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Density results for the $W^{1/2}$ energy of maps into a manifold

Mariano Giaquinta[1], **Domenico Mucci**[2]

[1] Scuola Normale Superiore, Piazza dei Cavalieri 7, I-56100 Pisa (e-mail: giaquinta@sns.it)

[2] Dipartimento di Matematica dell'Università di Parma, Parco Area delle Scienze 53/A, I-43100 Parma (e-mail: domenico.mucci@unipr.it)

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Abstract. We show that maps from B^n to a smooth compact boundaryless manifold $\mathcal{Y}$ which are smooth out of a singular set of dimension $n-2$ are dense for the strong topology in $W^{1/2}(B^n, \mathcal{Y})$. We also prove that for $n \geq 2$ smooth maps from B^n to $\mathcal{Y}$ are dense in $W^{1/2}(B^n, \mathcal{Y})$ if and only if $\pi_1(\mathcal{Y}) = 0$, i.e. the first homotopy group of $\mathcal{Y}$ is trivial.

In this paper we consider vector valued maps into a manifold which have finite $W^{1/2}$-energy and we discuss density properties with respect to the strong topology of $W^{1/2}$. Let B^n be the unit ball $\mathbb{R}^n$, $n \geq 2$, and let $\mathcal{Y}$ be a smooth oriented Riemannian manifold of dimension $M \geq 1$, isometrically embedded in $\mathbb{R}^N$ for some $N \geq 2$. We shall assume that $\mathcal{Y}$ is compact, connected and without boundary.

Let $W^{1/2}(B^n, \mathbb{R}^N)$ denote standard space of functions φ which are traces of functions u in $W^{1,2}(B^n \times I, \mathbb{R}^N)$, where $I =]-1, 1[$, with the norm given by

$$|\varphi|_{1/2} := |\varphi|_{L^2} + \inf\{\mathbf{D}(u) \,:\, u = \varphi \quad \text{on } B^n \times \{0\}\},$$

compare [1]. If $u \in W^{1,2}(B^n \times I, \mathbb{R}^N)$ we will denote by

$$\mathbf{D}(u) := \frac{1}{2} \int_{B^n \times I} |Du(z)|^2 \, dz$$

the *Dirichlet energy* of u. Also, let

$$W^{1/2}(B^n, \mathcal{Y}) := \{\varphi \in W^{1/2}(B^n, \mathbb{R}^N) \mid \varphi(x) \in \mathcal{Y} \quad \text{for a.e. } x \in B^n\}.$$

Finally let $R^{\infty}_{1/2}(B^n, \mathcal{Y})$ be the set of all maps $u \in W^{1/2}(B^n, \mathcal{Y})$ which are smooth except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u) = \bigcup_{i=1}^{r} \Sigma_i \,, \qquad r \in \mathbb{N},$$

where Σ_i is a smooth $(n-2)$-dimensional subset of B^n with smooth boundary, if $n \geq 3$, and Σ_i is a point if $n = 2$. It is well-known that if $n = 1$ maps in $C^1(B^1, \mathcal{Y})$ are dense in $W^{1/2}(B^1, \mathcal{Y})$, compare [4]. If $n \geq 2$, our first result is

Theorem 1. *The class* $R^\infty_{1/2}(B^n, \mathcal{Y})$ *is dense in* $W^{1/2}(B^n, \mathcal{Y})$.

Theorem 1 was proved in [7], compare also [5], in dimension $n = 2$ and in the case $\mathcal{Y} = S^1$, the standard unit circle. Moreover, in [3] it is pointed out that if the first homotopy group of the target manifold is nontrivial, $\pi_1(\mathcal{Y}) \neq 0$, then there exist functions $\varphi \in W^{1/2}(B^n, \mathcal{Y})$ which cannot be approximated in $W^{1/2}$ by smooth maps in $W^{1/2}(B^n, \mathcal{Y})$. Our second result proves that the converse holds true. More precisely, we will show that if $\pi_1(\mathcal{Y}) = 0$, then in any dimension n smooth maps in $W^{1/2}(B^n, \mathcal{Y})$ are dense in $W^{1/2}(B^n, \mathcal{Y})$.

Theorem 2. *The class* $C^\infty(B^n, \mathcal{Y})$ *is dense in* $W^{1/2}(B^n, \mathcal{Y})$ *if and only if* $\pi_1(\mathcal{Y}) = 0$.

We remark that in [3, Lemma 4] it is claimed that if $n \leq p < n+1$, and $\pi_{[p]-1}(\mathcal{Y}) = 0$, then maps in $W^{1-1/p,p}(B^n, \mathcal{Y})$, which are smooth except at a finite number of point, can be approximated in $W^{1-1/p,p}$ by smooth maps in $C^\infty(B^n, \mathcal{Y})$. Actually the proof is not clear to us and we argue in a different way.

Since B^n is bilipschitz homeomorphic to the unit open n-cube

$$\mathcal{C}^n :=]0, 1[^n \,,$$

we will prove the theorems in the case of maps defined in $\mathcal{C}^n$. We point out that it is possible to modify the proofs of Theorems 1 and 2 to handle the case of maps defined in the unit n-sphere S^n or in the boundary of an $(n+1)$-cube. Moreover the proofs extend to cover the case of maps with fixed boundary data. More precisely, if $\widetilde{B}^n$ denotes a bounded domain in $\mathbb{R}^n$ such that $B^n \subset\subset \widetilde{B}^n$, $\psi : \widetilde{B}^n \to \mathcal{Y}$ is a given smooth $W^{1/2}$ function, and for $X = W^{1/2}$, $R^\infty_{1/2}$ or C^∞ we set

$$X_\psi(\widetilde{B}^n, \mathcal{Y}) := \{\varphi \in X(\widetilde{B}^n, \mathcal{Y}) \mid \varphi = \psi \quad \text{on } \widetilde{B}^n \setminus \overline{B}^n\} \,,$$

we can then also state the following density result, the proof of which is omitted.

Theorem 3. *The class* $R^\infty_{1/2,\psi}(\widetilde{B}^n, \mathcal{Y})$ *is dense in* $W^{1/2}_\psi(\widetilde{B}^n, \mathcal{Y})$. *Moreover, the class* $C^\infty_\psi(\widetilde{B}^n, \mathcal{Y})$ *is dense in* $W^{1/2}_\psi(\widetilde{B}^n, \mathcal{Y})$ *if and only if* $\pi_1(\mathcal{Y}) = 0$.

Before giving the proofs we fix some notation. We will always denote

$$z = (x, t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^n \times \mathbb{R}$$

a point in the cylinder $\mathcal{C}^n \times I$. If $u \in W^{1,2}(\mathcal{C}^n \times I, \mathbb{R}^N)$ and A is a "smooth" $\mathcal{H}^k$-measurable k-dimensional subset of $\mathcal{C}^n \times I$, we denote

$$\mathbf{D}(u, A) := \frac{1}{2} \int_A |Du_{|A}|^2 \, d\mathcal{H}^k \,, \qquad \mathbf{D}(u) := \mathbf{D}(u, \mathcal{C}^n \times I) \,,$$

the k-dimensional Dirichlet integral of the restriction $u_{|A}$ of u to A. Moreover we will write $T(u) = \varphi$ if $\varphi \in W^{1/2}(\mathcal{C}^n, \mathbb{R}^N)$ is the trace of u on $\mathcal{C}^n \times \{0\}$. If $p = (p_1, \dots, p_k) \in \mathbb{R}^k$, we set

$$\|p\|_k := \max_{1 \le i \le k} |p_i| .$$

Also, for $i = 1, \dots, n+1$ and $\lambda \in \mathbb{R}$, we denote by $P(\lambda, i)$ the restriction to $\mathcal{C}^n \times I$ of the hyperplane of $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ containing the point $\lambda\, e_i$ and orthogonal to e_i, where $(e_1, \dots, e_{n+1})$ is the canonical basis of $\mathbb{R}^{n+1}$, i.e.,

$$P(\lambda, i) := \{z \in \mathcal{C}^n \times I \mid (z - \lambda\, e_i \mid e_i)_{\mathbb{R}^{n+1}} = 0\} .$$

For $m \in \mathbb{N}^*$ and $a = (a_1, \dots, a_n) \in [1/4m, 3/4m]^n$ we denote by $\mathcal{L}_m$ the grid

$$\mathcal{L}_m := \bigcup_{i=1}^{n} \bigcup_{j=0}^{m-1} P(a_i + j/m, i) \tag{1}$$

and by $C_m^{(k)}$ the k-skeleton of the grid of $\mathcal{C}^n$ given by the intersection of $\mathcal{L}_m$ with the n-space $\mathbb{R}^n \times \{0\}$. Moreover we define by

$$\begin{aligned} \mathcal{C}_m^n &:= a + [0, (m-1)/m]^n \\ \Sigma_m^{(k)} &:= C_m^{(k)} \cap \mathcal{C}_m^n , \qquad \forall\, k = 1, \dots, n \end{aligned} \tag{2}$$

the closed n-cube of side $(m-1)/m$ inside $\mathcal{C}^n$ and the part of the k-skeleton $C_m^{(k)}$ which is contained in $\mathcal{C}_m^n$. We finally denote by $u^{(m)}$ the restriction $u^{(m)} := u_{|C_m^{(1)} \times I}$ of u to the 2-skeleton $C_m^{(1)} \times I$.

Remark 1. For future use, we denote by

$$\mathcal{Y}_\varepsilon := \{y \in \mathbb{R}^N \mid \operatorname{dist}(y, \mathcal{Y}) < \varepsilon\}$$

the ε-neighborhood of $\mathcal{Y}$ and we observe that, since $\mathcal{Y}$ is smooth, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ the nearest point projection Π_ε of $\mathcal{Y}_\varepsilon$ onto $\mathcal{Y}$ is a well defined Lipschitz map, with Lipschitz constant $\operatorname{Lip} \Pi_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$. Note that for $0 < \varepsilon \le \varepsilon_0$ the open set $\mathcal{Y}_\varepsilon$ is equivalent to $\mathcal{Y}$ in the sense of the algebraic topology. In particular, we have that

$$\pi_1(\mathcal{Y}_\varepsilon) = \pi_1(\mathcal{Y}) .$$

Proof of Theorem 1. Let $\varphi \in W^{1/2}(\mathcal{C}^n, \mathcal{Y})$ and $u \in W^{1,2}(\mathcal{C}^n \times I, \mathbb{R}^N)$ be the harmonic extension of φ, so that $T(u) = \varphi$. Since for $i = 1, \dots, n$ we have

$$\begin{aligned} \int_{1/4m}^{3/4m} \sum_{j=0}^{m-1} \mathbf{D}(u, P(t + j/m, i))\, dt &\le \sum_{j=0}^{m-1} \mathbf{D}(u, \{j/m \le x_i \le (j+1)/m\}) \\ &= \mathbf{D}(u, \mathcal{C}^n \times I) , \end{aligned}$$

we find a vector $a = a(m) \in [1/4m, 3/4m]^n$ such that $u_{|P(a_i+j/m,i)} \in W^{1,2}(P(a_i + j/m, i), \mathbb{R}^N)$ for every $i = 1, \dots, n$ and $j = 0, \dots, m-1$ and

$$\mathbf{D}(u, C_m^{(n-1)} \times I) \leq c\, m\, \mathbf{D}(u, \mathcal{C}^n \times I)\,. \tag{3}$$

We first make use of the argument of [3, 2.1], which in turn makes use of an idea from [8], by taking the 1-skeleton $C_m^{(1)}$ instead of the boundary of the unit square, and prove the following □

Proposition 1. *Let $\varepsilon_h \searrow 0$. There exists a sequence of continuous maps $\{u_h^{(m)}\}_h \subset W^{1,2}(\Sigma_m^{(1)} \times I, \mathbb{R}^N)$ such that $u_h^{(m)} \to u^{(m)}$ strongly in $W^{1,2}(\Sigma_m^{(1)} \times I, \mathbb{R}^N)$ and $\{T(u_h^{(m)})\}_h \subset W^{1/2}(\Sigma_m^{(1)}, \mathcal{Y}_{\varepsilon_h})$.*

Proof. If $z = (x, t) \in \Sigma_m^{(1)} \times I$ and $0 < h < 1/4m$ we denote by

$$C(z, h) := \overline{B}^n(x, h/2) \times [t - h/2, t + h/2]$$

the cylinder centered at z over the ball of diameter h and of height h, and by

$$\Sigma(z, h) := C(z, h) \cap (C_m^{(1)} \times I)$$

the intersection of the cylinder with the 2-skeleton $C_m^{(1)} \times I$. Setting then, for $z \in \Sigma_m^{(1)} \times I$,

$$u_h^{(m)}(z) := \frac{1}{\mathcal{H}^2\big(\Sigma(z, h)\big)} \int_{\Sigma(z,h)} u^{(m)}(y)\, d\mathcal{H}^2\,,$$

it is not difficult to show that $u_h^{(m)} \in W^{1,2}(\Sigma_m^{(1)} \times I, \mathbb{R}^N)$ is continuous and that $u_h^{(m)} \to u^{(m)}$ strongly in $W^{1,2}$ as $h \to 0^+$. It remains to show that if $\varphi_h^{(m)} := T(u_h^{(m)})$, possibly passing to a subsequence $\varphi_h^{(m)}(\Sigma_m^{(1)}) \subset \mathcal{Y}_{\varepsilon_h}$.

To this aim, for $\varepsilon > 0$ to be determined later, choose $h_\varepsilon > 0$ small so that for $h \leq h_\varepsilon$

$$\int_{\Sigma(z,h)} |Du^{(m)}(y)|^2\, d\mathcal{H}^2 \leq \varepsilon \qquad \forall\, z \in \Sigma_m^{(1)} \times I\,.$$

For fixed $P_0 \in \Sigma_m^{(1)} \times \{0\}$, we observe that the 2-dimensional set $\Sigma(P_0, h)$ always contains a square R_1 of side h. More precisely, suppose for example $P_0 = (x_0^1, \dots, x_0^n, 0)$, where $x_0^1 \in a_1 + [0, (m-1)/m]$ and $x_0^i = a_i + j_i/m$ for every $i = 2, \dots, n$, where $j_i \in \{0, \dots, m-1\}$. Then we have

$$\Sigma(P_0, h) = R_1 \cup \bigcup_{i=2}^{n} R_i$$

where R_1 is the square

$$R_1 := [x_0^1 - h/2, x_0^1 + h/2] \times \{(x_0^2, \dots, x_0^n)\} \times [-h/2, h/2]$$

and for $i = 2, \dots, n$ the (possibly degenerate) sets R_i are rectangles $R_i := \widetilde{R}_i \times [-h/2, h/2]$, where

$$\widetilde{R}_i := \{(a_1 + j_1/m, x_0^2, \dots, x_0^{i-1})\} \times [x_0^i - \overline{h}/2, x_0^i + \overline{h}/2] \times \{(x_0^{i+1}, \dots, x_0^n)\}$$

if $n \geq 3$, and

$$\widetilde{R}_2 := \{a_1 + j_1/m\} \times [x_0^2 - \overline{h}/2, x_0^2 + \overline{h}/2]$$

if $n = 2$, for some index j_1 and for some $\overline{h} \in [0, h]$, possibly $\overline{h} = 0$.

Slicing the square R_1 with hyperplanes orthogonal to the direction e_1, and taking $h \leq h_\varepsilon$, we find $h_1 \in [x_0^1 - h/2, x_0^1 + h/2]$ such that

$$\mathbf{D}(u^{(m)}, R_1 \cap P(h_1, 1)) \leq \frac{2}{h}\mathbf{D}(u^{(m)}, R_1) \leq \frac{1}{h}\int_{\Sigma(P_0,h)} |Du^{(m)}(y)|^2 \, d\mathcal{H}^2 \leq \frac{\varepsilon}{h}.$$

Choosing $z_0 \in R_1 \cap P(h_1, 1) \cap (\Sigma_m^{(1)} \times \{0\})$ and applying the Sobolev embedding theorem, since $R_1 \cap P(h_1, 1)$ is a line segment of length h, it follows that

$$\max_{z \in R_1 \cap P(h_1,1)} |u(z) - u(z_0)| \leq c\, \varepsilon^{1/2}.$$

Let now $\eta > 0$ to be determined later. Slicing the 2-dimensional set $\Sigma(P_0, h)$ with hyperplanes orthogonal to the "vertical" direction e_{n+1}, and setting

$$A_h := \{h' \in [-h/2, h/2] \, : \, \mathbf{D}(u^{(m)}, \Sigma(P_0, h) \cap P(h', n+1)) \leq \varepsilon\eta/h\}$$

and $B_h := [-h/2, h/2] \setminus A_h$, for every $h' \in A_h$, by the Sobolev theorem, since $\Sigma(P_0, h) \cap P(h', n+1)$ is the union of n line segments and $\operatorname{diam}(\Sigma(P_0, h) \cap P(h', n+1)) \leq c\, h$, we obtain

$$\max_{z \in \Sigma(P_0,h) \cap P(h',n+1)} |u(z) - u(z_0)| \leq c\, (\eta^{1/2} + 1)\, \varepsilon^{1/2}.$$

Consequently, since $\|u^{(m)}\|_\infty \leq K_\infty$, being $\mathcal{Y}$ compact, $\mathcal{L}^1(B_h) \leq h/\eta$ and $\mathcal{H}^2(\Sigma(P_0, h)) \geq h^2$, setting $y_h^{(m)} := u^{(m)}(z_0) \in \mathcal{Y}$, similarly to [3, 2.1] we infer that

$$|u_h^{(m)}(P_0) - y_h^{(m)}| \leq 4\, \frac{K_\infty}{\eta} + c\, (\eta^{1/2} + 1)\, \varepsilon^{1/2}.$$

Taking first η large so that $4\, K_\infty/\eta < \varepsilon_h/2$, and then ε small so that $c\, (\eta^{1/2} + 1)\, \varepsilon^{1/2} < \varepsilon_h/2$, we easily conclude that

$$\operatorname{dist}\big(\varphi_h^{(m)}(x_0), \mathcal{Y}\big) \leq |u_h^{(m)}(P_0) - y_h^{(m)}| < \varepsilon_h \qquad \forall\, x_0 \in \Sigma_m^{(1)}.$$

□

As a consequence of Proposition 1, we now prove the following

Proposition 2. *There exists a sequence of maps* $\{v_h^{(m)}\}_h \subset W^{1,2}(\mathcal{C}^n_m \times I, \mathbb{R}^N)$*, continuous out of* $\mathcal{C}^n_m \times \{0\}$*, such that* $v_h^{(m)} \to u_{|\mathcal{C}^n_m \times I}$ *strongly in* $W^{1,2}(\mathcal{C}^n_m \times I, \mathbb{R}^N)$*, with* ${v_h^{(m)}}_{|\Sigma_m^{(1)} \times I} = u_h^{(m)}$*. In particular we have*

$$T(v_h^{(m)})_{|\Sigma_m^{(1)}} \in W^{1/2}(\Sigma_m^{(1)}, \mathcal{Y}_{\varepsilon_h}) \qquad \forall\, h\,.$$

Proof. We first give the proof in the case $n = 2$.

The case $n = 2$. Let $\mathcal{Q}_m$ denote the family of all squares Q of side $1/m$ with boundary contained in the 1-grid $\Sigma_m^{(1)}$, i.e. $\partial Q \subset \Sigma_m^{(1)}$, so that

$$\cup \mathcal{Q}_m = \mathcal{C}^2_m\,.$$

For every h we let $0 < \varepsilon \ll 1$ to be fixed later. If $Q \in \mathcal{Q}_m$, we define $v_h^{(Q)} : Q \times I \to \mathbb{R}^N$ by setting for every $(x,t) \in Q \times I$

$$v_h^{(Q)} := \begin{cases} u\Big(p + \dfrac{x-p}{1-\varepsilon}, t\Big) & \text{if} \quad \rho \le \dfrac{1-\varepsilon}{2m} \\ S(\rho)\, u_h^{(m)}\big(y,t\big) + \big(1 - S(\rho)\big)\, u\big(y,t\big) & \text{if} \quad \dfrac{1-\varepsilon}{2m} \le \rho \le \dfrac{1}{2m}\,. \end{cases} \tag{4}$$

Here $\rho = \rho(x) := \|x - p\|_2$, where p is the center of Q, so that $\rho(x) = 1/2m$ if $x \in \partial Q$; moreover

$$y = y(x) := p + \frac{1}{2m}\,\frac{x-p}{\rho(x)}$$

and finally

$$S(\rho) := \frac{2m}{\varepsilon}\,\rho + \frac{\varepsilon - 1}{\varepsilon}\,, \tag{5}$$

so that $S\big(1/2m\big) = 1$ and $S\big((1-\varepsilon)/2m\big) = 0$. Trivially $v_h^{(Q)}$ is a function in $W^{1,2}(Q \times I, \mathbb{R}^N)$, continuous out of $Q \times \{0\}$, with $v_h^{(Q)} \to u_{|Q \times I}$ in $L^2(Q \times I, \mathbb{R}^N)$. Moreover, it is not difficult to prove that

$$\int_{\{\rho(x) \le (1-\varepsilon)/2m\} \times I} |Dv_h^{(Q)}|^2\, dx\, dt = 2\,\mathbf{D}(u,\, Q \times I)$$

and

$$\begin{aligned} \int_{\{(1-\varepsilon)/2m \le \rho(x) \le 1/2m\} \times I} |Dv_h^{(Q)}|^2\, dx\, dt &\le c\,(m)\,\frac{1}{\varepsilon} \int_{\partial Q \times I} |u - u_h^{(m)}|^2\, d\mathcal{H}^2 \\ &\quad + c\,(m)\,\varepsilon \int_{\partial Q \times I} \big(|D_\tau u|^2 + |D_\tau u_h^{(m)}|^2\big)\, d\mathcal{H}^2\,, \end{aligned}$$

where τ is an orthonormal frame to $\Sigma_m^{(1)} \times I$ and $c\,(m) > 0$ only depends on m. Define now $v_h^{(m)} : \mathcal{C}^2_m \times I \to \mathbb{R}^N$ by $v_h^{(m)}(x,t) := v_h^{(Q)}(x,t)$ if $x \in Q$ for some

$Q \in \mathcal{Q}_m$. Then $\{v_h^{(m)}\}_h$ is a sequence in $W^{1,2}(\mathcal{C}_m^2 \times I, \mathbb{R}^N)$, continuous out of $\mathcal{C}_m^2 \times \{0\}$, such that

$$\mathbf{D}(v_h^{(m)}, \mathcal{C}_m^2 \times I) \leq \mathbf{D}(u, \mathcal{C}_m^2 \times I) + c_1\,(m)\,\frac{1}{\varepsilon} \int_{\Sigma_m^{(1)} \times I} |u - u_h^{(m)}|^2 \, d\mathcal{H}^2$$
$$+ c_2\,(m)\,\varepsilon \int_{\Sigma_m^{(1)} \times I} \left(|D_\tau u|^2 + |D_\tau u_h^{(m)}|^2\right) d\mathcal{H}^2 \, .$$

Now, by Proposition 1, there exists $\overline{h}(m)$ such that for $h \geq \overline{h}(m)$

$$\int_{\Sigma_m^{(1)} \times I} |D_\tau u_h^{(m)}|^2 \, d\mathcal{H}^2 \leq 2 \int_{\Sigma_m^{(1)} \times I} |D_\tau u|^2 \, d\mathcal{H}^2$$

so that by (3) we have

$$\int_{\Sigma_m^{(1)} \times I} \left(|D_\tau u|^2 + |D_\tau u_h^{(m)}|^2\right) d\mathcal{H}^2 \leq 3\, c\, m\, \mathbf{D}(u, \mathcal{C}^2 \times I)\, .$$

Then, for every $j \in \mathbb{N}$ we first choose $\varepsilon = \varepsilon_j$ small so that

$$3\, c\, c_2\,(m)\,\varepsilon_j\, m\, \mathbf{D}(u, \mathcal{C}^2 \times I) \leq \frac{1}{j}\, .$$

Secondly, since by Proposition 1 we have $u_h^{(m)} \to u$ in $L^2(\Sigma_m^{(1)} \times I)$, we take $h = h_j \geq \overline{h}(m)$ large enough so that $h_{j+1} > h_j$ and

$$c_1\,(m)\,\frac{1}{\varepsilon_j} \int_{\Sigma_m^{(1)} \times I} |u - u_{h_j}^{(m)}|^2 \, d\mathcal{H}^2 \leq \frac{1}{j} \qquad \forall\, j \in \mathbb{N}\, .$$

Finally, since by the previous estimates

$$\mathbf{D}(v_{h_j}^{(m)}, \mathcal{C}_m^2 \times I) \leq \mathbf{D}(u, \mathcal{C}_m^2 \times I) + \frac{2}{j}\, ,$$

it suffices to relabel $\{v_j^{(m)}\}$ the subsequence $\{v_{h_j}^{(m)}\}$, where $\varepsilon = \varepsilon_j$ in (4).

The case $n \geq 3$. We first set $v_h^{(m)} = u_h^{(m)}$ on $\Sigma_m^{(1)} \times I$. Arguing by induction on the dimension $k = 2, \ldots, n$, by the inductive hypothesis we have already defined $v_h^{(m)} : \Sigma_m^{(k-1)} \times I \to \mathbb{R}^N$ in such a way that $v_h^{(m)} \to u_{|\Sigma_m^{(k-1)} \times I}$ strongly in $W^{1,2}(\Sigma_m^{(k-1)} \times I, \mathbb{R}^N)$.

We now extend $\{v_h^{(m)}\}$ to $\Sigma_m^{(k)} \times I$ as follows. Let F be a k-face of side $1/m$ of $\Sigma_m^{(k)}$, and hence with boundary contained in $\Sigma_m^{(k-1)}$. Without loss of generality, we may and will suppose F oriented by $e_1 \wedge \cdots \wedge e_k$, and we set

$$x = (\widetilde{x}, \widehat{x}) \in \mathbb{R}^k \times \mathbb{R}^{n-k}\, .$$

Similarly to (4), we define $v_h^{(F)} : F \times I \to \mathbb{R}^N$ by setting for $(x,t) \in F \times I$

$$v_h^{(F)} := \begin{cases} u\left(\widetilde{p} + \dfrac{\widetilde{x} - \widetilde{p}}{1-\varepsilon}, \widehat{p}, t\right) & \text{if} \quad \rho \le \dfrac{1-\varepsilon}{2m} \\ S(\rho)\, v_h^{(m)}\big(y, \widehat{p}, t\big) + \big(1 - S(\rho)\big)\, u\big(y, \widehat{p}, t\big) & \text{if} \quad \dfrac{1-\varepsilon}{2m} \le \rho \le \dfrac{1}{2m}\,. \end{cases}$$

Here $\rho = \rho(\widetilde{x}) := \|\widetilde{x} - \widetilde{p}\|_k$, where $p = (\widetilde{p}, \widehat{p})$ is the center of F; moreover

$$y = y(\widetilde{x}) := \widetilde{p} + \frac{1}{2m} \frac{\widetilde{x} - \widetilde{p}}{\rho(\widetilde{x})}$$

and $S(\rho)$ is given by (5). We then extend $v_h^{(m)} : \Sigma_m^{(k)} \times I \to \mathbb{R}^N$ by setting $v_h^{(m)}(x,t) := v_h^{(F)}(x,t)$ if $x \in F$ for some k-face F as before. Finally, similarly to the case $n = 2$, it is not difficult to show that $\{v_h^{(m)}\}_h$ is a sequence in $W^{1,2}(\Sigma_m^{(k)} \times I, \mathbb{R}^N)$, continuous out of $\Sigma_m^{(k)} \times \{0\}$, such that, possibly passing to a subsequence, $v_h^{(m)} \to u_{|\Sigma_m^{(k)} \times I}$ strongly in $W^{1,2}(\Sigma_m^{(k)} \times I, \mathbb{R}^N)$. The proof of Proposition 2 is complete. □

End of the proof of Theorem 1. We use an adaptation of [3, Lemma 5]. Slicing the cylinder $\mathcal{C}_m^n \times I$ with hyperplanes $P(t, n+1)$ orthogonal to the "vertical" direction e_{n+1}, since $\{v_h^{(m)}\}$ converges to $u_{|\mathcal{C}_m^n \times I}$ strongly in $W^{1,2}(\mathcal{C}_m^n \times I, \mathbb{R}^N)$, see Proposition 2, we may and do choose $a_{n+1} \in [1/4m, 3/4m]$ so that $v^{(m)}_{h|P(a_{n+1}+j/m,n+1)} \in W^{1,2}(P(a_{n+1} + j/m, n+1), \mathbb{R}^N)$ for every $h \in \mathbb{N}$ and $j \in \{-m, \dots, m-1\}$, with

$$\sum_{j=-m}^{m-1} \mathbf{D}(v_h^{(m)}, P(a_{n+1} + j/m, n+1)) \le c\, m\, \mathbf{D}(u, \mathcal{C}_m^n \times I) \qquad \forall h\,. \tag{6}$$

Let $\widetilde{\mathcal{Q}}_m$ denote the family of $(n+1)$-cubes of $\mathcal{C}_m^n \times I$, of side $1/m$, whose boundary lies in the n-skeleton

$$\mathcal{L}_m \cup \bigcup_{j=-m}^{m-1} P(a_{n+1} + j/m, n+1)\,,$$

compare (1). Also, let $\mathcal{F}_m$ be the family of the $(n+1)$-cubes in $\widetilde{\mathcal{Q}}_m$ which intersect the n-cube $\mathcal{C}^n \times \{0\}$, and let

$$G_m := \mathcal{C}^n \times] -10m^{-1}, 10m^{-1}[\,.$$

The case $n = 2$. Since $v^{(m)}_{h\ |\Sigma_m^{(1)} \times I} = u_h^{(m)}$, where $u_h^{(m)} \to u^{(m)}$ strongly in $W^{1,2}(\Sigma_m^{(1)} \times I, \mathbb{R}^N)$, compare Propositions 1 and 2, then by (3) and (6) we infer that for every h sufficiently large

$$\sum_{Q \in \widetilde{\mathcal{Q}}_m} \mathbf{D}(v_h^{(m)}, \partial Q) \le c\, m\, \mathbf{D}(u, \mathcal{C}^n \times I)\,.$$

As in [3, Lemma 5], by refining the slicing arguments in (3) and (6) we in fact may and do choose $(a_1, a_2, a_3) \in [1/4m, 3/4m]^3$ so that we also have

$$\sum_{l=1}^{(m-1)^2} \mathbf{D}(v_h^{(m)}, \partial C_l) \le c\, m\, \mathbf{D}(u, G_m) \qquad \forall\, h \ge \overline{h}\,, \tag{7}$$

where $\{C_l\}_{l=1}^{(m-1)^2}$ is a list of the cubes in $\mathcal{F}_m$. For every l let f_l be a diffeomorphism between C_l and $[-1/2m, 1/2m]^3$ such that

$$f_l(C_l \cap (\mathcal{C}^2 \times \{0\})) = [-1/2m, 1/2m]^2 \times \{0\}$$
$$f_l(\partial C_l \cap (\mathcal{C}^2 \times \{0\})) = \partial[-1/2m, 1/2m]^2 \times \{0\}$$

and

$$\|Df_l\|_\infty \le K\,, \qquad \|Df_l^{-1}\|_\infty \le K\,.$$

We then define $U_h^{(m)}$ on C_l by

$$U_h^{(m)}(z) = v_h^{(m)}\Big[f_l^{-1}\Big(\frac{f_l(z)}{2m\|f_l(z)\|_3}\Big)\Big],$$

so that

$$\mathbf{D}(U_h^{(m)}, C_l) \le \frac{c}{m}\,\mathbf{D}(v_h^{(m)}, \partial C_l)$$

for every l and hence, by (7),

$$\mathbf{D}(U_h^{(m)}, \cup\mathcal{F}_m) \le C\,\mathbf{D}(u, G_m)\,. \tag{8}$$

Set

$$U_h^{(m)}(z) = v_h^{(m)}(z) \qquad \forall\, z \in (\mathcal{C}_m^2 \times I) \setminus \cup\mathcal{F}_m\,,$$

so that $U_h^{(m)}$ is continuous on $\mathcal{C}_m^2 \times I$ except at one singularity on each C_l, which lies on $\mathcal{C}_m^2 \times \{0\}$. Moreover, $\{U_h^{(m)}\}$ is a sequence in $W^{1,2}(\mathcal{C}_m^2 \times I, \mathbb{R}^N)$ such that for h large enough

$$\mathbf{D}(U_h^{(m)} - v_h^{(m)}, \mathcal{C}_m^2 \times I) \le C\,\mathbf{D}(u, G_m)$$

and therefore, by Proposition 2,

$$\limsup_{h\to+\infty} \mathbf{D}(U_h^{(m)}, \mathcal{C}_m^2 \times I) \le \mathbf{D}(u, \mathcal{C}_m^2 \times I) + C\,\mathbf{D}(u, G_m)\,.$$

Remark 2. We also notice that for every cube C_l in $\mathcal{F}_m$, we have that ${U_h^{(m)}}_{|\partial C_l} = {v_h^{(m)}}_{|\partial C_l}$, where the traces $T(v_h^{(m)})_{|\Sigma_m^{(1)}} \in W^{1/2}(\Sigma_m^{(1)}, \mathcal{Y}_{\varepsilon_h})$, see Proposition 2. As a consequence, by the definition of f_l we infer that the traces $T(U_h^{(m)})$ are functions in $W^{1/2}(\mathcal{C}_m^2, \mathcal{Y}_{\varepsilon_h})$ for every h.

Now, let $\psi_m : \mathcal{C}^2 \to \mathcal{C}^2_m$ be an affine bijective function such that $\mathrm{Lip}\,\psi_m = (m-1)/m$ and $\psi_m \to Id_{\mathcal{C}^2}$ uniformly as $m \to +\infty$. Setting $u_m(x,t) := U^{(m)}_{h_m}(\psi_m(x), t)$ for some increasing sequence $h_m \nearrow \infty$, since $\mathrm{meas}(G_m) \to 0$ as $m \to +\infty$ we easily infer that $\{u_m\}_m$ is a sequence of maps in $W^{1,2}(\mathcal{C}^2 \times I, \mathbb{R}^N)$, continuous out of a finite number of points, such that $u_m \to u$ strongly in $W^{1,2}$. Moreover by Remark 2 it follows that the traces $T(u_m) \in W^{1/2}(\mathcal{C}^2, \mathcal{Y}_{\varepsilon_{h_m}})$ for every m. Therefore, taking $\varphi_m(x) := \Pi_{\varepsilon_{h_m}} \circ T(u_m)(x)$, compare Remark 1, clearly $\{\varphi_m\} \subset W^{1/2}(\mathcal{C}^2, \mathcal{Y})$ is continuous out of a discrete set of points and $\varphi_m \to \varphi$ in $W^{1/2}$. Finally, e.g. as in [2, Appendix], every function φ_m can be approximated by maps in $R^\infty_{1/2}(\mathcal{C}^2, \mathcal{Y})$.

The case $n \geq 3$. Let $\mathcal{F}^{(k)}_m$ be the k-dimensional skeleton of $\mathcal{F}_m$, i.e. the union of the k-faces of the $(n+1)$-cubes C_l of $\mathcal{F}_m$. Since $v^{(m)}_h \to u$ in $W^{1,2}(\mathcal{C}^n_m \times I, \mathbb{R}^N)$, by using a more refined slicing argument similar to the one in [6, Prop. 4], we may and do choose $(a_1, \dots, a_{n+1}) \in [1/4m, 3/4m]^{n+1}$ so that for every h sufficiently large the following holds:

(i) for every $k = 2, \dots, n$ the restriction of $v^{(m)}_h$ to any k-face Q of $\mathcal{F}^{(k)}_m$ is a function in $W^{1,2}(Q, \mathbb{R}^N)$;
(ii) there exists some absolute constant $c > 0$, not depending on h, such that

$$\mathbf{D}(v^{(m)}_h, \mathcal{F}^{(k)}_m) \leq c\, m^{n+1-k}\, \mathbf{D}(u, G_m) \qquad \forall\, k = 2, \dots, n\,. \tag{9}$$

First we let $U^{(m)}_h \equiv v^{(m)}_h$ on $\mathcal{F}^{(2)}_m$, and then we extend $U^{(m)}_h$ to $\mathcal{F}^{(k)}_m$ arguing by induction on the dimension $k = 3, \dots, n+1$. To this aim, for every k-face Q in $\mathcal{F}^{(k)}_m$ we distinguish two cases.

If Q is "horizontal", i.e. the direction e_{n+1} is not spanned by the vector space underlying Q, we let

$$U^{(m)}_h \equiv v^{(m)}_h \qquad \text{on } Q\,. \tag{10}$$

If Q is not "horizontal", as in the case $n = 2$ we let f_Q be a diffeomorphism between Q and $[-1/2m, 1/2m]^k$ such that

$$f_Q(Q \cap (\mathcal{C}^n \times \{0\})) = [-1/2m, 1/2m]^{k-1} \times \{0\}$$
$$f_Q(\partial Q \cap (\mathcal{C}^n \times \{0\})) = \partial[-1/2m, 1/2m]^{k-1} \times \{0\}$$

and

$$\|Df_Q\|_\infty \leq K\,, \qquad \|Df_Q^{-1}\|_\infty \leq K\,.$$

Since we have already defined $U^{(m)}_h$ on ∂Q, we extend $U^{(m)}_h$ to Q by setting

$$U^{(m)}_h(z) = U^{(m)}_h\Big[f_Q^{-1}\Big(\frac{f_Q(z)}{2m\|f_Q(z)\|_k}\Big)\Big], \tag{11}$$

so that

$$\mathbf{D}(U^{(m)}_h, Q) \leq \frac{c}{m}\, \mathbf{D}(U^{(m)}_h, \partial Q)\,. \tag{12}$$

Repeating the argument for $k = 3, \dots, n+1$, we then easily estimate

$$\mathbf{D}(U_h^{(m)}, \cup\mathcal{F}_m) \le C(n) \sum_{k=2}^{n} \frac{1}{m^{n+1-k}} \mathbf{D}(v_h^{(m)}, \mathcal{F}_m^{(k)}) \tag{13}$$

and hence, by (9), we obtain again (8). Setting then $U_h^{(m)}(z) = v_h^{(m)}(z)$ for every $z \in (\mathcal{C}_m^n \times I) \setminus \cup\mathcal{F}_m$, this way $U_h^{(m)}$ is continuous on $\mathcal{C}_m^n \times I$ except at an $(n-2)$-dimensional singular set, which lies on $\mathcal{C}_m^n \times \{0\}$, given by the union of a finite number (depending on n and m) of affine $(n-2)$-planes parallel to the coordinate directions in $\mathbb{R}^n \times \{0\}$. Moreover, by the construction we infer that the traces $T(U_h^{(m)}) \in W^{1/2}(\mathcal{C}_m^n, \mathcal{Y}_{\varepsilon_h})$ for every m. The rest of the proof follows as in the case $n = 2$.

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Proof of Theorem 2. We shall first give the proof in the case $n = 2$.

The case $n = 2$. Due to Theorem 1 it suffices to show that smooth maps in $C^\infty(\mathcal{C}^2, \mathcal{Y})$ are dense in $R^\infty_{1/2}(\mathcal{C}^2, \mathcal{Y})$. Let $\varphi \in R^\infty_{1/2}(\mathcal{C}^2, \mathcal{Y})$, so that φ is smooth out of a discrete set of points. Since the argument is local, without loss of generality we may and will suppose that φ is smooth out of the origin. For $0 < r < 1$ we denote

$$Q_r := [-r, r]^3 , \qquad F_r := Q_r \cap (\mathbb{R}^2 \times \{0\}) .$$

Let $u \in W^{1,2}(\mathcal{C}^2 \times I, \mathbb{R}^N)$ be the harmonic extension of φ. For every fixed $\varepsilon > 0$ let $0 < R = R(\varepsilon) \ll 1$ be such that

$$\mathbf{D}(u, Q_R) \le \varepsilon .$$

Since

$$\mathbf{D}(u, Q_R \setminus Q_{R/2}) = \frac{1}{2} \int_{R/2}^{R} dr \int_{\partial Q_r} |Du|^2 \, d\mathcal{H}^2 ,$$

then there exists $r = r(\varepsilon) \in [R/2, R]$ such that

$$\mathbf{D}(u, \partial Q_r) := \frac{1}{2} \int_{\partial Q_r} |Du|^2 \, d\mathcal{H}^2 \le \frac{2}{R} \mathbf{D}(u, Q_R \setminus Q_{R/2}) \le \frac{2\varepsilon}{R} . \tag{14}$$

Since $\varphi_{|\partial F_r} : \partial F_r \to \mathcal{Y}$ is a smooth map in $W^{1/2}(\partial F_r, \mathcal{Y})$, and the first homotopy group $\pi_1(\mathcal{Y}) = 0$, then there exists a smooth extension $\varphi_r : F_r \to \mathcal{Y}$ of φ with finite $W^{1,2}$-energy.

Let now $Q_r^\pm := \{z = (x, t) \in Q_r \mid \pm t \ge 0\}$ be the upper and lower half cubes of Q_r. Moreover, let $v_r^\pm : Q_r^\pm \to \mathbb{R}^N$ be the solution of the Dirichlet problem on $Q_r^\pm$ with boundary condition

$$\begin{cases} v_r^\pm = u & \text{on} \quad \partial Q_r^\pm \cap \{(x,t) \mid \pm t > 0\} \\ v_r^\pm = \varphi_r & \text{on} \quad F_r \end{cases}$$

and let $v_r : Q_r \to \mathbb{R}^N$ be given by $v_r(z) = v_r^\pm(z)$ if $z \in Q_r^\pm$. Define then $w_r : \mathcal{C}^2 \times I \to \mathbb{R}^N$ by

$$w_r(z) := \begin{cases} v_r\left(\dfrac{r}{\delta} z\right) & \text{if} \quad \|z\|_3 \le \delta \\ u\left(r \dfrac{z}{\|z\|_3}\right) & \text{if} \quad \delta \le \|z\|_3 \le r \\ u(z) & \text{if} \quad \|z\|_3 \ge r \end{cases}$$

for some $0 < \delta < r$, so that $w_r \in W^{1,2}(\mathcal{C}^2 \times I, \mathbb{R}^N)$ is continuous and with trace $T(w_r) \in W^{1/2}(\mathcal{C}^2, \mathcal{Y})$. We easily estimate

$$\mathbf{D}(w_r, \mathcal{C}^2 \times I) \le \mathbf{D}(u, \mathcal{C}^2 \times I) + c\, r\, \mathbf{D}(u, \partial Q_r) + \frac{\delta}{r} \mathbf{D}(v_r, Q_r)$$

for some absolute constant $c > 0$, so that by (14), and since $r < R$,

$$\begin{aligned} \mathbf{D}(w_r, \mathcal{C}^2 \times I) &\le \mathbf{D}(u, \mathcal{C}^2 \times I) + 2c\varepsilon + \frac{\delta}{r} \mathbf{D}(v_r, Q_r) \\ &\le \mathbf{D}(u, \mathcal{C}^2 \times I) + (2c+1)\,\varepsilon\,, \end{aligned}$$

taking δ sufficiently small. Letting $\varepsilon \to 0$ we infer that $w_{r_\varepsilon} \to u$ in $W^{1,2}(\mathcal{C}^2 \times I, \mathbb{R}^N)$ and hence that $T(w_{r_\varepsilon}) \to \varphi$ in $W^{1/2}(\mathcal{C}^2, \mathcal{Y})$. Since the trace $T(w_r) \in W^{1/2}(\mathcal{C}^2, \mathcal{Y})$ is continuous, then in a standard way it can be approximated by smooth maps, as required.

The case $n \ge 3$. We will modify the End of the proof of Theorem 1. Recall that the singular set of the approximating map $U_h^{(m)}$ is contained in $\mathcal{C}_m^n \times \{0\}$ and intersects every "horizontal" $(k+2)$-cube Q in $\mathcal{F}_m^{(k+2)}$, for $k = 1, \dots, n-1$, on a $(k-1)$-dimensional set obtained by the "homogeneous" extension (11) of the restriction of $U_h^{(m)}$ to the boundary of Q. To remove the singular set, working by induction on $k = 1, \dots, n-1$, it then suffices to modify the definition (11) to (17), where $v_Q : Q \to \mathbb{R}^N$ is a suitable smooth extension of the boundary datum. More precisely, let

$$F := Q \cap (\mathbb{R}^n \times \{0\}) \tag{15}$$

be the $(k+1)$-face in $\Sigma_m^{(k+1)}$ given by the intersection of Q with $\mathcal{C}^n \times \{0\}$, see (2). Moreover, let $\widetilde{\varphi} := T(U_h^{(m)})_{|\partial F}$ be the trace of $U_h^{(m)}$ on the boundary of F. Since $\pi_1(\mathcal{Y}) = 0$, if $k = 1$ there exists a smooth extension $\varphi_F : F \to \mathcal{Y}_{\varepsilon_h}$ of $\widetilde{\varphi}$ and therefore, as in the case $n = 2$, we define v_Q by solving the Dirichlet problem on the upper and lower half cubes $Q^\pm$ with boundary data given by (16). However, since we have no information on the higher order homotopy groups $\pi_k(\mathcal{Y})$ for $k \ge 2$, we cannot in general expect the existence of a smooth extension $\varphi_F : F \to \mathcal{Y}_{\varepsilon_h}$ of $\widetilde{\varphi}$. To overcome this difficulty, at the $(k-1)^{th}$ Step we will show how to modify the definition of the trace of $U_h^{(m)}$ on $\Sigma_m^{(k)}$ in such a way that

$\widetilde{\varphi} : \partial F \to \mathcal{Y}_{\varepsilon_h}$ is homotopically trivial. More precisely, we first let $U_h^{(m)} \equiv v_h^{(m)}$ on $\mathcal{F}_m^{(2)}$. Then for every $k = 1, \dots, n-1$ we give the following

k^{th} Step: definition of the trace on $\Sigma_m^{(k+1)}$ and extension to $\mathcal{F}_m^{(k+2)}$.

We first give a list $\{Q^{(i)}\}_i$ of the n-cubes of $\Sigma_m^{(n)}$ in such a way that $Q^{(i-1)}$ intersects $Q^{(i)}$ on an $(n-1)$-face, for every i.

In case $k \le n-2$, for every i we also give a list $\{\widetilde{F}_j^{(i)}\}_j$ of all the $(k+2)$-faces of $Q^{(i)}$ such that the following holds:

(i) $\widetilde{F}_j^{(i)}$ is not a $(k+2)$-face of the n-cubes $Q^{(l)}$, for every $l \le i-1$;

(ii) there exists a $(k+1)$-face $\widetilde{L}_j^{(i)}$ of $\widetilde{F}_j^{(i)}$ which is neither a face of the $\widetilde{F}_l^{(i)}$'s, for every $l \le j-1$, nor a face of the $Q^{(l)}$'s, for every $l \le i-1$.

We then relabel by $\{F^{(i)}\}_i$ the $(k+2)$-faces of $\Sigma_m^{(k+2)}$ by means of the lexicographic order given by the indices i and j of the $\widetilde{F}_j^{(i)}$'s. Note that $F^{(i)} = Q^{(i)}$ if $k = n-2$. Moreover, let $\{L_j^{(i)}\}_{j=1}^{j(k)}$ be a list of the $(k+1)$-faces of $F^{(i)}$ such that the last $(k+1)$-face $L_{j(k)}^{(i)}$ is exactly $\widetilde{L}_j^{(i)}$, if $F^{(i)} = \widetilde{F}_j^{(i)}$.

If $k = 1$, we let $\Psi^{(1)} : \Sigma_m^{(1)} \to \mathcal{Y}_{\varepsilon_h}$ be given by $\Psi^{(1)} := T(U_h^{(m)})_{|\Sigma_m^{(1)}}$, where $U_h^{(m)}$ is given by Theorem 1. Also, let $\widetilde{\varphi}_j^{(i)} := \Psi^{(1)}{}_{|\partial L_j^{(i)}}$ be the trace of $U_h^{(m)}$ on the boundary of the 2-face $L_j^{(i)}$. Since $\pi_1(\mathcal{Y}_{\varepsilon_h}) = 0$, compare Remark 1, then $\widetilde{\varphi}_j^{(i)}$ is homotopically trivial.

If $2 \le k \le n-2$, at the $(k-1)^{th}$ Step the function $\Psi^{(k)} : \Sigma_m^{(k)} \to \mathcal{Y}_{\varepsilon_h}$ is defined so that if $\widetilde{\varphi}_j^{(i)} := \Psi^{(k)}{}_{|\partial L_j^{(i)}}$, then $\widetilde{\varphi}_j^{(i)} : \partial L_j^{(i)} \to \mathcal{Y}_{\varepsilon_h}$ is a homotopically trivial smooth function in $W^{1/2}(\partial L_j^{(i)}, \mathcal{Y}_{\varepsilon_h})$ for all i and j.

For every $k \le n-2$, we let $\widetilde{\Phi}_j^{(i)} : L_j^{(i)} \to \mathcal{Y}_{\varepsilon_h}$ be a smooth map in $W^{1,2}(L_j^{(i)}, \mathcal{Y}_{\varepsilon_h})$ such that $\widetilde{\Phi}^{(i)}_{j|\partial L_j^{(i)}} = \widetilde{\varphi}_j^{(i)}$, and let $\widetilde{\Phi}^{(i)} : \partial F^{(i)} \to \mathcal{Y}_{\varepsilon_h}$ be such that $\widetilde{\Phi}^{(i)} = \widetilde{\Phi}_j^{(i)}$ on $L_j^{(i)}$. We now modify the maps $\widetilde{\Phi}^{(i)}$ to new maps $\Phi^{(i)}$ which are homotopically trivial. To this aim, let $V^{(i)} := \partial F^{(i)} \setminus L_{j(k)}^{(i)}$ and $g^{(i)} : \partial F^{(i)} \to V^{(i)}$ be a Lipschitz map such that $g^{(i)}{}_{|V^{(i)}} = Id_{|V^{(i)}}$ and $g^{(i)}{}_{|L_{j(k)}^{(i)}}$ is a 1 to 1 map onto $V^{(i)}$.

We first modify the function $\widetilde{\Phi}^{(1)}$ by setting $\Phi^{(1)} := \widetilde{\Phi}^{(1)} \circ g^{(1)}$. This way $\Phi^{(1)}$ is a homotopically trivial smooth map in $W^{1,2}(\partial F^{(i)}, \mathcal{Y}_{\varepsilon_h})$.

Arguing by iteration on the index i, once we have defined the functions $\Phi^{(l)}$, for $l = 1, \dots, i-1$, at the i^{th} step we first substitute $\widetilde{\Phi}^{(i)}$ by the map $\widehat{\Phi}^{(i)} : \partial F^{(i)} \to \mathcal{Y}_{\varepsilon_h}$ such that $\widehat{\Phi}^{(i)} = \Phi^{(l)}$ on $L_j^{(i)}$, if $L_j^{(i)}$ is a $(k+1)$-face of $F^{(l)}$ for some $l = 1, \dots, i-1$, and $\widehat{\Phi}^{(i)} = \widetilde{\Phi}^{(i)}$ elsewhere on $\partial F^{(i)}$. We remark that by the previous conditions (i) and (ii) we infer that $\widehat{\Phi}^{(i)} = \widetilde{\Phi}^{(i)}$ on $L_{j(k)}^{(i)}$. We then modify the function $\widehat{\Phi}^{(i)}$ by setting $\Phi^{(i)} := \widehat{\Phi}^{(i)} \circ g^{(i)}$. This way $\Phi^{(i)}$ is again a homotopically trivial smooth map in $W^{1,2}(\partial F^{(i)}, \mathcal{Y}_{\varepsilon_h})$. Finally, let $\widetilde{\Psi}^{(k+1)} : \Sigma_m^{(k+1)} \to \mathcal{Y}_{\varepsilon_h}$ be given by $\widetilde{\Psi}^{(k+1)} = \Phi^{(i)}$ on $\partial F^{(i)}$, for every i.

Remark 3. Note that, since $\Phi^{(i)} = \widehat{\Phi}^{(i)}$ on $V^{(i)}$, when defining $\Phi^{(i)}$ we do not modify the definition of $\widetilde{\Psi}^{(k+1)}$ on $\partial F^{(l)}$, for every $l \leq i-1$.

If $k = n-1$, at the $(n-2)^{th}$ Step the function $\Psi^{(n-1)} : \Sigma_m^{(n-1)} \to \mathcal{Y}_{\varepsilon_h}$ is defined in such a way that $\Psi^{(n-1)}{}_{|\partial Q^{(i)}} : \partial Q^{(i)} \to \mathcal{Y}_{\varepsilon_h}$ is a homotopically trivial smooth $W^{1,2}$ function for every n-cube $Q^{(i)}$ of $\Sigma_m^{(n)}$. Therefore, we let $\widetilde{\Psi}^{(n)} : \Sigma_m^{(n)} \to \mathcal{Y}_{\varepsilon_h}$ be a smooth $W^{1,2}$ extension of $\Psi^{(n-1)}$.

For every $k = 1, \dots, n-1$, we now extend the function $U_h^{(m)}$ to $\mathcal{F}_m^{(k+2)}$.

If Q is a "horizontal" $(k+2)$-cube in $\mathcal{F}_m^{(k+2)}$, we define $U_h^{(m)}$ as in (10).

If Q is not "horizontal", let F be the $(k+1)$-face given by (15) and let $\varphi_F : F \to \mathcal{Y}_{\varepsilon_h}$ be defined by $\varphi_F := \widetilde{\Psi}^{(k+1)}_{|F}$, so that φ_F is a smooth map in $W^{1,2}(F, \mathcal{Y}_{\varepsilon_h})$. Let now $Q^{\pm} := \{z = (x,t) \in Q \mid \pm t \geq 0\}$ be the upper and lower half $(k+2)$-cubes of Q. Moreover, let $v_Q^{\pm} : Q^{\pm} \to \mathbb{R}^N$ be the solution of the Dirichlet problem on $Q^{\pm}$ with boundary condition

$$\begin{cases} v_Q^{\pm} = U_h^{(m)} & \text{on} \quad \partial Q^{\pm} \cap \{(x,t) \mid \pm t > 0\} \\ v_Q^{\pm} = \varphi_F & \text{on} \quad F \end{cases} \tag{16}$$

and let $v_Q : Q \to \mathbb{R}^N$ be given by $v_Q(z) = v_Q^{\pm}(z)$ if $z \in Q^{\pm}$. If f_Q is the diffeomorphism between Q and $[-1/2m, 1/2m]^{k+2}$ given by Theorem 1, we modify the definition (11) of $U_h^{(m)}$ by setting for every $z \in Q$

$$U_h^{(m)} := \begin{cases} v_Q\Big[f_Q^{-1}\Big(\dfrac{f_Q(z)}{2m\delta}\Big)\Big] & \text{if } \|f_Q(z)\|_{k+2} \leq \delta \\ U_h^{(m)}\Big[f_Q^{-1}\Big(\dfrac{f_Q(z)}{2m\|f_Q(z)\|_{k+2}}\Big)\Big] & \text{if } \delta \leq \|f_Q(z)\|_{k+2} \leq \dfrac{1}{2m} \end{cases}. \tag{17}$$

Similarly to the case $n = 2$, we easily infer that (12) holds again if $0 < \delta < 1/2m$ is sufficiently small, whereas this time $U_h^{(m)}$ is continuous on Q and with trace $T(U_h^{(m)})$ in $W^{1/2}(F, \mathcal{Y}_{\varepsilon_h})$. We conclude the k^{th} Step by setting $\Psi^{(k+1)} := T(U_h^{(m)})_{|\Sigma_m^{(k+1)}}$.

After the $(n-1)^{th}$ Step, we obtain again (13) and hence, by (9), we conclude again with (8). The rest of the proof is similar to that of Theorem 1.

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