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Deformation theory of representable morphisms of algebraic stacks

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Abstract We study the relationship between the deformation theory of representable 1-morphisms between algebraic stacks and the cotangent complex defined by Laumon and Moret-Bailly.

1 Introduction

Let $x : X \to \mathcal{Y}$ be a representable morphism of algebraic stacks. In this paper we study the relationship between the deformation theory of x and the cotangent complex defined in ([13], 17.3).

Our interest in this relationship comes from two sources. First, a consequence of the discussion in this paper is a construction of a canonical obstruction theory (in the sense of ([2], 2.6)) for any algebraic stack locally of finite type over a noetherian base, and hence we obtain the converse to (loc. cit., 5.3) alluded to in (loc. cit., p. 182) (see (1.7) below). Secondly, the results of this paper are used in the theory of logarithmic geometry to understand the relationship between the logarithmic cotangent complex and deformation theory of log schemes ([14]).

In the subsequent sections we study in turn the following three problems (1), (2), and (3):

Problem 1 (Generalization of ([10], III.1.2.3)) Let I be a quasi-coherent sheaf on X and define a $\mathcal{Y}$*-extension of* X *by* I to be a 2-commutative diagram

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$$\begin{array}{ccc} X & \xrightarrow{j} & X' \\ {\scriptstyle x}\downarrow & \swarrow {\scriptstyle x'} & \\ \mathcal{Y}, & & \end{array} \tag{1.0.1}$$

where j is a closed immersion of algebraic stacks defined by a square zero ideal, together with an isomorphism $\iota : I \simeq \mathrm{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X)$. We often write $(j : X \hookrightarrow X', \iota)$, or just $X \hookrightarrow X'$, for such an extension, and denote by $\epsilon_j : x \to x' \circ j$ the specified isomorphism in $\mathcal{Y}(X)$. In fact, x' is necessarily representable (2.1), so the set of $\mathcal{Y}$-extensions of X by I form in a natural way a category $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$ (2.2). In fact, the category $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$ has a natural structure of a Picard category (2.12), and hence its set of isomorphism classes of objects $\mathrm{Exal}_{\mathcal{Y}}(X, I)$ has a structure of a group. In section 2 we prove the following (see (2.15)–(2.19) for the meaning of the right hand side):

Theorem 1.1 *There is a natural isomorphism*

$$Exal_{\mathcal{Y}}(X, I) \simeq Ext^1(L_{X/\mathcal{Y}}, I), \tag{1.1.1}$$

where $L_{X/\mathcal{Y}}$ denotes the cotangent complex of $x : X \to \mathcal{Y}$.

Remark 1.2 When X is a Deligne–Mumford stack, we also sometimes abuse notation and write $L_{X/\mathcal{Y}}$ for the restriction to the étale site of X. By ([15], 6.14) the restriction functor $D^+_{\mathrm{qcoh}}(X_{\mathrm{lis-et}}) \to D^+_{\mathrm{qcoh}}(X_{\mathrm{et}})$ (notation as in (loc. cit.)) is an equivalence of categories, and the construction in ([15], §9) of the derived pullback functor shows that this equivalence induced by restriction is compatible with derived pullbacks. We therefore hope that this abuse of notation does not cause confusion.

Remark 1.3 As pointed out by the referee, when X is a Deligne–Mumford stack (1.1) can be generalized as follows (see (2.26) for more details). Let $\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I)$ be the fibered category over the étale site of X which to any étale $U \to X$ associates the groupoid of $\mathcal{Y}$–extensions of U by $I|_U$. The fibered category $\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I)$ is naturally a Picard stack, and the proof of (1.1) can be generalized to show that $\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I)$ is equivalent to the Picard stack associated as in ([13], 14.4.5) to $(\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}, I))[1]$.

Problem 2 (Generalization of ([10], III.2.1.7)) Suppose $j : \mathcal{Y} \hookrightarrow \mathcal{Y}'$ is a closed immersion of algebraic stacks defined by a quasi-coherent square–zero ideal I on $\mathcal{Y}'$, and suppose that x is flat. A *flat deformation of X to $\mathcal{Y}'$* is a 2-cartesian square

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x'} \\ \mathcal{Y} & \xrightarrow{j} & \mathcal{Y}', \end{array}$$

where $x' : X' \to \mathcal{Y}'$ is flat (we often denote a flat deformation of X to $\mathcal{Y}'$ simply by $x' : X' \to \mathcal{Y}'$). The set of flat deformations of X to $\mathcal{Y}'$ form in a natural way a category (in fact a groupoid; (3.1)), and in section 3 we prove the following:

Theorem 1.4 (i). *There exists a canonical obstruction*

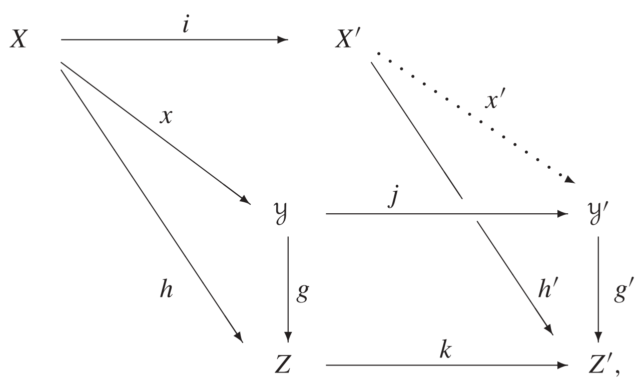
$$o(X, j) \in Ext^2(L_{X/\mathcal{Y}}, x^*I)$$

(sometimes abbreviated $o(X)$ or simply o) whose vanishing is necessary and sufficient for the existence of a flat deformation of X to $\mathcal{Y}'$.

(ii). *If $o(X, j) = 0$, then the set of isomorphism classes of flat deformations of X to $\mathcal{Y}'$ is a torsor under* $\mathrm{Ext}^1(L_{X/\mathcal{Y}}, x^*I)$.

(iii). *The automorphism group of any flat deformation of X to $\mathcal{Y}'$ is canonically isomorphic to* $\mathrm{Ext}^0(L_{X/\mathcal{Y}}, x^*I)$.

Problem 3 (Generalization of ([10], III.2.2.4)) Suppose $x : X \to \mathcal{Y}$ fits into a 2-commutative diagram of solid arrows between algebraic stacks



(1.4.1)

where Z and Z' are schemes, and i (resp. j, k) is a closed immersion defined by a square-zero ideal $I \subset \mathcal{O}_{X'}$ (resp. $J \subset \mathcal{O}_{\mathcal{Y}'}$, $K \subset \mathcal{O}_{Z'}$). The collection of maps $x' : X' \to \mathcal{Y}'$ filling in (1.4.1) form in a natural way a category (4.1), and in section 4 we prove the following:

Theorem 1.5 *Let $L_{\mathcal{Y}/Z}$ denote the cotangent complex of $g : \mathcal{Y} \to Z$.*

(i). *There is a canonical class $o(x, i, j) \in$* $\mathrm{Ext}^1(Lx^*L_{\mathcal{Y}/Z}, I)$ *(sometimes abbreviated $o(x)$ or simply o) whose vanishing is necessary and sufficient for the existence of an arrow $x' : X' \to \mathcal{Y}'$ filling in (1.4.1).*

(ii). *If $o(x, i, j) = 0$, then the set of isomorphism classes of maps $x' : X' \to \mathcal{Y}'$ filling in (1.4.1) is naturally a torsor under* $\mathrm{Ext}^0(Lx^*L_{\mathcal{Y}/Z}, I)$.

(iii). *For any morphism $x' : X' \to \mathcal{Y}'$ the group of automorphisms of x' (as a deformation of x) is canonically isomorphic to* $\mathrm{Ext}^{-1}(Lx^*L_{\mathcal{Y}/Z}, I)$.

Remark 1.6 It is perhaps worth remarking why the above results do not follow immediately from the general theory of Illusie ([10]). The point is that the cotangent complex of the morphism $x : X \to \mathcal{Y}$ is *not* defined as the cotangent complex of a morphism of ringed topoi (in the sense of loc. cit.), and there is not an interpretation of the category $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$ defined in problem (1) as a category of ring extensions in some topos associated to X. Hence one cannot apply directly the results of (loc. cit.).

Remark 1.7 Theorem (1.5) can be used to define an obstruction theory in the sense of ([2], 2.6) for every algebraic stack $\mathcal{X}$ locally of finite type over a noetherian base scheme S. Recall that such a theory consists of the following (in the following all rings are over S):

(1.7.1). For every surjection of noetherian rings $A \to A_0$ with nilpotent kernel and $a \in \mathcal{X}(A)$, a functor

$$\mathcal{O}_a : (A_0\text{-modules of finite type}) \longrightarrow (A_0\text{-modules of finite type}).$$

(1.7.2). For each surjection $A' \to A$ with kernel M, an A_0-module of finite type, a class $o_a \in \mathcal{O}_a(M)$ which is zero if and only if there exists a lifting of a to A'.

This data is furthermore required to be functorial and linear in (A_0, M).

We obtain such a theory for $\mathcal{X}$ by defining $\mathcal{O}_a(M) := \mathrm{Ext}^1(La^*L_{\mathcal{X}/S}, M)$, and taking for each $A' \to A$ as in (1.7.2) the class $o_a \in \mathrm{Ext}^1(La^*L_{\mathcal{X}/S}, M)$ to be the class obtained from (1.5 (i)) applied to (1.4.1) with $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$, $\mathcal{Y} = \mathcal{Y}' = \mathcal{X}$, and $Z = Z' = S$. Note that it follows from the construction of the cotangent complex and ([10], II.2.3.7) that the homology groups of $La^*L_{\mathcal{X}/S}$ are coherent. From this and standard properties of cohomology it follows that the A_0-modules $\mathrm{Ext}^1(La^*L_{\mathcal{X}/S}, M)$ are of finite type and that the additional conditions ([2], 4.1) on the obstruction theory hold (see (4.11)–(4.12) for functoriality).

Remark 1.8 It is natural to ask for a generalization of the work in this paper to all morphisms of algebraic stacks $\mathcal{X} \to \mathcal{Y}$. Unfortunately, the methods of this paper do not seem applicable in this level of generality. For example, $\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I)$ will in general be a 2-category as objects of $\mathcal{X}$ may have infinitesimal automorphisms inducing the identity in $\mathcal{Y}$. Recently, some progress in this direction has been made by Aoki ([1]) who studies the case when $\mathcal{X}$ is an algebraic stack and $\mathcal{Y}$ is a scheme.

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1.10 Conventions

Throughout this paper we follow the conventions of ([13]) except that we do not assume that our stacks are quasi-separated (this is important in the application to log geometry ([14])). More precisely, by an algebraic stack we mean a stack $\mathcal{X}$ in the sense of ([13], 3.1) satisfying the following:

(1.10.1). The diagonal

$$\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$$

is representable and of finite type;

(1.10.2). There exists a surjective smooth morphism $X \to \mathcal{X}$ from a scheme.

The reader should verify that the results from ([13]) used in this paper still hold with this slightly more general notion of an algebraic stack.

If $\mathcal{Z}$ and $\mathcal{X}$ are algebraic stacks, we write $\mathcal{X}(\mathcal{Z})$ for the groupoid of morphisms of stacks $\mathcal{Z} \to \mathcal{X}$.

The reader should be aware that there is a problem with the functoriality of the lisse-étale topos as defined in ([13]). The difficulties this causes in chapters 12–18 in (loc. cit.) have been worked out in ([15]).

We assume that the reader is familiar with the construction given in ([13], proof of 17.3) and ([15], section 10) of the cotangent complex $L_{\mathcal{Y}/\mathcal{Z}}$ of a morphism of algebraic stacks $\mathcal{Y} \to \mathcal{Z}$.

We denote by Δ the category whose objects are the ordered sets $[i] := \{0, \ldots, i\}$ ($i \in \mathbb{N}$), and whose morphisms are order preserving set maps. A simplicial algebraic space is a functor

$$X^\bullet : \Delta^{op} \longrightarrow (\text{category of algebraic spaces}).$$

For such a functor we denote by X^i the space $X^\bullet([i])$, and for a morphism $\delta : [i_1] \to [i_2]$ in Δ we denote by $X^\bullet(\delta) : X^{i_2} \to X^{i_1}$ the resulting morphism of algebraic spaces. For distinct integers $i_1, \ldots, i_\ell \in \{0, i\}$, we sometimes write

$$\mathrm{pr}^i_{01\cdots\hat{i}_1\cdots\hat{i}_2\cdots i} : X^i \longrightarrow X^{i-\ell}$$

for the map induced by the unique injective order preserving map $[i-\ell] \to [i]$ whose image does not intersect $\{i_1, \ldots, i_\ell\}$. For example, the map $X^2 \to X^1$ obtained from the map $[1] \to [2]$ sending 0 to 0 (resp. 1) and 1 to 1 (resp. 2) will be denoted pr^2_{01} (resp. pr^2_{12}).

2 Problem (1)

We proceed with the notation of problem (1).

Lemma 2.1 *For any $\mathcal{Y}$–extension $X \hookrightarrow X'$ of X by I, the morphism $x' : X' \to \mathcal{Y}$ is representable.*

Proof Let $Y \to \mathcal{Y}$ be a smooth cover with Y a scheme, and consider the algebraic stack $X'_Y := X' \times_{\mathcal{Y}} Y$. To prove that X'_Y is an algebraic space, it suffices to show that the objects of X'_Y admit no non–trivial automorphisms ([13], 8.1.1). Let $A' \to X'_Y$ be the stack which to any scheme T associates the groupoid of pairs (x', α), where $x' \in X'_Y(T)$ and α is an automorphism of x' in $X'_Y(T)$ (the "inertia stack"). The stack A' is isomorphic to the fiber product of the diagram

$$\begin{array}{ccc} & & X'_Y \\ & & \downarrow{\scriptstyle\Delta} \\ X'_Y & \xrightarrow{\Delta} & X'_Y \times X'_Y, \end{array}$$

and in particular is algebraic, and the morphism $A' \to X'_Y$ is representable. To prove the lemma it suffices to show that the map $A' \to X'_Y$ is an isomorphism.

Set $X_Y := X \times_{\mathcal{Y}} Y$ so that there is a closed immersion defined by a square–zero ideal $X_Y \hookrightarrow X'_Y$. Since x is representable, the pullback $A' \times_{X'_Y} X_Y \to X_Y$ is an isomorphism. Let $T' \to X'_Y$ be a smooth cover with T' a scheme and set

$A'_{T'} := T' \times_{X'_Y} A'$ and $T := X_Y \times_{X'_Y} T'$. By descent theory, it suffices to show that $A'_{T'} \to T'$ is an isomorphism. Since $A' \to X'_Y$ is representable, $A'_{T'}$ is an algebraic space, and since $A'_{T'} \times_{T'} T$ is a scheme (in fact isomorphic to T) it follows from ([12], III.3.6) that $A'_{T'}$ is a scheme. To prove the lemma it therefore suffices to show that if $A'_{T'} \to T'$ is a morphism of schemes which admits a section such that the base change to T is an isomorphism, then $A'_{T'} \to T'$ is an isomorphism. Since $T \hookrightarrow T'$ is defined by a nilpotent ideal, the underlying topological spaces of T and T' are equal as are the topological spaces of $A_{T'}$ and A_T. In particular, the map $A_{T'} \to T'$ induces an isomorphism between the underlying topological spaces. Therefore $A_{T'}$ is equal to the relative spectrum over T' of a quasi–coherent sheaf of $\mathcal{O}_{T'}$–algebras $\mathcal{A}$ on T'. The section of $A_{T'} \to T'$ induces a decomposition $\mathcal{A} \simeq \mathcal{O}_{T'} \oplus \mathcal{F}$ for some quasi–coherent sheaf $\mathcal{F}$ on T' such that the pullback of $\mathcal{F}$ to T is zero. It follows that $\mathcal{F} = \cap_n I^n \mathcal{F}$, and hence $\mathcal{F} = 0$ since I is a nilpotent ideal. □

2.2 If $(j_i : X \hookrightarrow X'_i, \iota_i)$ $(i = 1, 2)$ are two $\mathcal{Y}$-extensions of X by I, then we define a morphism

$$(j_1 : X \hookrightarrow X'_1, \iota_1) \longrightarrow (j_2 : X \hookrightarrow X'_2, \iota_2)$$

to be a pair (ψ, φ), where $\psi : X'_1 \to X'_2$ is a morphism of stacks and $\varphi : x'_1 \to x'_2 \circ \psi$ is an isomorphism in $\mathcal{Y}(X'_1)$ such that the following hold:

(2.2 (i)). $j_2 = \psi \circ j_1$ and if ρ denotes the induced isomorphism

$$\mathrm{Ker}(\mathcal{O}_{X'_1} \longrightarrow \mathcal{O}_X) \longrightarrow \mathrm{Ker}(\mathcal{O}_{X'_2} \longrightarrow \mathcal{O}_X),$$

then $\rho \circ \iota_1 = \iota_2$.

(2.2 (ii)). The composite

$$x \xrightarrow{\epsilon_{j_1}} x'_1 \circ j_1 \xrightarrow{\varphi} x'_2 \circ \psi \circ j_1 \xrightarrow{=} x'_2 \circ j_2$$

in $\mathcal{Y}(X)$ equals ϵ_{j_2}.

In this way the $\mathcal{Y}$-extensions of X by I form a category which we denote by $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$. Note that (2.2 (i)) implies that all morphisms in $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$ are isomorphisms.

2.3 If $u : I \to J$ is a morphism of quasi-coherent sheaves, then there is a natural pullback functor

$$u^* : \underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I) \longrightarrow \underline{\mathrm{Exal}}_{\mathcal{Y}}(X, J) \qquad (2.3.1)$$

obtained as follows. Given an extension $(j : X \hookrightarrow X', \iota)$, define X'_u to be the relative spectrum over X' ([13], 14.2.3) of the sheaf of $\mathcal{O}_{X'}$–algebras $\mathcal{O}_{X'} \oplus_I J$ ([7], 0_{IV}.18.2.8). Then there is a natural map $i : X'_u \to X'$ which induces a 2-commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{j_u} & X'_u \\ {\scriptstyle x}\downarrow & \swarrow {\scriptstyle x' \circ i} & \\ \mathcal{Y}. & & \end{array} \qquad (2.3.2)$$

This construction is compatible with morphisms in $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$, and hence we obtain the functor (2.3.1).

2.4 In order to proceed further, it is necessary to have some facts about simplicial algebraic spaces. Suppose

$$p : \mathcal{S} \longrightarrow (\text{Schemes})$$

is an algebraic stack, and let $s : S \to \mathcal{S}$ be a smooth cover. Let $S^\bullet$ denote the simplicial algebraic space obtained from s by applying the 0-coskeleton functor ([5], 5.1.1). For $i \in \mathbb{N}$, S^i is the $(i+1)$-fold fiber product $S \times_{\mathcal{S}} S \times_{\mathcal{S}} \cdots \times_{\mathcal{S}} S$ which we view as the space representing the functor which to any scheme T associates the isomorphism classes of collections of data $(t, \{f_j\}_{j=0}^i, \{\epsilon_j\}_{j=0}^i)$, where $t \in \mathcal{S}(T)$, $f_j : T \to S$ is a morphism of algebraic spaces, and $\epsilon_j : f_j^* s \to t$ is an isomorphism in $\mathcal{S}(T)$. If $\delta : [i'] \to [i]$ is a morphism in Δ, then the map $S^\bullet(\delta) : S^i \to S^{i'}$ is the map induced by the morphism of functors

$$(t, \{f_j\}_{j=0}^i, \{\epsilon_j\}_{j=0}^i) \longmapsto (t, \{f_{\delta(j)}\}_{j=0}^{i'}, \{\epsilon_{\delta(j)}\}_{j=0}^{i'}).$$

2.5 If $x : X \to \mathcal{S}$ is any representable morphism of algebraic stacks, we can base change $S^\bullet$ to X to get a simplicial space $X^\bullet$ over X. The space X^i is equal to $X \times_{\mathcal{S}} S^i$ which we think of as representing the functor which to any scheme T associates the set of isomorphism classes of collections $(f, \epsilon, t, \{f_j\}_{j=0}^i, \{\epsilon_j\}_{j=0}^i)$, where $(t, \{f_j\}_{j=0}^i, \{\epsilon_j\}_{j=0}^i) \in S^i(T)$, $f : T \to X$ is a morphism of algebraic stacks, and $\epsilon : f^*x \to t$ is an isomorphism in $\mathcal{S}(T)$. For $\delta : [i'] \to [i]$ the morphism $X^\bullet(\delta) : X^i \to X^{i'}$ is that induced by the map

$$(f, \epsilon, t, \{f_j\}_{j=0}^i, \{\epsilon_j\}_{j=0}^i) \longmapsto (f, \epsilon, t, \{f_{\delta(j)}\}_{j=0}^{i'}, \{\epsilon_{\delta(j)}\}_{j=0}^{i'}).$$

If $p^\bullet : X^\bullet \to S^\bullet$ denotes the map induced by the maps $X^i \to S^i$ sending a collection $(f, \epsilon, t, \{f_j\}_{j=0}^i, \{\epsilon_j\}_{j=0}^i)$ to $(t, \{f_j\}_{j=0}^i, \{\epsilon_j\}_{j=0}^i)$, and if $s_i \in \mathcal{S}(S^i)$ denotes the tautological element corresponding to t in a triple

$$(t, \{f_j\}_{j=0}^i, \{\epsilon_j\}_{j=0}^i),$$

then for each i there is a tautological isomorphism $\gamma_i : p^{i*}(s_i) \to \pi^{i*}x$, where $\pi^i : X^i \to X$ denotes the projection. These isomorphisms are compatible in the following sense. For each $\delta : [i'] \to [i]$ in Δ, there is a tautological isomorphism $\sigma_\delta : S^\bullet(\delta)^*(s_{i'}) \to s_i$, and it follows from the construction that the diagram

$$\begin{array}{ccc} X^\bullet(\delta)^*\pi^{i'*}x & \xrightarrow{X^\bullet(\delta)^*(\gamma_{i'})} & X^\bullet(\delta)^*p^{i'*}s_{i'} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \sigma_\delta} \\ \pi^{i*}x & \xrightarrow{\gamma_i} & p^{i*}s_i \end{array}$$

commutes.

2.6 One can extend the stack $\mathcal{S}$ to the category of simplicial spaces. If

$$T^\bullet : \Delta^{op} \longrightarrow \text{(Algebraic spaces)}$$

is a simplicial algebraic space, define $\mathcal{S}(T^\bullet)$ to be the groupoid of functors $F : \Delta^{op} \to \mathcal{S}$ for which $p \circ F = T^\bullet$ (where $\mathcal{S}$ is extended to the category of algebraic spaces in the usual way). To give an object s of $\mathcal{S}(T^\bullet)$ is equivalent to giving for each i and object $s_i \in \mathcal{S}(T^i)$, together with an isomorphisms $\rho_\delta : T^\bullet(\delta)^* s_{i'} \simeq s_i$ on T^i for each morphism $\delta : [i'] \to [i]$ in Δ, such that if $\delta' : [i''] \to [i']$ is a second morphism in Δ then the two isomorphisms

$$\rho_{\delta\circ\delta'}, \quad \rho_\delta \circ T^\bullet(\delta)^*(\rho_{\delta'}) : T^\bullet(\delta\circ\delta')^* s_{i''} \longrightarrow s_i \tag{2.6.1}$$

are equal. If $s, t \in \mathcal{S}(T^\bullet)$ are two objects then to give an isomorphism $s \to t$ is equivalent to giving a collection of isomorphisms $s^i \to t^i$ in $\mathcal{S}(T^i)$ compatible with the ρ_δ.

If $f^\bullet : V^\bullet \to T^\bullet$ is a morphism of simplicial algebraic spaces, then there is a natural pullback functor

$$f^{\bullet *} : \mathcal{S}(T^\bullet) \longrightarrow \mathcal{S}(V^\bullet)$$

obtained by sending a collection $(\{s_i\}, \{\rho_\delta\})$ to the family $(\{f^{i*}s_i\}, \{f^{\bullet *}(\delta)\})$, where for $\delta : [i'] \to [i]$ we denote by $f^{\bullet *}(\delta)$ the map

$$V^\bullet(\delta)^* f^{i'*} s_{i'} = f^{i*} T^\bullet(\delta)^* s_{i'} \xrightarrow{f^{i*}(\rho_\delta)} f^{i*} s_i.$$

Lemma 2.7 *If $\mathcal{Z}$ is an algebraic stack and $Z \to \mathcal{Z}$ a smooth cover with associated simplicial algebraic space $Z^\bullet$, then there is a natural equivalence of categories*

$$\mathcal{S}(Z^\bullet) \longrightarrow \mathcal{S}(\mathcal{Z}). \tag{2.7.1}$$

Proof Let δ_0 (resp. δ_1) denote the map $[0] \to [1]$ sending 0 to 0 (resp. 1). By ([13], 3.2), the category of 1-morphisms $\mathcal{Z} \to \mathcal{S}$ is equivalent to the category $\mathcal{T}$ of pairs (s, σ), where $s \in \mathcal{S}(Z)$ and $\sigma : \mathrm{pr}_0^{1*}(s) \to \mathrm{pr}_1^{1*}(s)$ is an isomorphism in $\mathcal{S}(Z^1)$ such that the two isomorphisms

$$\mathrm{pr}_{02}^{2*}(\sigma), \mathrm{pr}_{12}^{2*}(\sigma) \circ \mathrm{pr}_{01}^{2*}(\sigma) : \mathrm{pr}_0^{2*} s \longrightarrow \mathrm{pr}_2^{2*} s \tag{2.7.2}$$

in $\mathcal{S}(Z^2)$ are equal. Now for any object $(s_i, \rho_\delta) \in \mathcal{S}(Z^\bullet)$, the pair $(s_0, \rho_{\delta_1}^{-1} \circ \rho_{\delta_0})$ is an object of $\mathcal{T}$; the equality of the two morphisms in (2.7.2) holds because of the cocycle condition (2.6.1) (exercise). The functor (2.7.1) is that defined by the association $(s_i, \rho_\delta) \mapsto (s_0, \rho_{\delta_1}^{-1} \circ \rho_{\delta_0})$.

To see that (2.7.1) is fully faithful, let (s_i, ρ_δ) and (s'_i, ρ'_δ) be two objects of $\mathcal{S}(Z^\bullet)$. A morphism $(s_i, \rho_\delta) \to (s'_i, \rho'_\delta)$ is a collection of isomorphisms $\{\sigma_i : s_i \to s_{i'}\}$ such that for every morphism $\delta : [i_1] \to [i_2]$ the induced diagram

$$\begin{array}{ccc} Z^\bullet(\delta)^* s_{i_1} & \xrightarrow{\rho_\delta} & s_{i_2} \\ {\scriptstyle Z^\bullet(\delta)^*(\sigma_{i_1})}\downarrow & & \downarrow{\scriptstyle \sigma_{i_2}} \\ Z^\bullet(\delta)^* s'_{i_1} & \xrightarrow{\rho'_\delta} & s'_{i_2} \end{array}$$

commutes. In particular, the collection $\{\sigma_i\}$ is determined by σ_0: if $\delta : [0] \to [i]$ is any morphism, then σ_i must be the composite

$$s_i \xrightarrow{\rho_\delta^{-1}} Z^\bullet(\delta)^*(s_0) \xrightarrow{Z^\bullet(\delta)^*(\sigma_0)} Z^\bullet(\delta)^*(s'_0) \xrightarrow{\rho'_\delta} s'_i. \tag{2.7.3}$$

Conversely, given a morphism $\sigma_0 : s_0 \to s'_0$ compatible with the isomorphisms $\rho_{\delta_1}^{-1} \circ \rho_{\delta_0}$ and $\rho'^{-1}_{\delta_1} \circ \rho'_{\delta_0}$, we can use (2.7.3) to define the collection $\{\sigma_i\}$. To check that the collection $\{\sigma_i\}$ defines a morphism in $\mathcal{S}(Z^\bullet)$, it suffices to show that the morphism σ_i defined by (2.7.3) is independent of the choice of δ. If $\delta' : [0] \to [i]$ is a second morphism, then we can after possibly interchanging δ and δ' find a morphism $\tilde{\delta} : [1] \to [i]$ such that δ (resp. δ') equals $\tilde{\delta} \circ \delta_0$ (resp. $\tilde{\delta} \circ \circ\delta_1$). We then have a commutative diagram

$$\begin{array}{ccc}
s_i & \xrightarrow{\mathrm{id}} & s_i \\
\downarrow{\scriptstyle \rho_\delta^{-1}} & & \downarrow{\scriptstyle \rho_{\delta'}^{-1}} \\
Z^\bullet(\tilde{\delta})^*(Z^\bullet(\delta_0)^*(s_0)) & \xrightarrow{Z^\bullet(\tilde{\delta})^*(\rho_{\delta_1}^{-1}\circ\rho_{\delta_0})} & Z^\bullet(\tilde{\delta})^*(Z^\bullet(\delta_1)^*(s_0)) \\
\downarrow{\scriptstyle Z^\bullet(\tilde{\delta})^*(Z^\bullet(\delta_0)^*(\sigma_0))} & & \downarrow{\scriptstyle Z^\bullet(\tilde{\delta})^*(Z^\bullet(\delta_1)^*(\sigma_0))} \\
Z^\bullet(\tilde{\delta})^*(Z^\bullet(\delta_0)^*(s'_0)) & \xrightarrow{Z^\bullet(\tilde{\delta})^*(\rho'^{-1}_{\delta_1}\circ\rho'_{\delta_0})} & Z^\bullet(\tilde{\delta})^*(Z^\bullet(\delta_1)^*(s'_0)) \\
\downarrow{\scriptstyle \rho'_\delta} & & \downarrow{\scriptstyle \rho'_{\delta'}} \\
s'_i & \xrightarrow{\mathrm{id}} & s'_i,
\end{array}$$

where the top and bottom squares commute by (2.6.1) and the middle square commutes by assumption. From this it follows that (2.7.1) is fully faithful.

To see that (2.7.1) is essentially surjective, let (s, σ) be an object of $\mathcal{T}$. For each $i > 1$, define s_i to be $\mathrm{pr}_0^{i*}(s)$. If $\delta : [i_1] \to [i_2]$ is any morphism, then there exists a unique morphism $\tilde{\delta} : [1] \to [i_2]$ such that $Z^\bullet(\tilde{\delta} \circ \delta_0)$ (resp. $Z^\bullet(\tilde{\delta} \circ \delta_1)$) equals $\mathrm{pr}_0^{i_2}$ (resp. $\mathrm{pr}_0^{i_1} \circ Z^\bullet(\delta)$). Define ρ_δ to be the map

$$Z^\bullet(\delta)^* s_i = Z^\bullet(\tilde{\delta})^* \mathrm{pr}_1^{1*}(s) \xrightarrow{Z^\bullet(\tilde{\delta})^*(\sigma^{-1})} Z^\bullet(\tilde{\delta})^* \mathrm{pr}_0^{1*}(s) = \mathrm{pr}_0^{i_2*}(s).$$

To see that (s_i, ρ_δ) defines an object of $\mathcal{S}(Z^\bullet)$ (i.e. that (2.6.1) holds), let $\delta : [i'] \to [i]$ and $\delta' : [i''] \to [i']$ be morphisms in Δ, and let $\tilde{\delta} : [2] \to [i]$ be the unique morphism sending 0 (resp. 1, 2) to 0 (resp. $\delta(0)$, $\delta(\delta'(0))$). The fact that the two maps in (2.6.1) are equal then amounts the equality of the map

$$Z^\bullet(\delta \circ \delta')^* \mathrm{pr}_0^{i''*} s = Z^\bullet(\tilde{\delta})^*(\mathrm{pr}_{02}^{2*}(\mathrm{pr}_1^{1*} s)) \xrightarrow{Z^\bullet(\tilde{\delta})^*(\mathrm{pr}_{02}^{2*}(\sigma^{-1}))} Z^\bullet(\tilde{\delta})^*(\mathrm{pr}_{02}^{2*}(\mathrm{pr}_0^{1*} s))$$

with the composite of

$$Z^\bullet(\delta \circ \delta')^* \mathrm{pr}_0^{i''*} s = Z^\bullet(\tilde{\delta})^*(\mathrm{pr}_{12}^{2*}(\mathrm{pr}_1^{1*} s)) \xrightarrow{Z^\bullet(\tilde{\delta})^*(\mathrm{pr}_{12}^{2*}(\sigma^{-1}))} Z^\bullet(\tilde{\delta})^*(\mathrm{pr}_{12}^{2*}(\mathrm{pr}_0^{1*} s))$$

and

$$Z^\bullet(\tilde{\delta})^*(\mathrm{pr}_{12}^{2*}(\mathrm{pr}_0^{1*}s)) = Z^\bullet(\tilde{\delta})^*(\mathrm{pr}_{01}^{2*}(\mathrm{pr}_1^{1*}s)) \xrightarrow{Z^\bullet(\tilde{\delta})^*(\mathrm{pr}_{01}^{2*}(\sigma^{-1}))} Z^\bullet(\tilde{\delta})^*(\mathrm{pr}_{01}^{2*}(\mathrm{pr}_0^{1*}s)).$$

This follows from the cocycle condition (2.7.2). Moreover, it follows from the construction that the resulting object of $\mathcal{S}(Z^\bullet)$ induces (s, σ). □

2.8 Fix a smooth cover $Y \to \mathcal{Y}$ and let $Y^\bullet$ denote the associated simplicial algebraic space (2.4). For any morphism $T^\bullet \to Y^\bullet$ of simplicial algebraic spaces and quasi-coherent sheaf J on $T^\bullet$ ([13], page 125), we can define a category $\underline{\mathrm{Exal}}_{Y^\bullet}(T^\bullet, J)$. The objects of $\underline{\mathrm{Exal}}_{Y^\bullet}(T^\bullet, J)$ are closed immersions $j^\bullet : T^\bullet \hookrightarrow T'^\bullet$ of simplicial spaces over $Y^\bullet$ together with an isomorphism $\epsilon_{j^\bullet} : J \simeq \mathrm{Ker}(\mathcal{O}_{T'^\bullet} \to \mathcal{O}_{T^\bullet})$. A morphism $(j_1^\bullet : T^\bullet \to T_1'^\bullet) \to (j_2^\bullet : T^\bullet \to T_2'^\bullet)$ is a $Y^\bullet$-morphism $\psi^\bullet : T_1'^\bullet \to T_2'^\bullet$ such that $j_2^\bullet = \psi^\bullet \circ j_1^\bullet$ which is compatible with the $\epsilon_{j_i^\bullet}$.

In particular, if $X^\bullet$ denotes the simplicial space obtained from $x : X \to \mathcal{Y}$ by base change to $Y^\bullet$, then for any quasi-coherent sheaf I on X we obtain a natural functor

$$F : \underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I) \longrightarrow \underline{\mathrm{Exal}}_{Y^\bullet}(X^\bullet, \pi_X^* I), \tag{2.8.1}$$

where $\pi_X^* I$ denotes the quasi–coherent sheaf on $X^\bullet_{\mathrm{et}}$ obtained by pulling back I as in ([13], 13.2.4) (where the functor π_X^* is denoted ϵ_*). The functor (2.8.1) takes a $\mathcal{Y}$-extension $(j : X \hookrightarrow X', \iota)$ to the $Y^\bullet$-extension $(j^\bullet : X^\bullet \to X'^\bullet)$ obtained by base change.

Proposition 2.9 *The functor (2.8.1) is an equivalence of categories.*

Proof First we show that F is fully faithful. Suppose given two $\mathcal{Y}$-extensions $j_i : X \hookrightarrow X_i'$ and let I_F denote the map

$$\mathrm{Hom}(j_1 : X \hookrightarrow X_1', j_2 : X \hookrightarrow X_2') \longrightarrow \mathrm{Hom}(j_1^\bullet : X^\bullet \hookrightarrow X_1'^\bullet, j_2^\bullet : X^\bullet \hookrightarrow X_2'^\bullet) \tag{2.9.1}$$

induced by F.

By definition, the fiber product $X_i' \times_{\mathcal{Y}} Y$ represents the functor which to any scheme T associates the set of triples (t, s, ρ), where $t : T \to X_i'$ and $s : T \to Y$ are morphisms of algebraic stacks and $\rho : t^*(x_i') \to s^*(y)$ is an isomorphism in $\mathcal{Y}(T)$. If (ψ, φ) is a morphism from j_1 to j_2, then the induced map

$$X_1' \times_{\mathcal{Y}} Y \longrightarrow X_2' \times_{\mathcal{Y}} Y \tag{2.9.2}$$

sends (t, s, ρ) to $(\psi \circ t, s, \rho \circ t^*(\varphi^{-1}))$.

If (ψ_i, φ_i) $(i = 1, 2)$ are two morphisms with the same image $\psi^\bullet$ under I_F, then $\psi_1 = \psi_2$ since by (2.7) we have $X_2(X_1) \simeq X_2(X_1^\bullet)$. Thus we obtain an automorphism α of x_1' in $\mathcal{Y}(X_1')$ defined to be the composite

$$x_1' \xrightarrow{\varphi_2} x_2' \circ \psi_2 \xrightarrow{=} x_2' \circ \psi_1 \xrightarrow{\varphi_1^{-1}} x_1'.$$

Moreover from the functorial description of the map (2.9.2), we see that the pullback of α to $X_1' \times_{\mathcal{Y}} Y$ is the identity. Since $X_1' \times_{\mathcal{Y}} Y$ is a smooth cover of X_1' it follows that α is the identity, and so I_F is injective.

Conversely we show that every morphism $\psi^\bullet : X_1'^\bullet \to X_2'^\bullet$ of $Y^\bullet$-extensions of $X^\bullet$ is in the image of I_F. By (2.7) applied with $\mathcal{S} = X_2'$ and $Z^\bullet = X_1'^\bullet$, there exists a unique morphism $\psi : X_1' \to X_2'$ such that (2.2 (i)) holds and such that the diagram

$$\begin{array}{ccc} X_1'^\bullet & \xrightarrow{\psi^\bullet} & X_2'^\bullet \\ \downarrow & & \downarrow \\ X_1' & \xrightarrow{\psi} & X_2' \end{array}$$

commutes. Let $y^\bullet \in \mathcal{Y}(Y^\bullet)$ denote the tautological object corresponding to the identity functor $\mathcal{Y} \to \mathcal{Y}$ under (2.7), and let $\tau_i : \pi_{X_i'}^* x_i' \to p_i^* y^\bullet$ denote the isomorphism discussed in (2.5), where $\pi_{X_i'} : X_i'^\bullet \to X_i'$ and $p_i : X_i'^\bullet \to Y^\bullet$ denote the natural maps. Let $\sigma^\bullet : \pi_{X_1'}^* x_1' \to \psi^{\bullet *}\pi_{X_2'}^* x_2'$ be the isomorphism in $\mathcal{Y}(X_1'^\bullet)$

$$x_1' \xrightarrow{\tau_1} p_1^* y^\bullet = \psi^{\bullet *} p_2^* y^\bullet \xrightarrow{\psi^{\bullet *}(\tau_2^{-1})} \psi^{\bullet *} x_2'.$$

Then by (2.7) there exists a unique isomorphism $\varphi : x_1' \to \psi^* x_2'$ inducing $\sigma^\bullet$. Once again by (2.7) the condition (2.2 (ii)) holds since it holds after pulling back to $X^\bullet$, and hence the pair (ψ, φ) defines a morphism in $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$ which induces $\psi^\bullet$. Hence I_F is bijective.

To prove that the functor (2.8.1) is essentially surjective, we show that any object $j^\bullet : X^\bullet \hookrightarrow X'^\bullet$ of $\underline{\mathrm{Exal}}_{Y^\bullet}(X^\bullet, \pi_X^* I)$ is in the essential image of F. Let $d_i : X'^1 \to X'^0$ $(i = 0, 1)$ denote the projections pr_i^1 $(i = 0, 1)$, and note that we have commutative squares

$$\begin{array}{ccc} X \times_{\mathcal{Y}} (Y \times_{\mathcal{Y}} Y) & \xrightarrow{j^1} & X'^1 \\ {\scriptstyle \mathrm{pr}_i^1}\downarrow & & \downarrow{\scriptstyle d_i} \\ X \times_{\mathcal{Y}} Y & \xrightarrow{j^0} & X'^0. \end{array} \tag{2.9.3}$$

It follows from the fact that the kernel of j^{1*} is equal to $d_i^*(\pi_X^* I|_{X'^0})$ that the square (2.9.3) is cartesian. Since pr_i^1 is smooth, the same fact combined with the local criterion for flatness ([7], 0_{III}.10.2.1) implies that the d_i are flat, and hence by ([7], IV.17.5.1) they are smooth. Similarly, the maps

$$\begin{aligned} \mathrm{pr}_{01}^2 \times \mathrm{pr}_{12}^2 : X'^2 &\longrightarrow X'^1 \times_{d_1, X'^0, d_0} X'^1, \\ \mathrm{pr}_{01}^2 \times \mathrm{pr}_{12}^2 \times \mathrm{pr}_{23}^2 : X'^3 &\longrightarrow X'^1 \times_{d_1, X'^0, d_0} X'^1 \times_{d_1, X'^0, d_0} X'^1 \end{aligned} \tag{2.9.4}$$

are isomorphisms since they are morphisms of spaces flat over X'^0 which becomes isomorphisms over X^0. Let

$$\mu : X'^1 \times_{d_1, X'^0, d_0} X'^1 \longrightarrow X'^1$$

be the map obtained from $\mathrm{pr}^2_{02} : X'^2 \to X'^1$ and the first isomorphism in (2.9.4), and let $\epsilon : X'^0 \to X'^1$ be the morphism induced by the unique morphism $[1] \to [0]$ in Δ.

We claim that the data $(X^{0'}, X^{1'}, d_0, d_1, \mu, \epsilon)$ defines a groupoid in algebraic spaces ([13], 2.4.3). To see this, it suffices by ([6], V.1) to verify that the squares

$$\begin{array}{ccc} X'^1 \times_{d_1, X'^0, d_0} X'^1 & \xrightarrow{\mu} & X'^1 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle d_0} \\ X'^1 & \xrightarrow{d_0} & X'^0 \end{array} \qquad \begin{array}{ccc} X'^1 \times_{d_1, X'^0, d_0} X'^1 & \xrightarrow{\mu} & X'^1 \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle d_1} \\ X'^1 & \xrightarrow{d_1} & X'^0 \end{array} \tag{2.9.5}$$

are cartesian, where π_1 (resp. π_2) denotes the first (resp. second) projection, and that the diagram

$$\begin{array}{ccc} X'^1 \times_{d_1, X'^0, d_0} X'^1 \times_{d_1, X'^0, d_0} X'^1 & \xrightarrow{\mu \times id} & X'^1 \times_{d_1, X'^0, d_0} X'^1 \\ \downarrow{\scriptstyle id \times \mu} & & \downarrow{\scriptstyle \mu} \\ X'^1 \times_{d_1, X'^0, d_0} X'^1 & \xrightarrow{\mu} & X'^1 \end{array} \tag{2.9.6}$$

commutes. But the commutativity of (2.9.5) and (2.9.6) follows immediately from the identifications (2.9.4) and the definitions of μ and d_i. Moreover, the squares (2.9.5) are cartesian since all the spaces are flat over X'^0 and the squares become cartesian over X^0. Thus $(X^{0'}, X^{1'}, d_0, d_1, \mu, \epsilon)$ is a groupoid in algebraic spaces with the d_i smooth.

Let X' denote the algebraic stack obtained from $(X^{0'}, X^{1'}, d_0, d_1, \mu, \epsilon)$ ([13], 4.3.1), and let $q : X'^0 \to X'$ be the projection. By ([13], 13.2.4), the closed immersion $j^\bullet : X^\bullet \hookrightarrow X'^\bullet$ descends to a closed immersion $j : X \hookrightarrow X'$ defined by a square zero ideal in $\mathcal{O}_{X'}$ which is identified with I using the given isomorphism $\pi_X^* I \simeq \mathrm{Ker}(j^\bullet)$.

If $Z^\bullet$ denotes the 0–coskeleton of q, then there is by the universal property of 0–coskeleton a map

$$X'^\bullet \longrightarrow Z^\bullet. \tag{2.9.7}$$

For each i, the map $X'^i \to Z^i$ is a morphism of X'-extensions of X^i by the pullback of I, and hence the map (2.9.7) is an isomorphism. If $p : X'^\bullet \to Y^\bullet$ denotes the specified map, then there is by (2.5) a tautological isomorphism $\epsilon^\bullet : \pi_X^* x \to j^{\bullet *}(p^{\bullet *} y^\bullet)$ in $\mathcal{Y}(X^\bullet)$, where $y^\bullet \in \mathcal{Y}(Y^\bullet)$ denotes the tautological object. Hence by (2.7) we obtain an object $x' \in \mathcal{Y}(X')$ together with an isomorphism $\epsilon : x \to x' \circ j$ defining an object in $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$ which induces $j^\bullet : X^\bullet \hookrightarrow X'^\bullet$. □

2.10 Consider a 2-commutative diagram of algebraic stacks

$$\begin{array}{ccc} V & \xrightarrow{g} & X \\ {\scriptstyle v}\downarrow & \swarrow{\scriptstyle x} & \\ \mathcal{Y}, & & \end{array} \tag{2.10.1}$$

where v and x are representable, and let I be a quasi-coherent sheaf on V such that

$R^1 g_* I = 0$ (this holds for example if g is affine). Then there is a natural functor

$$g_* : \underline{\mathrm{Exal}}_{\mathcal{Y}}(V, I) \longrightarrow \underline{\mathrm{Exal}}_{\mathcal{Y}}(X, g_* I) \tag{2.10.2}$$

obtained as follows. First note that if $Y \to \mathcal{Y}$ is a smooth cover with associated simplicial algebraic space $Y^\bullet$, then by (2.9) there are equivalences

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(V, I) \simeq \underline{\mathrm{Exal}}_{Y^\bullet}(V^\bullet, \pi_V^* I), \quad \underline{\mathrm{Exal}}_{\mathcal{Y}}(X, g_* I) \simeq \underline{\mathrm{Exal}}_{Y^\bullet}(X^\bullet, \pi_X^* g_* I),$$

where $\pi_V : V^\bullet \to V$ (resp. $\pi_X : X^\bullet \to X$) is the base change to $Y^\bullet$ of V (resp. X). It therefore suffices to define a natural functor

$$g_*^\bullet : \underline{\mathrm{Exal}}_{Y^\bullet}(V^\bullet, \pi_V^* I) \to \underline{\mathrm{Exal}}_{Y^\bullet}(X^\bullet, \pi_X^* g_* I),$$

and for this in turn it suffices to define for each i a functor

$$\underline{\mathrm{Exal}}_{Y^i}(V^i, I|_{V^i}) \to \underline{\mathrm{Exal}}_{Y^i}(X^i, g_*^i I)$$

functorial with respect to the base change maps $Y^{i'} \to Y^i$. This reduces the construction of g_* to the case when V, X, and $\mathcal{Y}$ are all algebraic spaces (note that $R^1 g_*^i I|_{V^i} = 0$ on $X^i_{\text{lis-et}}$).

In this case, let $\underline{\mathrm{Exal}}_{v^{-1}\mathcal{O}_{\mathcal{Y}_{\mathrm{et}}}}(\mathcal{O}_X, g_* I)$ be the groupoid of $v^{-1}\mathcal{O}_{\mathcal{Y}_{\mathrm{et}}}$–extensions of $\mathcal{O}_X$ by $g_* I$ in X_{et} ([10], III.1.1). There is a natural functor

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, g_* I) \to \underline{\mathrm{Exal}}_{v^{-1}\mathcal{O}_{\mathcal{Y}_{\mathrm{et}}}}(\mathcal{O}_X, g_* I) \tag{2.10.3}$$

which sends a $\mathcal{Y}$–extension $j : X \hookrightarrow X'$ of X by $g_* I$ to

$$0 \longrightarrow g_* I \longrightarrow j^{-1}\mathcal{O}_{X'_{\mathrm{et}}} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

By (2.11) below, the functor (2.10.3) is an equivalence of categories. Hence to define (2.10.2) it suffices to define a functor

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(V, I) \to \underline{\mathrm{Exal}}_{v^{-1}\mathcal{O}_{\mathcal{Y}_{\mathrm{et}}}}(\mathcal{O}_X, g_* I). \tag{2.10.4}$$

For this let $i : V \hookrightarrow V'$ be a $\mathcal{Y}$–extension of V by I inducing an exact sequence on V_{et}

$$0 \longrightarrow I \longrightarrow i^{-1}\mathcal{O}_{V'_{\mathrm{et}}} \longrightarrow \mathcal{O}_V \longrightarrow 0.$$

Since $R^1 g_* I = 0$ the sequence

$$0 \longrightarrow g_* I \longrightarrow g_* i^{-1}\mathcal{O}_{V'_{\mathrm{et}}} \longrightarrow g_*\mathcal{O}_V \longrightarrow 0 \tag{2.10.5}$$

is exact, and the functor (2.10.4) is by definition the functor sending $i : V \hookrightarrow V'$ to the extension

$$0 \longrightarrow g_* I \longrightarrow g_* i^{-1}\mathcal{O}_{V'_{\mathrm{et}}} \times_{g_*\mathcal{O}_V} \mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow 0$$

obtained from (2.10.5) by pullback via the map $\mathcal{O}_X \to g_*\mathcal{O}_V$.

Lemma 2.11 *If $p : X \to Y$ is a morphism of algebraic spaces and I is a quasi-coherent sheaf on X, then the functor*

$$\underline{Exal}_Y(X, I) \longrightarrow \underline{Exal}_{p^{-1}\mathcal{O}_Y}(\mathcal{O}_X, I), \tag{2.11.1}$$

$$(j : X \hookrightarrow X') \longmapsto (0 \to I \to j^{-1}\mathcal{O}_{X'} \to \mathcal{O}_X \to 0)$$

is an equivalence of categories.

Proof Let $\underline{\mathrm{Exal}}_Y(-, I)$ (resp. $\underline{\mathrm{Exal}}_{p^{-1}\mathcal{O}_Y}(-, I)$) denote the fibered category over the small étale site of X whose fiber over any $U \to X$ is the groupoid $\underline{\mathrm{Exal}}_Y(U, I|_U)$ (resp. $\underline{\mathrm{Exal}}_{p^{-1}\mathcal{O}_Y|_U}(\mathcal{O}_U, I|_U)$). The pullback of an object $U \hookrightarrow U'$ of $\underline{\mathrm{Exal}}_Y(U, I)$ via an étale morphism $V \to U$ is defined to be the unique lift V' of V to an étale U'-scheme ([8], I.8.3) together with the natural map $V \hookrightarrow V'$. The fibered category $\underline{\mathrm{Exal}}_Y(-, I)$ is in fact a stack over X_{et}. Indeed if $V \to U$ is an étale cover and $j_V : V \hookrightarrow V'$ an object of $\underline{\mathrm{Exal}}_Y(V, I)$ together with descent datum relative to V/U, then the pullback $V \times_U V \hookrightarrow V_2'$ of V to $V \times_U V$ comes equipped with two maps $d_0, d_1 : V_2' \to V'$ which are étale. By the same reasoning as in (proof of full faithfulness in (2.9)) these maps define an étale equivalence relation on V', and the resulting algebraic space U' comes equipped with a canonical closed immersion $U \hookrightarrow U'$ defining an object of $\underline{\mathrm{Exal}}_Y(U, I)$ which pulls back to $V \hookrightarrow V'$.

The functor (2.11.1) extends naturally to a morphism of stacks

$$\underline{\mathrm{Exal}}_Y(-, I) \longrightarrow \underline{\mathrm{Exal}}_{p^{-1}\mathcal{O}_Y}(-, I), \tag{2.11.2}$$

and hence to prove that (2.11.2) is an equivalence it suffices to show that it is locally an equivalence. Hence we may replace X by an étale cover. Thus we may assume that X and Y are schemes in which case the result follows from the invariance of the étale site under infinitesimal thickenings ([8], I.8.3). □

2.12 Just as in the classical case ([10], III.1.1.5), the constructions (2.3) and (2.10) enable us to give $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$ the structure of a Picard category ([3], XVIII.1.4). Let us just sketch the construction of the "sum" functor. Given two objects $j_i : X \hookrightarrow X_i'$ ($i = 1, 2$) of $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$, we obtain an object

$$(X_1' \hookrightarrow X'') = j_{1*}(j_2 : X \hookrightarrow X_2') \in \underline{\mathrm{Exal}}_{\mathcal{Y}}(X_1', I),$$

and the composition $X \xrightarrow{j_1} X_1' \longrightarrow X''$ is then an object of $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I \oplus I)$. Denoting by $s : I \oplus I \to I$ the summation map, we get an object $s^*(X \hookrightarrow X'')$ of $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$.

A consequence of the fact that $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$ is a Picard category, is that the set of isomorphism classes of objects in $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$, denoted $\mathrm{Exal}_{\mathcal{Y}}(X, I)$, has a structure of an abelian group. The category $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$ also has an action of the ring $\Gamma(X, \mathcal{O}_X)$ for which $f \in \Gamma(X, \mathcal{O}_X)$ acts via the functor induced by $\times f : I \to I$ and (2.3). This action makes $\mathrm{Exal}_{\mathcal{Y}}(X, I)$ a $\Gamma(X, \mathcal{O}_X)$–module.

Remark 2.13 The definition of the sum functor given here is a priori asymmetric. However, as in ([10], III.1.1.5) interchanging j_1 and j_2 gives canonically isomorphic functors. This is because $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$ is an "additive cofibered category" over the additive category of quasi–coherent $\mathcal{O}_X$–modules in the sense of ([9], 1.2). This amounts to the following two statements:

(i). The category $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, 0)$ is equivalent to the punctual category. This is immediate.

(ii). For any two quasi–coherent $\mathcal{O}_X$–modules I and J the natural functor

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I \times J) \to \underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I) \times \underline{\mathrm{Exal}}_{\mathcal{Y}}(X, J)$$

is an equivalence of categories. This follows from (2.9) and the corresponding result for ringed topoi.

2.14 As in the proof of (2.11), when X is a Deligne–Mumford stack the Picard category $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I)$ is even the value on X of a Picard stack ([13], 14.4.2) $\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I)$ on the étale site $\mathrm{Et}(X)$. Namely, for any étale $U \to X$ define

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I)(U) := \underline{\mathrm{Exal}}_{\mathcal{Y}}(U, I|_U).$$

If $V \to U$ is an étale morphism and $(U \hookrightarrow U') \in \underline{\mathrm{Exal}}_{\mathcal{Y}}(U, I|_U)$, the invariance of the étale site under infinitesimal thickenings ([8], I.8.3) implies that $V \to U$ lifts uniquely to an étale $V' \to U'$. We therefore obtain a functor

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(U, I|_U) \to \underline{\mathrm{Exal}}_{\mathcal{Y}}(V, I|_V)$$

sending $(U \hookrightarrow U')$ to $(V \hookrightarrow V')$. This makes $\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I)$ a fibered category over $\mathrm{Et}(X)$ which is in fact a stack. In fact, as we discuss in (2.26) below, the Picard category structure on each $\underline{\mathrm{Exal}}_{\mathcal{Y}}(U, I|_U)$ gives $\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I)$ the structure of a Picard stack.

2.15 Next we turn to the meaning of the right hand side of (1.1.1). If $\mathcal{A}$ is an abelian category, we denote by $D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$, $D^b(\mathcal{A})$) the derived category of complexes bounded below (resp. bounded above, bounded), and for $n \in \mathbb{Z}$ we denote by $\tau_{\geq n}$ the "canonical truncation in degree $\geq n$" functor ([3], XVII.1.1.13). Let $D'(\mathcal{A})$ denote the category of projective systems

$$K = (\cdots \longrightarrow K_{\geq -n-1} \longrightarrow K_{\geq -n} \longrightarrow \cdots \longrightarrow K_{\geq 0}),$$

where each $K_{\geq -n} \in D^+(\mathcal{A})$ and the maps

$$K_{\geq -n} \longrightarrow \tau_{\geq -n} K_{\geq -n}, \quad \tau_{\geq -n} K_{\geq -n-1} \longrightarrow \tau_{\geq -n} K_{\geq -n}$$

are all isomorphisms. We denote by $D'^b(\mathcal{A})$ the full subcategory of $D'(\mathcal{A})$ consisting of objects K for which $K_{\geq -n} \in D^b(\mathcal{A})$ for all n.

2.16 The shift functor (–)[1] : $D(\mathcal{A}) \to D(\mathcal{A})$ is extended to $D'(\mathcal{A})$ by defining

$$K[1] := (\cdots \to K_{\geq -n-1}[1] \to K_{\geq -n}[1] \to \cdots \to K_0[1] \to (\tau_{\geq 1} K_0)[1])$$

for $K \in D'(\mathcal{A})$ ([13], 17.4 (3)). We say that a triangle of $D'(\mathcal{A})$

$$K_1 \xrightarrow{u} K_2 \xrightarrow{v} K_3 \xrightarrow{w} K_1[1] \tag{2.16.1}$$

is *distinguished* if for every $n \geq 0$, there exists a commutative diagram

$$\begin{array}{ccccccc} K_{1,\geq -n} & \xrightarrow{u} & K_{2,\geq -n} & \xrightarrow{v'} & L & \xrightarrow{w'} & K_{1,\geq -n}[1] \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \beta} & & \downarrow \\ K_{1,\geq -n} & \xrightarrow{u} & K_{2,\geq -n} & \xrightarrow{v} & K_{3,\geq -n} & \xrightarrow{w} & K_{1,\geq -n+1}[1] \end{array} \tag{2.16.2}$$

where the top row is a distinguished triangle in $D(\mathcal{A})$ and the map

$$\tau_{\geq -n}L \longrightarrow K_{3,\geq -n} \tag{2.16.3}$$

induced by β is an isomorphism. For example, if

$$L_1 \longrightarrow L_2 \longrightarrow L_3 \longrightarrow L_1[1]$$

is a distinguished triangle in $D(\mathcal{A})$, then

$$(\tau_{\geq -n}L_1) \longrightarrow (\tau_{\geq -n}L_2) \longrightarrow (\tau_{\geq -n}L_3) \longrightarrow (\tau_{\geq -n}L_1)[1]$$

is distinguished in $D'(\mathcal{A})$, where $(\tau_{\geq -n}L_i)$ denotes the system with

$$(\tau_{\geq -n}L_i)_{\geq -n} := \tau_{\geq -n}L_i.$$

Remark 2.17 The above definition of a distinguished triangle in $D'(\mathcal{A})$ differs from that in ([13], 17.4 (3)). In (loc. cit.), a triangle (2.16.1) is said to be distinguished if for all $n \geq 0$ and for all commutative diagrams

$$\begin{array}{ccccccc} K_1 & \xrightarrow{u} & K_2 & \xrightarrow{v} & K_3 & \xrightarrow{w} & K_1[1] \\ {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & \downarrow{\scriptstyle\beta_3} & & \downarrow{\scriptstyle\alpha_1[1]} \\ K_{1,\geq -n} & \xrightarrow{u} & K_{2,\geq -n} & \xrightarrow{v'} & L & \xrightarrow{w'} & K_{1,\geq -n}[1], \end{array} \tag{2.17.1}$$

where the bottom row is a distinguished triangle in $D(\mathcal{A})$ and α_1 and α_2 denote the natural maps, the map $K_{3,\geq -n} \to \tau_{\geq -n}L$ is an isomorphism. This definition is not suitable for this paper. For example, suppose $\mathcal{A}$ is the category of R-modules for some ring R, and let $M \in \mathcal{A}$ be a nonzero object. Defining $K_1 = 0$, $K_2 = K_3 = M$ (placed in degree 0), and v to be the zero map, we obtain a triangle in $D'(\mathcal{A})$

$$0 \longrightarrow M \xrightarrow{0} M \longrightarrow 0[1] \tag{2.17.2}$$

for which there does not exist a diagram as in (2.17.1) for any n (exercise). Hence the triangle (2.17.2) is distinguished in $D'(\mathcal{A})$ according to ([13], 17.4 (3)) even though it is not distinguished in $D(\mathcal{A})$. The reader should note, however, that the statement of ([13], 17.4 (3)) remains correct with the notion of distinguished triangle defined in (2.16).

2.18 Suppose now that $\mathcal{A}$ has enough injectives and let M be an object of $\mathcal{A}$. Then the functor

$$\mathrm{RHom}_{D^b(\mathcal{A})}(-, M) : D^b(\mathcal{A}) \longrightarrow D^+(\mathrm{Ab}),$$

where Ab denotes the category of abelian groups, extends naturally to a functor $\mathrm{RHom}_{D'^b(\mathcal{A})}(-, M)$ from $D'^b(\mathcal{A})$ to the category of ind–objects in $D^b(\mathrm{Ab})$ satisfying the condition dual to that defining $D'(\mathcal{A})$. . For $K \in D'^b(\mathcal{A})$, define

$$\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n} := \tau_{\leq n}\mathrm{RHom}_{D^b(\mathcal{A})}(K_{\geq -n}, M).$$

There is a natural map

$$\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n} \to \mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n+1}$$

which induces an isomorphism

$$\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n} \simeq \tau_{\leq n}\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n+1}.$$

We define $\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)$ to be the ind–object $\{\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n}\}$ in $D^b(\mathrm{Ab})$. In the case when $K = (\tau_{\geq -n}L)$ for some $L \in D^b(\mathcal{A})$, the object $\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)$ is simply the inductive system $\{\tau_{\leq n}\mathrm{RHom}_{D(\mathcal{A})}(L, M)\}$.

For any integer i, the ind–group $H^i(\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n})$ is essentially constant and we define $\mathrm{Ext}^i(K, M) \in \mathrm{Ab}$ to be the limit. For any $n \geq i$ the natural map

$$\mathrm{Ext}^i_{D^b(\mathcal{A})}(K_{\geq -n}, M) \to \mathrm{Ext}^i(K, M)$$

is an isomorphism.

It follow from the definition of a distinguished triangle in $D'(\mathcal{A})$ (2.16) that a distinguished triangle (2.16.1) induces a long exact sequence

$$\dashrightarrow \mathrm{Ext}^{i+1}(K_3, M) \to \mathrm{Ext}^i(K_2, M) \to \mathrm{Ext}^i(K_1, M) \to \mathrm{Ext}^{i+1}(K_3, M) \dashrightarrow .$$

2.19 In the situation of problem (1), we take $\mathcal{A}$ equal to the category of $\mathcal{O}_X$-modules on the lisse-étale site of X, and denote $D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$, etc.) by $D^+(\mathcal{O}_X)$ (resp. $D^-(\mathcal{O}_X)$, etc.). Since the cotangent complex $L_{X/\mathcal{Y}}$ by definition is an object in $D'^b(\mathcal{O}_X)$ ([15], section 10), we obtain groups $\mathrm{Ext}^i(L_{X/\mathcal{Y}}, I)$. Note that by construction, there is a natural isomorphism

$$\mathrm{Ext}^i(L^{\geq -n}_{X/\mathcal{Y}}, I) \longrightarrow \mathrm{Ext}^i(L_{X/\mathcal{Y}}, I)$$

for $n \geq i$.

2.20 With notation as in (2.8), let $X^\bullet_{\mathrm{et}}$ denote the simplicial topos defined in ([13], 12.4), and let $\underline{\mathrm{Exal}}_{p^{-1}\mathcal{O}_{Y^\bullet}}(\mathcal{O}_{X^\bullet}, \pi_X^*I)$ denote the category of $p^{-1}\mathcal{O}_{Y^\bullet}$-extensions of $\mathcal{O}_{X^\bullet}$ by π_X^*I in the topos $X^\bullet_{\mathrm{et}}$ ([10], III.1.1). Then there is a natural functor

$$\underline{\mathrm{Exal}}_{Y^\bullet}(X^\bullet, \pi_X^*I) \longrightarrow \underline{\mathrm{Exal}}_{p^{-1}\mathcal{O}_{Y^\bullet}}(\mathcal{O}_{X^\bullet}, \pi_X^*I). \tag{2.20.1}$$

sending an object $j^\bullet : X^\bullet \hookrightarrow X'^\bullet$ of $\underline{\mathrm{Exal}}_{Y^\bullet}(X^\bullet, \pi_X^*I)$ to

$$0 \longrightarrow \pi_X^*I \longrightarrow j^{-1}\mathcal{O}_{X'^\bullet} \longrightarrow \mathcal{O}_{X^\bullet} \longrightarrow 0.$$

Lemma 2.21 *The functor (2.20.1) is an equivalence of categories.*

Proof Let $\mathcal{C}$ denote the category whose objects are pairs $(i, j : X^i \hookrightarrow X^{i\prime})$, where $i \in \mathbb{N}$ and $j : X^i \hookrightarrow X^{i\prime}$ is an object of $\underline{\mathrm{Exal}}_{Y^i}(X^i, \pi_X^*I|_{X^i})$. A morphism $(i_1, j_1 : X^{i_1} \hookrightarrow X^{i_1\prime}) \to (i_2, j_2 : X^{i_2} \hookrightarrow X^{i_2\prime})$ is a pair (δ, ψ), where $\delta : [i_2] \to [i_1]$ is a morphism in Δ and $\psi : X^{i_1\prime} \to X^{i_2\prime} \times_{Y^{i_2}, Y^\bullet(\delta)} Y^{i_1}$ is a morphism of Y^{i_1}-extensions of X^{i_1}. The category $\underline{\mathrm{Exal}}_{Y^\bullet}(X^\bullet, \pi_X^*I)$ is naturally equivalent to the category of sections of $\mathcal{C} \to \Delta^{op}$.

Similarly, let $\mathcal{C}'$ denote the category whose objects are pairs $(i, \pi_X^*I|_{X^i} \to \mathcal{A} \to \mathcal{O}_{X^i})$, where $i \in \mathbb{N}$ and $\pi_X^*I|_{X^i} \to \mathcal{A} \to \mathcal{O}_{X^i}$ is an object of $\underline{\mathrm{Exal}}_{p_i^{-1}\mathcal{O}_{Y^i}}(\mathcal{O}_{X^i}, \pi_X^*I|_{X^i})$, and whose morphisms $(i_1, \pi_X^*I|_{X^{i_1}} \to \mathcal{A}_1 \to \mathcal{O}_{X^{i_1}}) \to (i_2, \pi_X^*I|_{X^{i_2}} \to$

$\mathcal{A}_2 \to \mathcal{O}_{X^{i_2}}$) are pairs (δ, ψ), where $\delta : [i_2] \to [i_1]$ is a morphism in Δ and ψ is an isomorphism between $\mathcal{A}_1 \to \mathcal{O}_{X^{i_1}}$ and

$$\begin{array}{ccc} X(\delta)^{-1}\pi_X^* I|_{X^{i_2}} \otimes_{\mathcal{O}_{Y^{i_2}}, Y(\delta)} \mathcal{O}_{Y^{i_1}} & \to X(\delta)^{-1}\mathcal{A}_2 \otimes_{\mathcal{O}_{Y^{i_2}}, Y(\delta)} \mathcal{O}_{Y^{i_1}} \to \mathcal{O}_{X^{i_1}} \\ \downarrow = & \\ \pi_X^* I|_{X^{i_1}} & \end{array}$$

in $\underline{\mathrm{Exal}}_{p_{i_1}^{-1}\mathcal{O}_{Y^{i_1}}}(X^{i_1}, \pi_X^* I|_{X^{i_1}})$. The category $\underline{\mathrm{Exal}}_{p^{-1}\mathcal{O}_{Y^\bullet}}(\mathcal{O}_{X^\bullet}, \pi_X^* I)$ is naturally equivalent to the category of sections of $\mathcal{C}' \to \Delta^{op}$, and the functor (2.20.1) is obtained by composing a sections $\Delta^{op} \to \mathcal{C}$ with the natural functor

$$\mathcal{C} \longrightarrow \mathcal{C}', \quad (i, j : X^i \hookrightarrow X^{i'}) \longmapsto (i, j^{-1}\mathcal{O}_{X^{i'}} \to \mathcal{O}_{X^i}). \tag{2.21.1}$$

Hence to prove (ii) it suffices to show that (2.21.1) is an equivalence, and for this it suffices to show that each of the functors

$$\underline{\mathrm{Exal}}_{Y^i}(X^i, \pi_X^* I|_{X^i}) \longrightarrow \underline{\mathrm{Exal}}_{p_i^{-1}\mathcal{O}_{Y^i}}(\mathcal{O}_{X^i}, \pi_X^* I|_{X^I})$$

is an equivalence which follows from (2.11). □

2.22 By ([10], III.1.1.5), the category $\underline{\mathrm{Exal}}_{p^{-1}\mathcal{O}_{Y^\bullet}}(\mathcal{O}_{X^\bullet}, \pi_X^* I)$ has a natural structure of a Picard category, and it follows from the construction that the composition of (2.8.1) and (2.20.1)

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I) \longrightarrow \underline{\mathrm{Exal}}_{p^{-1}\mathcal{O}_{Y^\bullet}}(\mathcal{O}_{X^\bullet}, \pi_X^* I)$$

is a morphism of Picard categories ([3], XVIII.1.4.6). In particular, if

$$\mathrm{Exal}_{p^{-1}\mathcal{O}_{Y^\bullet}}(\mathcal{O}_{X^\bullet}, \pi_X^* I)$$

denotes the isomorphism classes of objects in $\underline{\mathrm{Exal}}_{p^{-1}\mathcal{O}_{Y^\bullet}}(\mathcal{O}_{X^\bullet}, \pi_X^* I)$, then ([10], III.1.2.3) shows that there is a natural isomorphism

$$\mathrm{Exal}_{p^{-1}\mathcal{O}_{Y^\bullet}}(\mathcal{O}_{X^\bullet}, \pi_X^* I) \simeq \mathrm{Ext}^1(L_{X^\bullet/Y^\bullet}, \pi_X^* I), \tag{2.22.1}$$

where $L_{X^\bullet/Y^\bullet}$ denotes the cotangent complex of the morphism of topoi $X^\bullet_{\text{et}} \to Y^\bullet_{\text{et}}$. Hence the following two lemmas (2.23) and (2.24) complete the proof of (1.1).

Lemma 2.23 (i). *For each integer $n > 0$, there is a natural isomorphism $\pi_X^* L_{X/\mathcal{Y}}^{\geq -n} \simeq \tau_{\geq -n} L_{X^\bullet/Y^\bullet}$, where $\pi_X^* L_{X/\mathcal{Y}}^{\geq -n}$ denotes the restriction of $L_{X/\mathcal{Y}}^{\geq -n}$ to the étale site of $X^\bullet$.*

(ii). *The natural map induced by* (i) *and restriction*

$$\mathit{Ext}^i_X(L_{X/\mathcal{Y}}, I) \longrightarrow \mathrm{Ext}^i_{X^\bullet}(L_{X^\bullet/Y^\bullet}, \pi_X^* I) \tag{2.23.1}$$

is an isomorphism for all i. Here $\pi_X^* I$ denotes the sheaf on $X^\bullet_{\text{et}}$ defined by the sheaf I on X ([15], 6.12).

Proof Statement (i) follows from the construction of the cotangent complex $L_{X/\mathcal{Y}}$ ([15], section 10).

To see (ii), note that (2.23.1) is induced by the maps

$$\mathrm{Ext}^i_{D^b_{qcoh}(\mathcal{O}_X)}(L^{\geq -n}_{X/\mathcal{Y}}, I) \longrightarrow \mathrm{Ext}^i_{D^b_{qcoh}(\mathcal{O}_{X^\bullet})}(\tau_{\geq -n} L_{X^\bullet/Y^\bullet}, \pi_X^* I) \quad (2.23.2)$$

defined by (i) and the equivalence $D^b_{qcoh}(\mathcal{O}_X) \simeq D^b_{\mathrm{qcoh}}(\mathcal{O}_{X^\bullet_{\mathrm{et}}})$ ([15], 6.19), where $D^b_{qcoh}(\mathcal{O}_X)$ (resp. $D^b_{qcoh}(\mathcal{O}_{X^\bullet})$) denotes the triangulated sub-category of $D^b(\mathcal{O}_X)$ (resp. $D^b(\mathcal{O}_{X^\bullet})$) consisting of objects all of whose cohomology groups are quasi-coherent. It follows that (2.23.1) is an isomorphism. □

Lemma 2.24 *The isomorphism (1.1.1) induced by (2.23.1) and (2.22.1) is independent of the choice of $Y \to \mathcal{Y}$.*

Proof Let $y_2 : Y_2 \to \mathcal{Y}$ be a second smooth cover. Since any two covers of $\mathcal{Y}$ can be dominated by a third, we may assume that $y_2 = y \circ \ell$, where $\ell : Y_2 \to Y$ is a morphism of schemes. In this case, the lemma follows from the construction and ([10], III.1.2.2) which shows that the following diagram commutes

$$\begin{array}{ccc}
\mathrm{Exal}_{\mathcal{Y}}(X, I) & \xrightarrow{\mathrm{id}} & \mathrm{Exal}_{\mathcal{Y}}(X, I) \\
\Big\downarrow{\scriptstyle \rho_1} & & \Big\downarrow{\scriptstyle \rho_2} \\
\mathrm{Exal}_{p^{-1}\mathcal{O}^\bullet_Y}(\mathcal{O}_{X^\bullet}, \pi_X^* I) & \xrightarrow{t^{\bullet *}} & \mathrm{Exal}_{p_2^{-1}\mathcal{O}^\bullet_{Y_2}}(\mathcal{O}_{X_2^\bullet}, t^{\bullet *}\pi_X^* I) \\
\Big\downarrow{\scriptstyle (2.22.1)} & & \Big\downarrow{\scriptstyle (2.22.1)} \\
\mathrm{Ext}^1(L_{X^\bullet/Y^\bullet}, \pi_X^* I) & \xrightarrow{t^{\bullet *}} & \mathrm{Ext}^1(L_{X_2^\bullet/Y_2^\bullet}, t^{\bullet *}\pi_X^* I) \\
\Big\uparrow{\scriptstyle (2.23.1)} & & \Big\uparrow{\scriptstyle (2.23.1)} \\
\mathrm{Ext}^1(L_{X/\mathcal{Y}}, I) & \xrightarrow{\mathrm{id}} & \mathrm{Ext}^1(L_{X/\mathcal{Y}}, I),
\end{array}$$

where $p : X^\bullet \to Y^\bullet$ (resp. $p_2 : X_2^\bullet \to Y_2^\bullet$, $t^\bullet : X_2^\bullet \to X^\bullet$) denote the morphism of simplicial spaces obtained from $y : Y \to \mathcal{Y}$ (resp. $y_2 : Y_2 \to \mathcal{Y}$, $\ell : Y_2 \to Y$) and ρ_i ($i = 1, 2$) denote the maps induced by the composite of (2.8.1) and (2.20.1). □

Let us note the following corollary of the proof which will be used in the next sections:

Corollary 2.25 *The automorphism group of any object $(j : X \hookrightarrow X') \in \underline{Exal}_{\mathcal{Y}}(X, I)$ is canonically isomorphic to* $\mathrm{Ext}^0(L_{X/\mathcal{Y}}, I)$.

Proof Let $Y \to \mathcal{Y}$ be a smooth cover, and let $(j^\bullet : X^\bullet \hookrightarrow X'^\bullet)$ be the resulting object of $\underline{\mathrm{Exal}}_{Y^\bullet}(X^\bullet, \pi_X^* I)$. By (2.21) and ([10], II.1.2.4.3), the automorphism group of $(j^\bullet : X^\bullet \hookrightarrow X'^\bullet)$ is canonically isomorphic to $\mathrm{Ext}^0(L_{X^\bullet/Y^\bullet}, \pi_X^* I)$. Hence (2.9), (2.21), and (2.23 (ii)) yield an isomorphism

$$\mathrm{Aut}(j : X \hookrightarrow X') \simeq \mathrm{Ext}^0(L_{X/\mathcal{Y}}, I).$$

The fact that this automorphism is independent of the choice of $Y \to \mathcal{Y}$ follows from the argument used in the proof of (2.24). □

2.26 Following a suggestion of the referee, when X is a Deligne–Mumford stack (in particular a scheme or algebraic space), Theorem (1.1) can be generalized as follows.

Recall ([3], XVIII.1.4.11) that to any two–term complex $K^{-1} \to K^0$ of abelian sheaves on a site $\mathcal{S}$, one can associate a Picard prestack $\mathrm{pch}(K^\bullet)$ over $\mathcal{S}$. For any $U \in \mathcal{S}$ the groupoid $\mathrm{pch}(K^\bullet)$ has objects the elements of $K^0(U)$, and for two sections $x, y \in K^0(U)$ a morphism $x \to y$ is a section $g \in K^{-1}(U)$ with $dg = y - x$. We denote the associated Picard stack by $\mathrm{ch}(K^\bullet)$.

Let $Y \to \mathcal{Y}$ be a smooth cover by a scheme, and let $Y^\bullet$ be the associated simplicial algebraic space. Denote by $X^\bullet$ the base change of X to $Y^\bullet$. Recall ([13], 12.4) that the étale site $\mathrm{Et}(X^\bullet)$ of $X^\bullet$ can be described as follows. The objects are pairs (n, U), where $[n] \in \Delta$ and $U \to X^n$ is étale. A morphism $(n, U) \to (n', U')$ is a pair (δ, φ), where $\delta : [n'] \to [n]$ is a morphism in Δ and $\varphi : U \to U'$ is a morphism over $\delta^* : X^n \to X^{n'}$. Let $\pi : X^\bullet \to X$ be the projection, and define $\underline{\mathrm{Exal}}'_{Y^\bullet}(-, \pi^* I)$ to be the Picard stack over $\mathrm{Et}(X^\bullet)$ which to any (n, U) associates the groupoid $\underline{\mathrm{Exal}}_{Y^n}(U, I|_U)$. If $(\delta, \varphi) : (n, U) \to (n', U')$ is a morphism then the pullback functor

$$(\delta, \varphi)^* : \underline{\mathrm{Exal}}'_{Y^\bullet}(-, \pi^* I)(n', U') \to \underline{\mathrm{Exal}}'_{Y^\bullet}(-, \pi^* I)(n, U)$$

is the composite of the base change functor

$$\underline{\mathrm{Exal}}_{Y^{n'}}(U', I|_{U'}) \to \underline{\mathrm{Exal}}_{Y^n}(U' \times_{Y^{n'}} Y^n, I|_{U' \times_{Y^{n'}} Y^n})$$

with the isomorphism

$$\underline{\mathrm{Exal}}_{Y^n}(U' \times_{Y^{n'}} Y^n, I|_{U' \times_{Y^{n'}} Y^n}) \simeq \underline{\mathrm{Exal}}_{Y^n}(U' \times_{X^{n'}} X^n, I|_{U' \times_{X^{n'}} X^n})$$

and the pullback functor

$$\underline{\mathrm{Exal}}_{Y^n}(U' \times_{X^{n'}} X^n, I|_{U' \times_{X^{n'}} X^n}) \to \underline{\mathrm{Exal}}_{Y^n}(U, I|_U).$$

Let $\pi_* \underline{\mathrm{Exal}}'_{Y^\bullet}(-, \pi^* I)$ be the Picard stack over $\mathrm{Et}(X)$ which to any $V \to X$ associates $\underline{\mathrm{Exal}}'_{Y^\bullet}(-, \pi^* I)(V \times_X X^\bullet)$. Since $V \times_X X^\bullet$ is not an object of $\mathrm{Et}(X^\bullet)$ this should be interpreted as the groupoid of morphisms of stacks $V \times_X X^\bullet \to \underline{\mathrm{Exal}}'_{Y^\bullet}(-, \pi^* I)$, where $V \times_X X^\bullet$ denotes the sheaf obtained by pulling back the sheaf on X_{et} represented by V. In the notation of (2.8), the category $\underline{\mathrm{Exal}}'_{Y^\bullet}(-, \pi^* I)$ $(V \times_X X^\bullet)$ is equivalent to $\underline{\mathrm{Exal}}_{Y^\bullet}(V \times_X X^\bullet, \pi_V^* I)$. As in (2.8), pullback defines a morphism of Picard stacks over $\mathrm{Et}(X)$

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I) \to \pi_* \underline{\mathrm{Exal}}'_{Y^\bullet}(-, \pi^* I) \tag{2.26.1}$$

which by (2.9) is an equivalence.

By (A.7), there is a natural equivalence of Picard stacks

$$\underline{\mathrm{Exal}}'_{Y^\bullet}(-, \pi^* I) \simeq \mathrm{ch}((\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X^\bullet/Y^\bullet}, \pi^* I))[1]).$$

Applying π_* and using (2.26.1), we obtain an equivalence of Picard stacks over the étale site of X

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I) \simeq \pi_*(\mathrm{ch}((\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X^\bullet/Y^\bullet}, \pi^* I))[1])). \tag{2.26.2}$$

By ([3], XVIII.1.4.19) and the natural isomorphism

$$\tau_{\leq 1} R\pi_*(\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X^\bullet/Y^\bullet}, \pi^* I)) \simeq \tau_{\leq 1} R\pi_*(R\underline{\mathrm{Hom}}(L_{X^\bullet/Y^\bullet}, \pi^* I))$$

the right hand side of (2.26.2) is isomorphic to

$$\mathrm{ch}(\tau_{\leq 1} R\pi_*(R\underline{\mathrm{Hom}}(L_{X^\bullet/Y^\bullet}, \pi^* I))[1]).$$

Since for any n, $\tau_{\geq -n} L_{X^\bullet/Y^\bullet} \simeq \pi^*(L^{\geq -n}_{X/\mathcal{Y}}|_{X_{et}})$, there is by trivial duality ([3], XVII.2.3.7) a natural isomorphism

$$R\pi_* R\underline{\mathrm{Hom}}(L_{X^\bullet/Y^\bullet}, \pi^* I) \simeq R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}|_{X_{et}}, R\pi_*\pi^* I).$$

Since the map $I \to R\pi_*\pi^* I$ is an isomorphism (this can be seen using the argument of ([13], 13.5.5) which is still valid in the present context), this gives a quasi–isomorphism

$$\tau_{\leq 1} R\pi_* R\underline{\mathrm{Hom}}(L_{X^\bullet/Y^\bullet}, \pi^* I)[1] \simeq \tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}|_{X_{et}}, I)[1].$$

The equivalence (2.26.2) therefore induces an equivalence of Picard stacks

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I) \simeq \mathrm{ch}((\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}, I))[1]). \tag{2.26.3}$$

It follows from the construction of the isomorphism (1.1.1) that it is obtained by evaluating both sides of (2.26.3) on X and noting that for any two term complex $K^{-1} \to K^0$ on a site $\mathcal{S}$ the group of isomorphism classes of objects of $\mathrm{ch}(K^\bullet)$ over an object $U \in \mathcal{S}$ is naturally isomorphic to $H^0(U, K^\bullet|_U)$ (to see this last assertion it suffices to consider the case when K^{-1} is injective in which case it follows from ([3], XVIII.1.4.16 (I)) and the observation that $H^0(U, K^\bullet) \simeq \mathrm{Coker}(K^{-1}(U) \to K^0(U))$). Note also that here we are using the fact that Ext–groups on a Deligne–Mumford stack can be computed using either the étale or lisse-étale topology (this is a special case of ([15], 6.19)).

Remark 2.27 In the above it seems necessary to work with the étale site of X instead of the lisse-étale site (and hence restrict to the case when X is a Deligne–Mumford stack). This is because the construction of pullback in (2.14) is not valid for arbitrary (even smooth) morphisms $V \to U$.

2.28 If $\Lambda : P \to P'$ is a morphism of Picard stacks, define the *kernel* K *of* Λ, sometimes written $\mathrm{Ker}(\Lambda : P \to P')$, as follows. For any object $V \in \mathcal{S}$, the groupoid $K(V)$ is defined to be the groupoid of pairs (p, ι), where $p \in P(V)$ and $\iota : 0 \to \Lambda(p)$ is an isomorphism in $P'(V)$. For two objects (p, ι) and (p', ι') define $(p, \iota) + (p', \iota')$ to be $p + p' \in P(V)$ with the isomorphism

$$0 \xrightarrow{\iota+\iota'} \Lambda(p) + \Lambda(p') \xrightarrow{\simeq} \Lambda(p + p'), \tag{2.28.1}$$

where the second isomorphism is the one provided by the structure of a morphism of Picard categories on Λ.

Lemma 2.29 *Let* $K^\bullet, H^\bullet \in C^{[-1,0]}(\mathcal{O}_S)$ *be two complexes, let* $f : K^\bullet \to H^\bullet$ *be a morphism of complexes, and let* $\mathrm{Cone}(f)$ *denote the cone of* f. *Then the kernel of the induced morphism* $\mathrm{ch}(f) : \mathrm{ch}(K^\bullet) \to \mathrm{ch}(H^\bullet)$ *is canonically isomorphic to* $\mathrm{ch}(\tau_{\leq 0}\mathrm{Cone}(f)[-1])$.

Proof Let $\mathcal{K}$ denote the kernel. The complex $\tau_{\leq 0}\mathrm{Cone}(f)[-1]$ is isomorphic to the complex

$$K^{-1} \to K^0 \times_{H^0} H^{-1},$$

where K^{-1} is placed in degree -1 and the differential is given by the maps $d : K^{-1} \to K^0$ and $f^{-1} : K^{-1} \to H^{-1}$. It follows that $\mathrm{pch}(\tau_{\leq 0}\mathrm{Cone}(f)[-1])$ is the prestack which to any V associates the groupoid of pairs (k, ι), where $k \in \mathrm{pch}(K^\bullet)(V)$ and $\iota : 0 \to f(k)$ is an isomorphism in $\mathrm{pch}(H^\bullet)$. From this it follows that there is a natural fully faithful morphism of Picard stacks $\mathrm{ch}(\tau_{\leq 0}\mathrm{Cone}(f)[-1]) \to \mathcal{K}$. To see that it is essentially surjective it suffices to show that every object of $\mathcal{K}$ is locally in the image which is clear since every object of $\mathcal{K}$ can locally be represented by a pair (k, ι) with $k \in K^0(V)$. □

2.30 The equivalence of stacks (2.26.3) is functorial in I in the sense that if $u : I \to I'$ is a morphism of quasi–coherent sheaves on X, then the induced square

$$\begin{array}{ccc} \underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I) & \xrightarrow{(2.3)} & \underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I') \\ {\scriptstyle (2.26.3)}\downarrow & & \downarrow{\scriptstyle (2.26.3)} \\ \mathrm{ch}((\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}|_{X_{et}}, I))[1]) & \longrightarrow & \mathrm{ch}((\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}|_{X_{et}}, I'))[1]) \end{array} \tag{2.30.1}$$

is naturally commutative. Define

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I \to I') := \mathrm{Ker}(\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I) \to \underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I')). \tag{2.30.2}$$

For any étale $U \to X$, the groupoid $\underline{\mathrm{Exal}}_{\mathcal{Y}}(U, I \to I')$ is the groupoid of pairs (U', s), where $U' \in \underline{\mathrm{Exal}}_{\mathcal{Y}}(U, I)$ and $s : u^*U' \to U$ is a retraction over $\mathcal{Y}$ of the inclusion $U \hookrightarrow u^*U'$ (see (2.3) for the definition of u^*U').

The commutativity of (2.30.1) implies that (2.26.3) induces an equivalence between $\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I \to I')$ and

$$\mathrm{Ker}(\mathrm{ch}((\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}|_{X_{et}}, I))[1]) \to \mathrm{ch}((\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}|_{X_{et}}, I'))[1])).$$

Since in the derived category the shifted cone

$$\mathrm{Cone}(R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}|_{X_{et}}, I) \to R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}|_{X_{et}}, I'))[-1]$$

is $R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}|_{X_{et}}, I \to I')$, it follows from (2.29) that there is a natural equivalence

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I \to I') \simeq \mathrm{ch}((\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}|_{X_{et}}, I \to I'))[1]). \tag{2.30.3}$$

This shows in particular that if $\mathrm{Exal}_{\mathcal{Y}}(X, I \to I')$ denotes the group of isomorphism classes in $\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, I \to I')$, then $\mathrm{Exal}_{\mathcal{Y}}(X, I \to I')$ depends only on the image of $I \to I'$ in the derived category and there is a canonical isomorphism

$$\mathrm{Exal}_{\mathcal{Y}}(X, I \to I') \simeq \mathrm{Ext}^1(L_{X/\mathcal{Y}}|_{X_{et}}, I \to I'). \tag{2.30.4}$$

In general, if $I^\bullet$ is an object of the derived category $D^{[0,\infty[}_{qcoh}(\mathcal{O}_X)$, set

$$\mathrm{Exal}_{\mathcal{Y}}(X, I^\bullet) := \mathrm{Exal}_{\mathcal{Y}}(X, \tau_{\leq 1} I^\bullet).$$

Remark 2.31 Recall ([3], XVIII.1.4.17) that $\mathrm{ch}(-)$ induces an equivalence of categories between $D^{[-1,0]}(\mathrm{Ab}_X)$ (the derived category of abelian sheaves on X), and the category whose objects are Picard stacks on X_{et} and whose morphisms are isomorphism classes of morphisms of Picard stacks. If $\mathcal{P}ic$ denotes this second category, then it follows from the above discussion that for $I^\bullet \in C^{[-1,0]}(\mathcal{O}_X)$ the object of $\mathcal{P}ic$ defined by the Picard stack $\underline{\mathrm{Exal}}_{\mathcal{Y}}(-, I^\bullet)$ depends up to canonical isomorphism only on the image of $I^\bullet$ in $D^{[-1,0]}(\mathcal{O}_X)$, and is canonically isomorphic to the Picard stack $\mathrm{ch}((\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{X/\mathcal{Y}}|_{X_{et}}, I))[1])$.

Remark 2.32 With a suitable notion of $\mathcal{O}_X$–linear Picard stack due to Deligne (see (A.2) and (A.8)), the above discussion can be generalized to $\mathcal{O}_X$–linear Picard stacks. In particular the equivalence (2.26.3) extends naturally to an equivalence of $\mathcal{O}_X$–linear Picard stacks. Because there is no published reference for the notion of $\mathcal{O}_X$–linear Picard stack, however, we limit ourselves to the above equivalence on the level of Picard stacks.

2.33 The above discussion enables us to understand the functoriality of the isomorphism in (1.1) as follows. Suppose given a 2-commutative diagram of algebraic stacks

$$\begin{array}{ccc} Z & \xrightarrow{a} & X \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle x} \\ \mathcal{W} & \longrightarrow & \mathcal{Y} \end{array}$$

with Z and X Deligne–Mumford stacks and z and x representable. Let I (resp. J) be a quasi-coherent sheaf on X (resp. Z), and suppose given a map $\epsilon : I \to a_* J$.

There is a natural map

$$a_* : \mathrm{Exal}_{\mathcal{W}}(Z, J) \longrightarrow \mathrm{Exal}_{\mathcal{Y}}(X, Ra_* J) \tag{2.33.1}$$

defined as follows. First there is a natural forgetful map

$$\mathrm{Exal}_{\mathcal{W}}(Z, J) \longrightarrow \mathrm{Exal}_{\mathcal{Y}}(Z, J)$$

so it suffices to consider the case when $\mathcal{W} = \mathcal{Y}$.

Let $J \to E^\bullet$ be an injective resolution and let $u : E^0 \to \overline{E}^1$ be $\tau_{\leq 1} E^\bullet$. We have $\mathrm{Exal}_{\mathcal{Y}}(Z, J) \simeq \mathrm{Exal}_{\mathcal{Y}}(Z, E^0 \to \overline{E}^1)$ and $\tau_{\leq 1} Ra_* J \simeq a_*(E^0) \to a_*(\overline{E}^1)$. Hence to define (2.33.1) it suffices to define a map

$$\mathrm{Exal}_{\mathcal{Y}}(Z, E^0 \to \overline{E}^1) \to \mathrm{Exal}_{\mathcal{Y}}(X, a_* E^0 \to a_* \overline{E}^1).$$

For this let (Z', s) be an object of $\underline{\mathrm{Exal}}_{\mathcal{Y}}(Z, E^0 \to \overline{E}^1)$. Since E^0 is injective, we can apply (2.10) to get an object $a_*Z' \in \underline{\mathrm{Exal}}_{\mathcal{Y}}(X, a_*E^0)$. Moreover, it follows from the construction in (2.10) that the section s induces a trivialization a_*s of the image of a_*Z' under the map

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(X, a_*E^0) \to \underline{\mathrm{Exal}}_{\mathcal{Y}}(X, a_*\overline{E}^1)$$

induced by $E^0 \to \overline{E}^1$. The map (2.33.1) is defined to be the map obtained from the functor $(Z', s) \mapsto (a_*Z', a_*s)$.

There is by ([13], 17.3 (2)) (see also ([15], 10.1)) a natural map $La^*L_{X/\mathcal{Y}} \to L_{Z/\mathcal{W}}$ which induces a map

$$\mathrm{Ext}^i(L_{Z/\mathcal{W}}, J) \longrightarrow \mathrm{Ext}^i(La^*L_{X/\mathcal{Y}}, J), \tag{2.33.2}$$

and hence we obtain a diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^1(L_{Z/\mathcal{W}}, J) & \xrightarrow{(2.33.2)} & \mathrm{Ext}^1(La^*L_{X/\mathcal{Y}}, J) & \xleftarrow{\tau} & \mathrm{Ext}^1(L_{X/\mathcal{Y}}, I) \\
{\scriptstyle (1.1)}\downarrow & & {\scriptstyle (2.30.4)}\downarrow & & \downarrow{\scriptstyle (1.1)} \\
\mathrm{Exal}_{\mathcal{W}}(Z, J) & \xrightarrow{a_*} & \mathrm{Exal}_{\mathcal{Y}}(X, Ra_*J) & \xleftarrow{q} & \mathrm{Exal}_{\mathcal{Y}}(X, I),
\end{array} \tag{2.33.3}$$

where the map τ is the composite of

$$\epsilon_* : \mathrm{Ext}^1(L_{X/\mathcal{Y}}, I) \longrightarrow \mathrm{Ext}^1(L_{X/\mathcal{Y}}, Ra_*J)$$

with the isomorphism

$$\mathrm{Ext}^1(L_{X/\mathcal{Y}}, Ra_*J) \simeq \mathrm{Ext}^1(La^*L_{X/\mathcal{Y}}, J)$$

obtained from "trivial duality" ([3], XVII.2.3.7) and q is the map induced by ϵ and functoriality as in (2.3). We leave to the reader the task of verifying that the diagram (2.33.3) commutes.

3 Problem (2)

3.1 We proceed with the notation of problem (2). If $x' : X' \to \mathcal{Y}'$ is a flat deformation of X to $\mathcal{Y}'$, then since x' is flat, the kernel of $\mathcal{O}_{X'} \to \mathcal{O}_X$ is canonically isomorphic to x^*I, and hence a flat deformation of X defines an object of $\underline{\mathrm{Exal}}_{\mathcal{Y}'}(X, x^*I)$. A morphism of flat deformations $(j_1 : X \hookrightarrow X_1') \to (j_2 : X \hookrightarrow X_2')$ is defined to be a morphism between the resulting objects of $\underline{\mathrm{Exal}}_{\mathcal{Y}'}(X, x^*I)$. By the local criterion for flatness ([7], 0_{III}.10.2.1), if

$$\begin{array}{ccc}
X & \xrightarrow{j} & X' \\
{\scriptstyle x}\downarrow & \swarrow{\scriptstyle x'} & \\
\mathcal{Y}', & &
\end{array} \tag{3.1.1}$$

is an object of $\underline{\mathrm{Exal}}_{\mathcal{Y}'}(X, x^*I)$ for which the induced map

$$x^*I = x^*(\mathrm{Ker}(\mathcal{O}_{\mathcal{Y}'} \to \mathcal{O}_{\mathcal{Y}})) \xrightarrow{x'^b} \mathrm{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X) = x^*I \tag{3.1.2}$$

is the identity, then $x' : X' \to \mathcal{Y}'$ is flat. Therefore, there is an equivalence between the category of flat deformations of X to $\mathcal{Y}'$ and the full subcategory of $\underline{\mathrm{Exal}}_{\mathcal{Y}'}(X, x^*I)$ whose objects are extension for which (3.1.2) is the identity.

3.2 With these observations in hand, we can prove (1.4) using the method of ([10], proof of III.2.1.7). The distinguished triangle

$$Lx^*L_{\mathcal{Y}/\mathcal{Y}'} \longrightarrow L_{X/\mathcal{Y}'} \longrightarrow L_{X/\mathcal{Y}} \longrightarrow Lx^*L_{\mathcal{Y}/\mathcal{Y}'}[1], \tag{3.2.1}$$

induced by the composite $X \to \mathcal{Y} \to \mathcal{Y}'$ and ([15], 10.1 (iii)), induces a long exact sequence

$$0 \longrightarrow E^0_{X/\mathcal{Y}} \xrightarrow{(a)} E^0_{X/\mathcal{Y}'} \xrightarrow{(b)} E^0_{\mathcal{Y}/\mathcal{Y}'} \longrightarrow E^1_{X/\mathcal{Y}} \xrightarrow{(c)} E^1_{X/\mathcal{Y}'} \xrightarrow{(d)} E^1_{\mathcal{Y}/\mathcal{Y}'} \xrightarrow{(e)} E^2_{X/\mathcal{Y}},$$

where we have written $E^i_{X/\mathcal{Y}}$ (resp. $E^i_{X/\mathcal{Y}'}$, $E^i_{\mathcal{Y}/\mathcal{Y}'}$) for $\mathrm{Ext}^i(L_{X/\mathcal{Y}}, x^*I)$ (resp. $\mathrm{Ext}^i(L_{X/\mathcal{Y}'}, x^*I), \mathrm{Ext}^i(Lx^*L_{\mathcal{Y}/\mathcal{Y}'}, x^*I)$). Here we use the derived pullback functor defined in ([15], section 9).

Note that since $\mathcal{Y} \to \mathcal{Y}'$ is representable, $Lx^*L_{\mathcal{Y}/\mathcal{Y}'}$ has no homology in positive degrees, and this is why the map (a) is injective. We claim that the following lemma (3.3) proves (1.4). Indeed (3.3 (ii)) shows that (a) is an isomorphism, and hence (1.4 (iii)) follows from (2.25). Moreover, if $o \in \mathrm{Ext}^2(L_{X/\mathcal{Y}}, x^*I)$ denotes the image under (e) of the class in $\mathrm{Ext}^1(Lx^*L_{\mathcal{Y}/\mathcal{Y}'}, x^*I)$ corresponding to $id : x^*I \to x^*I$ via (3.3.1), then the discussion (3.1) together with (3.3 (iii)) shows that there exists a flat deformation of $x : X \to \mathcal{Y}$ if and only if $o = 0$ and hence we obtain (1.4 (i)). Finally combining (3.3 (iii)) with (3.3 (ii)) we see that if $o = 0$ then the map (c) makes the set of isomorphism classes of flat deformations of X to $\mathcal{Y}'$ a torsor under $\mathrm{Ext}^1(L_{X/\mathcal{Y}}, x^*I)$.

Lemma 3.3 (i). *There is a canonical isomorphism* $H_1(L_{\mathcal{Y}/\mathcal{Y}'}) \simeq I$ *and* $H_0(L_{\mathcal{Y}/\mathcal{Y}'}) = 0$.

(ii). *If J is any sheaf of $\mathcal{O}_X$–modules,* $\mathrm{Ext}^0(Lx^*L_{\mathcal{Y}/\mathcal{Y}'}, J) = 0$ *and there is a natural isomorphism*

$$\mathrm{Ext}^1(Lx^*L_{\mathcal{Y}/\mathcal{Y}'}, J) \simeq \mathrm{Hom}(x^*I, J). \tag{3.3.1}$$

(iii). *The composite*

$$Exal_{\mathcal{Y}'}(X, x^*I) \xrightarrow{(1.1.1)} Ext^1(L_{X/\mathcal{Y}'}, x^*I) \xrightarrow{(d)} Ext^1(Lx^*L_{\mathcal{Y}/\mathcal{Y}'}, x^*I) \xrightarrow{(3.3.1)} \mathrm{Hom}(x^*I, x^*I)$$

is the map which sends an extension $(j : X \hookrightarrow X')$ to the map (3.1.2).

Proof For (i), let $Y' \to \mathcal{Y}'$ be a smooth cover and let $\pi_{Y'} : Y'^{\bullet} \to \mathcal{Y}'$ be the associated simplicial algebraic space. Denote by $\pi_Y : Y^{\bullet} \to \mathcal{Y}$ the base change of $Y'^{\bullet}$ to $\mathcal{Y}$. By the construction of the cotangent complex ([15], section 10), for any $n \geq 0$ there is a natural isomorphism $\pi^*L^{\geq -n}_{\mathcal{Y}/\mathcal{Y}'} \simeq \tau_{\geq -n}L_{Y^{\bullet}/Y'^{\bullet}}$, where $\pi^*L^{\geq -n}_{\mathcal{Y}/\mathcal{Y}'}$ denotes the restriction of $L^{\geq -n}_{\mathcal{Y}/\mathcal{Y}'}$ to $Y^{\bullet}_{\mathrm{et}}$. On the other hand, by ([10], III.1.2.8.1) we have

$H_0(L_{Y^\bullet/Y'^\bullet}) = 0$ and $H_1(L_{Y^\bullet/Y'^\bullet}) \simeq \pi_Y^* I$. From this and ([15], 6.19) assertion (i) follows (we leave to the reader the task of verifying that the isomorphism is independent of the choice of $Y' \to \mathcal{Y}'$).

To see (ii), note that $H_0(x^* L_{\mathcal{Y}/\mathcal{Y}'}) \simeq x^* H_0(L_{\mathcal{Y}/\mathcal{Y}'}) = 0$ by (i), which in turn implies that $H_1(x^* L_{\mathcal{Y}/\mathcal{Y}'}) \simeq x^* H_1(L_{\mathcal{Y}/\mathcal{Y}'}) \simeq x^* I$. In particular, $\tau_{\leq -1} x^* L_{\mathcal{Y}/\mathcal{Y}'} \to x^* L_{\mathcal{Y}/\mathcal{Y}'}$ is a quasi–isomorphism and

$$\mathrm{Ext}^0(x^* L_{\mathcal{Y}/\mathcal{Y}'}, J) \simeq \mathrm{Hom}(H_0(x^* L_{\mathcal{Y}/\mathcal{Y}'}), J) = 0$$

and

$$\mathrm{Ext}^1(x^* L_{\mathcal{Y}/\mathcal{Y}'}, J) \simeq \mathrm{Hom}(H_1(x^* L_{\mathcal{Y}/\mathcal{Y}'}), J) \simeq \mathrm{Hom}(x^* I, J).$$

For (iii), let $Y' \to \mathcal{Y}'$ be a smooth cover as above, and let $\pi_X : X^\bullet \to X$ be the simplicial space obtained from $Y'^\bullet$ by base change. By construction of $L_{X/\mathcal{Y}}$ and $L_{X/\mathcal{Y}'}$, $\pi_X^* L_{X/\mathcal{Y}} \simeq (\tau_{\geq -n} L_{X^\bullet/Y^\bullet})$ and $\pi_X^* L_{X/\mathcal{Y}'} \simeq (\tau_{\geq -n} L_{X^\bullet/Y'^\bullet})$, where $(\tau_{\geq -n} L_{X^\bullet/Y^\bullet})$ and $(\tau_{\geq -n} L_{X^\bullet/Y'^\bullet})$ denote the systems in $D'(\mathcal{O}_{X^\bullet})$ (2.15) obtained from $L_{X^\bullet/Y^\bullet}$ and $L_{X^\bullet/Y'^\bullet}$ by truncation. Hence the pullback by π_X of the distinguished triangle (3.2.1) is naturally identified with the triangle induced by the distinguished triangle in $D(\mathcal{O}_{X^\bullet})$ ([10], II.2.1.5.6)

$$Lp^* L_{Y^\bullet/Y'^\bullet} \longrightarrow L_{X^\bullet/Y'^\bullet} \longrightarrow L_{X^\bullet/Y^\bullet} \longrightarrow Lp^* L_{Y^\bullet/Y'^\bullet}[1],$$

where $p : X^\bullet \to Y^\bullet$ denotes the map induced by x. Note also that by the same argument as in the proof of (2.23), the natural map

$$\mathrm{Ext}^i(Lx^* L_{\mathcal{Y}/\mathcal{Y}'}, x^* I) \longrightarrow \mathrm{Ext}^i(Lp^* L_{Y^\bullet/Y'^\bullet}, \pi_X^* x^* I)$$

is an isomorphism for all i. By the construction of the isomorphisms (1.1.1) and (3.3.1), there is a natural commutative diagram

$$\begin{array}{ccc}
\mathrm{Exal}_{\mathcal{Y}'}(X, x^* I) & \xrightarrow{\pi_X^*} & \mathrm{Exal}_{Y'^\bullet}(X^\bullet, \pi_X^* x^* I) \\
{\scriptstyle (1.1.1)}\downarrow & & \downarrow{\scriptstyle (2.11.1) \text{ and } (2.22.1)} \\
\mathrm{Ext}^1(L_{X/\mathcal{Y}'}, x^* I) & \xrightarrow{\pi_X^*} & \mathrm{Ext}^1(L_{X^\bullet/Y'^\bullet}, \pi_X^* x^* I) \\
{\scriptstyle (3.3.1)\circ(d)}\downarrow & & \downarrow{\scriptstyle ([8],\ \mathrm{III}.1.2.8)} \\
\mathrm{Hom}(x^* I, x^* I) & \xrightarrow{\pi_X^*} & \mathrm{Hom}(\pi_X^* x^* I, \pi_X^* x^* I),
\end{array}$$

and hence (iii) follows from ([10], III.2.1.2). □

3.4 The obstruction in (1.4 (i)) is functorial in the following sense. Consider a commutative diagram of algebraic stacks

$$\begin{array}{ccc}
Z & \xrightarrow{f} & X \\
{\scriptstyle z}\downarrow & & \downarrow{\scriptstyle x} \\
\mathcal{W} & \xrightarrow{g} & \mathcal{Y} \\
{\scriptstyle j_{\mathcal{W}}}\downarrow & & \downarrow{\scriptstyle j_{\mathcal{Y}}} \\
\mathcal{W}' & \xrightarrow{h} & \mathcal{Y}',
\end{array}$$

where x and z are representable and flat, and $j_{\mathcal{Y}}$ (resp. $j_{\mathcal{W}}$) is a closed immersion defined by a quasi–coherent square–zero ideal I (resp. J) on $\mathcal{Y}'$ (resp. $\mathcal{W}'$). Let $o_X \in \mathrm{Ext}^2(L_{X/\mathcal{Y}}, x^*I)$ (resp. $o_Z \in \mathrm{Ext}^2(L_{Z/\mathcal{W}}, z^*J)$) be the obstruction defined in (1.4 (i)) to the existence of a flat deformation of X (resp. Z) to $\mathcal{Y}'$ (resp. $\mathcal{W}'$). The natural maps $f^*x^*I \to z^*J$ and $f^*L_{X/\mathcal{Y}} \to L_{Z/\mathcal{W}}$ induce a diagram

$$\begin{array}{ccc}
\mathrm{Ext}^2(L_{X/\mathcal{Y}}, x^*I) & & \\
\downarrow & & \\
\mathrm{Ext}^2(f^*L_{X/\mathcal{Y}}, f^*x^*I) & & \\
\downarrow & & \\
\mathrm{Ext}^2(f^*L_{X/\mathcal{Y}}, z^*J) & \longleftarrow & \mathrm{Ext}^2(L_{Z/\mathcal{W}}, z^*J).
\end{array}$$

Lemma 3.5 *The images of o_X and o_Z in $\mathrm{Ext}^2(f^*L_{X/\mathcal{Y}}, z^*J)$ are equal.*

Proof From the distinguished triangle

$$x^*L_{\mathcal{Y}/\mathcal{Y}'} \longrightarrow L_{X/\mathcal{Y}'} \longrightarrow L_{X/\mathcal{Y}} \longrightarrow x^*L_{\mathcal{Y}/\mathcal{Y}'}[1]$$

and (3.3 (ii)) we obtain a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(x^*I, x^*I) & \simeq & \mathrm{Ext}^1(x^*L_{\mathcal{Y}/\mathcal{Y}'}, x^*I) & \to & \mathrm{Ext}^2(L_{X/\mathcal{Y}}, x^*I) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(f^*x^*I, f^*x^*I) & \simeq & \mathrm{Ext}^1(f^*x^*L_{\mathcal{Y}/\mathcal{Y}'}, f^*x^*I) & \to & \mathrm{Ext}^2(f^*L_{X/\mathcal{Y}}, f^*x^*I) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(f^*x^*I, z^*J) & \simeq & \mathrm{Ext}^1(f^*x^*L_{\mathcal{Y}/\mathcal{Y}'}, z^*J) & \xrightarrow{\partial} & \mathrm{Ext}^2(f^*L_{X/\mathcal{Y}}, z^*J).
\end{array}$$

There is also a morphism of triangles

$$\begin{array}{ccccccc}
f^*x^*L_{\mathcal{Y}/\mathcal{Y}'} & \longrightarrow & f^*L_{X/\mathcal{Y}'} & \longrightarrow & f^*L_{X/\mathcal{Y}} & \longrightarrow & f^*x^*L_{\mathcal{Y}/\mathcal{Y}'}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
z^*L_{\mathcal{W}/\mathcal{W}'} & \longrightarrow & L_{Z/\mathcal{W}'} & \longrightarrow & L_{Z/\mathcal{W}} & \longrightarrow & z^*L_{\mathcal{W}/\mathcal{W}'}[1]
\end{array}$$

which induces a commutative diagram

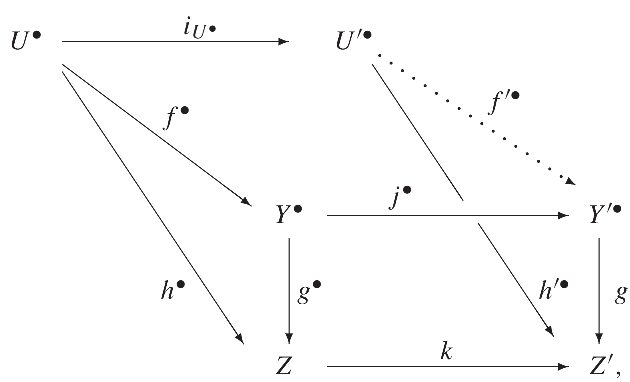
$$\begin{array}{ccccc}
\mathrm{Hom}(z^*J, z^*J) & \xrightarrow{\simeq} & \mathrm{Ext}^1(z^*L_{\mathcal{W}/\mathcal{W}'}, z^*J) & \longrightarrow & \mathrm{Ext}^2(L_{Z/\mathcal{W}}, z^*J) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(f^*x^*I, z^*J) & \xrightarrow{\simeq} & \mathrm{Ext}^1(f^*x^*L_{\mathcal{Y}/\mathcal{Y}'}, z^*J) & \xrightarrow{\partial} & \mathrm{Ext}^2(f^*L_{X/\mathcal{Y}}, z^*J).
\end{array}$$

From this it follows that the images of both o_Z and o_X in $\mathrm{Ext}^2(f^*L_{X/\mathcal{Y}}, z^*J)$ are equal to the image under ∂ of the class corresponding to the morphism $f^*x^*I \to z^*J$ induced by h. □

4 Problem (3)

4.1 Suppose given a commutative diagram of solid arrows as in (1.4.1). Define the *category of deformations of* x, denoted $\underline{\mathrm{Def}}(x)$, as follows. The objects of $\underline{\mathrm{Def}}(x)$ are pairs (x', ϵ), where $x' : X' \to \mathcal{Y}'$ is a 1-morphism over Z' and $\epsilon : x' \circ i \simeq j \circ x$ is an isomorphism in $\mathcal{Y}'(X)$. Any object $(x', \epsilon) \in \underline{\mathrm{Def}}(x)$ defines an object of $\underline{\mathrm{Exal}}_{\mathcal{Y}'}(X, I)$, and we define a morphism $(x_1', \epsilon_1) \to (x_2', \epsilon_2)$ to be a morphism between the resulting objects of $\underline{\mathrm{Exal}}_{\mathcal{Y}'}(X, I)$ whose underlying morphism of algebraic stacks $X' \to X'$ is the identity. That is, a morphism in $\underline{\mathrm{Def}}(x)$ is an isomorphism $\varphi : x_1' \to x_2'$ in $\mathcal{Y}'(X')$ such that the two isomorphisms $\epsilon_1, \epsilon_2 \circ i^*(\varphi) : x_1' \circ i \longrightarrow j \circ x$ are equal.

4.2 Let $Y' \to \mathcal{Y}'$ be a smooth cover of $\mathcal{Y}'$ by a scheme, and let $Y \to \mathcal{Y}$ denote the pullback to $\mathcal{Y}$. Denote by $\pi_{Y'} : Y'^{\bullet} \to \mathcal{Y}'$ (resp. $\pi_Y : Y^{\bullet} \to \mathcal{Y}$) the associated simplicial space, and choose an étale surjective morphism $\gamma : U \to X \times_{\mathcal{Y}} Y$, with U an affine scheme. Since the composite $u : U \to X$ is smooth, we can by ([8], III.5.5) lift U to a smooth morphism $u' : U' \to X'$ with U' affine. Let $u^{\bullet} : U^{\bullet} \to X$ (resp. $u'^{\bullet} : U'^{\bullet} \to X'$) denote the 0–coskeleton of the morphism u (resp. u'). The map γ induces a morphism $U^{\bullet} \to X^{\bullet}$, where $X^{\bullet}$ denotes the 0–coskeleton of the projection $X \times_{\mathcal{Y}} Y \to X$. Since $X^{\bullet}$ is canonically isomorphic to the base change over $\mathcal{Y}$ of $Y^{\bullet}$ to X, we obtain a natural map $f^{\bullet} : U^{\bullet} \to Y^{\bullet}$. We thus have a commutative diagram of solid arrows



(4.2.1)

where $i_U^{\bullet}$ (resp. $j^{\bullet}$) is a closed immersion with square-zero kernel isomorphic to $u^{\bullet *}I$ (resp. $\pi_Y^* J$). Denote by $\mathrm{Def}(f^{\bullet})$ the set of dotted arrows $f'^{\bullet} : U'^{\bullet} \to Y'^{\bullet}$ filling in (4.2.1). We will prove (1.5) by comparing $\underline{\mathrm{Def}}(x)$ with $\mathrm{Def}(f^{\bullet})$.

4.3 Let $y'^{\bullet} \in \mathcal{Y}'(Y'^{\bullet})$ be the tautological object (2.5), and let $x^{\bullet} \in \mathcal{Y}(U^{\bullet})$ denote the pullback of x. Since the morphism $f^{\bullet} : U^{\bullet} \to Y^{\bullet}$ factors through the simplicial space $X^{\bullet}$ obtained by base change to X from $Y^{\bullet}$, there is a canonical isomorphism $\tau : f^{\bullet *} j^{\bullet *} y'^{\bullet} \simeq x^{\bullet}$ (2.5).

If $x' : X' \to \mathcal{Y}'$ is an object of $\underline{\mathrm{Def}}(x)$, denote by $x'^{\bullet} \in \mathcal{Y}'(U^{\bullet})$ the object obtained from x' via the equivalence (2.7), and let $\underline{\widetilde{\mathrm{Def}}}(x)$ be the category whose objects are triples $(x', f'^{\bullet}, \tau)$, where x' is an object of $\underline{\mathrm{Def}}(x)$, $f'^{\bullet} : U'^{\bullet} \to Y'^{\bullet}$ is an object of $\mathrm{Def}(f^\bullet)$, and $\tau' : f'^{\bullet *}(y'^\bullet) \simeq x'^\bullet$ is an isomorphism in $\mathcal{Y}'(U'^\bullet)$ such that the induced isomorphism

$$f^{\bullet *} j^{\bullet *} y'^\bullet \simeq i^{\bullet *} f'^\bullet(y'^\bullet) \xrightarrow{i^{\bullet *}(\tau')} i^{\bullet *} x'^\bullet \simeq x^\bullet \tag{4.3.1}$$

equals τ.

Lemma 4.4 *Let $x' \in \underline{\mathrm{Def}}(x)$ be an object. To give a pair $(f'^\bullet, \tau')$ making $(x', f'^\bullet, \tau')$ an object of $\widetilde{\underline{\mathrm{Def}}}(f^\bullet)$ is equivalent to giving a morphism $U' \to X' \times_{\mathcal{Y}'} Y'$ over X' such that the composite $U \hookrightarrow U' \to X' \times_{\mathcal{Y}'} Y'$ is the composite $U \to X \times_{\mathcal{Y}} Y \to X' \times_{\mathcal{Y}'} Y'$, where the first map is γ.*

Proof Let $X'^\bullet$ be the simplicial algebraic space obtained by base change to $Y'^\bullet$ from $x' : X' \to \mathcal{Y}'$. It follows from the description of $X'^\bullet$ in (2.5), that giving a pair $(f'^\bullet, \tau')$ as in the lemma is equivalent to giving a morphism of simplicial spaces $U^{\bullet'} \to X'^\bullet$ over X' inducing the map $U^\bullet \to X^\bullet$ obtained from γ. On the other hand, the space $X'^\bullet$ is canonically isomorphic to the 0–coskeleton of the morphism $X'^0 \to X'$, and hence the lemma follows from the universal property of 0–coskeleton ([5], 5.1.1). □

Lemma 4.5 (i). *The category $\widetilde{\underline{\mathrm{Def}}}(x)$ is discrete. That is, it is equivalent to the category defined by its set of isomorphism classes of objects $\widetilde{\mathrm{Def}}(x)$.*

(ii). *The functor $\widetilde{\underline{\mathrm{Def}}}(x) \to \underline{\mathrm{Def}}(x)$ sending $(x', f'^\bullet, \tau')$ to x' is essentially surjective.*

(iii). *The map $\widetilde{\mathrm{Def}}(x) \to \mathrm{Def}(f^\bullet)$ sending $(x', f'^\bullet, \tau')$ to $f'^\bullet$ is bijective.*

Proof To see (i) it suffices to show that the objects of $\widetilde{\underline{\mathrm{Def}}}(x)$ admit no non-trivial automorphisms. But an automorphism of an object $(x', f'^\bullet, \tau') \in \widetilde{\underline{\mathrm{Def}}}(x)$ is simply an automorphism σ of $x' \in \underline{\mathrm{Def}}(x)$ for which the induced automorphism of $x'^\bullet$ is trivial. By (2.7) there are no non-trivial such automorphisms.

As for (ii), note that by (4.4) it suffices to show that for any $x' \in \mathcal{Y}'(X')$ the resulting diagram of solid arrows

$$\begin{array}{ccc} U & \longrightarrow & X' \times_{\mathcal{Y}'} Y' \\ \downarrow & \nearrow & \downarrow \\ U' & \longrightarrow & X', \end{array} \tag{4.5.1}$$

can be filled in by a dotted arrow. Since U is affine, this follows from the fact that $Y'/\mathcal{Y}'$ is smooth.

As for (iii), note that by (2.7) the category $\widetilde{\underline{\mathrm{Def}}}(x)$ is equivalent to the category of quadruples $(x'^\bullet, \epsilon, f'^\bullet, \tau')$, where $x'^\bullet \in \mathcal{Y}'(U'^\bullet)$, ϵ is an isomorphism between the pullback of $x'^\bullet$ to $U^\bullet$ and $x^\bullet$, $f'^\bullet : U^\bullet \to Y^\bullet$ is an object of $\mathrm{Def}(f^\bullet)$, and $\tau' : f'^{\bullet *}(y^\bullet) \simeq x'^\bullet$ is an isomorphism in $\mathcal{Y}'(U'^\bullet)$ such that the induced map (4.3.1) equals τ. This category is in turn tautologically equivalent to the category defined by $\mathrm{Def}(f^\bullet)$. □

4.6 Let $\Omega^1_{Y^\bullet/\mathcal{Y}}$ denote the sheaf on $Y^\bullet$ whose restriction to Y^i is $\Omega^1_{Y^i/\mathcal{Y}}$ and whose transition morphisms are those given by functoriality. By the construction of the cotangent complex ([15], section 10), the composite $Y^\bullet \to \mathcal{Y} \to Z$ induces a distinguished triangle

$$Lf^{\bullet *}\pi_Y^* L_{\mathcal{Y}/Z} \longrightarrow Lf^{\bullet *}L_{Y^\bullet/Z} \longrightarrow f^{\bullet *}\Omega^1_{Y^\bullet/\mathcal{Y}} \longrightarrow Lf^{\bullet *}\pi_Y^* L_{\mathcal{Y}/Z}[1]. \quad (4.6.1)$$

Lemma 4.7 (i). *For all $j \geq 0$, the natural map*

$$\mathcal{E}xt^j(f^{\bullet *}\Omega^1_{Y^\bullet/\mathcal{Y}}, u^{\bullet *}I) \longrightarrow \mathrm{Ext}^j(f^{0*}\Omega^1_{Y/\mathcal{Y}}, u^* I) \quad (4.7.1)$$

is an isomorphism, and for $j > 0$ these groups are zero.

(ii). *For $j > 0$, the natural map*

$$\mathrm{Ext}^j(Lf^{\bullet *}L_{Y^\bullet/Z}, u^{\bullet *}I) \longrightarrow \mathrm{Ext}^j(u^{\bullet *}Lx^*L_{\mathcal{Y}/Z}, u^{\bullet *}I)$$

is an isomorphism.

(iii). *The sequence*

$$\begin{aligned} &\mathcal{E}xt^0(f^{0*}\Omega^1_{Y/\mathcal{Y}}, u^* I) \to \mathcal{E}xt^0(Lf^{\bullet *}L_{Y^\bullet/Z}, u^{\bullet *}I) \\ &\to \mathrm{Ext}^0(u^{\bullet *}Lx^*L_{\mathcal{Y}/Z}, u^{\bullet *}I) \to 0 \end{aligned} \quad (4.7.2)$$

is exact.

Proof Note first that the functor

$$e^* : (\mathcal{O}_{U^\bullet}\text{-modules}) \longrightarrow (\mathcal{O}_U\text{-modules}), \quad \mathcal{F}^\bullet \longmapsto \mathcal{F}^0$$

has an exact left adjoint $e_!$ (this is a special case of ([11], VI.5.3 and VI.5.7 (a)). If $\mathcal{G}$ is a $\mathcal{O}_U$-module, then the restriction of $e_!\mathcal{G}$ to U^i is defined to be

$$(e_!\mathcal{G})^i := \oplus_{m \in \mathrm{Hom}_\Delta([0],[i])} U^\bullet(m)^*\mathcal{G},$$

and for $\delta : [i_1] \to [i_2]$ the transition map $U^\bullet(\delta)^*(e_!\mathcal{G})^{i_2} \to (e_!\mathcal{G})^{i_1}$ is the one induced by the maps

$$U^\bullet(\delta)^*U^\bullet(m)^*G \xrightarrow{\simeq} U^\bullet(\delta \circ m)^*G.$$

We leave it to the reader to verify that $e_!$ really defines a left adjoint to e^*.

The key observation is that the natural map

$$e_!e^*f^{\bullet *}\Omega^1_{Y^\bullet/\mathcal{Y}} \longrightarrow f^{\bullet *}\Omega^1_{Y^\bullet/\mathcal{Y}} \quad (4.7.3)$$

is an isomorphism. To see this, note that the restriction of $f^{\bullet *}\Omega^1_{Y^\bullet/\mathcal{Y}}$ to X^i is $f^{i*}\Omega^1_{Y^i/\mathcal{Y}}$ and hence the fact that (4.7.3) is an isomorphism follows from ([13], 17.3 (5)) and ([13], 17.5.8).

To deduce (i) from this, note that the case $j = 0$ follows from the universal property of $e_!$. Moreover, since U is affine and $\Omega^1_{Y/\mathcal{Y}}$ is locally free, the right hand side of (4.7.1) is zero, and hence to complete the proof of (i) it suffices to show that any extension

$$0 \longrightarrow u^{\bullet *}I \longrightarrow E_1^\bullet \longrightarrow \cdots \longrightarrow E_j^\bullet \longrightarrow f^{\bullet *}\Omega^1_{Y^\bullet/\mathcal{Y}} \longrightarrow 0 \quad (4.7.4)$$

is split (here we think of $\mathrm{Ext}^j(f^{\bullet *}\Omega^1_{Y^\bullet/\mathcal{Y}}, u^{\bullet *}I)$ as classifying Yoneda extensions). But because of the isomorphism (4.7.3), such an extension (4.7.4) is isomorphic to the pushout via the map $e_!e^*u^{\bullet *}I \to u^{\bullet *}I$ of the extension

$$0 \longrightarrow u^*I \longrightarrow E_1^0 \longrightarrow \cdots \longrightarrow E_j^0 \longrightarrow f^{0*}\Omega^1_{Y/\mathcal{Y}} \longrightarrow 0 \qquad (4.7.5)$$

and hence is trivial.

Statements (ii) and (iii) follow from (i) and the long exact sequence of Ext-groups obtained from (4.6.1). □

4.8 We are now ready to complete the proof of (1.5). To construct the class o in (1.5 (i)), note that by ([10], III.2.2.4) there exists a canonical class $\tilde{o} \in \mathrm{Ext}^1(f^{\bullet *}L_{Y^\bullet/Z}, u^{\bullet *}I)$ whose vanishing is necessary and sufficient for $\mathrm{Def}(f^\bullet)$ to be non-empty. We let o be the image of $\tilde{o}$ under the isomorphism

$$\mathrm{Ext}^1(Lf^{\bullet *}L_{Y^\bullet/Z}, u^{\bullet *}I) \overset{(4.7.1\ (ii))}{\longrightarrow} \mathrm{Ext}^1(u^{\bullet *}L_{\mathcal{Y}/Z}, u^{\bullet *}I) \overset{([15],6.19)}{\longrightarrow} \mathrm{Ext}^1(Lx^*L_{\mathcal{Y}/Z}, I).$$

By (4.5), the class o is zero if and only if $\mathrm{Def}(x)$ is non-empty.

To see (1.5 (ii)), we use the following lemma:

Lemma 4.9 *Let G be an abelian group, $H \subset G$ a subgroup, and T a G-torsor (i.e. a set with a simply transitive G-action). Then the set of H-orbits T/H is naturally a G/H-torsor.*

Proof If $\bar{g} \in G/H$ and $\bar{t} \in T/H$, let $\bar{g} \cdot \bar{t}$ be the class of $g \cdot t$ for any liftings $g \in G$ and $t \in T$. It is immediate that this is well defined, and that it makes T/H a G/H-torsor. □

By ([10], III.2.2.4) and (4.5 (iii)), the set $\widetilde{\mathrm{Def}}(x)$ is a torsor under

$$\mathrm{Ext}^0(Lf^{\bullet *}L_{Y^\bullet/\mathcal{Y}}, u^{\bullet *}I),$$

and by the functoriality of ([10], III.2.2.4), the action of an element $\partial \in \mathrm{Ext}^0(f^{0*}\Omega^1_{Y/\mathcal{Y}}, u^*I)$ on $\widetilde{\mathrm{Def}}(x)$ via the map in (4.7.2) is given by sending a pair (x', γ'), where $x' \in \underline{\mathrm{Def}}(x)$ and $\gamma' : U' \to X' \times_{x',\mathcal{Y}'} Y'$ is a map as in (4.4), to $(x', \partial * \gamma')$, where $\partial * \gamma'$ denotes the map $U' \to X' \times_{\mathcal{Y}'} Y'$ obtained from ∂ and the torsorial action of $\mathrm{Ext}^0(f^{0*}\Omega^1_{Y/\mathcal{Y}}, u^*I)$ on the set of such maps ([10], III.2.2.4). Since the set of orbits of $\widetilde{\mathrm{Def}}(x)$ under $\mathrm{Ext}^0(f^{0*}\Omega^1_{Y/\mathcal{Y}}, u^*I)$ is $\mathrm{Def}(x)$, Lemma (4.9) combined with (4.7) implies that $\mathrm{Def}(x)$ is naturally a torsor under $\mathrm{Ext}^0(u^{\bullet *}L_{\mathcal{Y}/Z}, u^{\bullet *}I)$. From this and the isomorphism

$$\mathrm{Ext}^0(u^{\bullet *}L_{\mathcal{Y}/Z}, u^{\bullet *}I) \longrightarrow \mathrm{Ext}^0(Lx^*L_{\mathcal{Y}/Z}, I).$$

obtained from ([15], 6.9) we obtain (1.5 (ii)).

Finally to see (1.5 (iii)), note that by (2.25) the group $\mathrm{Aut}(x')$ is canonically isomorphic to the kernel of the natural map

$$\mathrm{Ext}^0(L_{X/\mathcal{Y}'}, I) \longrightarrow \mathrm{Ext}^0(L_{X/Z'}, I),$$

and from the distinguished triangle

$$Lx^*Lj^*L_{\mathcal{Y}'/Z'} \longrightarrow L_{X/Z'} \longrightarrow L_{X/\mathcal{Y}'} \longrightarrow Lx^*Lj^*L_{\mathcal{Y}'/Z'}[1]$$

we see that this kernel is canonically isomorphic to $\mathrm{Ext}^{-1}(Lx^*Lj^*L_{\mathcal{Y}'/Z'}, I)$. Hence the following lemma completes the proof of (1.5).

Lemma 4.10 *The natural map*

$$\mathit{Ext}^{-1}(Lx^*L_{\mathcal{Y}/Z}, I) \longrightarrow \mathrm{Ext}^{-1}(Lx^*Lj^*L_{\mathcal{Y}'/Z'}, I) \qquad (4.10.1)$$

is an isomorphism.

Proof Let $Y' \to \mathcal{Y}'$ be a smooth cover, and let

$$\begin{array}{ccc} X^\bullet & \xrightarrow{i^\bullet} & X'^\bullet \\ {\scriptstyle f^\bullet}\downarrow & & \downarrow{\scriptstyle f'^\bullet} \\ Y^\bullet & \xrightarrow{j^\bullet} & Y'^\bullet \end{array}$$

be the commutative diagram over Z' obtained by base change. By the construction of the cotangent complex and ([15], 6.9), there is a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{-1}(Lx^*L_{\mathcal{Y}/Z}, I) & \xrightarrow{\simeq} & \mathrm{Ext}^{-1}(Lf^{\bullet *}(L_{Y^\bullet/Z} \to \Omega^1_{Y^\bullet/\mathcal{Y}}), \pi_X^* I) \\ {\scriptstyle (4.10.1)}\downarrow & & \downarrow{\scriptstyle \gamma} \\ \mathrm{Ext}^{-1}(Lx^*Lj^*L_{\mathcal{Y}'/Z'}, I) & \xrightarrow{\simeq} & \mathrm{Ext}^{-1}(Lf^*Lj^*(L_{Y'^\bullet/Z'} \to \Omega^1_{Y'^\bullet/\mathcal{Y}'}), \pi_X^* I), \end{array}$$

where the map γ is the one induced by functoriality ([10], II.1.2.3). Now by (loc. cit., II.1.2.4.2), the map γ is naturally identified with the map

$$\begin{array}{c} \mathrm{Hom}(f^{\bullet *}\mathrm{Coker}(\Omega^1_{Y^\bullet/Z} \to \Omega^1_{Y^\bullet/\mathcal{Y}}), \pi_X^* I) \\ \downarrow \\ \mathrm{Hom}(f'^{\bullet *}\mathrm{Coker}(\Omega^1_{Y'^\bullet/Z'} \to \Omega^1_{Y'^\bullet/\mathcal{Y}'}), \pi_X^* I), \end{array}$$

which is an isomorphism since the formation of differentials commutes with base change. □

4.11 The obstruction in (1.5 (i)) is functorial in the following sense.

With the notation of (1.5), consider a 2–commutative diagram of representable morphisms of algebraic stacks

$$\begin{array}{ccc} T & \xrightarrow{\ell} & T' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a'} \\ X & \xrightarrow{i} & X', \end{array}$$

where ℓ is a closed immersion defined by a square–zero ideal $M \subset \mathcal{O}_{T'}$. The morphism a' induces a map $a^*I \to M$. If t denotes the composite $T \to X \to \mathcal{Y}$, then (1.5 (i)) gives an obstruction $o_T \in \text{Ext}^1(Lt^*L_{\mathcal{Y}/Z}, M)$ whose vanishing is necessary and sufficient for the existence of a Z'–morphism $t' : T' \to \mathcal{Y}'$ extending t. Let $o_X \in \text{Ext}^1(Lx^*L_{\mathcal{Y}/Z}, I)$ denote the obstruction of (1.5 (i)).

Lemma 4.12 *The image of o_X under the natural map*

$$Ext^1(Lx^*L_{\mathcal{Y}/Z}, I) \to Ext^1(Lt^*L_{\mathcal{Y}/Z}, a^*I) \to Ext^1(Lt^*L_{\mathcal{Y}/Z}, M) \quad (4.12.1)$$

is o_T.

Proof Let $\pi_{Y'} : Y'^\bullet \to \mathcal{Y}'$, $\pi_Y : Y^\bullet \to \mathcal{Y}$, $u^\bullet : U^\bullet \to X$, and $U'^\bullet \to X'$ be as in (4.2). Choose a smooth cover $T'^0 \to U'^0 \times_{X'} T'$ with T'^0 an affine scheme, and let $T'^\bullet$ by the 0–coskeleton of the map $T'^0 \to T'$ so there is a morphism $T'^\bullet \to U'^\bullet$ over a'. Denote by $T^\bullet$ the base change of $T'^\bullet$ to T, and let $k^\bullet : T^\bullet \to Y^\bullet$ be the projection. If $v^\bullet : T^\bullet \to T$ is the projection, then the natural maps $u^{\bullet *}Lx^*L_{\mathcal{Y}/Z} \to Lf^{\bullet *}L_{Y^\bullet/Z}$ and $v^{\bullet *}Lt^*L_{\mathcal{Y}/Z} \to Lk^{\bullet *}L_{Y^\bullet/Z}$ induce for every j a commutative diagram

$$\begin{array}{ccc}
\text{Ext}^j(Lf^{\bullet *}L_{Y^\bullet/Z}, u^{\bullet *}I) & \xrightarrow{p} & \text{Ext}^j(u^{\bullet *}Lx^*L_{\mathcal{Y}/Z}, u^{\bullet *}I) \\
\downarrow & & \downarrow s \\
\text{Ext}^j(Lk^{\bullet *}L_{Y^\bullet/Z}, v^{\bullet *}M) & \xrightarrow{q} & \text{Ext}^j(v^{\bullet *}Lt^*L_{\mathcal{Y}/Z}, v^{\bullet *}M).
\end{array}$$

By ([15], 6.9) there are natural isomorphisms

$$\text{Ext}^j(u^{\bullet *}Lx^*L_{\mathcal{Y}/Z}, u^{\bullet *}I) \simeq \text{Ext}^j(Lx^*L_{\mathcal{Y}/Z}, I),$$

$$\text{Ext}^j(v^{\bullet *}Lt^*L_{\mathcal{Y}/Z}, v^{\bullet *}M) \simeq \text{Ext}^j(Lt^*L_{\mathcal{Y}/Z}, M)$$

which identify s with (4.12.1) and o_X (resp. o_T) with the image under p (resp. q) of the obstruction

$$o_{X^\bullet} \in \text{Ext}^1(Lf^{\bullet *}L_{Y^\bullet/Z}, u^{\bullet *}I) \ \ (\text{resp. } o_{T^\bullet} \in \text{Ext}^1(Lk^{\bullet *}L_{Y^\bullet/Z}, v^{\bullet *}M))$$

defined by Illusie ([10], III.2.2.4) to extending $f^\bullet$ (resp. $k^\bullet$) to a morphism $f'^\bullet : U'^\bullet \to Y'^\bullet$ (resp. $k'^\bullet : T'^\bullet \to Y'^\bullet$). Hence the lemma follows from the functoriality of Illusie's construction (loc. cit.). □

Appendix A. Illusie's Theorem

Let $(\mathcal{S}, \mathcal{O}_\mathcal{S})$ be a ringed site, and let $\text{Mod}(\mathcal{O}_\mathcal{S})^{\Delta^o}$ denote the category of simplicial $\mathcal{O}_\mathcal{S}$–modules.

For basic facts about Picard stacks we refer to ([3], XVIII.1.4).

A.1 Let $\Omega_\bullet$ be a simplicial $\mathcal{O}_{\mathcal{S}}$–module, and let I be an $\mathcal{O}_{\mathcal{S}}$–module.

Let $\underline{\mathrm{Ext}}(\Omega_\bullet, I)$ be the stack over $\mathcal{S}$ which to any object U of the site associates the groupoid of extensions of simplicial $\mathcal{O}_{\mathcal{S}|_U}$–modules on the site $\mathcal{S}|_U$ of objects of $\mathcal{S}$ over U

$$\mathcal{E} = (0 \longrightarrow I|_U \xrightarrow{\alpha} \widetilde{\Omega}_\bullet \xrightarrow{\beta} \Omega_\bullet|_U \longrightarrow 0), \tag{A.1.1}$$

where I is viewed as a constant simplicial module. If

$$\mathcal{E}' = (0 \longrightarrow I|_U \xrightarrow{\alpha'} \widetilde{\Omega}'_\bullet \xrightarrow{\beta'} \Omega_\bullet|_U \longrightarrow 0).$$

is a second object then a morphism $\mathcal{E} \to \mathcal{E}'$ in $\underline{\mathrm{Ext}}(\Omega_\bullet, I)$ is a morphism $\rho : \widetilde{\Omega}_\bullet \to \widetilde{\Omega}'_\bullet$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I|_U & \xrightarrow{\alpha} & \widetilde{\Omega}_\bullet & \xrightarrow{\beta} & \Omega_\bullet|_U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & I|_U & \xrightarrow{\alpha'} & \widetilde{\Omega}'_\bullet & \xrightarrow{\beta'} & \Omega_\bullet|_U & \longrightarrow & 0 \end{array}$$

commutes. The stack $\underline{\mathrm{Ext}}(\Omega_\bullet, I)$ has a natural Picard stack structure with addition law given by Baer summation.

Remark A.2 It is possible to define a notion of $\mathcal{O}_{\mathcal{S}}$–linear Picard stack which is a Picard stack with an action of the sheaf of rings $\mathcal{O}_{\mathcal{S}}$, generalizing the notion of $\mathcal{O}_{\mathcal{S}}$–module. This notion is due to Deligne (private notes), and has also been alluded to in ([4]). The stack $\underline{\mathrm{Ext}}(\Omega_\bullet, I)(U)$ is naturally an $\mathcal{O}_{\mathcal{S}}$–linear Picard stack. For $U \in \mathcal{S}$ and $f \in \mathcal{O}_{\mathcal{S}}(U)$, the functor giving the action of f on $\underline{\mathrm{Ext}}(\Omega_\bullet, I)(U)$ sends an extension (A.1.1) to the pushout of this extension via the map $\times f : I|_U \to I|_U$. Because the notion of $\mathcal{O}_{\mathcal{S}}$–linear Picard stack has not appeared in the literature, however, we do not emphasize this additional structure.

Proposition A.3 *There is a natural equivalence of Picard stacks*

$$\underline{Ext}(\Omega_\bullet, I) \to \mathrm{ch}(\tau_{\leq 1} R\underline{\mathrm{Hom}}(N(\Omega_\bullet), I)[1]), \tag{A.3.1}$$

where $N(\Omega_\bullet)$ denotes the normalized complex of $\Omega_\bullet$.

Proof Consider first the case when I is an injective $\mathcal{O}_{\mathcal{S}}$–module, and let $\underline{\mathrm{Ext}}'(\Omega_\bullet, I)$ be the prestack over I which to any U associates the groupoid of pairs $(\widetilde{\Omega}_\bullet, s)$, where $\widetilde{\Omega}_\bullet$ is in $\underline{\mathrm{Ext}}(\Omega_\bullet, I)$ and $s : \Omega_0 \to \widetilde{\Omega}_0$ is a section of the projection $\widetilde{\Omega}_0 \to \Omega_0$. A morphism $t : (\widetilde{\Omega}_\bullet, s) \to (\widetilde{\Omega}'_\bullet, s')$ is a morphism $\widetilde{\Omega}_\bullet \to \widetilde{\Omega}'_\bullet$ in $\underline{\mathrm{Ext}}(\Omega_\bullet, I)$. There is a natural morphism of prestacks

$$\underline{\mathrm{Ext}}'(\Omega_\bullet, I) \to \underline{\mathrm{Ext}}(\Omega_\bullet, I), \quad (\widetilde{\Omega}_\bullet, s) \mapsto \widetilde{\Omega}_\bullet \tag{A.3.2}$$

which induces an equivalence between the associated stacks.

We define a morphism of Picard prestacks

$$\underline{\mathrm{Ext}}'(\Omega_\bullet, I) \to \mathrm{pch}(\tau_{\leq 1} R\underline{\mathrm{Hom}}(N(\Omega_\bullet), I)[1]) \tag{A.3.3}$$

as follows.

Lemma A.4 *For any $\widetilde{\Omega}_\bullet \in \underline{Ext}(\Omega_\bullet, I)$, the map*

$$N(\widetilde{\Omega}_\bullet)_{-n} \to N(\Omega_\bullet)_{-n} \tag{A.4.1}$$

is an isomorphism for $n > 0$.

Proof Recall ([10], I.1.3.1) that

$$N(\widetilde{\Omega}_\bullet)_{-n} := \cap_{i>0}\mathrm{Ker}(d_i : \widetilde{\Omega}_n \to \widetilde{\Omega}_{n-1}), \tag{A.4.2}$$

where $d_i : \widetilde{\Omega}_n \to \widetilde{\Omega}_{n-1}$ denotes the map induced by the unique injective order preserving map $[n-1] \to [n]$ whose image does not contain i, and $N(\Omega_\bullet)$ is defined similarly. Fix an integer $n > 0$. If ω and ω' are in $\mathrm{Ker}(N(\widetilde{\Omega}_\bullet)_{-n} \to N(\Omega_\bullet)_{-n})$, then $\omega - \omega'$ is in $\mathrm{Ker}(\widetilde{\Omega}_n \to \Omega_n) \simeq I$ and since $d_i(\omega - \omega') = 0$ for $i > 0$ and $d_i|_I = \mathrm{id}$ it follows that $\omega - \omega' = 0$. Thus (A.4.1) is injective.

For surjectivity, consider first the case when $n = 1$. If $\omega \in \mathrm{Ker}(d_1 : \Omega_1 \to \Omega_0)$, then we can choose locally a lifting $\tilde{\omega} \in \widetilde{\Omega}_1$ of ω and $d_1(\tilde{\omega}) \in I$. Therefore, $\tilde{\omega} - d_1(\tilde{\omega})$ is a lift of ω and in $N(\widetilde{\Omega})_{-1}$. This proves the case $n = 1$.

For $n > 1$, note that if $\tilde{\omega} \in \widetilde{\Omega}_n$ is a lifting of an element $\omega \in N(\Omega_\bullet)_n$, then for each $i > 0$ the element $d_i(\tilde{\omega})$ is in I. Furthermore, for $0 < i < n$ we have $d_i(d_i(\tilde{\omega})) = d_i(d_{i+1}(\tilde{\omega}))$. Since $d_i|_I = \mathrm{id}$ it follows that $d_i(\tilde{\omega}) = d_{i+1}(\tilde{\omega})$ for all $0 < i < n$. Let $\iota \in I$ be this element. Then $\tilde{\omega} - \iota$ is a lifting of ω which is in $N(\widetilde{\Omega}_\bullet)_{-n}$. □

Since I is injective

$$\tau_{\leq 1} R\underline{\mathrm{Hom}}(N(\Omega_\bullet), I) \simeq ([\Omega_0, I] \to \mathrm{Ker}([N(\Omega_\bullet)_{-1}, I] \to [N(\Omega_\bullet)_{-2}, I])), \tag{A.4.3}$$

where we write $[-,-]$ for $\underline{\mathrm{Hom}}(-,-)$. If $(\widetilde{\Omega}_\bullet, s)$ is an object of $\underline{\mathrm{Ext}}'(\Omega_\bullet, I)$ we obtain a section of $\mathrm{Ker}([N(\Omega_\bullet)_{-1}, I] \to [N(\Omega_\bullet)_{-2}, I])$ from the difference of the composite

$$N(\Omega_\bullet)_{-1} \xrightarrow{\simeq} N(\widetilde{\Omega}_\bullet)_{-1} \xrightarrow{d_0} \widetilde{\Omega}_0 \tag{A.4.4}$$

and the composite

$$N(\Omega_\bullet)_{-1} \xrightarrow{d_0} \Omega_0 \xrightarrow{s} \widetilde{\Omega}_0. \tag{A.4.5}$$

If $\lambda : (\widetilde{\Omega}_\bullet, s) \to (\widetilde{\Omega}'_\bullet, s')$ is a morphism in $\underline{\mathrm{Ext}}'(\Omega_\bullet, I)$, the difference $s' - \lambda \circ s$ defines an element of $[\Omega_0, I]$. From this we obtain the functor (A.3.3).

To verify that the induced morphism of Picard stacks (A.3.1) is an equivalence, note that the functor induces an isomorphism between the sheaves associated to the presheaves of isomorphism classes of objects, and that the morphism of sheaves

$$\underline{\mathrm{Aut}}_{\underline{\mathrm{Ext}}(\Omega_\bullet, I)}(0) \to \underline{\mathrm{Aut}}_{\mathrm{ch}(\tau_{\leq 1} R\underline{\mathrm{Hom}}(N(\Omega_\bullet), I)[1])}(0) \tag{A.4.6}$$

is an isomorphism. By the construction, these statements are equivalent to the statement that for every $U \in \mathcal{S}$ the natural map

$$\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{O}_{\mathcal{S}|_U})^{\Delta^o}}(\Omega_\bullet|_U, I|_U) \to \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{O}_{\mathcal{S}|_U})}(N(\Omega_\bullet)|_U, I|_U), \quad i = 0, 1 \tag{A.4.7}$$

is an isomorphism, which follows from ([10], I.3.2.1.15).

This completes the proof in the case when I is injective. Note also that even if I is not injective, then the above discussion shows that the full substack $\underline{\mathrm{Ext}}(\Omega_\bullet, I)'$ (which is again a Picard stack) of $\underline{\mathrm{Ext}}(\Omega_\bullet, I)$ consisting of extensions $\mathcal{E}$ for which the map $\beta^0 : \widetilde{\Omega}_0 \to \Omega_0$ locally admits a section is isomorphic to $\mathrm{ch}(\underline{\mathrm{Hom}}(N(\Omega_\bullet), I)[1])$.

For general I, choose an injective resolution in the category of $\mathcal{O}_{\mathcal{S}}$–modules $I \to J^\bullet$ and set $\overline{J}^1 := \mathrm{Ker}(J^1 \to J^2)$. Pushout along the map $I \to J^0$ defines an equivalence of Picard categories

$$\underline{\mathrm{Ext}}(\Omega_\bullet, I) \simeq \mathrm{Ker}(\underline{\mathrm{Ext}}(\Omega_\bullet, J^0) \to \underline{\mathrm{Ext}}(\Omega_\bullet, \overline{J}^1)'),$$

where the right hand side is the kernel in the sense of (2.28). From above we therefore obtain an isomorphism

$$\underline{\mathrm{Ext}}(\Omega_\bullet, I) \simeq \mathrm{Ker}(\mathrm{ch}(\underline{\mathrm{Hom}}(N(\Omega_\bullet), J^0)[1]) \to \mathrm{ch}(\underline{\mathrm{Hom}}(N(\Omega_\bullet), \overline{J}^1)[1])).$$

Since $\tau_{\leq 1} R\underline{\mathrm{Hom}}(N(\Omega_\bullet), I)[2]$ is isomorphic to

$$\tau_{\leq 0}\mathrm{Cone}(\underline{\mathrm{Hom}}(N(\Omega_\bullet), J^0)[1] \to \underline{\mathrm{Hom}}(N(\Omega_\bullet), \overline{J}^1)[1]),$$

the result for general I follows from (2.29). □

Remark A.5 The equivalence in (A.3) is even an equivalence of $\mathcal{O}_{\mathcal{S}}$–linear Picard stacks.

A.6 Consider a morphism $A \to B$ of sheaves of rings on $\mathcal{S}$, and let I be a sheaf of B–modules on $\mathcal{S}$. Let $\underline{\mathrm{Exal}}_A(B, I)$ be the stack over $\mathcal{S}$ which to any U associates the groupoid of A–algebra extensions $B' \to B|_U$ of $B|_U$ by $I|_U$ ([10], III.1.1.1). The stack $\underline{\mathrm{Exal}}_A(B, I)$ has a natural structure of a Picard stack with additive structure as in ([10], III.1.1.5).

Theorem A.7 ([10], III.1.2.2) *Let $L_{B/A}$ denote the cotangent complex of $A \to B$. Then there is a natural equivalence of Picard stacks*

$$F : \underline{\mathrm{Exal}}_A(B, I) \to \mathrm{ch}(\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{B/A}, I)[1]). \tag{A.7.1}$$

Proof Let $P_\bullet \to B$ be the canonical free resolution of the A–algebra B ([10], I.1.5.5.6). Recall that $P_\bullet$ is a simplicial A–algebra. Any object

$$X : 0 \to I \to B' \to B \to 0$$

of $\underline{\mathrm{Exal}}_A(B, I)$ defines an I–extension of simplicial algebras

$$0 \to I \to P_\bullet \times_B B' \to P_\bullet \to 0,$$

where $P_\bullet \times_B B'$ denotes the simplicial A–algebra $[n] \mapsto P_n \times_B B'$ and I is viewed as a constant simplicial module. By ([10], III.1.1.7.1) this sequence induces an exact sequence of simplicial $P_\bullet$–modules

$$0 \to I \to \Omega^1_{P_\bullet \times_B B'/A} \otimes P_\bullet \to \Omega^1_{P_\bullet/A} \to 0$$

which after applying $(-)\otimes_{P_\bullet} B$ (with B viewed as a constant simplicial A–algebra) gives an exact sequence of simplicial B–modules

$$0 \to I \to E \to \Omega^1_{P_\bullet/A} \otimes B \to 0.$$

In this way we obtain a functor

$$\underline{\mathrm{Exal}}_A(B, I) \to \underline{\mathrm{Ext}}(\Omega^1_{P_\bullet/A} \otimes B, I).$$

It follows from the construction that this extends in a natural way to a morphism of Picard stacks. From (A.3), we therefore obtain a morphism of Picard stacks

$$\underline{\mathrm{Exal}}_A(B, I) \to \mathrm{ch}(\tau_{\leq 1} R\underline{\mathrm{Hom}}(N(\Omega^1_{P_\bullet/A} \otimes B), I)[1]).$$

Since by definition $L_{B/A} = N(\Omega^1_{P_\bullet/A} \otimes B)$ this defines the desired morphism of Picard stacks (A.7.1).

To verify that this morphism is an equivalence, it suffices to show that for any object $U \in \mathcal{S}$ the functor induces a bijection between the sets of isomorphism classes of objects over U, and that the map of sheaves

$$\underline{\mathrm{Aut}}_{\underline{\mathrm{Exal}}_A(B,I)}(0) \to \underline{\mathrm{Aut}}_{\mathrm{ch}(\tau_{\leq 1} R\underline{\mathrm{Hom}}(L_{B/A}, I)[1])}(0)$$

is an isomorphism. This follows from ([10], III.1.2.3 and II.1.2.4.3). □

Remark A.8 The stack $\underline{\mathrm{Exal}}_A(B, I)$ is naturally a B–linear Picard stack. The multiplicative structure is given by associating to any local section $f \in B$ the direct image functor ([10], III.1.1.3)

$$\underline{\mathrm{Exal}}_A(B, I) \to \underline{\mathrm{Exal}}_A(B, I)$$

induced by the B–module homomorphism $\times f : I \to I$. The right hand side of (A.7.1) is also a B–linear Picard stack with B–action induced by the action of B on $R\underline{\mathrm{Hom}}(L_{B/A}, I)$, and the equivalence in (A.7) is then naturally an equivalence of B–linear Picard stacks.

Using a suitable generalization of ([3], XVIII.1.4.17), this enables one to recover $\tau_{\geq -1} L_{B/A}$ from the B–linear Picard stack $\underline{\mathrm{Exal}}_A(B, I)$.

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