$ and $x_m := x_{\tau_m} \in Y$. Now there is the following fundamental result [7, Lemma 4.4] (with a different sign convention):

Lemma 3.7 (Cox 95) *In this notation there is a well-defined bijection*

$$h : \mathcal{R} \to \{(x_\tau, f) \in Y \times \mathcal{M}, : x_\tau \neq f, \deg(x_\tau) = \deg(f)\},$$

$$m \mapsto (x_m, \prod_{\tau' \neq \tau_m} x_{\tau'}^{\langle v_{\tau'}, m\rangle}).$$

For $m \in \mathcal{R}$ we have

$$m \in \mathcal{S} \Longleftrightarrow h(m) \in Y \times Y,$$

in this case $h(m) = (x_m, x_{-m})$.

Hence semisimple roots correspond to ordered pairs of indeterminates in S of the same degree.

The next result can be used to 'orthogonalize' pairs of roots:

Lemma 3.8 *Let $v, w \in \mathcal{R}$, $v \neq -w$, $\langle \eta_v, w\rangle > 0$.*
Then $\langle \eta_w, v\rangle = 0$ and $v + w \in \mathcal{R}$.
Moreover $p(v, w) := \langle \eta_v, w\rangle v + w \in \mathcal{R}$, $v \perp p(v, w)$, $\eta_{p(v,w)} = \eta_{v+w} = \eta_w$.

$$p(v, w) \in \mathcal{S} \text{ iff } v + w \in \mathcal{S} \text{ iff } v \in \mathcal{S} \text{ and } w \in \mathcal{S}.$$

Proof Let v correspond to (x_v, f) and w to (x_w, g) as in Lemma 3.7. Hence $x_v \neq x_w$. The assumption implies that x_v appears in the monomial g. Assume $\langle \eta_w, v\rangle > 0$. Then x_w would appear in the monomial f. However since $v \neq -w$ this is a contradiction to the antisymmetry of the order relation defined in [7, Lemma 1.3]. The remaining statements are easy to see. □

Corollary 3.9 *$v \in \mathcal{U}$ and $w \in \mathcal{S}_1$ implies $\langle \eta_v, w\rangle = 0$.*

Lemma 3.8 defines a partial addition on $\mathcal{R}$ and generalizes parts of [5, Prop. 3.3] in a paper on polytopal linear groups due to Bruns and Gubeladze. The setting there is that of so called 'column structures' of polytopes where 'column vectors' correspond to roots. Most parts of this lemma were however already independently known and proven by the author as an application of Corollary 4.9 below in the case of a reflexive polytope.

For an unambiguous description of $\mathcal{S}$ it is now convenient to define an equivalence relation on the set of semisimple roots:

Definition 3.10 Let $v \equiv w$ (v is *equivalent* to w), if $v, w \in \mathcal{S}$ and $\eta_{-v} = \eta_{-w}$. In particular this yields $\langle \eta_{-v}, w\rangle = -\langle \eta_{-v}, -w\rangle = 1$.

Proposition 3.11 *Let $A \subseteq \mathcal{R}$ and $B \subseteq \mathcal{S}$ an A-root basis partitioned into t equivalence classes of order $c_1, \ldots, c_t$. Then:*

$$A = \{\pm b \;:\; b \in B\} \cup \{b - b' \;:\; b, b' \in B,\; b \neq b',\; b \equiv b'\} \subseteq \mathcal{S},$$
$$|\,A\,| = |\,B\,| + \textstyle\sum_{i=1}^{t} c_i^2 \leq |\,B\,| + |\,B\,|^2,$$
$$\eta(A)s = \{\eta_{\pm b} \;:\; b \in B\},\;\; |\,\eta(A)\,| = |\,B\,| + t \leq 2|\,B\,|.$$

Proof Only the first equation has to be proven: Let $m \in A$, by 3.4 we have $m = \sum_{b \in B} \lambda_b b$ for $\lambda_b \in \mathbb{Z}$. Let $l := \sum_{b \in B} |\,\lambda_b\,|$. Proceed by induction on l, let $l > 1$. By orthogonality we have $-1 \leq \langle \eta_b, m \rangle = -\lambda_b$, hence $\lambda_b \leq 1$ for all $b \in B$. Assume there is an element $b \in B$ with $\lambda_b < 0$. Lemma 3.8 implies $b+m \in \operatorname{lin}(B) \cap \mathcal{R} = A$, so $b+m \in \mathcal{S}$ by induction hypothesis. Now Lemma 3.8 yields $-m \in A$. This yields $\lambda_b = -1$. Therefore $\lambda_b \in \{1, 0, -1\}$ for all $b \in B$. Assume $l > 2$. By possibly replacing m with $-m$ we can achieve that there are two elements $b, b' \in B$ with $\lambda_b = 1 = \lambda_{b'}$, hence $\eta_b = \eta_m = \eta_{b'}$, a contradiction. Therefore $l = 2$, and there are two elements $b, b' \in B$ with $m = b - b'$. Assume $b \not\equiv b'$. Then necessarily $\langle \eta_{-b'}, b \rangle = 0$, so $\eta_b = \eta_m = \eta_{-b'}$, a contradiction. □

Definition 3.12 The natural grading of the polynomial ring $S = \mathbb{C}[x_\tau \;:\; \tau \in \Delta(1)]$ by the class group $\operatorname{Cl}(X)$ induces a partition of $Y = \{x_\tau \;:\; \tau \in \Delta(1)\}$ into equivalence classes of indeterminates of the same degree:

1. Let $Y_1, \ldots, Y_p$ be the equivalence classes of order at least two such that there exists no monomial in $\mathcal{M} \backslash Y$ of the same degree.
2. Let $Y_{p+1}, \ldots, Y_q$ be the remaining classes of order at least two.
3. Let $Y_{q+1}, \ldots, Y_r$ be the the equivalence classes of order one such that there exists an monomial in $\mathcal{M} \backslash Y$ of the same degree.
4. Let $Y_{r+1}, \ldots, Y_s$ be the remaining classes of order one.

Remark The equivalence relation on Y given by the grading by $\operatorname{Cl}(X)$ should not be confused with the equivalence relation $\equiv$ on $\mathcal{S}$ defined in 3.10.

By Lemma 3.7 ordered pairs of indeterminates contained in one of the sets $Y_1, \ldots, Y_p$ correspond to roots in $\mathcal{S}_1$, ordered pairs in $Y_{p+1}, \ldots, Y_q$ correspond to roots in $\mathcal{S}_2$. As changing $m \leftrightarrow -m$ for $m \in \mathcal{S}$ just means to reverse the corresponding pair of monomials, we immediately see that $-\mathcal{S}_1 = \mathcal{S}_1$ and $-\mathcal{S}_2 = \mathcal{S}_2$. Moreover Lemma 3.7 yields that *any root in $\mathcal{S}_1$ is orthogonal and not equivalent to any root in $\mathcal{S}_2$*.

Using Lemma 3.7 it also evident how to determine the invariants $|\,\eta(\mathcal{S}_i)\,|$, $|\,\eta(\mathcal{U}_i)\,|$, $|\mathcal{S}_i\,|$, $|\mathcal{U}_i\,|$ for $i = 1, 2$ from the given data in Definition 3.12.

Example 3.13 For illustration we consider the three-dimensional reflexive simplex $P := \operatorname{conv}((1, 0, 0), (1, 3, 0), (1, 0, 3), (-5, -6, -3))$. The vertices of P^* are $(-1, 0, 0)$, $(-1, 0, 2)$, $(2, -1, -1)$, $(-1, 1, 0)$. For $X = X_P$ we have $|\,\mathcal{S}\,| = 4$, $\dim_{\mathbb{R}} \mathcal{S} = 2$. F_1 and F_2 contain one antipodal pair of semisimple roots, while F_3 and F_4 contain the other pair. F_3, F_4 each contain three unipotent roots, pairs of unipotent roots in different facets are orthogonal. We can read this off the data $S = \mathbb{C}[x_1, x_2, x_3, x_4]$, $\operatorname{Cl}(X) \cong \mathbb{Z}$, $\deg(x_1) = \deg(x_2) = 1$ and $\deg(x_3) = \deg(x_4) = 2$. Hence $Y_1 = \{x_1, x_2\}$, $Y_2 = \{x_3, x_4\}$, $p = 1$, $q = r = s = 2$. $\{x_1^2, x_1 x_2, x_2^2\}$ are the elements in $\mathcal{M} \backslash Y$ of degree 2. X is just the weighted projective space with weights $(1, 1, 2, 2)$.

It is now easy to construct root bases:

Proposition 3.14 *Let $I \subseteq \{1, \ldots, q\}$. For $i \in I$ we choose $y_i \in Y_i$ and let R_i denote the set of $|Y_i| - 1$ semisimple roots corresponding to ordered pairs in Y_i with second element y_i. We set $B := \cup_{i \in I} R_i$ and $A := \mathrm{lin}(B) \cap \mathcal{R}$.*

Then B is an A-root basis partitioned into equivalence classes $\{R_i\}_{i \in I}$, and any root in A corresponds uniquely to an ordered pair in Y_i for some $i \in I$.

Moreover any A-root basis is given by this construction.

Proof By construction and Lemma 3.7 $\langle \eta_v, w \rangle = 0 = \langle \eta_w, v \rangle$ for $v, w \in B$, $v \neq w$, hence B is an A-root basis with given equivalence classes. Using Lemma 3.7 and the description of A in Prop. 3.11 the other statements are easily seen. □

By choosing $I = \{1, \ldots, q\}$ we get (see also Remark 3.6):

Corollary 3.15 *$\mathcal{S}$-root bases exist, in particular $\mathcal{R} \cap \mathrm{lin}(\mathcal{S}) = \mathcal{S}$. Moreover*

$$\dim_{\mathbb{R}} \mathrm{lin}(\mathcal{S}) = \sum_{i=1}^{q} (|Y_i| - 1).$$

Different choices of y_i in Prop. 3.14 yield different $\mathcal{S}$-root bases:

Example 3.16 Let's look at $X = \mathbb{P}^d$: We let E_d denote the d-dimensional simplex $\mathrm{conv}(-e_1 - \cdots - e_d, e_1, \ldots, e_d)$, where $e_1, \ldots, e_d$ is a $\mathbb{Z}$-basis of N. Hence $P := E_d^*$ is the reflexive polytope corresponding to d-dimensional projective space $X_P = \mathbb{P}^d$. The homogeneous coordinate ring $S = \mathbb{C}[x_0, x_1, \ldots, x_d]$ is trivially graded, so $q = 1$. $\mathrm{Aut}(X)$ is reductive with $d^2 + d$ roots. Choosing $y_1 := x_1$ yields the $\mathcal{S}$-root basis $B = R_1 = \{e_1^*, e_1^* - e_2^*, \ldots, e_1^* - e_d^*\}$, where $e_1^*, \ldots, e_d^*$ denotes the dual basis of M. All elements of B are mutually equivalent. The possible choices of y_1 as $x_1, \ldots, x_n$ lead to all $d + 1$ different $\mathcal{S}$-root bases.

Remark 3.17 $P' := \mathrm{conv}(\mathcal{S})$ is a centrally symmetric reflexive polytope with $\mathcal{S} = \mathcal{V}(P') = \partial P' \cap M$. More precisely due to 3.11 there is an isomorphism of lattice polytopes (with respect to lattices $\mathrm{lin}(\mathcal{S}) \cap M$ and $\mathbb{Z}^{c_1 + \cdots + c_q}$)

$$\mathrm{conv}(\mathcal{S}) \cong (\mathcal{Z}_{c_1} \oplus \cdots \oplus \mathcal{Z}_{c_q})^*,$$

where $c_i := |Y_i| - 1$ for $i = 1, \ldots, q$, and $\mathcal{Z}_n := \mathrm{conv}([0, 1]^n, -[0, 1]^n)$. See Theorem 4.10 for a stronger statement.

The existence of an $\mathcal{S}$-root basis yields:

Proposition 3.18 *A d-dimensional complete toric variety is isomorphic to a product of projective spaces iff there are d linearly independent semisimple roots.*

In this case

$$X \cong \mathbb{P}^{|Y_1|-1} \times \cdots \times \mathbb{P}^{|Y_q|-1}.$$

Proof Let $q = 1$, so there is an $\mathcal{S}$-root basis $b_1, \ldots, b_d$ with $\eta_{-b_1} = \cdots = \eta_{-b_d}$. Assume there exists $\tau \in \Delta(1)$ with $\tau \notin \{\tau_{b_1}, \ldots, \tau_{b_d}, \tau_{-b_1}\}$. Then $\langle v_\tau, b_i \rangle = 0$ for $i = 1, \ldots, d$, since $b_i \in \mathcal{S}$. This implies $v_\tau = 0$, a contradiction. Therefore $\Delta(1)$ is determined. Since no cone in Δ contains a linear subspace, this already implies $X \cong \mathbb{P}^d$. The general case is treated similarly and left to the reader. □

As a corollary we get from the existence of an $\mathcal{S}$-root basis and Prop. 3.11:

Corollary 3.19 $|\mathcal{S}| \leq d^2 + d$, *with equality iff* $X \cong \mathbb{P}^d$.

Above results yield now the following existence theorem:

Proposition 3.20

1. *There exists an* $\mathbb{R}$*-linearly independent family* B *of roots that can be partitioned into three pairwise disjoint subsets* B_1, B_2, B_3 *such that* B_1 *is an* $\mathcal{S}_1$*-root basis,* B_2 *is an* $\mathcal{S}_2$*-root basis,* $B_1 \cup B_2$ *is an* $\mathcal{S}$*-root basis and* B_3 *is a* $\mathcal{U}_1$*-facet basis such that* $\langle \eta_b, b' \rangle = 0$ *for all* $b \in B_1 \cup B_2$ *and* $b' \in B_3$.
   Hence $\dim_{\mathbb{R}} \operatorname{lin}(\mathcal{S}) + |\eta(\mathcal{U}_1)| = |B| \leq d$.
2. *There exists an* $\mathcal{R}$*-facet basis* D *that can be partitioned into three pairwise disjoint subsets* D_1, D_2, D_3 *such that* D_1 *is a* $\mathcal{U}_1$*-facet basis,* D_2 *is a* $\mathcal{U}_2$*-facet basis,* $D_1 \cup D_2$ *is a* $\mathcal{U}$*-facet basis and* D_3 *is an* $\mathcal{S}_1$*-root basis.*
   Hence $|\eta(\mathcal{U}_1)| + |\eta(\mathcal{U}_2)| + \dim_{\mathbb{R}} \operatorname{lin}(\mathcal{S}_1) = |D| \leq d$.

The details of the proof can be found in [14] (for the existence of root bases one uses Prop. 3.14 and for the existence of facet bases Lemma 3.8).

Corollary 3.21
1. $|\eta(\mathcal{U})| \leq d$, *where equality implies that* $\eta(\mathcal{R}) = \eta(\mathcal{U})$.
2. $|\eta(\mathcal{U}) \backslash \eta(\mathcal{S})| \leq \operatorname{codim}_{\mathbb{R}} \operatorname{lin}(\mathcal{S})$.
3. $|\eta(\mathcal{R})| \leq 2d$, *with equality iff* $X \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$.

Proof 1. Follows from 3.20(2). 2. Follows from 3.20(1).

3. Let D be the $\mathcal{R}$-facet basis from 3.20(2), we have $|D| \leq d$. By definition $\eta(\mathcal{R}) = \{\eta_x : x \in D_1 \cup D_2\} \cup \{\eta_{\pm x} : x \in D_3\}$, this gives the upper bound. Equality implies $|D| = d$ and $D = D_3$, i.e., $\mathcal{R} = \mathcal{S}$, with no element in D equivalent to any other. Applying Prop. 3.18 yields the desired result. □

While the case when $M_{\mathbb{R}}$ is spanned by semisimple roots is completely classified, there are partial results in the case of codimension one.

Proposition 3.22 *Let* $\dim_{\mathbb{R}} \operatorname{lin}(\mathcal{S}) = d - 1$.

1. *If* $|\Delta(1)| \neq \eta(\mathcal{S})$, *then there exists* $\tau \in \Delta(1) \backslash \eta(\mathcal{S})$ *with* $\Delta(1) \backslash \eta(\mathcal{S}) \subseteq \{\pm \tau\}$, *and we have* $\mathcal{V}_\tau \cong \mathbb{P}^{|Y_1|-1} \times \cdots \times \mathbb{P}^{|Y_q|-1}$.
2. *If* $q = 1$, *i.e.,* $|\mathcal{S}| = d^2 - d$, *then* $|\eta(\mathcal{U})| = 1$ *and* $\eta(\mathcal{S}) \cap \eta(\mathcal{U}) = \emptyset$.

Proof Let $b_1, \ldots, b_{d-1}$ be an $\mathcal{S}$-root basis. By 3.4 we can find a lattice point $b_d \in M$ such that $b_1, \ldots, b_d$ is a $\mathbb{Z}$-basis of M. Let $e_1, \ldots, e_d$ denote the dual $\mathbb{Z}$-basis.

1. Let $\tau \in \Delta(1) \backslash \eta(\mathcal{S})$. Then $\langle v_\tau, b_i \rangle = 0$ for all $i = 1, \ldots, d-1$, hence $v_\tau \in \{\pm e_d\}$. The set $\mathcal{S}$ is by construction canonically the set of roots of $\mathcal{V}_\tau$, so we can apply Prop. 3.18.

2. Let $q = 1$. By 3.11 this is equivalent to $|\mathcal{S}| = (d-1)^2 + d - 1 = d^2 - d$. For $i = 1, \ldots, d-1$ there exist $k_i \in \mathbb{Z}$ such that $\eta_i := \eta_{b_i} = -e_i + k_i e_d$. There exists $k_d \in \mathbb{Z}$ such that $\eta_d := \eta_{-b_1} = e_1 + \cdots + e_{d-1} + k_d e_d$.

Since $|\eta(\mathcal{S})| = d$, there exists $\tau \in \Delta(1)\backslash\eta(\mathcal{S})$, we may assume $v_\tau = e_d$. Let $x = \lambda_1 b_1 + \cdots + \lambda_d b_d \in M$. We have $x \in \mathcal{R}$ with $\eta_x = e_d$ iff $\langle x, e_d\rangle = -1$ and $\langle x, \eta_i\rangle \geq 0$ for $i = 1, \ldots, d$. This is equivalent to $\lambda_d = -1$, $\lambda_i \leq -k_i$ for $i = 1, \ldots, d-1$ and $\lambda_1 + \cdots + \lambda_{d-1} \geq k_d$. Hence there exists a root $x \in \mathcal{R}$ with $\eta_x = e_d$ if and only if $k_1 + \cdots + k_d \leq 0$.

On the other hand let $u := k_1 b_1 + \cdots + k_{d-1} b_{d-1} + b_d \in M$. Then $u^\perp$ is a hyperplane spanned by $\eta_1, \ldots, \eta_{d-1}$. We have $\langle u, e_d\rangle = 1$ and $\langle u, \eta_d\rangle = k_1 + \cdots + k_d$. Therefore when $|\Delta(1)| = d+1$, we get $\langle u, \eta_d\rangle < 0$, so there exists $x \in \mathcal{R}$ with $\eta_x = e_d$, necessarily $e_d \in \eta(\mathcal{U})$. Otherwise for $\Delta(1)\backslash\eta(\mathcal{S}) = \{\pm e_d\}$, the analogous computation for $-e_d$ yields that either e_d or $-e_d$ is in $\eta(\mathcal{U})$.

Assume $\eta(\mathcal{S}) \cap \eta(\mathcal{U}) \neq \emptyset$, so $\mathcal{S}_2 \neq \emptyset$. Use the family B in Prop. 3.20(1): Since by assumption all elements in $B_1 \cup B_2$ are mutually equivalent, however no element in $\mathcal{S}_1$ is equivalent to one in $\mathcal{S}_2$, we have $B = B_2$, i.e., $\mathcal{S} = \mathcal{S}_2$. This yields $|\eta(\mathcal{U}_2)| = d$. Since $|\eta(\mathcal{U}_1)| = 1$, we get a contradiction to 3.21(1). □

This result yields sharp upper bounds on $\dim \mathrm{Aut}(X)$ in the reductive case:

Theorem 3.23 *Let X be a d-dimensional complete toric variety with reductive automorphism group. Let $n := \dim \mathrm{Aut}(X)$. Then*

$$n \leq d^2 + 2d, \quad \textit{with equality only in the case of projective space.}$$

If $d = 2$ and X is not a product of projective spaces, then $n = 2$.
If $d \geq 3$ and X is not a product of projective spaces, then

$$n \leq d^2 - 2d + 4,$$

where equality holds iff $q = 2$ with $|Y_1| = 2$ and $|Y_2| = d - 1$.

Proof Let $c_i := |Y_i| - 1$ for $i = 1, \ldots, q$. By 3.11 and 3.15, we have $l := c_1 + \cdots + c_q = \dim_{\mathbb{R}} \mathcal{S}$ and $|\mathcal{S}| = c_1^2 + \cdots + c_q^2 + l \leq l^2 + l$. Recall from 3.2 that $n = |\mathcal{S}| + d$. From 3.18 we get the first statement for $l = d$ (or see 3.19). Moreover for the second statement we can assume $l = d - 1$, since $(d-2)^2 + (d-2) < d^2 - 3d + 4$.

By 3.22(2) we have $q > 1$, since Δ is semisimple; in particular $d > 2$.

We may assume $c_1 \leq \cdots \leq c_q$.

If $q = 2$, then $c_1 + c_2 = d - 1$, hence either $c_1 = 1$ and $c_2 = d - 2$ (this yields $c_1 c_2 = d - 2$), or $c_1 \geq 2$ and $c_2 \geq (d-1)/2$ (this yields $c_1 c_2 \geq d - 1$).

If $q \geq 3$, then $\sum_{i<j} c_i c_j \geq c_1(c_2 + \cdots + c_q) + c_2 c_3 = c_1(c_1 + \cdots + c_q) + c_2 c_3 - c_1 c_1 \geq c_1(d-1) \geq d - 1$.

In any case $|\mathcal{S}| = c_1^2 + \cdots c_q^2 + d - 1 = (c_1 + \ldots + c_q)^2 + d - 1 - 2\sum_{i<j} c_i c_j \leq (d-1)^2 + d - 1 + 2(2-d) = d^2 - 3d + 4$, with equality only for $q = 2$ with $c_1 = 1$ and $c_2 = d - 2$. □

The following example shows that the last bound is sharp for any $d \geq 3$:

Example 3.24 (due to C. Haase, Duke University)
Let $P \subseteq \mathbb{R}^d$ be the d-dimensional reflexive polytope defined as the convex hull of $(2E_1^*) \times \{0\} \times \{1\}$ and $\{0\} \times (2E_{d-2}^*) \times \{-1\}$, where $E_k^* \subseteq \mathbb{R}^k$ denotes as in 3.16 the k-dimensional reflexive polytope corresponding to $\mathbb{P}^k$. This implies that $P \cap (\mathbb{R}^1 \times \mathbb{R}^{d-2} \times \{0\}) \cong E_1^* \times E_{d-2}^*$, $\mathcal{N}_P$ is semisimple with $\dim_{\mathbb{R}} \mathcal{S} = d - 1$, and the last upper bound in the previous theorem is attained by X_P.

4 The set of roots of a reflexive polytope

Throughout the section let P be a d-dimensional reflexive polytope in $M_{\mathbb{R}}$.

In this section we will focus on Gorenstein toric Fano varieties, these varieties correspond to reflexive polytopes as described in the second section. When P is reflexive, we have by duality that *the set of roots $\mathcal{R}$ of the normal fan $\mathcal{N}_P$ is exactly the set of lattice points in the relative interior of facets of P.*

Definition 4.1 The set $\mathcal{R}$ of roots of P is defined as the set of roots of $\mathcal{N}_P$. For $m \in \mathcal{R}$ we denote by $\mathcal{F}_m$ the unique facet of P that contains m, and we again define $\eta_m = \eta_{\mathcal{F}_m}$ to be the unique primitive inner normal with $\langle \eta_m, \mathcal{F}_m \rangle = -1$. For a subset $A \subseteq \mathcal{R}$ it is convenient to define $\mathcal{F}(A) := \{\mathcal{F}_m \, : \, m \in A\}$. We say P is *semisimple*, if $\mathcal{N}_P$ is semisimple, i.e., $\mathcal{R} = -\mathcal{R}$; equivalently $\mathrm{Aut}(X_P)$ is reductive.

Most results of the previous section have now a direct geometric interpretation. Here three examples shall be explicitly stated (just use Corollary 3.21(1), the basic fact $-\mathcal{S}_1 = \mathcal{S}_1$, and Corollary 3.21(3)):

Corollary 4.2 *There are at most d facets of P containing unipotent roots.*

Corollary 4.3 *If a facet of P contains an unipotent root and a semisimple root x, then the facet containing $-x$ also contains an unipotent root.*

Corollary 4.4 *There are at most $2d$ facets containing roots; equality holds if and only if $P \cong [-1, 1]^d$ (isomorphic as lattice polytopes).*

For another application we apply Prop. 3.18 and Prop. 3.22(2) in the case of $d = 2$ and get a characterization of semisimple reflexive polygons (for this use the well-known lemma [13, Lemma 1.17(1)]):

Corollary 4.5 *Let P be a two-dimensional reflexive polytope.*
Then P is semisimple iff X_{P^*} is nonsingular or $X_P \cong \mathbb{P}^2$ or $X_P \cong \mathbb{P}^1 \times \mathbb{P}^1$.

There is a nice property of pairwise orthogonal families of roots:

Proposition 4.6 *Let B be a non-empty set of pairwise orthogonal roots.*
Then $F := \bigcap_{b \in B} \mathcal{F}_b$ is a non-empty face of P of codimension $|B|$, and the sum over all elements in B is a lattice point in the relative interior of F.

Proof Let $B = \{b_1, \ldots, b_l\}$ with $|B| = l$. For $i \in \{1, \ldots, l\}$ we define $s_i := \sum_{j=1}^{i} b_j$ and $F_i := \cap_{j=1}^{i} \mathcal{F}_{b_j}$. Orthogonality implies that $\{\mathcal{F}_{b_1}, \ldots, \mathcal{F}_{b_l}\}$ is exactly the set of facets containing s_l. Therefore $s_l \in \mathrm{relint} F_l$, and since any l-codimensional face of P is contained in at least l facets, we have $\mathrm{codim} F_l \leq l$. On the other hand $s_i \notin F_{i+1}$ for all $i = 1, \ldots, l$, so $F_1 \supsetneq \cdots \supsetneq F_l$, hence we obtain $\mathrm{codim} F_l = l$. □

This proposition can be applied to a $\mathcal{U}$-facet basis (see 3.20(2)):

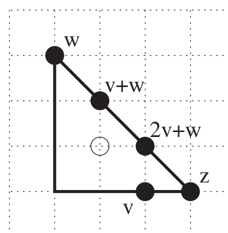
Corollary 4.7 *If $\mathcal{U} \neq \emptyset$, then $\bigcap_{F \in \mathcal{F}(\mathcal{U})} F \neq \emptyset$ is a face of codimension $|\mathcal{F}(\mathcal{U})|$.*

To further sharpen the results of the previous section we need an elementary but fundamental property of pairs of lattice points on the boundary of a reflexive polytope (for a proof see [13, Prop. 4.1]).

Lemma 4.8 *Let $v, w \in \partial P \cap M$ with $w \neq -v$ such that v, w are not contained in a common facet.*

Then $v + w \in \partial P \cap M$, and there exists $z := z(v, w) \in \partial P \cap M$ such that $z = av + bw$ for $a, b \in \mathbb{N}_{\geq 1}$ with $a = 1$ or $b = 1$, and v, z (as well as w, z) are contained in a common facet.

The result shall be illustrated for $P := E_2^*$, i.e., $X_P \cong \mathbb{P}^2$:



This partial addition on $\partial P \cap M$ extends the partial addition of roots in Lemma 3.8 (see also [5, Def. 3.2]).

Corollary 4.9 *Let $v \in \mathcal{R}$, $w \in \partial P \cap M$ with $w \neq -v$ and $w \notin \mathcal{F}_v$.*
Then $v + w \in \partial P \cap M$ and $z(v, w) \in \mathcal{F}_v$. Moreover

$$\langle \eta_v, w \rangle > 0 \text{ iff } z(v, w) = av + w \text{ for } a \geq 2.$$

In this case $z(v, w) = (\langle \eta_v, w \rangle + 1)v + w$.

Now we can improve Prop. 3.18 by taking the ambient space of semisimple roots into account (recall the definition of E_d in Example 3.16).

Theorem 4.10 *Let $B \subseteq \mathcal{S}$ be an A-root basis for a subset $A \subseteq \mathcal{R}$, and $R_1, \ldots, R_t$ the partition of B into equivalence classes of order $c_1, \ldots, c_t$. Then there are isomorphisms of lattice polytopes (with respect to lattices* $\text{lin}(A) \cap M$ *and* $\mathbb{Z}^{c_1+\cdots+c_t}$*)*

$$P \cap \text{lin}(A) \;\cong\; \bigoplus_{i=1}^{t} P \cap \text{lin}(R_i) \;\cong\; \bigoplus_{i=1}^{t} E_{c_i}^*.$$

In particular the intersection of P with the space spanned by all semisimple roots is again a reflexive polytope corresponding to a product of projective spaces.

Proof Let $t = 1$, i.e., all elements in B are mutually equivalent. The general case can be found in [14]. Let $l = |B| \geq 2$, $B = \{b_1, \ldots, b_l\}$, $b := b_1 + \cdots + b_l$.

Claim: $P \cap \text{lin}(b_1, \ldots, b_l) = \text{conv}(b, b - (l+1)b_i \,:\, i = 1, \ldots, l) \cong E_l^*$.

Denote by Q the simplex on the right hand side of the claim, so $Q \cong E_l^*$.

By 4.6 $b \in \bigcap_{i=1}^{l} \mathcal{F}_{b_i}$. Since by assumption $\langle \eta_{-b_i}, b \rangle = \sum_{j=1}^{l} \langle \eta_{-b_i}, b_j \rangle = l$, it follows from 4.9 that $z(-b_i, b) = b - (l+1)b_i \in \mathcal{F}_{-b_i}$ for $i = 1, \ldots, l$. Hence

$Q \subseteq P \cap \mathrm{lin}(b_1, \ldots, b_l)$. On the other hand the previous calculation and orthogonality also implies that $Q \cap \mathcal{F}_{b_1}, \ldots, Q \cap \mathcal{F}_{b_l}, Q \cap \mathcal{F}_{-b_1}$ are exactly the facets of the simplex Q. This proves the claim. □

5 Criteria for a reductive automorphism group

In this section we give several criteria for the automorphism group of a complete toric variety, respectively a Gorenstein toric Fano variety, to be reductive.

Definition 5.1 For a polytope $Q \subseteq M_{\mathbb{R}}$ we let b_Q denote the *barycenter* of Q. When Q is a lattice polytope, we denote by $\mathrm{vol}(Q)$ the *lattice volume* of Q, i.e., $\mathrm{vol}(\Pi) = 1$ for a fundamental paralleloped Π of the lattice $\mathrm{aff}(Q) \cap M$.

Theorem 5.2

1. *Let $X = X(N, \triangle)$ be a complete toric variety.*

   The following conditions (a) – (c) are equivalent*:*

   (a) *$\triangle$ is semisimple, i.e.,* $\mathrm{Aut}(X)$ *is reductive*
   (b) $\sum_{x \in \mathcal{R}} x = 0$
   (c) $\sum_{\tau \in \triangle(1)} \langle v_\tau, x \rangle = 0$ *for all* $x \in \mathcal{R}$

   If $\sum_{\tau \in \triangle(1)} v_\tau = 0$*, then $\triangle$ is semisimple.*

2. *Let X_P be a Gorenstein toric Fano variety for $P \subseteq M_{\mathbb{R}}$ reflexive.*

   The following conditions (a) – (e) are equivalent*:*

   (a) *P is semisimple, i.e.,* $\mathrm{Aut}(X_P)$ *is reductive*
   (b) $\sum_{y \in P^* \cap N} \langle y, x \rangle = 0$ *for all* $x \in \mathcal{R}$
   (c) $\langle b_{P^*}, x \rangle = 0$ *for all* $x \in \mathcal{R}$
   (d) $\mathrm{vol}(F') = \mathrm{vol}(\mathcal{F}_x)$ *for all* $x \in \mathcal{R}$, $F' \in \mathcal{F}(P)$ *with* $\langle \eta_{F'}, x \rangle > 0$
   (e) $|F' \cap M| = |\mathcal{F}_x \cap M|$ *for all* $x \in \mathcal{R}$, $F' \in \mathcal{F}(P)$ *with* $\langle \eta_{F'}, x \rangle > 0$

   Any one of the following conditions is sufficient *for P to be semisimple:*

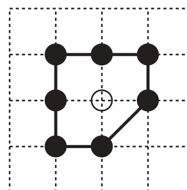
   i. $b_P = 0$
   ii. $b_{P^*} = 0$
   iii. $\sum_{m \in P \cap M} m = 0$
   iv. $\sum_{y \in P^* \cap N} y = 0$
   v. $\sum_{v \in \mathcal{V}(P^*)} v = 0$
   vi. *All facets of P have the same relative lattice volume*
   vii. *All facets of P have the same number of lattice points*

   Condition vi *implies* v*, e.g., if X_{P^*} is nonsingular.*

Remark 5.3 Using the list of d-dimensional reflexive polytopes for $d \leq 4$ and the computer program PALP due to Kreuzer and Skarke (see [10,11]) we found examples showing that in the second part of the theorem the sufficient conditions i–v are pairwise independent, i.e., in general no condition implies any other. These examples can be found in [14]. For instance the following seven column vectors are the vertices of a four-dimensional reflexive polytope P that satisfies $b_P = 0$, $\sum_{m \in P \cap M} m = 0$, $\sum_{v \in \mathcal{V}(P)} v = 0$, however P^* does not satisfy *any* of these three conditions.

$$\begin{matrix} 1&0&0&0&-1&-1&1\\ 0&1&0&0&0&-1&0\\ 0&0&1&0&-1&0&0\\ 0&0&0&1&0&0&-1 \end{matrix}$$

Example 5.4 The "dual" of condition v is not a sufficient condition: The following reflexive polygon is not semisimple, however the sum of the five vertices is zero.



Remark 5.5 Using the existence of an Einstein-Kähler metric (see Theorem 1.1) Batyrev and Selivanova deduced in [2] that P has to be semisimple, if X_P is nonsingular and P is *symmetric*, i.e., the group $\mathrm{Aut}_M(P)$ of linear lattice automorphisms leaving P invariant has only the origin 0 as a fixpoint. Now the second part of the previous theorem yields several direct proofs of this combinatorial result: A symmetric reflexive polytope P obviously satisfies $b_P = 0$ and $\sum_{m \in P \cap M} m = 0$. Moreover one can show that also P^* has to be symmetric (see [14]), so even conditions i–v are satisfied. Yet another proof can be derived from Corollary 4.7, since, if P were not semisimple, then the sum of all lattice points in the non-empty face $\bigcap_{F \in \mathcal{F}(\mathcal{U})} F$ would be a non-zero fixpoint. Furthermore the first part of the theorem immediately yields a generalization to complete toric varieties, see Theorem 1.4 in the introduction.

For the proof of Theorem 5.2 we need some lemmas. The first is just a simple observation:

Lemma 5.6 *Let $\triangle$ be a complete fan in $N_{\mathbb{R}}$.*

$$m \in \mathcal{R} \Longrightarrow \sum_{\tau \in \triangle(1)} \langle v_\tau, m \rangle \in \mathbb{N},$$

in this case

$$m \in \mathcal{S} \Longleftrightarrow \sum_{\tau \in \triangle(1)} \langle v_\tau, m \rangle = 0.$$

Lemma 5.7 *Let $\triangle$ be a complete fan in $N_{\mathbb{R}}$.*

Let $A \subseteq \mathcal{R}$ be a subset such that

$$\sum_{m \in A} k_m m = 0$$

for some positive integers $\{k_m\}_{m \in A}$. Then $A \subseteq \mathcal{S}$.

Proof Assume $A \cap \mathcal{U} \neq \emptyset$. Then by 5.6
$0 = \sum_{\tau \in \Delta(1)} \langle v_\tau, \sum_{m \in A} k_m m \rangle = \sum_{m \in A \cap \mathcal{U}} k_m \sum_{\tau \in \Delta(1)} \langle v_\tau, m \rangle \geq 1$, a contradiction. □

In the case of a reflexive polytope the following result is fundamental:

Lemma 5.8 *Let P be a d-dimensional reflexive polytope in $M_\mathbb{R}$.*
Let $m \in \mathcal{R}$. Define the canonical projection map along m

$$\pi_m \; : \; M_\mathbb{R} \to M_\mathbb{R}/\mathbb{R}m.$$

Then π_m induces an isomorphism of lattice polytopes

$$\mathcal{F}_m \to \pi_m(P),$$

with respect to the lattices $\mathrm{aff}(F) \cap M$ *and $M/m\mathbb{Z}$.*

Proof [13, Prop. 3.2] immediately implies that $\pi_m : \mathcal{F}_m \to \pi_m(P)$ is a bijection. It is even an isomorphism of lattice polytopes by 2.4.

Another proof can be easily done using only the definition of a root. □

Using this lemma we get a reformulation of 5.6. Note that $A - B := \{a - b : a \in A,\ b \in B\}$ for arbitrary sets $A, B \subseteq \mathbb{R}^d$; a facet F of a d-dimensional polytope $Q \subseteq M_\mathbb{R}$ is said to be *parallel* to $\mathbb{R}x$ for some $x \in M_\mathbb{R}$, if $\langle \eta_F, x \rangle = 0$.

Lemma 5.9 *Let P be a d-dimensional reflexive polytope in $M_\mathbb{R}$.*
Let $m \in \mathcal{R}$. We set $F := \mathcal{F}_m$.

1. $P \subseteq F - \mathbb{R}_{\geq 0}x$, $P \cap M \subseteq (F \cap M) - \mathbb{N}x$, $\{n \in P^* \cap N \; : \; \langle n, m \rangle < 0\} = \{\eta_m\}$.
2. $P = \mathrm{conv}(F, F')$ *for a facet F' if and only if there is only one facet $F' \neq F$ not parallel to $\mathbb{R}m$, i.e., $\langle \eta_{F'}, m \rangle > 0$.*
3. *$m \in \mathcal{S}$ iff the previous condition is satisfied and $\langle \eta_{F'}, m \rangle = 1$.*
   In this case $F' = \mathcal{F}_{-m}$. Furthermore F and F' are naturally isomorphic as lattice polytopes and $\{n \in P^* \cap N \; : \; \langle n, m \rangle \neq 0\} = \{\eta_m, \eta_{-m}\}$.

Lemma 5.10 *Let P be a d-dimensional reflexive polytope in $M_\mathbb{R}$.*
For $v \in \mathcal{V}(P^*)$ we denote by $v^* \in \mathcal{F}(P)$ the corresponding facet of P. Then

$$\sum_{v \in \mathcal{V}(P^*)} \mathrm{vol}(v^*)\, v = 0.$$

Proof Having chosen a fixed lattice basis of M we denote by $\mathrm{vol}_{\mathbb{R}^d}$ the associated differential-geometric volume in $M_\mathbb{R} \cong \mathbb{R}^d$. Let $F \in \mathcal{F}(P)$ arbitrary. Since η_F is primitive, it is a well-known fact that the determinant of the lattice $\mathrm{aff}(F) \cap M$, i.e., the volume of a fundamental paralleloped, is exactly $\|\eta_F\|$, hence we get $\mathrm{vol}_{\mathbb{R}^d}(F) = \mathrm{vol}(F) \cdot \|\eta_F\|$. The easy direction of the so called existence theorem of Minkowski (see [3, no. 60]) yields $\sum_{F \in \mathcal{F}(P)} \mathrm{vol}(F)\, \eta_F = 0$. □

The approximation approach in the next proof is based upon an idea of Batyrev.

Lemma 5.11 *Let $Q \subseteq M_{\mathbb{R}}$ be a d-dimensional polytope with a facet F and $x \in \operatorname{aff}(F)$ such that $Q \subseteq F - \mathbb{R}_{\geq 0}x$. For $q \in Q$ with $q = y - \lambda x$, where $y \in F$ and $\lambda \in \mathbb{R}_{\geq 0}$, we define $A(q) := y - 2\lambda x$. This definition extends uniquely to an affine map A of $M_{\mathbb{R}}$.*

Then $A(b_Q)$ is either in the interior of Q or in the relative interior of a facet of Q not parallel to $\mathbb{R}x$. The last case happens exactly iff there exists only one facet $F' \neq F$ not parallel to $\mathbb{R}x$.

Proof First assume there is exactly one facet $F' \neq F$ not parallel to $\mathbb{R}x$. This implies $Q = \operatorname{conv}(F, F')$. Choose an $\mathbb{R}$-basis $e_1, \dots, e_d$ of $M_{\mathbb{R}}$ such that $e_d = x$ and $\mathbb{R}e_1, \dots, \mathbb{R}e_{d-1}$ are parallel to F. Now let $y \in F$ and define $h(y) \in \mathbb{R}_{\geq 0}$ such that $y - h(y)x \in F'$. For $k \in \mathbb{N}_{>0}$ let $F_k(y) := y + \cup_{i=1}^{d-1} [-1/(2k), 1/(2k)]e_i$ and $Q_k(y) := F_k(y) - [0, h(y)]x$. Then $b_{Q_k(y)} = y - h(y)/2x$ and $A(b_{Q_k(y)}) = y - h(y)x \in F'$. Let $M' := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{d-1}$ and $z \in \operatorname{relint} F$. For any $k \in \mathbb{N}_{>0}$ we define $G_k := (z + M'/k) \cap F$ and $F_k := \cup_{y \in G_k} F_k(y)$. For $k \to \infty$ the sets F_k converge uniformly to F. Therefore also $Q_k := \cup_{y \in G_k} Q_k(y)$ converges uniformly to Q for $k \to \infty$. This implies that b_{Q_k} converges to b_Q for $k \to \infty$. Since A is affine and b_{Q_k} is a finite convex combination of $\{b_{Q_k(y)} : y \in G_k\}$ for any $k \in \mathbb{N}_{>0}$, also $A(b_{Q_k})$ is a finite convex combination of $\{A(b_{Q_k(y)} : y \in G_k\} \subseteq F'$ for any $k \in \mathbb{N}_{>0}$. This implies $A(b_{Q_k}) \in F'$ for any $k \in \mathbb{N}_{>0}$. Since A is continuous and F' is closed, this yields $A(b_Q) \in F'$. Moreover obviously $A(b_Q) \in \operatorname{relint} F'$.

Now let there be more than one facet different from F and not parallel to $\mathbb{R}x$. Then we can choose a polyhedral subdivision of Q into finitely many polytopes $\{K_j\}$ such that any K_j satifies the assumption of the previous case. Hence, since b_Q is a proper convex combination of $\{b_{K_j}\}$, also $A(b_Q)$ is a proper convex combination of $\{A(b_{K_j})\} \subseteq \partial Q$. However since not all $A(b_{K_j})$ are contained in one facet, $A(b_Q)$ is in the interior of Q. □

Proof of Theorem 5.2 The first part of the theorem, when X is a complete toric variety, follows from 5.6 and 5.7. So let $X = X_P$ for $P \subseteq M_{\mathbb{R}}$ a d-dimensional reflexive polytope, and we consider the second part of the theorem.

The equivalences of (a), (b), (c) and the sufficiency of ii, iv, v follow from 5.6 and 5.9. (d) and (e) are necessary conditions for semisimplicity due to 5.9.

Let (d) be satisfied and $x \in \mathcal{R}$. By 5.9(1) and 5.10 we have

$$\operatorname{vol}(\mathcal{F}_x) = \sum_{v \in \mathcal{V}(P^*),\, \langle v, x\rangle > 0} \operatorname{vol}(v^*)\langle v, x\rangle.$$

By assumption there is only one vertex $v \in \mathcal{V}(P^*)$ with $\langle v, x\rangle > 0$, furthermore $\langle v, x\rangle = 1$. Hence 5.9 implies $x \in \mathcal{S}$.

Let (e) be satisfied. Let $x \in \mathcal{R}$, $F := \mathcal{F}_x$ and $F' \in \mathcal{F}(P)$ with $\langle \eta_{F'}, x\rangle > 0$. Due to 5.9(1) and by assumption there is a bijective map $h : F' \to F$ of lattice polytopes, i.e., $h(F' \cap M) = F \cap M$. Hence there exists a lattice point $x' \in F'$ with $h(x') = x$, necessarily $x' = -x \in \operatorname{relint} F'$, so $x \in \mathcal{S}$.

The sufficiency of vi, vii is now trivial, 5.10 shows that vi implies v.

From now on let $x \in \mathcal{R}$ and A be the affine map defined in 5.11 for $Q := P$ and $F := \mathcal{F}_x$.

Let i be satisfied. By 5.9(1) we can apply Lemma 5.11 to get $-x = x - 2x = A(0) = A(b_P) \in \mathcal{R}$, since $\operatorname{int} P \cap M = \{0\}$.

Eventually let iii be satisfied. For any $y \in F \cap M$ define $x_y \in P \cap M$ such that $x_y := y - kx$ for $k \in \mathbb{N}$ maximal, and let $T_y := [y, x_y]$. Then 5.9(1) implies that

$$-x = A(0) = A\left(\frac{1}{|P\cap M|}\sum_{m\in P\cap M} m\right) = A\left(\sum_{y\in F\cap M}\frac{|T_y\cap M|}{|P\cap M|}\frac{1}{|T_y\cap M|}\sum_{m\in T_y\cap M} m\right)$$
$$= \sum_{y\in F\cap M}\frac{|T_y\cap M|}{|P\cap M|} A\left(\frac{1}{|T_y\cap M|}\sum_{m\in T_y\cap M} m\right) = \sum_{y\in F\cap M}\frac{|T_y\cap M|}{|P\cap M|} x_y.$$

Hence $-x$ is a proper convex combination of $\{x_y\}_{y\in F\cap M}$, so $-x \in \mathcal{R}$. □

6 Centrally symmetric reflexive polytopes

In this section the following result is going to be proved (again $E_1 := [-1, 1]$).

Theorem 6.1 *Let $P \subseteq M_\mathbb{R}$ be a centrally symmetric reflexive polytope.*

1. *$P \cong E_1^{|\mathcal{R}|/2} \times G$ for a $|\mathcal{R}|/2$-codimensional face G of P that is a centrally symmetric reflexive polytope (with respect to* $\operatorname{aff}(G) \cap M$ *and a unique lattice point in* $\operatorname{relint} G$*) and has no roots itself.*
2. *Any facet contains at most 3^{d-1} lattice points and at most one root of P. P contains at most 3^d lattice points and has at most $2d$ roots. Hence*

$$\dim \operatorname{Aut}(X_P) \leq 3d.$$

3. *The following statements are equivalent:*
   (a) *P contains 3^d lattice points*
   (b) *P has $2d$ roots, i.e.,* $\dim \operatorname{Aut}(X_P) = 3d$
   (c) *Every facet of P has 3^{d-1} lattice points*
   (d) *Every facet of P contains a root of P*
   (e) *$P \cong E_1^d$, i.e., $X_P \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$*

The first property immediately implies:

Corollary 6.2 *Let $P \subseteq M_\mathbb{R}$ be a centrally symmetric reflexive polytope.*

If P contains no facet that is centrally symmetric with respect to a root of P, or there are at most $d - 1$ facets of P that can be decomposed as a product of lattice polytopes $E_1 \times F'$, then P has no roots.

Hence if $d \geq 3$ and P is simplicial, or $d \geq 4$ and any facet of P is simplicial, then

$$\dim \operatorname{Aut}(X_P) = d.$$

For the proof of Theorem 6.1 we need the following lemma that is an easy corollary of 5.9 and 2.4:

Lemma 6.3 *Let $P \subseteq M_\mathbb{R}$ be a centrally symmetric reflexive polytope. Let $F \in \mathcal{F}(P)$. Then*

$$P \cong E_1 \times F \text{ iff } F \text{ contains a root } x \text{ of } P.$$

In this case F is a centrally symmetric reflexive polytope (with respect to the lattice $\operatorname{aff}(F) \cap M$ *with origin x).*

Proof of Theorem 6.1 1. Apply the previous lemma inductively.

2. The bounds on the lattice points were proven in [13, Thm. 6.4]. Since as just seen any facet of P containing a root is reflexive, it contains precisely one lattice point in its relative interior. Now we apply 3.2 and 1. (or 4.4).

3. (b) $\Leftrightarrow$ (e) $\Leftrightarrow$ (d): Since P as a centrally symmetric polytope contains at least $2d$ facets, this follows from 1., alternatively use 2. and 4.4.

For the remaining equivalences we need the canonical map

$$\alpha : P \cap M \to M/3M \cong (\mathbb{Z}/3\mathbb{Z})^d.$$

In the proof of [13, Thm. 6.4] it was shown that α is injective.

Let $F \in \mathcal{F}(P)$ be arbitrary but fixed. Define $u := \eta_F \in \mathcal{V}(P^*)$ and also the $\mathbb{Z}/3\mathbb{Z}$-extended map $\alpha(u) : M/3M \to \mathbb{Z}/3\mathbb{Z}$. For $m \in P \cap M$ we have $\langle u, m\rangle \in \{-1, 0, 1\}$, in particular

$$m \in F \Longleftrightarrow \langle \alpha(u), \alpha(m)\rangle = -1 \in \mathbb{Z}/3\mathbb{Z}.$$

(d) $\Rightarrow$ (a): Trivial, since (d) $\Leftrightarrow$ (e).

(a) $\Rightarrow$ (c): If P contains 3^d lattice points, then α is a bijection, and therefore $|F \cap M| = |\{z \in M/3M : \langle \alpha(\eta_F), z\rangle = -1\}| = 3^{d-1}$.

(c) $\Rightarrow$ (d): The assumption implies that for any facet $F' \in \mathcal{F}(P)$ the map

$$\alpha|_{F'} : F' \cap M \to \{z \in M/3M : \langle \alpha(\eta_{F'}), z\rangle = -1\}$$

is a bijection. Define $x := (1/3^{d-1}) \sum_{m \in F \cap M} m \in \operatorname{relint} F$.

It remains to prove $x \in M$.

Choose a facet $G \in \mathcal{F}(P^*)$ and an $\mathbb{R}$-linearly independent family $w_1, \ldots, w_d$ of vertices of G such that $w_1 = u$ and $w_2, \ldots, w_d$ are contained in a $(d-2)$-dimensional face of P^*.

Denote the corresponding facets of P by $F_1, F_2, \ldots, F_d$ with $\eta_{F_j} = w_j$ for $j = 1, \ldots, d$, so $F_1 = F$. Then $Q := \cap_{j=2}^d F_j$ is a one-dimensional face of P. Therefore also the affine span of $\alpha(Q \cap M)$ is a one-dimensional affine subspace of $M/3M$. Since $|F \cap Q| = 1$ there exists an element $b \in M/3M$ such that $\langle \alpha(u), b\rangle = 0$ and $\langle \alpha(w_j), b\rangle = -1$ for all $j = 2, \ldots, d$. Applying the assumption to F_2 yields a lattice point $v \in P \cap M$ with $\alpha(v) = b$. Hence also $\langle u, v\rangle = 0$ and $\langle w_j, v\rangle = -1$ for $j = 2, \ldots, d$.

By 2.4 we find a $\mathbb{Z}$-basis $e_1^* = u, e_2^*, \ldots, e_d^*$ of N such that for any $j = 2, \ldots, d$ there exist $\lambda_{j,k} \in \mathbb{R}$ with $e_j^* = \lambda_{j,2}(w_2 - u) + \cdots + \lambda_{j,d}(w_d - u)$.

Fact 1: $\langle w_k, \sum_{m \in F \cap M} m\rangle = 0$ for $k = 2, \ldots, d$.

(*Proof:* Since $F \cap F_k \neq \emptyset$, the assumption implies for $i = -1, 0, 1 \in \mathbb{Z}/3\mathbb{Z}$ that $|\{z \in M/3M : \langle \alpha(u), z\rangle = -1, \langle \alpha(w_k), z\rangle = i\}| = 3^{d-2}$.)

Fact 2: $\sum_{k=2}^d \lambda_{j,k} \in \mathbb{Z}$ for $j = 2, \ldots, d$.

(*Proof:* $\langle e_j^*, v\rangle = (-\sum_{k=2}^d \lambda_{j,k})\langle u, v\rangle + \sum_{k=2}^d \lambda_{j,k}\langle w_k, v\rangle = -\sum_{k=2}^d \lambda_{j,k}$ by the choice of v.)

Using these two facts we can finish the proof:

$$\langle e_1^*, x\rangle = \langle u, x\rangle = -1 \in \mathbb{Z},$$

$$\langle e_j^*, x\rangle = (1/3^{d-1})\left((-\sum_{k=2}^{d} \lambda_{j,k})\langle u, \sum_{m\in F\cap M} m\rangle + \sum_{k=2}^{d} \lambda_{j,k}\langle w_k, \sum_{m\in F\cap M} m\rangle\right)$$

$$= \sum_{k=2}^{d} \lambda_{j,k} \in \mathbb{Z} \text{ for } j = 2, \ldots, d.$$

Hence $x \in M$. □

Remark 6.4 Dropping the assumption of reflexivity and regarding just a complete toric variety $X = X(N, \triangle)$ with centrally symmetric $\triangle(1)$ we still get immediately from 3.1, 3.2 and 3.21(3) that $\dim \mathrm{Aut}(X) \leq 3d$, with equality iff $X \cong (\mathbb{P}^1)^d$.

For X as before, assume that X is also Gorenstein, i.e., the anticanonical divisor $-K_X$ is a Cartier divisor. In this case we can still show by slightly modifiying the proof of [13, Thm. 6.4] that $h^0(X, -K_X) \leq 3^d$ (see [14]).

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