Math. Z. 252, 473–496 (2006) **Mathematische Zeitschrift**

DOI: 10.1007/s00209-005-0812-y

C^k-estimates for the $\overline{\partial}$-equation on convex domains of finite type

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Received: 26 July 2004; in final form: 8 December 2004 /
Published online: 19 November 2005 –

Abstract. For a bounded convex domain $D \subset \mathbb{C}^n$ with C^∞ smooth boundary of finite type m and $q = 1, \dots, n-1$, we construct a $\overline{\partial}$-solving integral operator T_q^* such that for all $k \in \mathbb{N}$ and the usual C^k and $C^{k+\frac{1}{m}}$-norms the operator $T_q^* : C^k_{0,q}(\overline{D}) \cap \ker \overline{\partial} \to C^{k+\frac{1}{m}}_{0,q-1}(\overline{D})$ is continuous.

Mathematics Subject Classification (2000): 32F18, 32F32, 32T25, 32W05, 32A26

1. Introduction

For a convex domain of finite type, K. Diederich, B. Fischer and J.E. Fornæss constructed in [4] a linear $\overline{\partial}$-solving operator which satisfies the following.

Theorem 1. *Let D be a bounded convex domain in $\mathbb{C}^n$ with C^∞-smooth boundary of finite type m, $q = 1, \dots, n-1$. We denote by $C^0_{0,q}(\overline{D})$ the Banach space of $(0, q)$-forms with continuous coefficients on $\overline{D}$ and by $C^{\frac{1}{m}}_{0,q-1}(\overline{D})$ the Banach space of $(0, q-1)$-forms whose coefficients are uniformly Hölder continuous of order $\frac{1}{m}$ on $\overline{D}$. Then there are bounded linear operators $T_q : C^0_{0,q}(\overline{D}) \to C^{\frac{1}{m}}_{0,q-1}(\overline{D})$ such that $\overline{\partial} T_q f = f$ for all $f \in C^0_{0,q}(\overline{D})$ with $\overline{\partial} f = 0$.*

For the construction of T_q they used Cauchy-Fantappiè kernels with the support function constructed in [3]. Using the ε-extremal basis of McNeal they estimated each terms of the kernel to prove the continuity of T_q. The techniques they introduced are the first step to generalise their result to the C^k-estimates. As have done I. Lieb and R.M. Range in the strictly pseudoconvex case (see [7]), we modify T_q and show the following result.

Theorem 2. *Let D be a bounded convex domain in $\mathbb{C}^n$ with C^∞-smooth boundary of finite type m and $q = 1, \dots, n-1$. Then there exists a linear operator $T_q^* : C_{0,q}(\overline{D}) \to C_{0,q-1}(D)$ such that for all $k \in \mathbb{N}$ and all $\overline{\partial}$-closed $f \in C^k_{0,q}(\overline{D})$, we have*

i) $\overline{\partial} T_q^* f = f$,

ii) $T_q^* f$ *belongs to* $C^{k+\frac{1}{m}}_{0,q-1}(\overline{D})$ *and there exists a constant* $c_k > 0$, *not depending on* f, *such that* $\|T_q^* f\|_{\overline{D},k+\frac{1}{m}} \leq c_k \|f\|_{\overline{D},k}$.

For the notion of C^k estimates and C^k norms we adopt the definition of [7]. The hard part of the estimates of T_q in [4] was the control of a boundary integral, but to take advantage of the higher regularity of the $(0, q)$-form f, as in [7], we shall integrate over a small annulus G around D. This confronts us with new problems. For example, the normal component of the kernel in the integration variable has a bad behavior, but since K. Diederich, B. Fischer and J.E. Fornæss integrate only over the boundary only the tangential part of the kernel plays a role. By integrating on G however we have to take care of this component. Therefore we show new estimates for the derivatives in the normal direction of the defining function of the domain D. The main difficulty here consists in the uniformity of these inequalities in a neighborhood of bD, the boundary of D. After many integrations by parts as in [8] we can control the integrals by analysing them with respect to ε-extremal bases of McNeal.

This article is organized as follows. In section 2, we recall the support function F of [3], the Hefer decomposition Q and the Cauchy-Fantappiè kernel constructed with it. In section 3 we show the new estimates for the normal derivatives of the defining function and link them to the ε-extremal basis. This is used in section 4 to estimate the derivatives up to order 2 of the Hefer section and achieve the proof of the theorem 2.

2. Integral operator

We recall the definition of the support function F of [3]. Let D be a bounded convex domain in $\mathbb{C}^n$ with C^∞ smooth boundary of finite type m and r a defining function of D. For $\alpha \in \mathbb{R}$ we set $D_\alpha := \{z \in \mathbb{C}^n, r(z) < \alpha\}$ and we assume that r has been chosen to be C^∞ and convex on $\mathbb{C}^n$ and such that $grad\, r(\zeta) \neq 0$ for all ζ in a bounded neighborhood $\mathcal{V}$ of bD. We fix some ζ in $\mathcal{V}$ and denote by $T_\zeta^{\mathbb{C}} bD_{r(\zeta)}$ the complex tangent space to $bD_{r(\zeta)}$ at ζ and by η_ζ the outer unit normal at ζ to $bD_{r(\zeta)}$. We choose an orthonormal basis $w'_1, \dots, w'_n$ such that $w'_1 = \eta_\zeta$ and set $r_\zeta(\omega) = r(\zeta + \omega_1 w'_1 + \dots + \omega_n w'_n)$ and

$$F_\zeta(\omega) := 3\omega_1 + K\omega_1^2 - K' \sum_{j=2}^{m} \kappa_j M^{2^j} \sum_{\substack{|\beta|=j \\ \beta_1=0}} \frac{1}{\beta!} \frac{\partial^j r_\zeta}{\partial \omega^\beta}(0)\omega^\beta$$

where $K,\ K',\ M$ are positive real numbers, $\kappa_j = 1$ when $j \equiv 0\, mod\, 4$, -1 when $j \equiv 2\, mod\, 4$ and 0 otherwise.

We write $z \in \mathbb{C}^n$ as $z = \zeta + \omega_{1,z} w'_1 + \dots + \omega_{n,z} w'_n$ and define $F(\zeta, z)$ by

$$F(\zeta, z) := F_\zeta(\omega_{1,z}, \dots, \omega_{n,z}).$$

Theorem 3. *The neighborhood $\mathcal{V}$ of bD and the constants M, K and K' in the definition of F can be chosen such that F satisfies for some positive real numbers k', c and R and any $\zeta \in \mathcal{V}$, any unit vector $v \in T_\zeta^{\mathbb{C}} bD_{r(\zeta)}$ and any $w = (w_1, w_2) \in \mathbb{C}^2$, with $|w| < R$ and $r(\zeta + w_1\eta_\zeta + w_2 v) - r(\zeta) \le 0$*

$$\begin{aligned}&\Re F(\zeta, \zeta + w_1\eta_\zeta + w_2 v)\\ &\quad \le -\left|\frac{\Re w_1}{2}\right| - \frac{K}{2}(\Im w_1)^2 - \frac{K'k'}{4}\sum_{j=2}^{m}\sum_{\alpha+\beta=j}\left|\left.\frac{\partial^j r(\zeta+\lambda v)}{\partial\lambda^\alpha\partial\overline{\lambda}^\beta}\right|_{\lambda=0}\right||w_2|^j\\ &\qquad + c(r(\zeta + w_1\eta_\zeta + w_2 v) - r(\zeta)).\end{aligned}$$

This theorem was shown in [3]. However we may have $F(\zeta, z) = 0$ when $|\zeta - z| > R$ so we should use a global version of this support function. For example we can construct such a function S as in [1]. This construction does not require other ideas than those of [11]. As in the strictly pseudoconvex case (see [11], p. 224, proof of theorem 1.13) S satisfies

i) S is of regularity C^∞ in $\mathcal{V} \times \mathcal{U}$, $\mathcal{U}$ a neighborhood of $\overline{D}$ and $S(\zeta, \cdot)$ is holomorphic on $\mathcal{U}$.
ii) $S(\zeta, \zeta) = 0$ for $\zeta \in \mathcal{U} \cap \mathcal{V}$.
iii) There exists a constant $c > 0$ such that $\Re S(\zeta, z) \le -c|\zeta - z|^m$ for all $(\zeta, z) \in \mathcal{V} \times \mathcal{U}$ with $r(\zeta) \ge r(z)$.
iv) On $\{(\zeta, z) \in \mathcal{V} \times \mathcal{U}, |\zeta - z| < \frac{R}{2}\}$, there is a C^∞ function A with $\frac{1}{2} \le |A(\zeta, z)| \le \frac{3}{2}$, such that $S = A \cdot F$.

We cannot define a Hefer section as in [4] because they only used the local explicitly known support function F. Therefore we choose an arbitrary unitary matrix U of $\mathbb{C}^{n\times n}$ and set

$$\Sigma(\zeta, \omega) = S(\zeta, \zeta + U\omega), \tag{1}$$

$$\sigma_j(\zeta, \omega) = \int_0^1 \frac{\partial \Sigma}{\partial \omega_j}(\zeta, t\omega)dt, \tag{2}$$

$$Q(\zeta, z) = -\overline{U}(\sigma_1(\zeta, \overline{U}^t(z - \zeta)), \dots, \sigma_n(\zeta, \overline{U}^t(z - \zeta))). \tag{3}$$

A simple calculation shows that $\Sigma(\zeta, \omega) = \sum_{j=1}^n \omega_j\sigma_j(\zeta, \omega)$, that Q does not depend on U and satisfies $S(\zeta, z) = \sum_{j=1}^n Q_j(\zeta, z)(\zeta_j - z_j)$.

Later on we will choose $U = U(\zeta)$ such that $\overline{U}^t\eta_\zeta = (1, 0, \dots, 0)$. With that choice the σ_j will locally have the same behavior than the Q_ζ^j of [4].

Now we define the Cauchy-Fantappiè kernel. Set $\eta_0(\zeta, z) = \sum_{j=1}^n \overline{\zeta_j - z_j}d\zeta_j$, $\eta_1(\zeta, z) = \sum_{j=1}^n Q_j(\zeta, z)d\zeta_j$, $\eta(\zeta, \lambda, z) = (1 - \lambda)\frac{\eta_0(\zeta,z)}{|\zeta-z|^2} + \lambda\frac{\eta_1(\zeta,z)}{S(\zeta,z)}$.

For $0 \le q \le n - 1$, set

$$\Omega_{n,q}(\eta) = \frac{(-1)^{\frac{q(q-1)}{2}}}{(2i\pi)^n}\binom{n-1}{q}\eta \wedge (\overline{\partial}_{\zeta,\lambda}\eta)^{n-q-1} \wedge (\overline{\partial}_z\eta)^q,$$

and if $q = -1, n$, $\Omega_{n,-1}(\eta) = \Omega_{n,n}(\eta) = 0$. We denote by $B_{n,q}$ the component of the Bochner-Martinelli kernel of bidegree $(0, q)$ in z and $(n, n-q-1)$ in ζ. The operator T_q from theorem 1 is defined for $f \in C^0_{0,q}(\overline{D})$ and $z \in D$ by

$$T_q f(z) := \int_{bD\times[0,1]} f(\zeta) \wedge \Omega_{n,q-1}(\eta)(\zeta, \lambda, z) - \int_D f(\zeta) \wedge B_{n,q-1}(\zeta, z).$$

We modify T_q as I. Lieb and R.M. Range have done in [7]. To do so, we need a Seeley type operator (see [7] or [12] for details). We set $G := \mathcal{V} \setminus D$, $\mathcal{V}$ given by the theorem 1.

Lemma 1. *There exists a linear extension operator $E : C(\overline{D}) \to C(G \cup D)$ such that*

i) $Eu|_{\overline{D}} = u$ for all $u \in C(\overline{D})$ and Eu has a compact support in $G \cup D$,
ii) for all $k \in \mathbb{N}$ and $u \in C^k(\overline{D})$, Eu belongs to $C^k(G \cup D)$ and there exists a constant $c_k > 0$, not depending on u, such that $\|Eu\|_{G\cup D,k} \le c_k\|u\|_{\overline{D},k}$.

We set $\iota_1 : \begin{cases} \mathbb{C}^n \times \{1\} \times \mathbb{C}^n \to \mathbb{C}^n \times [0,1] \times \mathbb{C}^n \\ (\zeta, \lambda, z) \mapsto (\zeta, \lambda, z) \end{cases}$, and $K_{n,q} = \iota_1^*(\Omega_{n,q}(\eta))$. For all $z \in D$ and all $(0, q)$-form f we define

$$M_q(f)(z) = \overline{\partial}_z \int_{G\times[0,1]} Ef(\zeta) \wedge \Omega_{n,q-2}(\eta)(\zeta, \lambda, z), \qquad \text{if } 2 \le q \le n-1,$$
$$= \int_G Ef(\zeta) \wedge K_{n,0}(\zeta, z), \qquad \text{if } q = 1.$$

At last we define T_q^* by $T_q^* := T_q - M_q$.

Since $K_{n,0}$ is holomorphic with respect to z, $\overline{\partial}_z T_1^* = \overline{\partial}_z T_1$. For $q > 1$, $M_q(f)$ is obviously $\overline{\partial}_z$-closed so $\overline{\partial}_z T_q^* f = \overline{\partial}_z T_q f$ thus (i) of theorem 2 holds for $q = 1, \dots, n-1$.

Because T_q already satisfies C^0-estimates (see theorem 1), C^0-estimates for T_q^* will be proved if we show that $M_q f$ belongs to $C^{\frac{1}{m}}_{0,q-1}(\overline{D})$ and satisfies $\|M_q f\|_{\overline{D},\frac{1}{m}} \lesssim \|f\|_{\overline{D},0}$ uniformly with respect to $f \in C^0_{0,q}(\overline{D})$.

In order to prove C^k-estimates for $k > 0$, we use Stokes theorem and get as in [7] for all $\overline{\partial}$-closed $f \in C^k_{0,q}(\overline{D})$, $k \ge 1$, and all z in D

$$T_q^* f(z) = -\int_{G\times[0,1]} \overline{\partial}_\zeta (Ef)(\zeta) \wedge \Omega_{n,q-1}(\eta)(\zeta, \lambda, z) - \int_{G\cup D} Ef(\zeta) \wedge B_{n,q-1}(\zeta, z). \tag{4}$$

Since Ef has compact support in $G \cup D$, $\int_{G\cup D} Ef(\zeta) \wedge B_{n,q-1}(\zeta, \cdot)$ belongs to $C^{k+\varepsilon}_{0,q-1}(\overline{D})$ and satisfies $\left\|\int_{G\cup D} Ef(\zeta) \wedge B_{n,q-1}(\zeta, \cdot)\right\|_{\overline{D},k+\varepsilon} \lesssim \|f\|_{\overline{D},k}$, for all $\varepsilon \in [0, 1[$, uniformly with respect to f. Therefore it suffices to prove that $T'_q f := -\int_{G\times[0,1]} \overline{\partial}_\zeta (Ef) \wedge \Omega_{n,q-1}(\eta)$ belongs to $C^{k+\frac{1}{m}}_{0,q-1}(\overline{D})$. We will prove that

$\frac{\partial T'_q f}{\partial \overline{z}_l}$ and $\frac{\partial T'_q f}{\partial z_l}$ belong to $C^{k-1+\frac{1}{m}}_{0,q-1}(\overline{D})$ for all $\overline{\partial}$-closed $f \in C^k_{0,q}(\overline{D})$, $k \geq 1$, and $l = 1, \dots, n$.

In order to show this result, as in [4], we fix some point z_0 near bD, a sufficiently small $\varepsilon > 0$ and denote by $w_1^*, \dots, w_n^*$ an ε-extremal basis at z_0. $\zeta^* = (\zeta_1^*, \dots, \zeta_n^*)$ will denote the ε-extremal coordinates at z_0 of a point ζ and Φ_* the unitary transformation such that $\zeta^* = \Phi_*(\zeta - z_0)$. We want to get estimates of the Hefer section in terms of the following complex directional level distances

$$\tau(\zeta, v, \varepsilon) := \sup\{\tau, r(\zeta + \lambda v) - r(\zeta) < \varepsilon \quad \text{for all } \lambda \in \mathbb{C},\ |\lambda| < \tau\}$$

(see [10]). To do so we choose for ζ in G a unitary matrix $\Psi(\zeta)$ such that $\Psi(\zeta)\Phi_*\eta_\zeta = (1, 0, \dots, 0)$ and in (1), (2) and (3) we set $U = \overline{\Psi(\zeta)\Phi_*}^t$ and we express the kernel in the ε-extremal basis by setting $Q^*(\zeta, z) := \overline{\Phi}_* Q(\zeta, z)$. Thus we have $\eta_1(\zeta, z) = \sum_{i=1}^n Q_i^*(\zeta, z) d\zeta_i^*$ and $\overline{\partial}_\zeta \eta_1(\zeta, z) = \sum_{i,j=1}^n \frac{\partial Q_i^*}{\partial \overline{\zeta}_j^*}(\zeta, z) d\overline{\zeta}_j^* \wedge d\zeta_i^*$.

We write $\tau_i(z_0, \varepsilon) = \tau(z_0, w_i^*, \varepsilon)$, $i = 1, \dots, n$, and set $\mathcal{P}_\varepsilon(z_0) := \{\zeta \in \mathbb{C}^n, |\zeta_i^*| < \tau_i(z_0, \varepsilon),\ i = 1, \dots n\}$. Then we should use properties of the ε-extremal basis summarized in [4] to estimate Q^*.

For $\zeta \in \mathcal{P}_\varepsilon(z_0)$ K. Diederich, B. Fischer and J.E. Fornæss obtained the estimate $\left|\frac{\partial Q_2^*}{\partial \overline{\zeta}_1^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon}{\tau_1(z_0,\varepsilon)\tau_2(z_0,\varepsilon)}$. However, when ε tends to 0, $\frac{\varepsilon}{\tau_1(z_0,\varepsilon)\tau_2(z_0,\varepsilon)}$ goes to infinity. This estimation does not matter when integration is over bD because $d\overline{\zeta}_1^*$ is the normal component in ζ of the kernel and does not play any role. When the domain of integration is G it is impossible to conclude as in [4]. It turns out that a factor ε is missing, even if $\frac{\partial Q_i^*}{\partial \overline{\zeta}_1^*}$ is only estimated by a constant. In order to improve the estimates of $\frac{\partial Q_i^*}{\partial \overline{\zeta}_1^*}$ we have to generalise the estimates of the tangential derivatives of r given in the proposition 3.1 (vii) of [4] also to normal derivatives.

3. Normal derivatives

As in [4], for real numbers A and B, maybe depending on some parameters, we write $A \lesssim B$ if there exists a constant $c > 0$ such that $A \leq cB$ and $A \eqsim B$ if $A \lesssim B$ and $B \lesssim A$. Each time, we specify on which parameters c depends.

We begin with a lemma in $\mathbb{C}^2$ that we generalize later on to $\mathbb{C}^n$, $n \geq 2$. For $w = (w_1, w_2) \in \mathbb{C}^2$, we set $x_j = \Re w_j$ and $y_j = \Im w_j$, $j = 1, 2$ and for $Q(z) = \sum_{j=0}^N \sum_{k+l=j} q_{kl} z^k \overline{z}^l$ we define $\|Q\| := \sum_{j=0}^N \sum_{k+l=j} |q_{kl}|$.

Lemma 2. *Let ρ_0 be C^∞ convex function defined on a neighborhood of $\overline{B(0,1)} \subset \mathbb{C}^2$. We assume that $\rho_0(w) = \rho_0(0) + \frac{\partial \rho_0}{\partial x_1}(0) x_1 + P_{2r_0}(w_2) + R_0'(w)$ where $P_{2r_0} \not\equiv 0$ is an homogeneous polynomial of even degree $2r_0 > 0$ and R_0' satisfies*

$$|R_0'(w)| \leq C(|w_1|^2 + |w_1 w_2| + |w_2|^{2r_0+1}) \quad \forall w \in \overline{B(0,1)}.$$

For any integer $m' \geq 2r_0$, there exist $s, c > 0$ such that for all C^∞ convex functions $\tilde{\rho}$ defined on a neighborhood of $\overline{B(0,1)}$ with $\|\tilde{\rho} - \rho_0\|_{\overline{B(0,1)}, m'+3} < s$ and

$\tilde{\rho}(w) = \tilde{\rho}(0) + \frac{\partial\tilde{\rho}}{\partial x_1}(0)x_1 + \tilde{R}(w)$ where $\tilde{R}(0) = 0$ and $grad\,\tilde{R}(0) = 0$, the following inequalities hold

$$\left|\frac{\partial\tilde{\rho}}{\partial x_1}(w) - \frac{\partial\tilde{\rho}}{\partial x_1}(0)\right| + \left|\frac{\partial\tilde{\rho}}{\partial y_1}(w)\right| \leq c\left(|w_1| + \sqrt{\sum_{j=2}^{m'} \|\tilde{P}_j\| |w_2|^j}\right)$$

where $w = (w_1, w_2) \in \overline{B(0,1)}$ and $\tilde{P}_j(w_2) = \sum_{\substack{0\leq k,l\leq j\\ k+l=j}} \frac{1}{k!l!}\frac{\partial^j\tilde{\rho}}{\partial w_2^k\partial\overline{w_2}^l}(0)w_2^k\overline{w}_2^l$.

Remark 1. The condition on R'_0 only means that P_{2r_0} is the first non zero term of the Taylor expansion at 0 of $R_0(w_1,\cdot) = \rho_0(w_1,\cdot) - \rho_0(0) - x_1\frac{\partial\rho_0}{\partial x_1}(0)$ and that $R'_0(w) = R_0(w_1, w_2) - P_{2r_0}(w_2)$. This condition will be fulfilled if ρ_0 is the C^∞ defining function of a convex domain of finite type $2r_0$ in $\mathbb{C}^2$. m' is needed for the generalization to higher dimensions.

Proof of lemma 2. $\left|\frac{\partial\tilde{\rho}}{\partial x_1}(w) - \frac{\partial\tilde{\rho}}{\partial x_1}(0)\right|$ and $\left|\frac{\partial\tilde{\rho}}{\partial y_1}(w)\right|$ can be estimated by the same method. We only estimate $\left|\frac{\partial\tilde{\rho}}{\partial x_1}(w) - \frac{\partial\tilde{\rho}}{\partial x_1}(0)\right|$ with all details.
We set $s = \frac{\|P_{2r_0}\|}{2} > 0$ and choose a C^∞ convex function $\tilde{\rho}$ defined in a neighborhood of $\overline{B(0,1)}$ such that $\|\tilde{\rho} - \rho_0\|_{\overline{B(0,1)},m'+3} < s$ and $\tilde{\rho}(w) = \tilde{\rho}(0) + \frac{\partial\tilde{\rho}}{\partial x_1}(0)x_1 + \tilde{R}(w)$ with $\tilde{R}(0) = 0$ and $grad\,\tilde{R}(0) = 0$.
For $j \geq 2$ we set $\tilde{P}_j(w_2) = \sum_{k+l=j}\frac{1}{k!l!}\frac{\partial^j\tilde{\rho}}{\partial w_2^k\partial\overline{w_2}^l}(0)w_2^k\overline{w_2}^l$ and

$$R_1(w) = \frac{\partial\tilde{R}}{\partial x_1}(0, w_2) + \int_0^1 (1-t)\left(x_1\frac{\partial^2\tilde{R}}{\partial x_1^2}(tw_1, w_2) + y_1\frac{\partial^2\tilde{R}}{\partial x_1\partial y_1}(tw_1, w_2)\right)dt,$$

$$\tilde{R}_1(w) = \frac{\partial\tilde{R}}{\partial y_1}(0, w_2) + \int_0^1 (1-t)\left(x_1\frac{\partial^2\tilde{R}}{\partial x_1\partial y_1}(tw_1, w_2) + y_1\frac{\partial^2\tilde{R}}{\partial y_1^2}(tw_1, w_2)\right)dt.$$

An integration by parts leads to $\tilde{R}(w) = x_1R_1(w) + y_1\tilde{R}_1(w) + \tilde{R}(0, w_2)$, so

$$\frac{\partial\tilde{\rho}}{\partial x_1}(w) = \frac{\partial\tilde{\rho}}{\partial x_1}(0) + R_1(w) + x_1\frac{\partial R_1}{\partial x_1}(w) + y_1\frac{\partial\tilde{R}_1}{\partial x_1}(w). \tag{5}$$

The derivatives of $\tilde{R}_1$ and R_1 are bounded in $\overline{B(0,1)}$ independently of $\tilde{\rho}$ because $\|\tilde{\rho} - \rho_0\|_{\overline{B(0,1)},m'+3} < s$. Therefore we have

$$\left|x_1\frac{\partial R_1}{\partial x_1}(w) + y_1\frac{\partial\tilde{R}_1}{\partial x_1}(w)\right| \lesssim |w_1|. \tag{6}$$

We estimate R_1 using the convexity of $\tilde{\rho}$.

We fix $v_2 \in \mathbb{C}$ such that $|v_2| = 1$. For $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $|\alpha_1|^2 + |\alpha_2|^2 \leq 1$, we set $\tilde{\rho}_{v_2}(\alpha_1, \alpha_2) := \tilde{\rho}(\alpha_1, \alpha_2 v_2)$. Since $\tilde{\rho}_{v_2}$ is convex we have

$$\left(\frac{\partial^2 \tilde{\rho}_{v_2}}{\partial \alpha_1^2}\right)\left(\frac{\partial^2 \tilde{\rho}_{v_2}}{\partial \alpha_2^2}\right) - \left(\frac{\partial^2 \tilde{\rho}_{v_2}}{\partial \alpha_1 \partial \alpha_2}\right)^2 \geq 0. \tag{7}$$

We compute and estimate each term of this inequality.

$$\left|\frac{\partial^2 \tilde{\rho}_{v_2}}{\partial \alpha_1^2}(0, \alpha_2)\right| \lesssim 1, \tag{8}$$

uniformly with respect to $\tilde{\rho}$ because $\|\tilde{\rho}\|_{\overline{B(0,1)},2} \leq \|\tilde{\rho}_0\|_{\overline{B(0,1)},2} + \frac{1}{2}\|P_{2r_0}\|$.

$$\left|\frac{\partial^2 \tilde{\rho}_{v_2}}{\partial \alpha_1 \partial \alpha_2}(0, \alpha_2)\right| = \left|\frac{\partial R_1(0, \alpha_2 v_2)}{\partial \alpha_2}\right|. \tag{9}$$

To estimate $\frac{\partial^2 \tilde{\rho}_{v_2}}{\partial \alpha_2^2}$, we first expand $\tilde{\rho}$ in Taylor series at 0 and get

$$\tilde{\rho}(w_1, w_2) = \sum_{j=0}^{2r_0} \sum_{|I|+|J|=j} \frac{1}{I!J!} \frac{\partial^j \tilde{\rho}}{\partial w^I \partial \overline{w}^J}(0) w^I \overline{w}^J + \tilde{R}'(w_1, w_2)$$

and

$$\frac{\partial^2 \tilde{\rho}_{v_2}}{\partial \alpha_2^2}(0, \alpha_2) = \sum_{j=2}^{2r_0} \sum_{k+l=j} \frac{j(j-1)}{k!l!} \alpha_2^{j-2} \frac{\partial^j \tilde{\rho}}{\partial w_2^k \partial \overline{w_2}^l}(0) v_2^k \overline{v_2}^l + \frac{\partial^2 \tilde{R}'(0, \alpha_2 v_2)}{\partial \alpha_2^2}. \tag{10}$$

Since the derivatives up to order $2r_0 + 3$ of $\tilde{\rho}$ are uniformly bounded we have $\left|\frac{\partial^2 \tilde{R}'(0,\alpha_2 v_2)}{\partial \alpha_2^2}\right| \lesssim |\alpha_2|^{2r_0-1}$ for all $\alpha_2 \in [0, 1]$. Since $\|\tilde{\rho} - \rho_0\|_{\overline{B(0,1)},m'+3} < \frac{\|P_{2r_0}\|}{2}$ we have $\|\tilde{P}_{2r_0}\| \geq \frac{\|P_{2r_0}\|}{2} > 0$ and $\left|\frac{\partial^2 \tilde{R}'(0,\alpha_2 v_2)}{\partial \alpha_2^2}\right| \lesssim \|\tilde{P}_{2r_0}\||\alpha_2|^{2r_0-2}$. Using (10) we get $\left|\frac{\partial^2 \tilde{\rho}_{v_2}}{\partial \alpha_2^2}(0, \alpha_2)\right| \lesssim \sum_{j=2}^{2r_0} \|\tilde{P}_j\||\alpha_2|^{j-2}$, which with (7), (8) and (9) gives

$$\left|\frac{\partial R_1(0, \alpha_2 v_2)}{\partial \alpha_2}\right| \lesssim \sum_{j=2}^{2r_0} \sqrt{\|\tilde{P}_j\|} |\alpha_2|^{\frac{j}{2}-1} \tag{11}$$

for all $\alpha_2 \in [0, 1]$, uniformly with respect to $\tilde{\rho}$ and v_2.

We integrate (11). Since $R_1(0) = \frac{\partial \tilde{R}}{\partial x_1}(0) = 0$ we have $|R_1(0, \alpha v_2)| \lesssim \sqrt{\sum_{j=2}^{m'} \|\tilde{P}_j\||\alpha|^j}$ for all $\alpha \in [0, 1]$. Since this inequality holds for all v_2 such that $|v_2| = 1$ we have uniformly with respect to $\tilde{\rho}$ and for all $w_2 \in \mathbb{C}$ with $|w_2| \leq 1$

$$|R_1(0,w_2)| \lesssim \sqrt{\sum_{j=2}^{m'} \|\tilde{P}_j\| |w_2|^j}. \tag{12}$$

Also we have uniformly with respect to $\tilde{\rho}$ $|R_1(w_1,w_2) - R_1(0,w_2)| \lesssim |w_1|$. Plugging this inequality with (6) and (12) into (5), we finally get

$$\left|\frac{\partial\tilde{\rho}}{\partial x_1}(w) - \frac{\partial\tilde{\rho}}{\partial x_1}(0)\right| \lesssim |w_1| + \sqrt{\sum_{j=2}^{m'} \|\tilde{P}_j\| |w_2|^j}$$

uniformly with respect to $\tilde{\rho}$ and w. □

We now extend lemma 2 to a convex domain $D \subset \mathbb{C}^n$. We recall that $\mathcal{V}$ is a bounded neighborhood of bD, that for $\alpha \in \mathbb{R}$, $D_\alpha = \{\zeta \in \mathbb{C}^n, r(\zeta) < \alpha\}$, $T^{\mathbb{C}}_\zeta bD_{r(\zeta)}$ is the complex tangent space in ζ to $bD_{r(\zeta)}$ and η_ζ the outer unit normal at ζ to $bD_{r(\zeta)}$.

Proposition 1. *There exists a constant $c > 0$ such that for all $\zeta \in \mathcal{V}$ and all $z = \zeta + w_1\eta_\zeta + w_2 v$, with v a unit vector in $T^{\mathbb{C}}_\zeta bD_{r(\zeta)}$, $w_j = x_j + iy_j \in \mathbb{C}$, $j = 1,2$, with $|w_1|^2 + |w_2|^2 \leq 1$, the following inequality holds*

$$\left|\frac{\partial r(\zeta + w_1\eta_\zeta + w_2 v)}{\partial y_1}\right| + \left|\frac{\partial r(\zeta + w_1\eta_\zeta + w_2 v)}{\partial x_1} - \left.\frac{\partial r(\zeta + w_1\eta_\zeta)}{\partial x_1}\right|_{w_1=0}\right|$$
$$\leq c\left(|w_1| + \sqrt{\sum_{j=2}^{m} \sum_{\substack{0\leq k,l\leq j\\ k+l=j}} \frac{1}{k!l!} \left|\left.\frac{\partial^j r(\zeta + w_2 v)}{\partial w_2^k \partial \overline{w_2}^l}\right|_{w_2=0}\right| |w_2|^j}\right). \tag{13}$$

Proof. We fix $\zeta_0 \in \overline{\mathcal{V}}$, $v_0 \in T^{\mathbb{C}}_{\zeta_0} bD_{r(\zeta_0)}$, $|v_0| = 1$ and set $\rho_{\zeta_0,v_0}(w_1,w_2) := r(\zeta_0 + w_1\eta_{\zeta_0} + w_2 v_0)$. Lemma 2 applied to ρ_{ζ_0,v_0} give us two constants c_{ζ_0,v_0} and s_{ζ_0,v_0}. Since $r \in C^\infty(\mathbb{C}^n)$, there exist a neighborhood $V_{\zeta_0,v_0}(\zeta_0)$ of ζ_0 and a neighborhood $V_{\zeta_0,v_0}(v_0)$ of v_0 in $\mathbb{C}^n$ such that for all $\zeta \in V_{\zeta_0,v_0}(\zeta_0)$ and all $v \in V_{\zeta_0,v_0}(v_0) \cap T^{\mathbb{C}}_\zeta bD_{r(\zeta)}$, the convex function $\rho_{\zeta,v} := r(\zeta + w_1\eta_\zeta + w_2 v)$ satisfies $\|\rho_{\zeta,v} - \rho_{\zeta_0,v_0}\|_{B(0,1),m+3} < s_{\zeta_0,v_0}$. According to lemma 2 the inequality (13) holds with $c = c_{\zeta_0,v_0}$ for all ζ in $V_{\zeta_0,v_0}(\zeta_0)$ and all v in $V_{\zeta_0,v_0}(v_0) \cap T^{\mathbb{C}}_\zeta bD_{r(\zeta)}$.

The compactness argument used to prove the theorem 2.3 of [3] achieves the proof. □

We now translate the inequality (13) in terms of ε-extremal basis.

Lemma 3. *For $\varepsilon > 0$ sufficiently small, $z_0 \in \mathcal{V}$ and $w_1^*, \dots, w_n^*$ an ε-extremal basis at z_0 and $\zeta \in \mathcal{P}_\varepsilon(z_0)$ we have uniformly in z_0, ε and ζ*

$$\left|\frac{\partial r}{\partial w_1^*}(\zeta) - \frac{\partial r}{\partial w_1^*}(z_0)\right| + \left|\frac{\partial r}{\partial \overline{w}_1^*}(\zeta) - \frac{\partial r}{\partial \overline{w}_1^*}(z_0)\right| \lesssim \varepsilon^{\frac{1}{2}}.$$

Proof. In order to use proposition 1, we write $\zeta \in \mathcal{P}_\varepsilon(z_0)$ as $\zeta = z_0 + \lambda\eta_{z_0} + \mu v$, $v \in T^{\mathbb{C}}_{z_0} bD_{r(z_0)}$, $|v| = 1$. Since $w_1^* = \eta_{z_0}$, we have $|\lambda| \lesssim \varepsilon$. Moreover since $z_0 + \mu v$ is also in $\mathcal{P}_\varepsilon(z_0)$, $z_0 + \frac{1}{2^{2n}}\mu v$ belongs to $\frac{1}{2^{2n}}\mathcal{P}_\varepsilon(z_0)$ which is included in $D_{r(z_0)+\varepsilon}$ (see [10], proposition 3.1), thus $\frac{|\mu|}{2^{2n}} \leq \tau(z_0, v, \varepsilon)$. Since $\sum_{j=2}^m \sum_{\substack{0\leq k,l\leq j\\ k+l=j}} \frac{1}{k!l!} \left|\left.\frac{\partial^j r(z_0+w_2 v)}{\partial w_2^k \partial \overline{w_2}^l}\right|_{w_2=0}\right| \tau(z_0, v, \varepsilon)^j \approx \varepsilon$ (see [4], proposition 3.1 (vi)), the proposition 1 gives $\left|\frac{\partial r}{\partial w_1^*}(\zeta) - \frac{\partial r}{\partial w_1^*}(z_0)\right| + \left|\frac{\partial r}{\partial \overline{w}_1^*}(\zeta) - \frac{\partial r}{\partial \overline{w}_1^*}(z_0)\right| \lesssim \varepsilon^{\frac{1}{2}}$. □

In [4], proposition 3.1 (vii) applied to ε-extremal bases has been a major tool to prove Hölder estimates. In order to make this method applicable we need to reformulate lemma 3 in the following way.

Corollary 1. *For all $z_0 \in \mathcal{V}$, all sufficiently small ε, all $\zeta \in \mathcal{P}_\varepsilon(z_0)$ and all multi-indices α and β with $|\alpha| + |\beta| \geq 2$ we have, uniformly in z_0, ε and ζ,*

$$\left|\frac{\partial^{|\alpha|+|\beta|} r}{\partial w^{*\alpha} \partial \overline{w^*}^\beta}(\zeta)\right| \lesssim \frac{\varepsilon}{\prod_{i=1}^n \tau_i'(z_0, \varepsilon)^{\alpha_i+\beta_i}},$$

where $\tau_i'(z_0, \varepsilon) = \tau_i(z_0, \varepsilon)$ if $i \neq 1$ and $\tau_1'(z_0, \varepsilon) = \varepsilon^{\frac{1}{2}}$.

Remark 2. When $\alpha_1 + \beta_1 = 1$ this corollary improves the estimates given in proposition 3.1 (vii) of [4] by the gain of a factor $\varepsilon^{\frac{1}{2}}$.

Proof of corollary 1. The case $\alpha_1 + \beta_1 > 1$ is obvious because $\frac{\partial^{|\alpha|+|\beta|} r}{\partial w^{*\alpha} \partial \overline{w^*}^\beta}(\zeta)$ is bounded and $\frac{\varepsilon}{\prod_{i=1}^n \tau_i'(z_0,\varepsilon)^{\alpha_i+\beta_i}}$ is bounded away from 0.

The case $\alpha_1 + \beta_1 = 0$ follows from the proposition 3.1 (vii) and (iv) of [4].

So we assume that $\alpha_1 = 1$ and $\beta_1 = 0$ (the case $\alpha_1 = 0$, $\beta_1 = 1$ is analogous). We expand $\frac{\partial r}{\partial w_1^*}$ in a Taylor series up to order $\frac{m}{2} + |\alpha| + |\beta| - 1$ at z_0

$$\frac{\partial r}{\partial w_1^*}(\zeta) - \frac{\partial r}{\partial w_1^*}(z_0) = \sum_{j=1}^{|\alpha|+|\beta|+\frac{m}{2}-1} \sum_{|\alpha'|+|\beta'|=j} \frac{\partial^{j+1} r}{\partial w_1^* \partial w^{*\alpha'} \partial \overline{w^*}^{\beta'}}(z_0)\zeta^{*\alpha'}\overline{\zeta^*}^{\beta'} + o(|\zeta^*|^{|\alpha|+|\beta|+\frac{m}{2}-1}).$$

If $\zeta \in \mathcal{P}_\varepsilon(z_0)$ $|\zeta - z_0| \lesssim \varepsilon^{\frac{1}{m}}$, so we have $|o(|\zeta^*|^{|\alpha|+|\beta|+\frac{m}{2}-1})| \lesssim \varepsilon^{\frac{1}{2}}$ uniformly in z_0, ζ and ε. Lemma 3 implies that for all $\zeta \in \mathcal{P}_\varepsilon(z_0)$, $\varepsilon > 0$ small enough,

$$\left|\sum_{j=1}^{|\alpha|+|\beta|+\frac{m}{2}-1} \sum_{|\alpha'|+|\beta'|=j} \frac{\partial^{j+1} r}{\partial w_1^* \partial w^{*\alpha'} \partial \overline{w^*}^{\beta'}}(z_0)\zeta^{*\alpha}\overline{\zeta^*}^{\beta'}\right| \lesssim \varepsilon^{\frac{1}{2}}.$$

By setting $\xi_i := \frac{\zeta_i^*}{\tau_i(z_0,\varepsilon)}$, we normalize and get

$$\left|\sum_{j=1}^{|\alpha|+|\beta|+\frac{m}{2}-1} \sum_{|\alpha'|+|\beta'|=j} \frac{\partial^{j+1} r}{\partial w_1^* \partial w^{*\alpha'} \partial \overline{w^*}^{\beta'}}(z_0)\xi^{\alpha'}\overline{\xi}^{\beta'} \prod_{i=1}^n \tau_i(z_0, \varepsilon)^{\beta_i'+\alpha_i'}\right| \lesssim \varepsilon^{\frac{1}{2}},$$

for all $\xi \in \mathbb{C}^n$ which satisfy $|\xi_i| \le 1$, $i = 1, \dots, n$.

For a polynomial $P(\xi) = \sum_{|\alpha'|+|\beta'|\le|\alpha|+|\beta|+\frac{m}{2}-1} p_{\alpha',\beta'}\xi^{\alpha'}\overline{\xi}^{\beta'}$, we set $\|P\|_\star = \sup_{|\xi_1|,\dots,|\xi_n|\le 1} |P(\xi_1,\dots,\xi_n)|$ and $\|P\|_{\star\star} = \sup_{|\alpha'|+|\beta'|\le|\alpha|+|\beta|+\frac{m}{2}-1} |p_{\alpha',\beta'}|$.

Since $\|\cdot\|_\star$ and $\|\cdot\|_{\star\star}$ are two equivalent norms on the vector space of polynomials of degree at most $|\alpha|+|\beta|+\frac{m}{2}-1$, this implies for all α' and β' with $1 \le |\alpha'|+|\beta'| \le |\alpha|+|\beta|+\frac{m}{2}-1$

$$\left|\frac{\partial^{|\alpha'|+|\beta'|+1} r}{\partial w_1^* \partial w^{*\alpha'} \partial \overline{w^*}^{\beta'}}(z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\prod_{i=1}^n \tau_i(z_0,\varepsilon)^{\beta_i'+\alpha_i'}}. \tag{14}$$

Next, we compute the Taylor expansion of $\frac{\partial^{|\alpha|+|\beta|} r}{\partial w^{*\alpha}\partial \overline{w^*}^{\beta}}$ at z_0 of order $\frac{m}{2}$

$$\begin{aligned}\frac{\partial^{|\alpha|+|\beta|} r}{\partial w^{*\alpha}\partial \overline{w^*}^{\beta}}(\zeta) \\ &= \sum_{0\le|\alpha'|+|\beta'|\le\frac{m}{2}} \frac{\partial^{|\alpha|+|\alpha'|+|\beta|+|\beta'|} r}{\partial w^{*\alpha'+\alpha}\partial \overline{w^*}^{\beta'+\beta}}(z_0)\zeta^{*\alpha'}\overline{\zeta^*}^{\beta'} + o(|\zeta^*|^{\frac{m}{2}}).\end{aligned}$$

Inequality (14) hields to $\left|\frac{\partial^{|\alpha|+|\beta|+|\alpha'|+|\beta'|} r}{\partial w^{*\alpha+\alpha'}\partial \overline{w^*}^{\beta+\beta'}}(\zeta)\right| \lesssim \frac{\varepsilon}{\prod_{i=1}^n \tau_i'(z_0,\varepsilon)^{\alpha_i+\alpha_i'+\beta_i+\beta_i'}}$ for all α' and β' with $|\alpha'|+|\beta'|+|\alpha|+|\beta| \le \frac{m}{2}$. Using $|\zeta_j^*| \le \tau_j(z_0,\varepsilon)$, $j = 1,\dots,n$ and $o(|\zeta^*|^{\frac{m}{2}}) \lesssim \varepsilon^{\frac{1}{2}}$ for all $\zeta \in \mathcal{P}_\varepsilon(z_0)$, we finally get $\left|\frac{\partial^{|\alpha|+|\beta|} r}{\partial w^{*\alpha}\partial \overline{w^*}^{\beta}}(\zeta)\right| \lesssim \frac{\varepsilon}{\prod_{i=1}^n \tau_i'(z_0,\varepsilon)^{\alpha_i+\beta_i}}$. □

4. Estimates of the Hefer-Leray section and conclusion

Corollary 1 will give us a gain of a factor $\varepsilon^{\frac{1}{2}}$ in the estimate of $\frac{\partial Q_i^*}{\partial \overline{\zeta_1^*}}$. Because S is holomorphic with respect to z, with this new factor $\varepsilon^{\frac{1}{2}}$ we will be able to prove that $\frac{\partial T_q' f}{\partial \overline{z}_l}$ belongs to $C_{0,q-1}^{k-1+\frac{1}{m}}(\overline{D})$ for all l and all $\overline{\partial}$-closed $f \in C_{0,q}^k(\overline{D})$, $k \ge 1$. We also have to prove that $\frac{\partial T_q'}{\partial z_l}$ belongs to $C_{0,q-1}^{k-1+\frac{1}{m}}(\overline{D})$. We can not directly succeed in showing this and we have to integrate by parts as J. Michel in [8]. In order to integrate by parts we set $\delta_l = \frac{\partial}{\partial \zeta_l} + \frac{\partial}{\partial z_l}$, $l = 1,\dots,n$. We have for all $\overline{\partial}$-closed $f \in C_{0,q}^1(\overline{D})$

$$\begin{aligned}\frac{\partial T_q' f}{\partial z_l} &= \int_{G\times[0,1]} \overline{\partial}_\zeta Ef \wedge \delta_l \Omega_{n,q-1}(\eta) - \int_{G\times[0,1]} \overline{\partial}_\zeta Ef \wedge \frac{\partial}{\partial \zeta_l}\Omega_{n,q-1}(\eta) \\ &= Y - X.\end{aligned}$$

Later on we will show that the action of δ_l to S or Q is comparable to that of $\frac{\partial}{\partial \overline{\zeta}_l}$. Thus Y will have good estimates. To treat X, as in [8], we use inner product, Stokes theorem and the hypothesis $\overline{\partial} f = 0$ on D to show that

$$X = \int_{bD\times[0,1]} \frac{\partial f}{\partial \zeta_l} \wedge \Omega_{n,q-1}(\eta) - \int_G \frac{\partial Ef}{\partial \zeta_l} \wedge K_{n,q-1} - \int_{G\times[0,1]} \frac{\partial Ef}{\partial \zeta_l} \wedge \overline{\partial}_z \Omega_{n,q-2}(\eta) + \int_G \frac{\partial Ef}{\partial \zeta_l} \wedge B_{n,q-1}.$$

This implies that for all $z \in D$

$$\begin{aligned} \frac{\partial T'_q f}{\partial z_l}(z) &= \int_G \left(\frac{\partial Ef}{\partial \zeta_l}(\zeta) - E\frac{\partial f}{\partial \zeta_l}(\zeta)\right) \wedge K_{n,q-1}(\zeta,z) - \int_{G\cup D} \frac{\partial Ef}{\partial \zeta_l}(\zeta) \wedge B_{n,q-1}(\zeta,z) \\ &+ \int_{G\times[0,1]} \left(\frac{\partial Ef}{\partial \zeta_l}(\zeta) - E\frac{\partial f}{\partial \zeta_l}(\zeta)\right) \wedge \overline{\partial}_z \Omega_{n,q-2}(\eta)(\zeta,\lambda,z) - T_q^*\left(\frac{\partial f}{\partial \zeta_l}\right)(z) \\ &+ \int_{G\times[0,1]} \overline{\partial}_\zeta Ef(\zeta) \wedge \delta_l \Omega_{n,q-1}(\eta)(\zeta,\lambda,z). \end{aligned} \tag{15}$$

In (15) we should notice that $\int_{G\times[0,1]}\left(\frac{\partial Ef}{\partial \zeta_l} - E\frac{\partial f}{\partial \zeta_l}\right) \wedge \overline{\partial}_z \Omega_{n,q-2}(\eta) = 0$ for $q = 1$. We also have $\int_G \left(\frac{\partial Ef}{\partial \zeta_l} - E\frac{\partial f}{\partial \zeta_l}\right) \wedge K_{n,q-1} = 0$ for $q \neq 1$ because, since S and Q are holomorphic with respect to z, $K_{n,q-1} = 0$ for all $q \neq 1$.

For $f \in C^k_{0,q}(\overline{D})$, $k > 0$, we will prove that each term in (15) belongs to $C^{k-1+\frac{1}{m}}_{0,q-1}(\overline{D})$. This result is already known for the Bochner-Martinelli term. For $T_q^*\left(\frac{\partial f}{\partial \zeta_l}\right)$ we use an induction argument. The other terms will be estimated with respect to an ε-extremal basis as in [4]. So we need estimates of $\delta_l S$ and $\delta_l Q$ in terms of ε-extremal bases.

We use the notations of the end of section 2 and fix some $z_0 \in D$ close enough to bD. When ζ in G is such that $|\zeta - z_0| \geq \varepsilon_0 > 0$, $S(\zeta, z_0)$ is bounded away from 0 so we just have to integrate on a small polydisc $\mathcal{P}_{\varepsilon_0}(z_0)$. We choose ε_0 sufficiently small so that for all $z \in D$ sufficiently close to bD, all $\varepsilon \in]0, \varepsilon_0]$ and all $\zeta \in \mathcal{P}_\varepsilon(z)$, we have $|\zeta - z| \leq \frac{R}{2}$, where R is given by theorem 3.

As in [4] we cover $\mathcal{P}_{\varepsilon_0}(z_0)$ with some polyannuli based on McNeal's polydiscs. For sufficiently small $\varepsilon > 0$ we set $\mathcal{P}^i_\varepsilon(z_0) := \mathcal{P}_{2^{-i}\varepsilon}(z_0) \setminus c_1\mathcal{P}_{2^{-i}\varepsilon}(z_0)$, where c_1 given by proposition 3.1 (i) of [4] is such that for all $\varepsilon > 0$ and all $i \in \mathbb{N}$ $c_1\mathcal{P}_{2^{-i}\varepsilon}(z_0) \subset \mathcal{P}_{2^{-1}(2^{-i}\varepsilon)}(z_0)$. This gives us the following covering

$$\mathcal{P}_{\varepsilon_0}(z_0) \subset \mathcal{P}_{|r(z_0)|}(z_0) \cup \bigcup_{i=0}^{j_0} \mathcal{P}^i_{\varepsilon_0}(z_0) \tag{16}$$

where j_0 satisfies $2^{-j_0}\varepsilon_0 \eqsim |r(z_0)|$, uniformly in z_0 and ε_0.

Now, we fix an ε in $]0, \varepsilon_0]$ and we choose an ε-extremal basis $w_1^*, \ldots, w_n^*$ at z_0 and assume that z_0 is close enough to the boundary and ε_0 small enough so that $\left|\frac{\partial r}{\partial w_1^*}(\zeta)\right| \geq c > 0$ for all $\zeta \in \mathcal{P}_\varepsilon(z_0)$. In [4] the support function S was only estimated for z in D and ζ in bD, but we need estimates of S when z belongs to D and ζ to G :

Lemma 4. *i) For all ζ in $\mathbb{C}^n$ such that $r(\zeta) \geq r(z_0)$ we have, uniformly in z_0 and ζ,*

$$|S(\zeta, z_0)| \gtrsim r(\zeta) - r(z_0).$$

ii) For sufficiently small ε and for $\zeta \in \mathcal{P}^0_\varepsilon(z_0)$ with $r(\zeta) \geq r(z_0)$ the following inequality holds uniformly in z_0, ε and ζ

$$|S(\zeta, z_0)| \gtrsim \varepsilon + r(\zeta) - r(z_0).$$

Proof. For $\zeta \in \mathcal{P}_\varepsilon(z_0)$ we have $S(\zeta, z_0) = A(\zeta, z_0)F(\zeta, z_0)$ and $|A(\zeta, z_0)| \geq \frac{1}{2}$. So it suffices to estimate $|F(\zeta, z_0)|$ and (i) is a straight forward consequence of the inequality satisfied by $-\Re F(\zeta, z_0)$. Now to prove (ii) we just have to show $|F(\zeta, z)| \gtrsim \varepsilon$. We assume for a moment the following.

Claim: Let $\zeta \in \mathcal{P}^0_\varepsilon(z_0)$ and $v \in T^{\mathbb{C}}_\zeta bD_{r(\zeta)}$, $|v| = 1$, such that $z_0 = \zeta + \mu v + \lambda\eta_\zeta$. Then for sufficiently small $\varepsilon > 0$ and sufficiently small $\tilde{c} > 0$, if $|\mu| < \tilde{c}\tau(\zeta, v, \varepsilon)$ we have $|\lambda| \eqsim \varepsilon$, uniformly with respect to z_0, ζ, ε and $\tilde{c}$.

Let ζ be in $\mathcal{P}^0_\varepsilon(z_0)$ such that $r(\zeta) \geq r(z_0)$. We first assume that $|\mu| < \tilde{c}\tau(\zeta, v, \varepsilon)$. The claim says that $|\lambda| \eqsim \varepsilon$ thus we have with the theorem 3 and the proposition 3.1 (vi) of [4]

$$\begin{aligned} |F(\zeta, z_0)| &\gtrsim |\Im F(\zeta, z)| + |\Re F(\zeta, z_0)| \\ &\gtrsim |\lambda| - |\lambda|^2 - K' \sum_{j=2}^{m} \sum_{\alpha+\beta=j} \left| \frac{\partial^j r(\zeta + \mu v)}{\partial\mu^\alpha \partial\overline{\mu}^\beta}\Big|_{\mu=0} \right| |\mu|^j \\ &\gtrsim \varepsilon(1 - \varepsilon - \tilde{c}), \end{aligned}$$

and if $\tilde{c}$ and ε are sufficiently small $|F(\zeta, z_0)| \gtrsim \varepsilon$.

Now, if $|\mu| \geq \tilde{c}\tau(\zeta, v, \varepsilon)$ by theorem 3 and proposition 3.1 (vi) of [4]

$$|F(\zeta, z_0)| \geq \frac{k'K'}{4} \sum_{j=2}^{m} \sum_{\alpha+\beta=j} \left| \frac{\partial^j r(\zeta + \mu v)}{\partial\mu^\alpha \partial\overline{\mu}^\beta}\Big|_{\mu=0} \right| |\mu|^j \gtrsim \varepsilon.$$

To conclude the proof of the lemma we prove the claim.

For ζ in $\mathcal{P}^0_\varepsilon(z_0)$ and $\tilde{c} > 0$ to be chosen in a moment, we write $z_0 = \zeta + \mu v + \lambda\eta_\zeta$ and assume that $|\mu| < \tilde{c}\tau(\zeta, v, \varepsilon)$. We denote by v^* and η^*_ζ the ε-extremal coordinates at z_0 of v and η_ζ respectively.

We first show that $|\lambda| \lesssim \varepsilon$. We have $\zeta^*_i = -\lambda(\eta_\zeta)^*_i - \mu v^*_i$. Since $(\eta_\zeta)^*_i = \frac{1}{|\partial r(\zeta)|} \frac{\partial r}{\partial w^*_i}(\zeta)$ according to proposition 3.1 (vii) and (iv) of [4], we have $|(\eta_\zeta)^*_i| \lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)}$ and $|\lambda| \lesssim \sum_{i=1}^n |\zeta^*_i||(\eta_\zeta)^*_i| \lesssim \varepsilon$.

For sufficiently small ε and $\tilde{c}$ and $i \neq 1$ we show that $|\zeta^*_i| < c_1\tau_i(z_0, \varepsilon)$.

On one hand, proposition 3.1 (iv) and (iii) of [4] lead to $\sum_{i=1}^n \frac{|\mu||v^*_i|}{\tau_i(z_0,\varepsilon)} \eqsim \frac{|\mu|}{\tau(\zeta,v,\varepsilon)}$, and if $|\mu| < \tilde{c}\tau(\zeta, v, \varepsilon)$, we have for all i $|\mu v^*_i| \lesssim \tilde{c}\tau_i(z_0, \varepsilon)$.

On the other hand, if $i \neq 1$, $\varepsilon^{\frac{1}{2}} \lesssim \tau_i(z_0, \varepsilon)$, so $|(\eta_\zeta)^*_i| \lesssim \tau_i(z_0, \varepsilon)$ and $|\lambda||(\eta_\zeta)^*_i| \lesssim \varepsilon\tau_i(z_0, \varepsilon)$.

Therefore we get $|\zeta_i^*| \lesssim (\varepsilon + \tilde{c})\tau_i(z_0, \varepsilon)$ for all $i \neq 1$. So, if we choose ε and $\tilde{c}$ sufficiently small, we have $|\zeta_i^*| < c_1\tau_i(z_0, \varepsilon)$ for all $i \neq 1$. Since $\zeta \notin c_1\mathcal{P}_\varepsilon(z_0)$ we must have $|\zeta_1^*| \geq c_1\tau_1(z_0, \varepsilon)$.

Since $|(\eta_\zeta)_1^*| \lesssim 1$ and $|\mu v_1^*| \lesssim \tilde{c}\tau_1(z_0, \varepsilon)$ uniformly in ζ, z_0 and ε, we have $|\lambda| \gtrsim c_1\tau_1(z_0, \varepsilon) - \tilde{c}\tau_1(z_0, \varepsilon)$. Since $\varepsilon \eqsim \tau_1(z_0, \varepsilon)$ (see proposition 3.1 (v) of [4]) we just have to choose $\tilde{c}$ sufficiently small again and the claim is true. □

It will be easier to study δ_l with respect to an ε-extremal basis. Therefore we put $\delta_j^* := \frac{\partial}{\partial z_j^*} + \frac{\partial}{\partial \zeta_j^*}$ and show estimates like those of lemma 5.4 in [4].

To study the Hefer decomposition for all ζ close enough to bD we need a unitary matrix $\Psi(\zeta)$ smoothly depending on ζ and such that $\Psi(\zeta)\Phi_*\eta_\zeta = (1, 0, \ldots, 0)$. In [4], such a matrix was already defined for all $\zeta \in bD$ but with the assumption that $|grad\ r| = 1$ on bD. We cannot assume this on a neighborhood of bD so we normalize and set

$$\begin{aligned}
\nu_j(\zeta) &:= \frac{1}{\sqrt{\sum_{i=1}^n \left|\frac{\partial r}{\partial \zeta_i^*}(\zeta)\right|^2}} \frac{\partial r}{\partial \zeta_j^*}(\zeta), \quad j = 1, \ldots, n,\\
A_j(\zeta) &:= 1 - \sum_{k=2}^j |\nu_k(\zeta)|^2, \quad j = 1, \ldots, n,\\
\Psi_{1i}(\zeta) &:= \nu_i(\zeta), \quad i = 1, \ldots, n,
\end{aligned}$$

and if $j > 1$

$$\Psi_{ji}(\zeta) := \frac{1}{\sqrt{A_{j-1}(\zeta)A_j(\zeta)}} \begin{cases} -\overline{\nu_j(\zeta)}\nu_i(\zeta) & \text{if } i = 1 \text{ or } i > j \\ 0 & \text{if } 1 < i < j \\ A_j(\zeta) & \text{if } i = j \end{cases}.$$

In spite of this normalization Ψ has the same properties than the matrix defined in [4]. For all $\zeta \in \mathcal{P}_\varepsilon(z_0)$ $\Psi(\zeta)$ is a unitary matrix such that $\Psi(\zeta)\Phi_*\eta_\zeta = (1, 0 \ldots, 0)$. Moreover $\Psi(\zeta)$ still satisfies estimates like those of lemma 5.2 in [4], that is

Proposition 2. *For all $\zeta \in \mathcal{P}_\varepsilon(z_0)$, $i = 2, \ldots, n$ and $j, k, l = 1, \ldots, n$, $j \neq i$, we have uniformly with respect to ζ, z_0 and ε*

$$1 \lesssim |\Psi_{jj}(\zeta)| \leq 1,$$

$$|\Psi_{jk}(\zeta)| \lesssim \frac{\varepsilon^2}{\tau_j(\zeta_0, \varepsilon)\tau_k(\zeta_0, \varepsilon)},$$

$$\left|\frac{\partial \Psi_{1j}}{\partial \zeta_k^*}(\zeta)\right| + \left|\frac{\partial \Psi_{1j}}{\partial \overline{\zeta_k^*}}(\zeta)\right| \lesssim \frac{\varepsilon}{\tau_j'(\zeta_0, \varepsilon)\tau_k'(\zeta_0, \varepsilon)},$$

$$\left|\frac{\partial \Psi_{ij}}{\partial \zeta_k^*}(\zeta)\right| + \left|\frac{\partial \Psi_{ij}}{\partial \overline{\zeta_k^*}}(\zeta)\right| \lesssim \frac{\varepsilon^2}{\tau_j(z_0, \varepsilon)\tau_i(z_0, \varepsilon)\tau_k'(z_0, \varepsilon)},$$

$$\left|\frac{\partial \Psi_{ii}}{\partial \zeta_k^*}(\zeta)\right| \left|\frac{\partial \Psi_{ii}}{\partial \overline{\zeta_k^*}}(\zeta)\right| \lesssim \frac{\varepsilon}{\tau_k'(z_0, \varepsilon)},$$

$$\left|\frac{\partial^2 \Psi_{1j}}{\partial \overline{\zeta_k^*}\partial \overline{\zeta_l^*}}(\zeta)\right| + \left|\frac{\partial^2 \Psi_{1j}}{\partial \zeta_k^*\partial \zeta_l^*}(\zeta)\right| + \left|\frac{\partial^2 \Psi_{1j}}{\partial \overline{\zeta_k^*}\partial \zeta_l^*}(\zeta)\right| \lesssim \frac{\varepsilon}{\tau_j'(\zeta_0, \varepsilon)\tau_k'(\zeta_0, \varepsilon)\tau_l'(\zeta_0, \varepsilon)},$$

$$\left|\frac{\partial^2\Psi_{ij}}{\partial\overline{\zeta_k^*}\partial\overline{\zeta_l^*}}(\zeta)\right|+\left|\frac{\partial^2\Psi_{ij}}{\partial\zeta_k^*\partial\zeta_l^*}(\zeta)\right|+\left|\frac{\partial^2\Psi_{ij}}{\partial\overline{\zeta_k^*}\partial\zeta_l^*}(\zeta)\right|\lesssim\frac{\varepsilon^2}{\tau_j(z_0,\varepsilon)\tau_i(z_0,\varepsilon)\tau_l'(z_0,\varepsilon)\tau_k'(z_0,\varepsilon)},$$

$$\left|\frac{\partial^2\Psi_{ii}}{\partial\overline{\zeta_k^*}\partial\overline{\zeta_l^*}}(\zeta)\right|+\left|\frac{\partial^2\Psi_{ii}}{\partial\zeta_k^*\partial\zeta_l^*}(\zeta)\right|+\left|\frac{\partial^2\Psi_{ii}}{\partial\overline{\zeta_k^*}\partial\zeta_l^*}(\zeta)\right|\lesssim\frac{\varepsilon}{\tau_k'(z_0,\varepsilon)\tau_l'(z_0,\varepsilon)}.$$

Proof. The inequality $|\Psi_{jj}(\zeta)|\leq 1$ holds because $\Psi(\zeta)$ is a unitary matrix.

The proposition 3.1 (vii) and (iv) of [4] give for all j

$$\left|\frac{\partial r}{\partial\zeta_j^*}(\zeta)\right|\lesssim\frac{\varepsilon}{\tau_j(z_0,\varepsilon)} \tag{17}$$

and therefore

$$|\nu_j(\zeta)|\lesssim\frac{\varepsilon}{\tau_j(z_0,\varepsilon)}.$$

We then estimate the derivatives of ν_j, $j=1,\dots,n$.

$$\begin{aligned}\frac{\partial\nu_j}{\partial\zeta_k^*}(\zeta)=&\frac{1}{\sqrt{\sum_{l=1}^n\left|\frac{\partial r}{\partial\zeta_l^*}(\zeta)\right|^2}}\frac{\partial^2 r}{\partial\zeta_j^*\partial\zeta_k^*}(\zeta)\\&-\frac{1}{2\left(\sum_{l=1}^n\left|\frac{\partial r}{\partial\zeta_l^*}(\zeta)\right|^2\right)^{\frac32}}\frac{\partial r}{\partial\zeta_j^*}(\zeta)\frac{\partial}{\partial\zeta_k^*}\sum_{l=1}^n\left|\frac{\partial r}{\partial\zeta_l^*}(\zeta)\right|^2\end{aligned}$$

(17) and the corollary 1 imply for $i>1$

$$\left|\frac{\partial\nu_i}{\partial\zeta_k^*}(\zeta)\right|\lesssim\frac{\varepsilon}{\tau_i'(z_0,\varepsilon)\tau_k'(z_0,\varepsilon)}.$$

Since for all l $\tau_l'(z_0,\varepsilon)\gtrsim\varepsilon^{\frac12}$, the corollary 1 implies $\left|\frac{\partial}{\partial\zeta_k^*}\sum_{l=1}^n\left|\frac{\partial r}{\partial\zeta_l^*}(\zeta)\right|^2\right|\lesssim\frac{\varepsilon^{\frac12}}{\tau_k'(z_0,\varepsilon)}$. Again with the corollary 1 this implies

$$\left|\frac{\partial\nu_1}{\partial\zeta_k^*}(\zeta)\right|\lesssim\frac{\varepsilon^{\frac12}}{\tau_k'(z_0,\varepsilon)}=\frac{\varepsilon}{\tau_1'(z_0,\varepsilon)\tau_k'(z_0,\varepsilon)}.$$

The same inequalities obviously hold for $\left|\frac{\partial\nu_i}{\partial\overline{\zeta}_k^*}(\zeta)\right|$. Moreover we could show as for the first order derivatives

$$\left|\frac{\partial^2\nu_j}{\partial\zeta_l^*\partial\zeta_k^*}(\zeta)\right|+\left|\frac{\partial^2\nu_j}{\partial\zeta_l^*\partial\overline{\zeta}_k^*}(\zeta)\right|+\left|\frac{\partial^2\nu_j}{\partial\overline{\zeta}_l^*\partial\overline{\zeta}_k^*}(\zeta)\right|\lesssim\frac{\varepsilon}{\tau_j'(z_0,\varepsilon)\tau_l'(z_0,\varepsilon)\tau_k'(z_0,\varepsilon)}.$$

Since for all $s \neq 1$ $\tau_s(z_0, \varepsilon) \gtrsim \varepsilon^{\frac{1}{2}}$, those estimates of ν_j and its derivatives imply

$$\left|\frac{\partial A_p}{\partial \overline{\zeta}^*_k}(\zeta)\right| + \left|\frac{\partial A_p}{\partial \zeta^*_k}(\zeta)\right| \lesssim \frac{\varepsilon}{\tau'_k(z_0, \varepsilon)}$$

$$\left|\frac{\partial^2 A_p}{\partial \zeta^*_l \partial \overline{\zeta}^*_k}(\zeta)\right| + \left|\frac{\partial^2 A_p}{\partial \zeta^*_l \partial \zeta^*_k}(\zeta)\right| + \left|\frac{\partial^2 A_p}{\partial \overline{\zeta}^*_l \partial \overline{\zeta}^*_k}(\zeta)\right| \lesssim \frac{\varepsilon}{\tau'_l(z_0, \varepsilon)\tau'_k(z_0, \varepsilon)}.$$

Moreover we have $|A_p(\zeta)| \gtrsim \left|\frac{\partial r}{\partial \zeta^*_1}(\zeta)\right| \gtrsim 1$ for all $\zeta \in \mathcal{P}_\varepsilon(z_0)$.

Now it suffices to use all those estimates and to distinguish the different cases to achieve the proof the proposition. □

We set $\omega(\zeta, z) = \Psi(\zeta)(z^* - \zeta^*)$ so that $F(\zeta, z) = F_\zeta(\omega(\zeta, z))$ and before we estimates Q^*_i and its derivatives we show

Lemma 5. *For $j, k = 1, \dots, n$, $l = 2, \dots, n$ and $\zeta, z \in \mathcal{P}_\varepsilon(z_0)$ the following inequalities hold uniformly with respect to ζ, z, z_0 and ε*

$$|\omega_j(\zeta, z)| \lesssim \tau_j(z_0, \varepsilon),$$

$$|\delta^*_j \omega_l(\zeta, z)| + \left|\frac{\partial \omega_l}{\partial \overline{\zeta}^*_j}(\zeta, z)\right| \lesssim \frac{\varepsilon}{\tau'_j(z_0, \varepsilon)}\tau_l(z_0, \varepsilon),$$

$$\left|\delta^*_k \frac{\partial \omega_l}{\partial \overline{\zeta}^*_j}(\zeta, z)\right| \lesssim \frac{\varepsilon}{\tau'_j(z_0, \varepsilon)\tau'_k(z_0, \varepsilon)}\tau_l(z_0, \varepsilon),$$

$$|\delta^*_j \omega_1(\zeta, z)| + \left|\frac{\partial \omega_1}{\partial \overline{\zeta}^*_j}(\zeta, z)\right| \lesssim \frac{\varepsilon}{\tau'_j(z_0, \varepsilon)},$$

$$\left|\delta^*_k \frac{\partial \omega_1}{\partial \overline{\zeta}^*_j}(\zeta, z)\right| \lesssim \frac{\varepsilon}{\tau'_j(z_0, \varepsilon)\tau'_k(z_0, \varepsilon)}.$$

Proof. Since $\tau_p(z_0, \varepsilon) \gtrsim \varepsilon$ for all p, the proposition 3.1 (v) of [4] and the proposition 2 give for all l and all $\zeta, z \in \mathcal{P}_\varepsilon(z_0)$

$$|\omega_l(\zeta, z)| \lesssim \tau_l(z_0, \varepsilon) + \sum_{\substack{p=1 \\ p \neq l}}^{n} \frac{\varepsilon^2}{\tau_l(z_0, \varepsilon)} \lesssim \tau_l(z_0, \varepsilon).$$

Using the proposition 3.1 (v) of [4] and proposition 2 one can analogously show the other estimates. □

Lemma 6. *For all $\zeta \in \mathcal{P}_\varepsilon(z_0)$, all multiindices β with $\beta_1 = 0$ and $j, k = 1, \dots, n$ we have uniformly with respect to ζ, z_0 and ε*

$$\left|\frac{\partial^{|\beta|} r_\zeta}{\partial \omega^\beta}(0)\right| \lesssim \frac{\varepsilon}{\prod_{p=1}^n \tau_p(z_0, \varepsilon)^{\beta_p}}$$

if $|\beta| \geq 2$ *and for* $|\beta| \geq 1$

$$\left|\frac{\partial}{\partial \overline{\zeta^*_j}} \frac{\partial^{|\beta|} r_\zeta}{\partial \omega^\beta}(0)\right| + \left|\frac{\partial}{\partial \zeta^*_j} \frac{\partial^{|\beta|} r_\zeta}{\partial \omega^\beta}(0)\right| \lesssim \frac{\varepsilon}{\tau'_j(z_0,\varepsilon)\prod_{p=1}^n \tau_p(z_0,\varepsilon)^{\beta_p}},$$

$$\left|\frac{\partial^2}{\partial \zeta^*_k \partial \overline{\zeta^*_j}} \frac{\partial^{|\beta|} r_\zeta}{\partial \omega^\beta}(0)\right| \lesssim \frac{\varepsilon}{\tau'_k(z_0,\varepsilon)\tau'_j(z_0,\varepsilon)\prod_{p=1}^n \tau_p(z_0,\varepsilon)^{\beta_p}}.$$

Proof. $r_\zeta(\omega) = r(\zeta + {}^t\overline{\Psi(\zeta)}\omega)$ therefore we have for all $\alpha_1, \dots, \alpha_p$, $p \geq 2$:

$$\frac{\partial^p r_\zeta}{\partial \omega_{\alpha_1} \dots \partial \omega_{\alpha_p}}(0) = \sum_{i_1,\dots,i_p=1}^n \frac{\partial^p r}{\partial \zeta^*_{i_1} \dots \partial \zeta^*_{i_p}}(\zeta) \prod_{l=1}^p \overline{\Psi}_{\alpha_l i_l}(\zeta).$$

If there exists s such that $\alpha_s \neq i_s$ we have by proposition 2 $|\Psi_{\alpha_s i_s}(\zeta)| \lesssim \frac{\varepsilon}{\tau_{\alpha_s}(z_0,\varepsilon)}$. If $\alpha_s = i_s$ for all s then the corollary 1 gives $\frac{\partial^p r}{\partial \zeta^*_{\alpha_1} \dots \partial \zeta^*_{\alpha_p}}(\zeta) \lesssim \frac{\varepsilon}{\prod_{s=1}^p \tau_{\alpha_p}(z_0,\varepsilon)}$.

The other estimates also follow from the corollary 1 and proposition 2. □

Lemma 7. *For all* $t \in [0,1]$, $\zeta, z \in \mathcal{P}_\varepsilon(z_0)$, $i, j, k = 1, \dots, n$, *we have uniformly in* ζ, z, z_0, t *and* ε

$$\left|F_\zeta(t\omega(\zeta,z))\right| \lesssim \varepsilon,$$

$$\left|\delta^*_j F_\zeta(t\omega(\zeta,z))\right| + \left|\frac{\partial}{\partial \overline{\zeta^*_j}}\left(F_\zeta(t\omega(\zeta,z))\right)\right| \lesssim \frac{\varepsilon}{\tau'_j(z_0,\varepsilon)},$$

$$\left|\delta^*_k \frac{\partial}{\partial \overline{\zeta^*_j}}\left(F_\zeta(t\omega(\zeta,z))\right)\right| \lesssim \frac{\varepsilon}{\tau'_j(z_0,\varepsilon)\tau'_k(z_0,\varepsilon)},$$

$$\left|\frac{\partial F_\zeta}{\partial \omega_i}(t\omega(\zeta,z))\right| \lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)},$$

$$\left|\delta^*_j \frac{\partial F_\zeta}{\partial \omega_i}(t\omega(\zeta,z))\right| + \left|\frac{\partial}{\partial \overline{\zeta^*_j}}\left(\frac{\partial F_\zeta}{\partial \omega_i}(t\omega(\zeta,z))\right)\right| \lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)\tau'_j(z_0,\varepsilon)},$$

$$\left|\delta^*_k \frac{\partial}{\partial \overline{\zeta^*_j}}\left(\frac{\partial F_\zeta}{\partial \omega_i}(t\omega(\zeta,z))\right)\right| \lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)\tau'_j(z_0,\varepsilon)\tau'_k(z_0,\varepsilon)}.$$

Proof. Since $\tau_1(z_0,\varepsilon) \eqsim \varepsilon$ the case $i = 1$ is obvious. Therefore only the cases $i > 1$ have to be considered. The estimates are then straightforwards consequences of the lemma 5 and 6. □

Lemma 8. *For all* $\zeta \in \mathcal{P}_\varepsilon(z_0)$, $i, j, k = 1, \dots, n$, *we have uniformly in* ζ, z_0 *and* ε

$$\left|Q^*_i(\zeta,z_0)\right| \lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)},$$

$$\left|\delta^*_j Q^*_i(\zeta,z_0)\right| + \left|\frac{\partial Q^*_i}{\partial \overline{\zeta^*_j}}(\zeta,z_0)\right| \lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)\tau'_j(z_0,\varepsilon)},$$

$$\left|\delta^*_k \frac{\partial Q^*_i}{\partial \overline{\zeta^*_j}}(\zeta,z_0)\right| \lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)\tau'_j(z_0,\varepsilon)\tau'_k(z_0,\varepsilon)}.$$

Proof. We have by definition $Q_i^* = -\sum_{l=1}^n \Psi_{li}\sigma_l$, thus we have to estimate the σ_l and its derivatives.

We set $A_\zeta(\omega) := A(\zeta, \zeta + \overline{\Psi(\zeta)}^t \omega)$ so that for $|\zeta - z| < \frac{R}{2}$

$$\sigma_l(\zeta, \omega(\zeta, z)) =$$
$$= \int_0^1 \frac{\partial A_\zeta}{\partial \omega_l}(t\omega(\zeta, z)) F_\zeta(t\omega(\zeta, z))dt + \int_0^1 A_\zeta(t\omega(\zeta, z)) \frac{\partial F_\zeta}{\partial \omega_l}(t\omega(\zeta, z))dt. \quad (18)$$

Since A and all its derivatives are bounded lemma 7 gives for all $l \neq 1$ $|\sigma_l(\zeta, \omega(\zeta, z_0))| \lesssim \frac{\varepsilon}{\tau_l(z_0,\varepsilon)}$. Now the proposition 2 give $\left|Q_i^*(\zeta, z_0)\right| \lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)}$.

The proposition 2 and the lemma 7 similarly give the other estimates. □

Corollary 2. *For all $i, j, k = 1, \dots, n$, $\zeta \in \mathcal{P}_\varepsilon(z_0)$ we have, uniformly in ζ, z_0 and ε,*

$$\left|\delta_j Q_i^*(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0, \varepsilon)}, \left|\delta_j \frac{\partial Q_i^*}{\partial \overline{\zeta}_k^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0, \varepsilon)\tau_k'(z_0, \varepsilon)}, \left|\delta_j S(\zeta, z_0)\right| \lesssim \varepsilon^{\frac{1}{2}}.$$

Proof. For all l we have $\tau_l'(z_0, \varepsilon) \gtrsim \varepsilon^{\frac{1}{2}}$. Lemma 8 gives $\left|\delta_l^* Q_i^*(\zeta, z)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}$, $\left|\delta_l^* \frac{\partial Q_i^*}{\partial \overline{\zeta}_k^*}(\zeta, z)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)\tau_k'(z_0,\varepsilon)}$. Since δ_j is a linear combination of $\delta_l^*, l = 1, \dots, n$, the two first inequalities are now obvious. The last inequality comes from the first one because $\delta_j S(\zeta, z) = \sum_{i=1}^n \delta_j Q_i^*(\zeta, z)(\zeta_i^* - z_i^*)$. □

At last, we prove similar estimations to lemma 5.5 of [4] for a differential operator of arbitrary order.

Lemma 9. *Let $\Delta_j = \frac{\partial^j}{\partial z^\alpha \partial \overline{z}^\beta}$ be a differentiation of order $j \geq 1$, $k = 0, \dots, n-1$, $l = 1, \dots, n$ and $\zeta \in (\mathcal{V} \setminus D) \cap \mathcal{P}_\varepsilon^0(z_0)$ if $\varepsilon \neq |r(z_0)|$ or $\zeta \in (\mathcal{V} \setminus D) \cap \mathcal{P}_\varepsilon(z_0)$ if $\varepsilon = |r(z_0)|$.*

$$\left|\Delta_j \frac{\eta_1(\zeta, z_0) \wedge (\overline{\partial}_\zeta \eta_1(\zeta, z_0))^k}{S^{k+1}(\zeta, z_0)}\right|$$

can be estimated by a sum of products of the form

$$\frac{\varepsilon^{-j}}{\prod_{i=0}^k \tau_{\nu_i}(z_0, \varepsilon) \prod_{i=1}^k \tau_{\mu_i}(z_0, \varepsilon)} \text{ and } \frac{\varepsilon^{-j-\frac{1}{2}}}{\prod_{i=0}^k \tau_{\nu_i}(z_0, \varepsilon) \prod_{i=1}^{k-1} \tau_{\mu_i}(z_0, \varepsilon)},$$

this last term appearing only when $k > 0$.

$$\left|\Delta_j \delta_l \frac{\eta_1(\zeta, z_0) \wedge (\overline{\partial}_\zeta \eta_1(\zeta, z_0))^k}{S^{k+1}(\zeta, z_0)}\right|$$

can be estimated by a sum of products of the form

$$\frac{\varepsilon^{-j-\frac{1}{2}}}{\prod_{i=0}^k \tau_{\nu_i}(z_0, \varepsilon) \prod_{i=1}^k \tau_{\mu_i}(z_0, \varepsilon)} \text{ and } \frac{\varepsilon^{-j-1}}{\prod_{i=0}^k \tau_{\nu_i}(z_0, \varepsilon) \prod_{i=1}^{k-1} \tau_{\mu_i}(z_0, \varepsilon)},$$

this last term appearing only when $k > 0$.

In both cases we have $\nu_i \neq \nu_{i'}$ and $\mu_i \neq \mu_{i'}$ if $i' \neq i$ and $\mu_i > 1$ for all i.

Proof. We fix $\zeta \in \mathcal{P}^0_\varepsilon(z_0) \cap (\mathcal{V} \setminus D)$ if $\varepsilon \neq |r(z_0)|$ and $\zeta \in \mathcal{P}_\varepsilon(z_0) \cap (\mathcal{V} \setminus D)$ otherwise. As in [4], we write $\frac{\eta_1 \wedge (\overline{\partial}_\zeta \eta_1)^k}{S^{k+1}}$ with respect to an ε-extremal basis and get a sum of $\Gamma^{\nu_0,\dots,\nu_k}_{\mu_1,\dots,\mu_k} := S^{-(k+1)} Q^*_{\nu_0} d\zeta^*_{\nu_0} \wedge \bigwedge_{i=1}^k \frac{\partial Q^*_{\nu_i}}{\partial \overline{\zeta}^*_{\mu_i}} d\overline{\zeta}^*_{\mu_i} \wedge d\zeta^*_{\nu_i}$, where necessarily the ν_i (respectively μ_i) are pairwise different. We apply lemma 4 (ii) if $\varepsilon \neq |r(z_0)|$ and lemma 4 (i) if $\varepsilon = |r(z_0)|$ and get in both case $|S(\zeta, z_0)| \gtrsim \varepsilon$. We notice that the derivatives of S are uniformly bounded with respect to ζ and z_0 and that the boundedness of the derivatives of S is the best estimate we have in general because one can show that $\left|\frac{\partial S}{\partial z^*_1}(\zeta, z_0)\right| \gtrsim 1$.

We use the estimates $|Q^*_{\nu_0}(\zeta, z_0)| \lesssim \frac{\varepsilon}{\tau_{\nu_0}(z_0,\varepsilon)}$, $\left|\frac{\partial Q^*_{\nu_i}}{\partial \overline{\zeta}^*_{\mu_i}}(\zeta, z_0)\right| \lesssim \frac{\varepsilon}{\tau_{\nu_i}(z_0,\varepsilon)\tau'_{\mu_i}(z_0,\varepsilon)}$ given by lemma 8. For $\Delta_{j'} = \frac{\partial^{j'}}{\partial z^{*\alpha'} \partial \overline{z^*}^{\beta'}}$, $j' \geq 1$, since $\Delta_{j'} Q^*_{\nu_0}(\zeta, z_0)$ and $\Delta_{j'} \frac{\partial Q^*_{\nu_i}}{\partial \overline{\zeta}^*_{\mu_i}}$ are bounded uniformly with respect to ζ and z_0 and $\frac{\varepsilon^{1-j'}}{\tau_{\nu_0}(z_0,\varepsilon)}$ and $\frac{\varepsilon^{1-j'}}{\tau_{\nu_i}(z_0,\varepsilon)\tau'_{\mu_i}(z_0,\varepsilon)}$ are bounded away from zero for small ε, we use the two estimates $|\Delta_{j'} Q^*_{\nu_0}(\zeta, z_0)| \lesssim \frac{\varepsilon^{1-j'}}{\tau_{\nu_0}(z_0,\varepsilon)}$ and $\left|\Delta_{j'} \frac{\partial Q^*_{\nu_i}}{\partial \overline{\zeta}^*_{\mu_i}}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{1-j'}}{\tau_{\nu_i}(z_0,\varepsilon)\tau'_{\mu_i}(z_0,\varepsilon)}$. The derivative of $Q^*_{\nu_0}$ and $\frac{\partial Q^*_{\nu_i}}{\partial \overline{\zeta}^*_{\mu_i}}$ may have better estimates, however this would not lead to better estimates because the derivatives of S are only bounded.

We now estimate $\Delta_j \Gamma^{\nu_0,\dots,\nu_k}_{\mu_1,\dots,\mu_k}(\zeta, z_0)$.

If $k = 0$ or if $\mu_i \neq 1$ for all i, $1 \leq i \leq k$, we have $\tau'_{\mu_i}(z_0, \varepsilon) = \tau_{\mu_i}(z_0, \varepsilon)$ for all i so

$$\left|\Delta_j \Gamma^{\nu_0,\dots,\nu_k}_{\mu_1,\dots,\mu_k}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{-j}}{\prod_{i=0}^k \tau_{\nu_i}(z_0, \varepsilon) \prod_{i=1}^k \tau_{\mu_i}(z_0, \varepsilon)}.$$

If $\mu_{i_0} = 1$ for a necessarily unique i_0, $1 \leq i_0 \leq k$, we have $\tau'_{\mu_{i_0}}(z_0, \varepsilon) = \varepsilon^{\frac{1}{2}}$ so

$$\left|\Delta_j \Gamma^{\nu_0,\dots,\nu_k}_{\mu_1,\dots,\mu_k}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{-j-\frac{1}{2}}}{\prod_{i=0}^k \tau_{\nu_i}(z_0, \varepsilon) \prod_{\substack{i=1 \\ i \neq i_0}}^k \tau_{\mu_i}(z_0, \varepsilon)}.$$

The last estimate can be shown by the same method. □

In order to estimate the different integrals, we also need this obvious lemma.

Lemma 10. *If ε is sufficiently small, then for all $j \in \mathbb{N}$ and all $g \in C^j(G \cup D)$, g identically zero on D and all $\zeta \in \mathcal{P}_\varepsilon(z_0)$, we have, uniformly with respect to z_0, ζ and g,*

$$|g(\zeta)| \lesssim \varepsilon^j \|g\|_{G \cup D, j}.$$

By using lemma 10 the regularity of the $\overline{\partial}$-closed form f will recover missing ε factors. We now are ready to estimate all the integrals. The method is based on the one of [4].

Proof of theorem 2 (ii) :. We first show theorem 2 (ii) for $k = 0$. Since $T_q^* = T_q - M_q$ and since T_q satisfies C^0-estimates (see theorem 1) we have to prove that $M_q f$ belongs to $C^{\frac{1}{m}}_{0,q-1}(\overline{D})$ and satisfies $\|M_q f\|_{\overline{D},\frac{1}{m}} \lesssim \|f\|_{\overline{D},0}$ uniformly with respect to $f \in C^0_{0,q}(\overline{D})$.

In order to use the Hardy-Littlewood lemma we set $\Delta = \frac{\partial}{\partial z_p}$ or $\Delta = \frac{\partial}{\partial \overline{z}_p}$, we fix $z_0 \in D$ close to bD and we use the covering (16).

For $q = 1$, $M_1 f(z_0) = \int_G Ef(\zeta) \wedge K_{n,0}(\zeta, z_0)$ and $K_{n,0} = (2i\pi S)^{-n}\eta_1 \wedge (\overline{\partial}_\zeta \eta_1)^{n-1}$. For all $j = 0, \dots, j_0$, all $\zeta \in \mathcal{P}^j_{\varepsilon_0}(z_0) \cap G$ the lemma 9 yields to

$$|\Delta f(\zeta) \wedge K_{n,0}(\zeta, z_0)| \lesssim \frac{\|f\|_{\overline{D},0}(2^{-j}\varepsilon_0)^{-1}}{\prod_{i=1}^n \tau_i(z_0, 2^{-j}\varepsilon_0) \prod_{i=2}^n \tau_i(z_0, 2^{-j}\varepsilon_0)} + \sum_{k=2}^n \frac{\|f\|_{\overline{D},0}(2^{-j}\varepsilon_0)^{-\frac{3}{2}}}{\prod_{i=1}^n \tau_i(z_0, 2^{-j}\varepsilon_0) \prod_{\substack{i=2 \\ i\neq k}}^n \tau_i(z_0, 2^{-j}\varepsilon_0)}.$$

For $l = 1, \dots, n$ we set $\zeta_l^* = u_l + iv_l$ with $u_l, v_l \in \mathbb{R}$. According to proposition 3.1 (v) of [4] we have $2^{-j}\varepsilon_0 \eqsim \tau_1(z_0, 2^{-j}\varepsilon_0)$ and therefore we get

$$\begin{aligned}
&\left|\Delta \int_{G\cap\mathcal{P}^j_{\varepsilon_0}(z_0)} Ef(\zeta) \wedge K_{n,0}(\zeta, z_0)\right| \\
&\lesssim \int_{\substack{|u_1|,|v_1|\leq\tau_1(z_0,2^{-j}\varepsilon_0) \\ \vdots \\ |u_n|,|v_n|\leq\tau_n(z_0,2^{-j}\varepsilon_0)}} \frac{\|f\|_{\overline{D},0}(2^{-j}\varepsilon_0)^{-1} du_1 dv_1 \dots du_n dv_n}{\prod_{i=1}^n \tau_i(z_0, 2^{-j}\varepsilon_0) \prod_{i=2}^n \tau_i(z_0, 2^{-j}\varepsilon_0)} \\
&\quad + \sum_{k=2}^n \int_{\substack{|u_1|,|v_1|\leq\tau_1(z_0,2^{-j}\varepsilon_0) \\ \vdots \\ |u_n|,|v_n|\leq\tau_n(z_0,2^{-j}\varepsilon_0)}} \frac{\|f\|_{\overline{D},0}(2^{-j}\varepsilon_0)^{-\frac{3}{2}} du_1 dv_1 \dots du_n dv_n}{\prod_{i=1}^n \tau_i(z_0, 2^{-j}\varepsilon_0) \prod_{\substack{i=2 \\ i\neq k}}^n \tau_i(z_0, 2^{-j}\varepsilon_0)} \\
&\lesssim \left(1 + \sum_{k=2}^n (2^{-j}\varepsilon_0)^{-\frac{1}{2}} \tau_k(z_0, 2^{-j}\varepsilon_0)\right) \|f\|_{\overline{D},0}.
\end{aligned}$$

We use the inequality $\tau_k(z_0, 2^{-j}\varepsilon_0) \lesssim (2^{-j}\varepsilon_0)^{\frac{1}{m}}$ and we get

$$\left|\Delta \int_{G\cap\mathcal{P}^j_{\varepsilon_0}(z_0)} Ef(\zeta) \wedge K_{n,0}(\zeta, z_0)\right| \lesssim (2^{-j}\varepsilon_0)^{\frac{1}{m}-\frac{1}{2}} \|f\|_{\overline{D},0}. \tag{19}$$

Using lemma 9 we get in the same way

$$\left|\Delta \int_{G\cap\mathcal{P}_{|r(z_0)|}(z_0)} Ef(\zeta) \wedge K_{n,0}(\zeta, z_0)\right| \lesssim |r(z_0)|^{\frac{1}{m}-\frac{1}{2}} \|f\|_{\overline{D},0}. \tag{20}$$

We add (19) for $j = 0, \dots, j_0$ and (20) and use $2^{-j_0}\varepsilon_0 \eqsim |r(z_0)|$. We get when $m = 2$

$$\left|\Delta \int_{G\cap\mathcal{P}_{\varepsilon_0}(z_0)} Ef(\zeta) \wedge K_{n,0}(\zeta, z_0)\right| \lesssim |\ln |r(z_0)|| \, \|f\|_{\overline{D},0}$$

and when $m > 2$

$$\left|\Delta \int_{G\cap\mathcal{P}_{\varepsilon_0}(z_0)} Ef(\zeta)\wedge K_{n,0}(\zeta,z_0)\right| \lesssim |r(z_0)|^{\frac{1}{m}-\frac{1}{2}}\|f\|_{\overline{D},0}.$$

Since z_0 is any point in D close to bD, in both case the Hardy-Littlewood lemma implies that $M_1 f$ belongs to $C^{\frac{1}{m}}_{0,0}(\overline{D})$ and satisfies $\|M_1 f\|_{\overline{D},\frac{1}{m}} \lesssim \|f\|_{\overline{D},0}$ uniformly with respect to f. With the theorem 1, this prove theorem 2 (ii) for $k = 0$ and $q = 1$.

For $\zeta \in \mathcal{P}^j_{\varepsilon_0}(z_0)\cap G$ we have $|\zeta - z_0| \gtrsim 2^{-j}\varepsilon_0$, thus for $k = 0,\dots,n-1$ and $\zeta \in \mathcal{P}^j_{\varepsilon_0}(z_0)\cap G$ we have $\left|\overline{\partial}_z \frac{\eta_0\wedge(\overline{\partial}_\zeta\eta_0)^k\wedge(\overline{\partial}_z\eta_0)^{q-2}}{|\zeta-z_0|^{2(n-k-1)}}\right| \lesssim \frac{1}{|\zeta-z_0|^{2(n-k-1)}}$ and $\left|\Delta\overline{\partial}_z \frac{\eta_0\wedge(\overline{\partial}_\zeta\eta_0)^k\wedge(\overline{\partial}_z\eta_0)^{q-2}}{|\zeta-z_0|^{2(n-k-1)}}\right| \lesssim \frac{\varepsilon^{-1}}{|\zeta-z_0|^{2(n-k-1)}}$. Since Q and S are holomorphic with respect to z, for $q = 2,\dots,n-1$ lemma 9 then gives

$$\begin{aligned}&\left|\Delta\int_{\lambda\in[0,1]} Ef(\zeta)\wedge\overline{\partial}_z\Omega_{n,q-2}(\eta)(\zeta,\lambda,z_0)\right|\\ &\lesssim \sum_{k=0}^{n-1}\sum_{\substack{1\le\nu_0<\ldots<\nu_k\le n\\ 1<\mu_1<\ldots<\mu_k\le n}} \frac{\|f\|_{\overline{D},0}(2^{-j}\varepsilon_0)^{-1}}{\prod_{i=0}^k\tau_{\nu_i}(z_0,2^{-j}\varepsilon_0)\prod_{i=1}^k\tau_{\mu_i}(z_0,2^{-j}\varepsilon_0)|\zeta-z_0|^{2(n-k-1)}}\\ &\quad+\sum_{k=1}^{n-1}\sum_{\substack{1\le\nu_0<\ldots<\nu_k\le n\\ 1<\mu_1<\ldots<\mu_{k-1}\le n}} \frac{\|f\|_{\overline{D},0}(2^{-j}\varepsilon_0)^{-\frac{3}{2}}}{\prod_{i=0}^k\tau_{\nu_i}(z_0,2^{-j}\varepsilon_0)\prod_{i=1}^{k-1}\tau_{\mu_i}(z_0,2^{-j}\varepsilon_0)|\zeta-z_0|^{2(n-k-1)}}\end{aligned}$$

uniformly with respect to $\zeta \in \mathcal{P}^j_{\varepsilon_0}(z_0)\cap G$. We estimate

$$\int_{G\cap\mathcal{P}^j_{\varepsilon_0}(z_0)} \frac{\|f\|_{\overline{D},0}(2^{-j}\varepsilon_0)^{-\frac{3}{2}}du_1dv_1\dots du_ndv_n}{\prod_{i=0}^k\tau_{\nu_i}(z_0,2^{-j}\varepsilon_0)\prod_{i=1}^{k-1}\tau_{\mu_i}(z_0,2^{-j}\varepsilon_0)|\zeta-z_0|^{2(n-k-1)}}.$$

Since $1\le\nu_0<\ldots<\nu_k\le n$ and $1<\mu_1<\ldots\mu_k<n$, we can integrate with respect to $u_{\nu_0},\dots,u_{\nu_k}, v_{\mu_1},\dots,v_{\mu_{k-1}}$ and v_1 and we get

$$\begin{aligned}&\int_{G\cap\mathcal{P}^j_{\varepsilon_0}(z_0)} \frac{\|f\|_{\overline{D},0}(2^{-j}\varepsilon_0)^{-\frac{3}{2}}du_1dv_1\dots du_ndv_n}{\prod_{i=0}^k\tau_{\nu_i}(z_0,2^{-j}\varepsilon_0)\prod_{i=1}^{k-1}\tau_{\mu_i}(z_0,2^{-j}\varepsilon_0)|\zeta-z_0|^{2(n-k-1)}}\\ &\lesssim (2^{-j}\varepsilon_0)^{-\frac{1}{2}}\int_{|\omega|\le\sup_{i=1,\dots,n}\tau_i(z_0,2^{-j}\varepsilon_0)}\frac{dV(\omega)}{|\omega|^{2(n-k-1)}}\|f\|_{\overline{D},0}\end{aligned}$$

where ω is a variable of dimension $2n-2k-1$. Since for all $i = 1,\dots,n$ $\tau_i(z_0,2^{-j}\varepsilon_0) \lesssim (2^{-j}\varepsilon_0)^{\frac{1}{m}}$, we get

$$\begin{aligned}&\int_{G\cap\mathcal{P}^j_{\varepsilon_0}(z_0)} \frac{\|f\|_{\overline{D},0}(2^{-j}\varepsilon_0)^{-\frac{3}{2}}du_1dv_1\dots du_ndv_n}{\prod_{i=0}^k\tau_{\nu_i}(z_0,2^{-j}\varepsilon_0)\prod_{i=1}^{k-1}\tau_{\mu_i}(z_0,2^{-j}\varepsilon_0)|\zeta-z_0|^{2(n-k-1)}}\\ &\lesssim (2^{-j}\varepsilon_0)^{-\frac{1}{2}}\int_{\rho\le(2^{-j}\varepsilon_0)^{\frac{1}{m}}}\frac{\rho^{2n-2k-2}d\rho}{\rho^{2(n-k-1)}}\|f\|_{\overline{D},0}\\ &\lesssim (2^{-j}\varepsilon_0)^{\frac{1}{m}-\frac{1}{2}}\|f\|_{\overline{D},0}.\end{aligned}$$

Analogously we show the following inequality

$$\int_{G\cap\mathcal{P}^j_{\varepsilon_0}(z_0)} \frac{\|f\|_{\overline{D},0}(2^{-j}\varepsilon_0)^{-1}du_1dv_1\dots du_ndv_n}{\prod_{i=0}^k \tau_{v_i}(z_0,2^{-j}\varepsilon_0)\prod_{i=1}^k \tau_{\mu_i}(z_0,2^{-j}\varepsilon_0)|\zeta-z_0|^{2(n-k-1)}} \lesssim \|f\|_{\overline{D},0}$$

and finally we get

$$\left|\Delta\int_{(G\cap\mathcal{P}^j_{\varepsilon_0}(z_0))\times[0,1]} Ef(\zeta)\wedge\overline{\partial}_z\Omega_{n,q-2}(\eta)(\zeta,\lambda,z_0)\right| \lesssim (2^{-j}\varepsilon_0)^{\frac{1}{m}-\frac{1}{2}}\|f\|_{\overline{D},0}. \tag{21}$$

We show in the same way

$$\left|\Delta\int_{(G\cap\mathcal{P}_{|r(z_0)|}(z_0))\times[0,1]} Ef(\zeta)\wedge\overline{\partial}_z\Omega_{n,q-2}(\eta)(\zeta,\lambda,z_0)\right| \lesssim |r(z_0)|^{\frac{1}{m}-\frac{1}{2}}\|f\|_{\overline{D},0}. \tag{22}$$

Now adding (21) for $j=0,\dots,j_0$ and (22) and using $2^{-j_0}\varepsilon_0 \eqsim |r(z_0)|$, we get $|\Delta M_q f(z_0)| \lesssim |r(z_0)|^{\frac{1}{m}-\frac{1}{2}}\|f\|_{\overline{D},0}$ when $m>2$ and $|\Delta M_q f(z_0)| \lesssim |\ln|r(z_0)||\,\|f\|_{\overline{D},0}$ when $m=2$. The Hardy-Littlewood lemma then implies that $M_q f$ belongs to $C^{\frac{1}{m}}_{0,q-1}(\overline{D})$ and satisfies $\|M_q f\|_{\overline{D},\frac{1}{m}} \lesssim \|f\|_{\overline{D},0}$ uniformly with respect to $f\in C^0_{0,q}(\overline{D})$. With the theorem 1, this prove the theorem 2 (ii) for $k=0$ and $q=2,\dots n-1$.

We may notice that, for $q=1,\dots,n-1$, $M_q f\in C^{\frac{1}{2}+\frac{1}{m}}_{0,q-1}(\overline{D})$ when $m>2$ and $M_q f\in C^{1-\alpha}_{0,q-1}(\overline{D})$ for all $\alpha\in]0,1]$ when $m=2$. However this is useless in this work because $T_q f$ is not as regular as $M_q f$.

Now we prove theorem 2 (ii) for $k>0$ and assume it shown for all $k'=0,\dots,k-1$. We fix some $\overline{\partial}$-closed $f\in C^k_{0,q}(\overline{D})$. We have to prove that $T'_q f = -\int_{G\times[0,1]}\overline{\partial}_\zeta Ef(\zeta)\wedge\Omega_{n,q-1}(\eta)(\zeta,\lambda,\cdot)$ belongs to $C^{\frac{1}{m}+k}_{0,q-1}(\overline{D})$. For $l=1,\dots,n$ we will prove that $\frac{\partial T'_q f}{\partial z_l}$ and $\frac{\partial T'_q f}{\partial \overline{z}_l}$ belong to $C^{k-1+\frac{1}{m}}_{0,q-1}(\overline{D})$. Let $\Delta_k=\frac{\partial^k}{\partial z^\alpha\partial\overline{z}^\beta}$ be a differentiation of order k.

We first prove that $\frac{\partial T'_q f}{\partial\overline{z}_l}$ belongs to $C^{k-1+\frac{1}{m}}_{0,q-1}(\overline{D})$. S and Q are holomorphic with respect to z and $|\zeta-z_0|\gtrsim 2^{-j}\varepsilon_0$ for $\zeta\in\mathcal{P}^j_{\varepsilon_0}(z_0)$. Therefore with lemma 10 and 9, for $\tilde{k}=0,\dots,n-q-1$ and $\zeta\in\mathcal{P}^j_{\varepsilon_0}(z_0)$, we can estimate

$$\Delta_k\Bigg(\overline{\partial}_\zeta Ef(\zeta)\wedge\frac{\partial}{\partial\overline{z}_l}\frac{\eta_1(\zeta,z_0)\wedge(\overline{\partial}_\zeta\eta_1(\zeta,z_0))^{\tilde{k}}}{S^{\tilde{k}+1}(\zeta,z_0)} \wedge\frac{\eta_0(\zeta,z_0)\wedge(\overline{\partial}_\zeta\eta_0(\zeta,z_0))^{n-q-\tilde{k}-1}\wedge(\overline{\partial}_z\eta_0(\zeta,z_0))^{q-1}}{|\zeta-z_0|^{2(n-\tilde{k}-1)}}\Bigg).$$

by a sum of terms such $\frac{\|f\|_{k,D}(2^{-j}\varepsilon_0)^{-1}}{\prod_{i=0}^{\tilde{k}}\tau_{\nu_i}(z_0,(2^{-j}\varepsilon_0)\prod_{i=1}^{\tilde{k}}\tau_{\mu_i}(z_0,(2^{-j}\varepsilon_0))|\zeta-z_0|^{2(n-\tilde{k}-1)}}$ and $\frac{\|f\|_{k,D}(2^{-j}\varepsilon_0)^{-\frac{3}{2}}}{\prod_{i=0}^{\tilde{k}}\tau_{\nu_i}(z_0,(2^{-j}\varepsilon_0))\prod_{i=1}^{\tilde{k}-1}\tau_{\mu_i}(z_0,(2^{-j}\varepsilon_0))|\zeta-z_0|^{2(n-\tilde{k}-1)}}$, this last term appearing only for $\tilde{k}>0$, and in both terms $\mu_i>1$ and $\mu_i\neq\mu_j$, $\nu_i\neq\nu_j$ for all $i,j,i\neq j$. Using $(2^{-j}\varepsilon_0)$-extremal coordinates we then integrate over $\mathcal{P}^j_{\varepsilon_0}(z_0)$ and get

$$\left|\Delta_k\int_{(G\cap\mathcal{P}^j_{\varepsilon_0}(z_0))\times[0,1]}\overline{\partial}_\zeta Ef(\zeta)\wedge\frac{\partial}{\partial\overline{z}_l}\Omega_{n,q-1}(\eta)(\zeta,\lambda,z_0)\right| \lesssim \|f\|_{D,k}(2^{-j}\varepsilon_0)^{\frac{1}{m}-1}. \tag{23}$$

Using lemma 10 and 9 on $\mathcal{P}_{|r(z_0)|}(z_0)$ we get

$$\left|\Delta_k\int_{(G\cap\mathcal{P}_{|r(z_0)|}(z_0))\times[0,1]}\overline{\partial}_\zeta Ef(\zeta)\wedge\frac{\partial}{\partial\overline{z}_l}\Omega_{n,q-1}(\eta)(\zeta,\lambda,z_0)\right| \lesssim \|f\|_{D,k}|r(z_0)|^{\frac{1}{m}-1}. \tag{24}$$

Adding (23) for $j=0,\dots,j_0$ and (24) and using $2^{-j_0}\varepsilon_0\eqsim|r(z_0)|$, we get $\left|\Delta_k\frac{\partial T'_qf}{\partial\overline{z}_l}(z_0)\right|\lesssim\|f\|_{D,k}|r(z_0)|^{\frac{1}{m}-1}$, where all the involved constants do not depend on z_0 and f.

The Hardy-Littlewood lemma then implies that $\frac{\partial T'_qf}{\partial\overline{z}_l}$ is in $C^{k-1+\frac{1}{m}}_{0,q-1}(\overline{D})$ and satisfies $\left\|\frac{\partial T'_qf}{\partial\overline{z}_l}\right\|_{\overline{D},k-1+\frac{1}{m}}\leq c_k\|f\|_{\overline{D},k}$, c_k depending only on k.

To prove that $\frac{\partial T'_qf}{\partial z_l}$ belongs to $C^{\frac{1}{m}+k-1}_{0,q-1}(\overline{D})$ we use (15).

Since Ef is compactly supported in $G\cup D$, $\int_{G\cup D}\frac{\partial Ef}{\partial\zeta_l}(\zeta)\wedge B_{n,q-1}(\zeta,\cdot)$ belongs to $C^{k-\varepsilon}_{0,q-1}(\overline{D})$ and satisfies $\left\|\int_{G\cup D}\frac{\partial Ef}{\partial\zeta_l}(\zeta)\wedge B_{n,q-1}(\zeta,\cdot)\right\|_{\overline{D},k-\varepsilon}\leq c_\varepsilon\|f\|_{\overline{D},k}$ for all $\varepsilon\in]0,1]$, c_ε depending only on ε.

By induction $T^*_q\left(\frac{\partial f}{\partial z_l}\right)$ belongs to $C^{k-1+\frac{1}{m}}_{0,q-1}(\overline{D})$ and $\left\|T^*_q\left(\frac{\partial f}{\partial z_l}\right)\right\|_{\overline{D},k-1+\frac{1}{m}}\lesssim\|f\|_{\overline{D},k}$ uniformly with respect to f.

Using lemma 9 and 10, exactly as we have studied M_qf, we show that, when $q=1$, $\int_G\left(\frac{\partial Ef}{\partial\zeta_l}(\zeta)-E\frac{\partial f}{\partial\zeta_l}(\zeta)\right)\wedge K_{n,0}(\zeta,\cdot)$ belongs to $C^{k-1+\frac{1}{m}}_{0,0}(\overline{D})$ and satisfies $\left\|\int_G\left(\frac{\partial Ef}{\partial\zeta_l}(\zeta)-E\frac{\partial f}{\partial\zeta_l}(\zeta)\right)\wedge K_{n,0}(\zeta,\cdot)\right\|_{\overline{D},k-1+\frac{1}{m}}\lesssim\|f\|_{\overline{D},k}$, and, when $q\neq1$, that $\int_{G\times[0,1]}\left(\frac{\partial Ef}{\partial\zeta_l}(\zeta)-E\frac{\partial f}{\partial\zeta_l}(\zeta)\right)\wedge\overline{\partial}_z\Omega_{n,q-2}(\eta)(\zeta,\lambda,\cdot)\in C^{k-1+\frac{1}{m}}_{0,q-1}(\overline{D})$ and $\left\|\int_{G\times[0,1]}\left(\frac{\partial Ef}{\partial\zeta_l}(\zeta)-E\frac{\partial f}{\partial\zeta_l}(\zeta)\right)\wedge\overline{\partial}_z\Omega_{n,q-2}(\eta)(\zeta,\lambda,\cdot)\right\|_{\overline{D},k-1+\frac{1}{m}}\lesssim\|f\|_{\overline{D},k}$. We also notice that for $q\neq1$ $\int_G\left(\frac{\partial Ef}{\partial\zeta_l}(\zeta)-E\frac{\partial f}{\partial\zeta_l}(\zeta)\right)\wedge K_{n,q-1}(\zeta,\cdot)=0$ and for $q=1$ $\int_{G\times[0,1]}\left(\frac{\partial Ef}{\partial\zeta_l}(\zeta)-E\frac{\partial f}{\partial\zeta_l}(\zeta)\right)\wedge\overline{\partial}_z\Omega_{n,q-2}(\eta)(\zeta,\lambda,\cdot)=0$.

In order to prove that the last term of (15) belongs to $C^{k-1+\frac{1}{m}}_{0,q-1}(\overline{D})$ we fix a differentiation $\Delta_k = \frac{\partial^k}{\partial z^\alpha \partial \overline{z}^\beta}$ of order k, $z_0 \in D$ close to bD and we use the covering (16). Lemma 9 and 10 give for $j = 0, \dots, j_0$:

$$\left|\Delta_k \int_{\lambda\in[0,1]} \overline{\partial}_\zeta Ef(\zeta) \wedge \delta_l \Omega_{n,q-1}(\eta)(\zeta,\lambda,z_0)\right|$$

$$\lesssim \sum_{k=0}^{n-1} \sum_{\substack{1\le\nu_0<\dots<\nu_k\le n\\ 1<\mu_1<\dots<\mu_k\le n}} \frac{\|f\|_{\overline{D},k}(2^{-j}\varepsilon_0)^{-\frac{3}{2}}}{\prod_{i=0}^k \tau_{\nu_i}(z_0,2^{-j}\varepsilon_0)\prod_{i=1}^k \tau_{\mu_i}(z_0,2^{-j}\varepsilon_0)|\zeta-z_0|^{2(n-k-1)-1}}$$

$$+\sum_{k=1}^{n-1} \sum_{\substack{1\le\nu_0<\dots<\nu_k\le n\\ 1<\mu_1<\dots<\mu_{k-1}\le n}} \frac{\|f\|_{\overline{D},0}(2^{-j}\varepsilon_0)^{-2}}{\prod_{i=0}^k \tau_{\nu_i}(z_0,2^{-j}\varepsilon_0)\prod_{i=1}^{k-1} \tau_{\mu_i}(z_0,2^{-j}\varepsilon_0)|\zeta-z_0|^{2(n-k-1)-1}}.$$

Using $(2^{-j}\varepsilon_0)$-extremal coordinates, we integrate over $G \cap \mathcal{P}^j_{\varepsilon_0}(z_0)$ and get

$$\left|\Delta_k \int_{G\cap\mathcal{P}^j_{\varepsilon_0}(z_0)\times[0,1]} \overline{\partial}_\zeta Ef(\zeta) \wedge \delta_l \Omega_{n,q-1}(\eta)(\zeta,\lambda,z_0)\right|$$
$$\lesssim \|f\|_{\overline{D},0}((2^{-j}\varepsilon_0)^{-\frac{1}{2}+\frac{1}{m}} + (2^{-j}\varepsilon_0)^{\frac{2}{m}-1}). \tag{25}$$

We also have

$$\left|\Delta_k \int_{G\cap\mathcal{P}_{|r(z_0)|}(z_0)\times[0,1]} \overline{\partial}_\zeta Ef(\zeta) \wedge \delta_l \Omega_{n,q-1}(\eta)(\zeta,\lambda,z_0)\right|$$
$$\lesssim \|f\|_{\overline{D},0}(|r(z_0)|^{-\frac{1}{2}+\frac{1}{m}} + |r(z_0)|^{\frac{2}{m}-1}). \tag{26}$$

Adding (26) and (25) for $j = 0, \dots, j_0$ and using $2^{-j_0}\varepsilon_0 \eqsim |r(z_0)|$ we get when $m > 2$

$$\left|\Delta_k \int_{G\cap\mathcal{P}_{\varepsilon_0}(z_0)\times[0,1]} \overline{\partial}_\zeta Ef(\zeta) \wedge \delta_l \Omega_{n,q-1}(\eta)(\zeta,\lambda,z_0)\right|$$
$$\lesssim \|f\|_{\overline{D},0}(|r(z_0)|^{-\frac{1}{2}+\frac{1}{m}} + |r(z_0)|^{\frac{2}{m}-1}).$$

and when $m = 2$

$$\left|\Delta_k \int_{G\cap\mathcal{P}_{\varepsilon_0}(z_0)\times[0,1]} \overline{\partial}_\zeta Ef(\zeta) \wedge \delta_l \Omega_{n,q-1}(\eta)(\zeta,\lambda,z_0)\right| \lesssim \|f\|_{\overline{D},0}|\ln|r(z_0)||.$$

The Hardy-Littlewood lemma then implies that $\int_{G\cap\mathcal{P}_{\varepsilon_0}(z_0)\times[0,1]} \overline{\partial}_\zeta Ef(\zeta)\wedge\delta_l\Omega_{n,q-1}(\eta)(\zeta,\lambda,\cdot)$ is in $C^{k-1+\frac{1}{m}}_{0,q-1}(\overline{D})$ and satisfies, uniformly with respect to f, $\left\|\int_{G\cap\mathcal{P}_{\varepsilon_0}(z_0)\times[0,1]} \overline{\partial}_\zeta Ef(\zeta) \wedge \delta_l \Omega_{n,q-1}(\eta)(\zeta,\lambda,\cdot)\right\|_{\overline{D},k-1+\frac{1}{m}} \lesssim \|f\|_{\overline{D},0}$.

Therefore equation (15) implies that $\frac{\partial T'_q f}{\partial z_l}$ belongs to $C^{k-1+\frac{1}{m}}_{0,q-1}(\overline{D})$ and satisfies, uniformly with repect to f, $\left\|\frac{\partial T'_q f}{\partial z_l}\right\|_{\overline{D},k-1+\frac{1}{m}} \lesssim \|f\|_{\overline{D},k}$. □

References

1. Alexandre, W.: Construction d'une fonction de support à la Diederich-Fornæss. PUB. IRMA, Lille 2001, Vol.**54**, N° III
2. Bruna, J., Charpentier, P., Dupain, Y.: Zero varieties for the Nevanlinna class in convex domains of finite type in $\mathbb{C}^n$. Ann. Math. **147**, 391–415 (1998)
3. Diederich, K., Fornæss, J.E.: Support functions for convex domains of finite type. Math. Z. **230**, 145–164 (1999)
4. Diederich, K., Fischer, B., Fornæss, J.E.: Hölder estimates on convex domains of finite type. Math. Z. **232**, 43–61 (1999)
5. Diederich, K., Mazzilli, E.: Zero varieties for the Nevanlinna class on all convex domains of finite type. Nagoya Math. J. **163**, 215–227 (2001)
6. Fischer, B.: L^p estimates on convex domains of finite type. Math. Z. **236**, 401–418 (2001)
7. Lieb, I., Range, R.M.: Lösungsoperatoren für den Cauchy-Riemann-Komplex mit C^k-Abschätzungen. Math. Ann. **253**, 145–164 (1980)
8. Michel, J.: $\overline{\partial}$-Problem für stückweise streng pseudokonvexe Gebiete in $\mathbb{C}^n$. Math. Ann. **280**, 45–68 (1988)
9. McNeal, J.D.: Convex domains of finite type. J. Functional Anal. **108**, 361–373 (1992)
10. McNeal, J.D.: Estimates on the Bergman kernels of convex domains. Adv. in Math. **109**(1), 108–139 (1994)
11. Range, R.M.: Holomorphic Functions and Integral Representations in Several Complex Variables. Springer-Verlag, New York, 1986
12. Seeley, R.T.: Extension of C^∞-functions defined in a half space. Proc. Amer. Soc. **15**, 625–626 (1964)