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The structure of F-pure rings

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Dedicated to Professor Melvin Hochster on the occasion of his sixtieth birthday

Abstract. For a reduced F-finite ring R of characteristic $p > 0$ and $q = p^e$ one can write $R^{1/q} = R^{a_q} \oplus M_q$, where M_q has no free direct summands over R. We investigate the structure of F-finite, F-pure rings R by studying how the numbers a_q grow with respect to q. This growth is quantified by the splitting dimension and the splitting ratios of R which we study in detail. We also prove the existence of a special prime ideal $\mathcal{P}(R)$ of R, called the splitting prime, that has the property that $R/\mathcal{P}(R)$ is strongly F-regular. We show that this ideal captures significant information with regard to the F-purity of R.

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a reduced, local ring of positive characteristic $p > 0$ and Krull dimension d. Throughout this paper, $q = p^e$ denotes a power of the characteristic, $R^{1/q}$ is the ring of qth roots of elements in R, and R will usually be assumed to be F-finite, i.e. $R^{1/p}$ is module finite over R. This implies that $R^{1/q}$ is module finite over R for all q. Write $R^{1/q} = R^{a_q} \oplus M_q$, which is a direct sum decomposition of $R^{1/q}$ over R such that M_q has no free direct summands. We would like to investigate the size of the numbers a_q, as q grows to infinity, by studying the splitting dimension of the ring R (for short the s-dimension of R, as it was called in [3]). (For precise definitions, we refer the reader to Section 2). The rings for which some $a_q > 0$ (equivalently, all $a_q > 0$) are called *F-pure*. The purpose of this paper is to demonstrate that there is much more structure to be discerned in such rings, and at least part of this structure is captured by the s-dimension and the splitting ratios (defined below).

The behavior of the sequence $\{a_q\}_q$ captures subtle information about the structure of the ring R with regard to the natural action of Frobenius on it. This behavior is intimately connected to the notions of F-purity and strong F-regularity. In fact, our work starts with and extends ideas originating in papers of Huneke and Leuschke ([13]), and Aberbach and Leuschke ([3]) where the relation between the strong F-regularity and the F-signature of Gorenstein rings ([13]) and Cohen-Macaulay rings ([3]) are analyzed. One can think of the concept of strong F-regularity as a very strong form of F-purity, as the definitions included below show. Our work shows that there is a spectrum of F-pure behavior, ranging from "merely F-pure," in the case that $\operatorname{sdim}(R) = 0$ (i.e., essentially $a_q = 1$ for all q), all the way up to strong F-regularity (i.e., $\operatorname{sdim}(R) = \dim(R)$). Moreover, the central notions of the paper, the splitting dimension and the Frobenius splitting ratios, will quantify this behavior in a clear fashion.

An important outcome of our analysis is the discovery, in any F-pure ring R, of a special prime ideal $\mathcal{P}(R)$, which we will call the *splitting prime* of R, whose dimension (i.e. $\dim(R/\mathcal{P})$) is an upper bound for the splitting dimension of R. A significant feature of this prime ideal $\mathcal{P}$ is that it defines a strongly F-regular quotient $R/\mathcal{P}$. In the case of a Gorenstein ring R this ideal is the only F-stable prime of R (the study of the F-stable primes of ring R was initiated in ([5]).

Let $(R, \mathfrak{m}, k)$ be an F-finite F-pure ring. For simplicity in making an initial statement, assume that the residue field of R is perfect. We let k be the largest integer such that $\liminf_q \frac{a_q}{q^k} > 0$ and call this number the s-dimension of R, $\operatorname{sdim}(R)$. We can then prove

Theorem 1.1. *There exists a prime ideal $\mathcal{P} = \mathcal{P}(R)$ such that R/P is strongly F-regular and* $\operatorname{sdim}(R) \leq \dim(R/\mathcal{P})$. *The prime $\mathcal{P}$ is* 0 *if and only if R is strongly F-regular.*

The proof of this theorem is contained in Theorem 3.3 and Theorem 4.8.

The first part of the paper will introduce the relevant concepts of the paper, such as the splitting dimension and the Frobenius splitting ratios. Section 2 of the paper contains some results about the splitting dimension, the splitting prime ideal $\mathcal{P}$, and the lower and upper Frobenius splitting ratios of a ring R. We also raise the question of whether or not $\operatorname{sdim}(R) = \mathcal{P}(R)$ and provide some evidence in support of an affirmative answer. The study of these concepts continues in Section 3 where we investigate the case of reduced rings that are images of regular local rings and state our main Theorem (Theorem 4.8). The last section of the paper extends some of the earlier results to the class of R-modules.

We would like to conclude our introduction by outlining some of the main facts about F-purity and strong F-regularity. The notion of F-purity was introduced in the work of Hochster and Roberts ([12]) and that of strong F-regularity in a paper by Hochster and Huneke ([9]). Both concepts have been studied by many other authors; the main facts about them, which are now considered standard, will be listed below without any proofs or references.

Let $F : R \to R$ be the Frobenius homomorphism $F(r) = r^p$. We denote by F^e the eth iteration of F, that is $F^e(r) = r^q$, $F^e : R \to R$. One can regard R as an

R-algebra via the homomorphism F^e. Although as an abelian group it equals R, it has a different scalar multiplication. We will denote this new algebra by $R^{(e)}$.

Definition 1.2. *R is* F-finite *if $R^{(1)}$ is module finite over R, or, equivalently (in the case that R is reduced), $R^{1/p}$ is module finite over R. R is called* F-pure *if the Frobenius homomorphism is a pure map, i.e, $F \otimes_R M$ is injective for every R-module M.*

If R is F-finite, then $R^{1/q}$ is module finite over R, for every q. Moreover, any quotient and localization of an F-finite ring is F-finite. Any finitely generated algebra over a perfect field is F-finite. An F-finite ring is excellent.

If R is F-finite, then R is F-pure if and only if the inclusion $R \hookrightarrow R^{1/p}$ splits over R. This is equivalent to the assertion that $R \hookrightarrow R^{1/q}$ splits, for every q. The reader should note that $R^{1/q} \simeq R^{(e)}$ as R-algebras, by sending $r^{1/q}$ to r.

Let R^o be the complement of the minimal primes of the ring R.

Definition 1.3. *A reduced Noetherian F-finite ring R is* strongly F-regular *if for every $c \in R^o$ there exists q such that the R-linear map $R \to R^{1/q}$ that sends 1 to $c^{1/q}$ splits over R, or equivalently $Rc^{1/q} \subset R^{1/q}$ splits over R.*

The notion is related to that of F-purity by the fact that, in the definition above, if a splitting exists for a choice of $c \in R^o$ and q, then the inclusion $R \hookrightarrow R^{1/q'}$ (sending $1 \mapsto 1$) also splits for every q'. Also, if a splitting of $Rc^{1/q} \subset R^{1/q}$ exists, then a splitting of $Rc^{1/q'} \subset R^{1/q'}$ exists too for all $q' \geq q$.

The notion of strong F-regularity localizes well, and all ideals are tightly closed in strongly F-regular rings. Regular rings are strongly F-regular and strongly F-regular rings are Cohen-Macaulay and normal. To show that a reduced Noetherian and F-finite ring is strongly F-regular it is necessary and sufficient to show that, for some $c \in R^o$ such that R_c is strongly F-regular, there exists q such that $Rc^{1/q} \subset R^{1/q}$ splits over R.

Let $E_R(k)$ be the injective hull of the residue field of R. Then R is strongly F-regular if and only if $0^*_{E_R} = 0$. The ideal $\mathrm{Ann}_R(0^*_{E_R})$ is called the CS-test ideal of R and is denoted by $\widetilde{\tau}(R)$. Note that $\widetilde{\tau}(R) = R$ if and only if R is strongly F-regular.

Definition 1.4. *The ring $(R, \mathfrak{m})$ is* approximately Gorenstein *if R has a sequence of $\mathfrak{m}$-primary irreducible ideals $\{I_t\}_t$ cofinal with the powers of $\mathfrak{m}$. See [8].*

By taking a subsequence, we may assume that $I_t \supset I_{t+1}$. For each t, let u_t be an element of R which represents a socle element modulo I_t. Then there is, for each t, a homomorphism $R/I_t \hookrightarrow R/I_{t+1}$ such that $u_t + I_t \mapsto u_{t+1} + I_{t+1}$. The direct limit of the system will be the injective hull $E = E_R(R/\mathfrak{m})$ and each u_t will map to the socle element of E, which we will denote by u. Hochster has shown that every excellent, reduced local ring is approximately Gorenstein ([8]). In particular, every F-finite reduced ring is approximately Gorenstein.

2. Notation and terminology

Let $(R, \mathfrak{m}, k)$ be a reduced, local, F-finite ring of positive characteristic $p > 0$ and Krull dimension d. As in the introduction, let

$$R^{1/q} = R^{a_q} \oplus M_q$$

be a direct sum decomposition of $R^{1/q}$ such that M_q has no free direct summands. If R is complete, such a decomposition is unique up to isomorphism, but it is always the case that the values of the a_q are independent of the decomposition. As mentioned in the introduction, we investigate here the size of the numbers a_q with respect to q.

For a local ring $(R, \mathfrak{m}, k)$, we set $\alpha(R) = \log_p[k : k^p]$. For I an ideal of R, let $I^{[q]} = (i^q : i \in I)R$. It is easy to see that, for an $\mathfrak{m}$-primary ideal I of R, $\lambda_R(R^{1/q}/IR^{1/q}) = \lambda(R/I^{[q]})q^{\alpha(R)}$. If P is a prime ideal of R, then $\alpha(P)$ will simply denote $\alpha(R_P)$. It is often convenient to understand many of the statements in this paper first in the case that k is perfect, i.e., $\alpha(R) = 0$. However, in order to use induction and have the concepts localize well we must keep track of the exponent $\alpha(R)$.

Before we proceed, we would like to remind the reader of the following fact ([14]).

Lemma 2.1. *Let R be an F-finite Noetherian ring of characteristic p. Then for any prime ideals $P \subset Q$ of R,*

$$\alpha(P) = \alpha(Q) + \operatorname{ht}(Q/P).$$

We would like to first define the notion of F-signature as it appears in [3] and [13].

Definition 2.2. *Let R be an F-finite local ring. The* F-signature *of R is*

$$s(R) = \lim_{q\to\infty} a_q/q^{d+\alpha(R)},$$

if it exists.

The following result, due to Aberbach and Leuschke, holds:

Theorem 2.3. *Let $(R, \mathfrak{m}, k)$ be a reduced Noetherian ring of positive characteristic p. Then* $\liminf_{q\to\infty} a_q/q^{d+\alpha(R)} > 0$ *if and only if* $\limsup_{q\to\infty} a_q/q^{d+\alpha(R)} > 0$ *if and only if R is strongly F-regular.*

Moreover, Huneke and Leuschke proved that if R is local, reduced and Gorenstein, then the F-signature exists. Yao has extended this result to rings that are Gorenstein on their punctured spectrum. Recently, the authors of this paper have shown the existence of the F-signature for $\mathbf{N}$-graded rings and isolated non $\mathbf{Q}$-Gorenstein rings, see [2].

We are now able to define the central concepts to be investigated in this paper.

Definition 2.4. *Let R be an F-finite local ring. The* s-dimension *of R,* $\operatorname{sdim}(R)$*, is the largest integer k such that*

$$\liminf_{q\to\infty} \frac{a_q}{q^{k+\alpha(R)}}$$

is not zero. If R is not F-pure then $\operatorname{sdim}(R) = -\infty$. *The* lower Frobenius splitting ratio *of R, $r_F^-(R)$, equals the value of the limit introduced above, that is*

$r_F^-(R) = \liminf_{q\to\infty} \frac{a_q}{q^{\mathrm{sdim}(R)+\alpha(R)}}$. *Similarly, the* upper Frobenius splitting ratio *of* R, $r_F^+(R)$, *equals* $\limsup_{q\to\infty} \frac{a_q}{q^{\mathrm{sdim}(R)+\alpha(R)}}$. *Whenever* $r_F^-(R) = r_F^+(R)$, *we call this number the* Frobenius splitting ratio *of* R, $r_F(R)$.

Whenever $\mathrm{sdim}(R) = \dim(R)$ *(equivalently,* R *is strongly* F*-regular), we recover the notion of* F*-signature of* R, *denoted by* $s(R)$, *and, we will use this terminology whenever we are in such a special case. In particular, whenever* $\mathrm{sdim}(R) = \dim(R)$, *we will be speaking of the* lower and upper signature *of* R, *denoted accordingly by* $s^-(R)$ *and* $s^+(R)$, *instead of its lower and, respectively, upper Frobenius splitting ratio.*

We show below that R is F-pure if and only if $\mathrm{sdim}(R) \geq 0$.

Remark 2.5. In [15] the notion of minimal relative Hilbert-Kunz multiplicity is defined, denoted $m_{HK}(R)$. Theorem 2.3 shows that this number is non-zero if and only if R is strongly F-regular. In our terminology, $m_{HK}(R) = 0$ if $\mathrm{sdim}(R) < \dim(R)$ and $m_{HK}(R) = s^-(R)$ otherwise.

The following result is classical and very helpful in understanding the number of free direct summands of $R^{1/q}$. We will quote it here in the form stated in [11].

Theorem 2.6. *Let* $(R, \mathfrak{m}, k)$ *be a local ring and* F *be a free finitely generated* R*-module. Let* M *be a finitely generated* R*-module and* $\phi : F \to M$ *an* R*-linear map. Then* ϕ *splits over* R *if and only if* $\phi \otimes E_R(k)$ *is an injective map.*

If M is a finite R-module that has no nonzero free direct summands, then for every $m \in M$, the map $R \to M$ that sends 1 to m does not split. By tensoring with $E_R(k)$ and using the above Theorem, we conclude that the map $E_R(k) \to E_R(k) \otimes M$ that sends e to $e \otimes m$ is not injective. Denote by u the socle generator for $E_R(k)$. The following holds:

Proposition 2.7. *Let* $(R, \mathfrak{m}, k)$ *be a local ring, let* M *be a finite* R*-module, and let* u *be the socle generator of* $E_R(k)$. *Then* M *has no nonzero free direct summands if and only if for every* $m \in M$, $u \otimes m = 0$ *in* $E_R(k) \otimes M$.

Proof. For each $m \in M$, define $f_m : R \to M$ by $f_m(r) = rm$. Then $f_m$does not split over R if and only if $f_m \otimes E_R(k)$ is not injective if and only if $\ker(f_m \otimes E_R(k))$ is nonzero (Theorem 2.6). This holds if and only if $\ker(f_m \otimes E_R(k))$ intersects the socle of $E_R(k)$ non trivially. But, $\mathrm{Soc}(E_R(k)) = Ru$. So, $f_m \otimes E_R(k)$ is not injective if and only if $u \otimes m = 0$. □

The following Corollary will prove to be of much use in what follows. Yao has also noted this result in his recent work [16].

Corollary 2.8. *Let* $(R, \mathfrak{m}, k)$ *be an* F*-finite reduced local ring of characteristic* p. *Let* $R^{1/q} = R^{a_q} \oplus M_q$ *be a direct sum decomposition of* $R^{1/q}$ *over* R, *where* M_q *has no* R*-free direct summands. Define*

$$J_q := \{r \in R^{1/q} : r \otimes u = 0 \text{ in } R^{1/q} \otimes E_R(k)\}.$$

Then, $a_q = \lambda_R(R^{1/q}/J_q)$.

Proof. $R^{1/q} = R^{a_q} \oplus M_q$; so, by tensoring with $Ru \subset E_R(k)$, we get

$$R^{1/q} \otimes R \cdot u = (R^{a_q} \otimes R \cdot u) \oplus (M_q \otimes R \cdot u).$$

Map this further to $R^{1/q} \otimes E_R(k)$ and apply Proposition 2.7 to M_q. We have that $\mathrm{Im}(R^{1/q} \otimes R \cdot u \to R^{1/q} \otimes E_R(k)) = \mathrm{Im}(g : R^{a_q} \otimes Ru \to R^{1/q} \otimes E_R(k))$, where $g(a \otimes ru) = a \otimes ru$. But g factors through $R^{a_q} \otimes E_R(k) \to R^{1/q} \otimes E_R(k)$ which is an injective map (again by Theorem 2.6), so g is also injective. But $\lambda(Ru) = 1$, and hence $\lambda(R^{a_q} \otimes Ru) = a_q$. In conclusion,

$$\lambda(\mathrm{Im}(R^{1/q} \otimes R \cdot u \to R^{1/q} \otimes E_R(k))) = a_q.$$

There is a natural R-linear map $R^{1/q} \to R^{1/q} \otimes E_R(k)$ that sends $r \in R^{1/q}$ to $r \otimes u$. The kernel of this map equals J_q and the theorem is proved. □

3. Splittings, s-dimension and strong F-regularity

Let $F^e : R \to R$ be the eth iteration of the Frobenius map. We can regard R as a new (right) R-algebra, under this map, which we denote by $R^{(e)}$. Corollary 2.8 leads us to define the following sequence of ideals.

Definition 3.1. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring. We define*

$$A_e = A_e(R) = \{r \in R : r \otimes u = 0 \in R^{(e)} \otimes_R E_R(k)\}.$$

If R is not reduced then the map $R \to R^{(e)}$ cannot be injective for $e > 0$, and hence by Theorem 2.6, $A_e = R$ for all $e \geq 1$. So assume that R is reduced (hence approximately Gorenstein). Then clearly, $A_e = (J_q)^{[q]}$ (note that $R^{1/q} \simeq R^{(e)}$ by $r^{1/q} \longmapsto r$). In fact, A_e is an ideal both of R and $R^{(e)}$. Using the notation in Definition 1.4, we see that $A_e = \cup_t (I_t^{[q]} : u_t^q)$. We claim that the sequence $\{A_e\}_e$ is non-increasing. If R is not F-pure then $A_e = R$ for all e. Otherwise, let $r \in A_e$. For $t \gg 0$, $ru_t^q \in I_t^{[q]}$ where $q = p^e$, hence $r^p u_t^q \in I_t^{[q]}$. Taking pth roots and applying the splitting yields, $ru_t^{q/p} \in I_t^{[q/p]}$, showing that $A_e \subseteq A_{e-1}$.

Definition 3.2. *Let $(R, \mathfrak{m}, k)$ be an F-finite reduced local ring. Let*

$$\mathcal{P}(R) = \{r \in R : r \otimes u = 0 \in R^{(e)} \otimes_R E_R(k) \text{ for all } e \gg 0\} = \cap_e A_e.$$

If the ring is understood we will sometimes refer to this ideal simply as $\mathcal{P}$.

As we will show in Theorem 3.3 below, the ideal $\mathcal{P}(R)$ is prime.

Since u is the socle generator for $E_R(k)$, we have $r^q \otimes u = 1 \otimes ru$, which is 0 if $r \in \mathfrak{m}$, so $m^{[q]} \subset A_e$ for every q. On the other hand, if $u \notin 0^*_{E_R}$, then $\mathcal{P} = 0$ (see [1], Proposition 2.4). If R is a domain, then $\mathcal{P}$ is nonzero only if R is not strongly F-regular, since $0^*_{E_R} = 0$ in strongly F-regular rings (see also Corollary 3.4). The reader is invited to note that there is a similarity between the ideal $\mathcal{P}(R)$ introduced here and the concept of F-stable primes of R introduced in [5]. In fact, when R is Gorenstein, then R admits a unique F-stable prime, which, indeed,

is $\mathcal{P}$. This similarity is underlined by our next Theorem that asserts that $\mathcal{P}(R)$ is in fact a prime ideal of R. We will refer to $\mathcal{P}(R)$ as the *splitting prime* for R.

Before we state the result, we need to introduce some notation. For every $c \in R$, one can define an R-linear map $\phi_{c,e} : R \to R^{1/q}$, $\phi_{c,e}(1) = c^{1/q}$, where $q = p^e$.

Theorem 3.3. *Let $(R, \mathfrak{m}, k)$ be an F-finite reduced local ring of characteristic p. Then, $A_e = \{c \in R : \phi_{c,q}$ does not split over $R\}$ and $\mathcal{P}(R)$ is a prime ideal or the unit ideal.*

Proof. By Theorem 2.6 we know that $\phi_{c,e}$ splits if and only $\phi_{c,e} \otimes E_R(k)$ is injective.

Take $c \in A_e$. Then $c^{1/q}$ belongs to J_q. Hence $c^{1/q} \otimes u = 0$, and therefore $\phi_{c,e}$ does not split. Now, assume that $c \in R$ is an element such that the map $\phi_{c,q}$ does not split. Then again $u \in \operatorname{Ker}(\phi_{c,e} \otimes E_R(k))$ which means that $c^{1/q} \in J_q$. So, $c \in A_e$. This shows the first assertion of the Theorem.

For the second assertion assume that $\mathcal{P}(R)$ is not the unit ideal and let us write

$$\mathcal{P} = \{c \in R : \phi_{c,e} \text{ does not split for any } e \gg 0\}.$$

If $\mathcal{P}$ is not prime, then there exist a and b not in $\mathcal{P}$ such that $ab \in \mathcal{P}$. Since a and b are not in $\mathcal{P}$ we can find e_1 and e_2 large enough such that the R-linear maps ϕ_{a,e_1} and ϕ_{b,e_2} split. Let $\theta_1 : R^{1/q_1} \to R$ an R-linear that maps a^{1/q_1} to 1. This map exists because ϕ_{a,e_1} splits over R. Similarly, we can find $\theta_2 : R^{1/q_1 q_2} \to R^{1/q_1}$, such that $\theta_2(b^{1/q_1 q_2}) = 1$ and θ_2 is R^{1/q_1}-linear (we can do this by taking the q_1-root of a splitting for ϕ_{b,e_2}).

Let $\theta = \theta_1\theta_2$. It is easy to check that θ is R-linear and $\theta((a^{q_2}b)^{1/q_1 q_2}) = 1$. So, for $e_0 = e_1 + e_2$, $\phi_{a^{q_2}b,e_0}$ splits.

We know that $ab \in \mathcal{P}$, and this implies that $a^{q_2}b \in \mathcal{P}$. So, if we fix e_2, this means that $\phi_{a^{q_2}b,e}$ does not split for any $e \gg 0$ large enough. This is a contradiction, since we have just shown the existence of a splitting for values q of the form $q = q_0 = p^{e_0} = p^{e_1+e_2}$, where $q_1 = p^{e_1}$ can be arbitrarily large.

In conclusion, $\mathcal{P}$ is a prime ideal. □

Corollary 3.4. *Let $(R, \mathfrak{m}, k)$ be an F-finite reduced local ring of characteristic p. Then, R is F-pure if and only if $\mathcal{P}(R)$ is a proper ideal if and only if* $\operatorname{sdim}(R) \geq 0$. *In this case,*

$$\operatorname{sdim}(R) \leq \dim(R/\mathcal{P}).$$

The ring R is strongly F-regular if and only if $\mathcal{P} = 0$. Also, $\dim(R/\mathcal{P}) = 0$ *if and only if $\mathcal{P} = \mathfrak{m}$ if and only if* $\operatorname{sdim}(R) = 0$, *in which case $r_F(R) = 1$.*

Proof. R is F-pure if and only if some $a_q > 0$ if and only if A_e is proper, and since the sequence $\{A_e\}_e$ is non-increasing this occurs if and only if $\mathcal{P}(R) \neq R$. By Theorem 2.8, $a_q = \lambda(R^{1/q}/J_q)$ and hence $a_q = \lambda(R/A_e) \cdot q^{\alpha(R)}$ with $\alpha(R) = \log_p[k : k^p]$. From this we can see that R is F-pure if and only if $\operatorname{sdim}(R) \geq 0$.

We next show that $\operatorname{sdim}(R) \leq \dim(R/\mathcal{P})$. Since $\mathfrak{m}^{[q]} \subset A_e$, for every $q = p^e$, we have that $\mathfrak{m}^{[q]} + \mathcal{P} \subset A_e$. Then $a_q/q^{k+\alpha(R)} = \lambda(R/A_q)/q^k \leq \lambda(R/\mathfrak{m}^{[q]} + \mathcal{P})/q^k$. If $k > \dim(R/\mathcal{P})$, then by the theory of Hilbert-Kunz functions, $\lim_{q\to\infty} \lambda(R/\mathfrak{m}^{[q]} + \mathcal{P})/q^k = 0$. So, $\operatorname{sdim}(R) \leq \dim(R/\mathcal{P})$.

The ideal $\mathcal{P} = 0$ if and only if for all $c \neq 0$ there exists a q such that $\phi_{c,q}$ splits; this is precisely the definition of a strongly F-regular ring.

Since $\mathcal{P}$ is prime, $\dim(R/\mathcal{P}) = 0$ if and only if $\mathcal{P} = \mathfrak{m}$. Clearly $\mathcal{P} \neq R$ if and only if R is F-pure if and only if $A_e \neq R$ for all e. So, $\mathcal{P} = \mathfrak{m}$ if and only if R is F-pure and $A_e = \mathfrak{m}$ for all e. In conclusion, $a_q/q^{\alpha(R)} = \lambda(R/A_e) = \lambda(R/\mathfrak{m}) = 1$. Hence, in particular, $r_F(R) = 1$. □

Remark 3.5. We note the following facts:

1) The proof of the above Corollary shows that, in fact, when R is F-pure and $\mathcal{P} = \mathfrak{m}$ (an important such case is that of an isolated non-strongly F-regular point), then $a_q = q^{\alpha(R)}$.
2) Assume that R is strongly F-regular. A result of Aberbach ([1]) shows that there exists e_o such that $A_e \subset \mathfrak{m}^{[q/q_o]}$ for every $e \geq e_o$. This shows that $a_q/q^{d+\alpha(R)} \geq \lambda(R/\mathfrak{m}^{[q/q_o]})/q^d$; since $e_{HK}(\mathfrak{m}) > 0$, we conclude that $\operatorname{sdim}(R) = d$ and $r_F(R) \geq e_{HK}(\mathfrak{m})/q_o^d$. This result was first established in [3] and our remark is along the lines of the argument given there.
3) Also, let us take $c \in \widetilde{\tau}(R) = \operatorname{Ann}_R(0^*_{E_R})$, a CS-test element. The ring R_c is strongly F-regular. If $c \notin A_e$, for some e, then R is strongly F-regular (as explained in the introduction). So, if R is not strongly F-regular, then $\widetilde{\tau}(R) \subseteq \mathcal{P}$.

The splitting prime of R does not localize well in general. The most obvious example is when $\mathcal{P}(R) = \mathfrak{m}$. Then for all $Q \in \operatorname{Spec}(R)$, R_Q is F-pure, so $\mathcal{P}(R_Q) \neq R_Q = \mathcal{P}(R)R_Q$. However, when $Q \supseteq \mathcal{P}(R)$ we do get a nice localization result.

Proposition 3.6. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic p. Then for every $Q \in \operatorname{Spec}(R)$ such that $\mathcal{P} \subset Q$ we have*

$$\mathcal{P}(R_Q) = \mathcal{P}(R)R_Q.$$

Proof. We first note that $(R^{1/q})_Q = (R_Q)^{1/q} = R^{1/q} \otimes R_Q$. Moreover, if $\phi_{c,e}$ splits, then $\phi_{c,e} \otimes R_Q$ splits, too.

Take $c/s \in \mathcal{P}(R_Q)$. Then $c/1$ belongs to $\mathcal{P}(R_Q)$. We would like to show that $c \in \mathcal{P}(R)$. Assume the contrary; then, $\phi_{c,e} : R \to R^{1/q}$ does split for $e \gg 0$. By tensoring with R_Q we get a splitting over R_Q, which is a contradiction.

Now, take $c \in \mathcal{P}(R)$. We want to show that $c/1$ belongs to $\mathcal{P}(R_Q)$. If not, for large q, the map $R_Q \to R_Q^{1/q}$ that sends 1 to $c^{1/q}/1$ splits. So, for each such $q = p^e$, there is an element $d_q \notin Q$ and a map $R^{1/q} \to R$ that sends $c^{1/q}$ to d_q. But, d_q is not in $\mathcal{P}(R)$, since $\mathcal{P}(R) \subset Q$. So, for $q' \gg 0$, there is an R-linear map $R^{1/q'} \to R$ that sends $d_q^{1/q'}$ to 1. Combining these last two assertions, we conclude that c is not in $\mathcal{P}(R)$ and this is a contradiction. □

Proposition 3.7. *Let R be an F-pure local ring. Then $Q \subset \mathcal{P}$, for every $Q \in \operatorname{Min}(R)$.*

Proof. Let $c \in R$. For every q, the map $\phi_{c,q}$ is not injective if c is a zero-divisor on R. Since R is reduced this means that c belongs to some minimal prime Q of R. So, if $c \in Q$, with Q a minimal prime, then $\phi_{c,q}$ cannot split over R for any q. The assertion of the Proposition follows. □

We would like to show that the splitting prime of the completion is extended. In order to do this we need the next Lemma.

Lemma 3.8. *Let $(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a faithfully flat map of F-finite local reduced rings such that the closed fiber $S/\mathfrak{m}S$ is regular of dimension n. Then $A_e(S)=A_e(R)S+\mathfrak{n}^{[q]}$. If $\operatorname{sdim} R=0$, then $\operatorname{sdim} S=n$, $\mathcal{P}(S)=\mathfrak{m}S$, and $r_F^-(S)=r_F^+(S)=1$.*

Proof. Both rings are approximately Gorenstein, and the closed fiber $S/\mathfrak{m}S$ is regular of dimension n. Let $z_1,\ldots,z_n$ be a sequence of elements in S that define a regular sequence on $S/\mathfrak{m}S$. Then for any $q=p^e$, $z_1^q,\ldots,z_n^q$ form a regular sequence on $S/\mathfrak{m}S$ and, in fact, $R\to S/(z_1^q,\ldots,z_n^q)S$ if a faithfully flat map.

If $\{I_t\}_t$ is a sequence of irreducible $\mathfrak{m}$-primary ideals of R, cofinal with the powers of $\mathfrak{m}$, then $\{J_t\}_t=\{I_tS+(z_1^t,\ldots,z_n^t)\}_t$ is a sequence of irreducible $\mathfrak{n}$-primary ideals of S, cofinal with the powers of $\mathfrak{n}$. Let u_t be elements of R which are socle elements modulo I_t. Then the images of the $u_t\cdot(z_1\cdots z_n)^{t-1}$ in S are socle elements modulo J_t. Moreover, $R/I_t^{[q]}\to S/I_t^{[q]}S$ is flat with regular closed fiber of dimension n. Since $R/I_t^{[q]}$ has depth 0, we see that $S/I_t^{[q]}S$ has depth n. It follows that $S/I_t^{[q]}S$ is Cohen-Macaulay. In particular, this shows that $z_1,\ldots,z_n$ form a regular sequence on $S/I_t^{[q]}S$, because their images in $S/I_t^{[q]}S$ form a system of parameters.

Let us compute $A_e(S)=\cup_t((I_t^{[q]}S+(z_1^{tq},\cdots,z_n^{tq})):u_t^q\cdot(z_1\cdots z_n)^{(t-1)q})$. If $a\in S$ such that

$$a\cdot u_t^q\cdot(z_1\cdots z_n)^{(t-1)q}\in I_t^{[q]}S+(z_1^{tq},\cdots,z_n^{tq})$$

then, by using that $z_1,\ldots,z_n$ form a regular sequence on $S/I_t^{[q]}S$, we have that

$$au_t^q\in I_t^{[q]}S+(z_1^q,\ldots,z_n^q).$$

This is equivalent to $a\in(I_t^{[q]}S+(z_1^q,\ldots,z_n^q)):_S u_t^q)$.

By flatness of $S/(z_1^q,\ldots,z_n^q)$ over R, $(I_t^{[q]}S+(z_1^q,\ldots,z_n^q)):_S u_t^q=(I_t^{[q]}:_R u_t^q)S+(z_1^q,\ldots,z_n^q)$. In conclusion, $A_e(S)=\cup_t((I_t^{[q]}:_R u_t^q)S+(z_1^q,\ldots,z_n^q))$ and hence $A_e(S)=A_e(R)S+(z_1^q,\ldots,z_n^q)$. So, if $\operatorname{sdim}(R)=0$, then $A_e(S)=\mathfrak{m}^{[q]}S+(z_1^q,\ldots,z_n^q)$. Hence, $a_q=\lambda_S(S/A_e(S))=\lambda_S(S/(\mathfrak{m}S+(z_1^q,\ldots,z_n^q))$ which shows that $\operatorname{sdim}(S)=n$. □

Proposition 3.9. *Let $(R,\mathfrak{m})$ and $(S,\mathfrak{n})$ be local, reduced, F-finite rings of characteristic p and let $f:(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a flat local map with regular fibers. Then $A_e(S)=A_e(R)S+\mathfrak{n}^{[q]}$ for every $q=p^e$, and $\mathcal{P}(S)=\mathcal{P}(R)S$. In particular, if R is local, reduced and F-finite, then $A_e(R)\widehat{R}=A_e(\widehat{R})$ and $\mathcal{P}(R)\widehat{R}=\mathcal{P}(\widehat{R})$.*

Proof. The assertion that $A_e(S)=A_e(R)+\mathfrak{n}^{[q]}$ has been shown in Lemma 3.8.

To see that $\mathcal{P}(S)=\mathcal{P}(R)S$ we first observe that $\mathcal{P}(S)=\cap_e A_e(S)$ is a prime ideal of S lying over $\mathcal{P}(R)$.

We first show that any prime of S that is minimal over $\mathcal{P}(R)S$ is in fact contained in $\mathcal{P}(S)$. Let $Q\neq\mathcal{P}(S)$ be a prime ideal of S that is minimal over $\mathcal{P}(R)S$. Since

$R \to S$ is flat then Q lies over $\mathcal{P}$. Thus the map $R_{\mathcal{P}(R)} \to S_Q$ is faithfully flat and $\mathcal{P}(R)S_Q = QS_Q$. By Lemma 3.8, $\operatorname{sdim}(S_Q) = 0$. In particular, $A_e(S_Q) = QS_Q$ for all e. However, if we take $c \in Q - \mathcal{P}(S)$, then for $q \gg 0$, $c^{1/q}S \subseteq S^{1/q}$ splits over S. This remains true after localizing, showing that $A_e(S_Q) \neq QS_Q$, a contradiction.

In conclusion, $Q \subseteq \mathcal{P}(S)$.

Note that $\mathcal{P}(R)$ is prime in R, and since $R \to S$ has regular fibers, then $\mathcal{P}(R)S$ is a radical ideal in S.

Now, we would like to argue that $\mathcal{P}(S)$ itself is a minimal prime of $\mathcal{P}(R)S$. The homomorphism $R_{\mathcal{P}(R)} \to S_{\mathcal{P}(S)}$ is flat because $\mathcal{P}(S)$ lies over $\mathcal{P}(R)$. Its closed fiber is regular and $\operatorname{sdim}(R_{\mathcal{P}(R)}) = 0$. If the dimension of its closed fiber is positive, then, by Lemma 3.8 below, $\operatorname{sdim}(S_{\mathcal{P}(S)}) > 0$. However, this is impossible. In conclusion, the closed fiber of $R_{\mathcal{P}(R)} \to S_{\mathcal{P}(S)}$ is 0-dimensional, so $\mathcal{P}(S)$ is minimal over $\mathcal{P}(R)S$.

Now, since $\mathcal{P}(R)S$ is radical and its only minimal prime is $\mathcal{P}(S)$, we obtain the equality $\mathcal{P}(R)S = \mathcal{P}(S)$.

The last assertion about the completion follows from the fact that, under our hypotheses, R is excellent, so all the fibers of the completion homomorphism are smooth, hence regular. □

Proposition 3.9 enables us to reduce the study of the s-dimension and the Frobenius splitting ratio of reduced excellent rings to the case of complete local rings. These rings are images of regular local rings. Following in the paths of Fedder, Glassbrenner, and Cowden-Vassilev we can analyze the finer structure of F-pure rings in this way.

4. Images of regular rings

In this section, we continue our analysis in the case of local rings that are images of regular local rings. The more general case of reduced F-finite rings can be reduced to this class of rings by Proposition 3.9.

Let us write $R = S/I$ where S is regular and local and $\pi : S \to R$ the natural projection. Assume that R is reduced, i.e., I is a radical ideal. There are two results that describe under what conditions such rings are F-pure or strongly F-regular. The first one is due to Fedder (Theorem 1.12 in [6]). It does not require R to be F-finite.

Theorem 4.1. *Let $(S, \mathfrak{m}, k)$ be a regular local ring of characteristic p and let $R = S/I$. Then R is F-pure if and only if $(I^{[p]} : I) \not\subset \mathfrak{m}^{[p]}$.*

The idea behind this criterion for F-purity has been used by Glassbrenner to give a similar criterion for strong F-regularity (Theorem 2.3 in [7]). We will state it here in the form we need it later.

Theorem 4.2. *Let $(S, \mathfrak{m}, k)$ be a local regular ring and let $R = S/I$ and $c \in S$. Then, for every $q = p^e$, the map $\phi_{\pi(c),e}$ splits if and only if $c \notin \mathfrak{m}^{[q]} :_S (I^{[q]} : I)$. Moreover, R is strongly F-regular if and only if $I = \cap_q \big(\mathfrak{m}^{[q]} :_S (I^{[q]} : I)\big)$.*

Using the notation introduced previously, we see that $A_e = \left(\mathfrak{m}^{[q]} :_S (I^{[q]} : I)\right)/I$. So, $a_q = q^{\alpha(R)}\, \lambda_S(S/(\mathfrak{m}^{[q]} :_S (I^{[q]} : I)))$. Note that $S/\mathfrak{m}^{[q]}$ is zero dimensional and Gorenstein. The quotient $S/\left(\mathfrak{m}^{[q]} :_S (I^{[q]} : I)\right)$ is an $S/\mathfrak{m}^{[q]}$-module and its Matlis dual is $\frac{(I^{[q]}:I)+\mathfrak{m}^{[q]}}{\mathfrak{m}^{[q]}}$. Hence, $a_q = q^{\alpha(R)}\, \lambda_S(\frac{(I^{[q]}:I)+\mathfrak{m}^{[q]}}{\mathfrak{m}^{[q]}})$.

Although Theorem 4.2 was stated in [7] under the assumption that S is local, regular and F-finite, one can note that the F-finiteness hypothesis can be removed along the lines of the argument used by Fedder in proving his criterion for F-purity (where we interpret "strongly F-regular" to mean that $0_E^* = 0$). The point is that one can make a flat base change, by enlarging the residue field to its perfect closure, to get to the case where S is F-finite. Strong F-regularity commutes with this base change (see, for example, Theorem 3.6 [1]).

Remark 4.3 (for proofs, see [6], 1.4 and 1.5). For every $q = p^e$, $\mathrm{Hom}_S(S^{1/q}, S)$ has an $S^{1/q}$- structure given by $s^{1/q} \cdot \phi(t^{1/q}) := \phi((st)^{1/q})$, for every $s, t \in S$ and $\phi \in \mathrm{Hom}_S(S^{1/q}, S)$. Moreover, $\mathrm{Hom}_S(S^{1/q}, S) \simeq S^{1/q}$. Let T be a generator for $\mathrm{Hom}_S(S^{1/q}, S)$ and $s^{1/q} \in S$. Then $s^{1/q}T$ defines an element in $\mathrm{Hom}_R(R^{1/q}, R)$ if and only if $s^{1/q} \in (IS^{1/q} : I^{1/q})$ if and only if $s \in (I^{[q]} : I)$.

The following result is similar to Theorem 3.1 in [4], in which Cowden-Vassilev showed that if R is F-pure then so is $R/\tau(R)$, where $\tau(R)$ is the test ideal of R. We give two proofs: one is similar to the proof of Theorem 3.1 given in [4], while the other one uses some ideas originating in Fedder's work, [6].

Proposition 4.4. *Let $(S, \mathfrak{m}, k)$ be an F-finite local regular ring and let $R = S/I$ be reduced and F-pure. Set $\mathcal{Q}$ to be the full preimage of $\mathcal{P}(R)$ in S. Then for all q, $(I^{[q]} : I) \subset (\mathcal{Q}^{[q]} : \mathcal{Q})$. In particular, if R is F-finite then* $\mathrm{sdim}(R/\mathcal{P}) \geq \mathrm{sdim}(R)$.

Proof. Proof 1. R is excellent and reduced, so it is approximately Gorenstein. Let $\{I_t\}$ be a collection of irreducible $\mathfrak{m}$-primary ideals in R cofinal with the powers of $\mathfrak{m}$, and denote by $\{J_t\}$ their full preimages in S. Let u_t be an element of S mapping to the socle in S/J_t. Then we may describe $\mathcal{Q}$ as $\mathcal{Q} = \cap_q(\cup_t(J_t^{[q]} + I) :_S u_t^q)$. Let $w \in (I^{[q]} : I)$ and $v \in \mathcal{Q}$. We want to show that $vw \in \mathcal{Q}^{[q]} = \left(\cap_{q'} \cup_t (J_t^{[q']} + I) : u_t^{q'}\right)^{[q]}$, which by flatness of the Frobenius endomorphism over S is $\cap_{q'} \cup_t \left((J_t^{[qq']} + I^{[q]}) : u_t^{qq'}\right)$. For all q' there is a t such that $vu_t^{qq'} \in J_t^{[qq']} + I$, hence $vwu_t^{qq'} \in J_t^{[qq']} + I^{[q]}$. This shows that $vw \in \mathcal{Q}^{[q]}$, as desired.

Proof 2. Let T be as in Remark 4.3. In the light of this Remark we need to show that if $\psi := s^{1/q}T \in \mathrm{Hom}_R(R^{1/q}, R)$, then $(s\mathcal{Q})^{1/q} \in \mathcal{Q}S^{1/q}$. By the same Remark, this is equivalent to the assertion that ψ induces an R-linear map $R^{1/q}/\mathcal{Q}^{1/q} \to R/\mathcal{Q}$.

We can define an $R^{1/q'}$-linear map by $\psi_{q'} : R^{1/qq'} \to R^{1/q'}$, $\psi_{q'}(a) = \psi(a^{q'})^{1/q'}$. This map is, in particular, R-linear.

Take $c \in \mathcal{Q}$; then, $c^{1/qq'} \otimes u = 0$ in $R^{1/qq'} \otimes E_R(k)$ where u is the socle generator in $E_R(k)$.

Clearly, $\psi_{q'}(c^{1/qq'}) \otimes u = 0$. That is, $\psi(c^{1/q})^{1/q'} \otimes u = 0$. So, $s^{1/q}T(c^{1/q}) = \psi(c^{1/q}) \in \mathcal{Q}$. Hence, $s^{1/q}T$ takes $\mathcal{Q}^{1/q}$ into $\mathcal{Q}$. So, $s^{1/q}T$ defines an element in $\mathrm{Hom}_R(R^{1/q}/\mathcal{Q}^{1/q}, R/\mathcal{Q})$, and hence $s \in (\mathcal{Q}^{[q]} : \mathcal{Q})$.

The last statement now follows, after noting that $\alpha(R) = \alpha(R/\mathcal{P})$, since

$$a_q(R) = q^{\alpha(R)}\,\lambda_S\left(\frac{I^{[q]} : I + \mathfrak{m}^{[q]}}{\mathfrak{m}^{[q]}}\right)$$
$$\leq q^{\alpha(R/\mathcal{P})}\,\lambda_S\left(\frac{\mathcal{Q}^{[q]} : \mathcal{Q} + \mathfrak{m}^{[q]}}{\mathfrak{m}^{[q]}}\right) = a_q(R/\mathcal{P}).$$

□

Remark 4.5. This result improves Theorem 3.4, since the s-dimension of a ring is bounded above by its dimension.

Corollary 4.6. *Let R be a characteristic p local F-pure ring that is a homomorphic image of a regular local ring $(S, \mathfrak{m}, k)$, that is $R = S/I$. Then* $\operatorname{sdim}(R_{\mathcal{P}}) = 0$.

Proof. Denote by $k(\mathcal{P})$ the residue field of the localization of R at $\mathcal{P}$. Apply Theorem 4.4 to $R_{\mathcal{P}}$. We get that $0 = \operatorname{sdim}(k(\mathcal{P})) \geq \operatorname{sdim}(R_{\mathcal{P}})$ and we are done. □

Theorem 4.7. *Let $R = S/I$ be a characteristic p local F-pure, F-finite ring that is a homomorphic image of a regular local ring S. Then $R/\mathcal{P}$ is strongly F-regular.*

Proof. If $\operatorname{sdim} R = 0$, i.e., $\mathcal{P} = \mathfrak{m}_R$ this is clear. Thus we may assume that $\operatorname{sdim} R > 0$. If $R/\mathcal{P}$ is not strongly F-regular we may localize a prime minimal among the set $\{P | (R/\mathcal{P})_P$ is not strongly F-regular$\}$. After relabeling this ring R, we note that the s-dimension is still positive. Thus we can assume that $(R/\mathcal{P})_Q$ is strongly F-regular for all Q, different from $\mathfrak{m}$, containing $\mathcal{P}$. We know that $R/\mathcal{P}$ is F-pure by Proposition 4.4.

If $R/\mathcal{P}$ is not strongly F-regular, then $\mathcal{P}(R/\mathcal{P})$ equals $\mathfrak{m}_{R/\mathcal{P}}$ since $\mathfrak{m}_{R/\mathcal{P}} = \widetilde{\tau}(R/\mathcal{P}) \subseteq \mathcal{P}(R/\mathcal{P})$. But then $\operatorname{sdim}(R/\mathcal{P}) = 0$, a contradiction. □

Now we are in a position to describe the s-dimension of a local ring of positive characteristic. We also give some information on the splitting ratios. We expect that the upper and lower splitting ratios always coincide. At present, we can only prove this if $\operatorname{sdim}(R) \leq 1$, however, we can give an upper bound on the ratio $r_F^+(R)/r_R^-(F)$, if $\operatorname{sdim}(R) = \dim(R/\mathcal{P})$.

Theorem 4.8. *Let R be an F-pure, F-finite local ring of positive characteristic. Then*

i) $R/\mathcal{P}$ *is strongly F-regular,* $\operatorname{sdim}(R) \leq \dim(R/\mathcal{P})$. *Moreover, if* $\operatorname{sdim}(R) = \dim(R/\mathcal{P})$, *then* $r_F^-(R) \leq s^-(R/\mathcal{P}) \leq 1$, $r_F^+(R) \leq s^+(R/\mathcal{P}) \leq 1$.

ii) If $\operatorname{sdim}(R) = \dim(R/\mathcal{P})$, *then* $r_F^+(R) \leq r_F^-(R) \cdot e_{HK}(R/\mathcal{P})$. *This shows that, in this case,* $r_F^+(R) > 0$ *if and only if* $r_F^-(R) > 0$.

iii) If $\dim(R/\mathcal{P}) = 1$, *then* $\operatorname{sdim}(R) = 1$ *and* $R/\mathcal{P}$ *is a DVR.*

iv) If $\dim(R/\mathcal{P}) \leq 1$ *then* $r_F^+(R) = r_F^-(R)$.

v) $\dim(R/\mathcal{P}) = \dim(R)$ *if and only if R is strongly F-regular. In this case,* $s^-(R) \leq s^+(R) e_{HK}(R)$.

vi) $\operatorname{depth} R \geq \operatorname{sdim} R$.

Proof. One can pass to the completion of R by Proposition 3.9, and hence assume that $R = S/I$ with S regular local.

For part i), let us note that since R is a homomorphic image of a regular ring, $R/\mathcal{P}$ is strongly F-regular by Theorem 4.7.

The inequalities stated in i) follow easily from the proof of Proposition 4.4.

To prove ii), let $j = \operatorname{sdim}(R)$ and choose two sequences of indices q and q' such that $\lambda(R/A_e)/q^j$ approaches r_F^- and $\lambda(R/A_{e+e'})/(qq')^j$ approaches r_F^+. Let us note that $\mathcal{P} + A_{e'}^{[q]} \subset A_{e+e'}$.

Claim:

$$\lambda(R/A_{e+e'}) \leq \lambda(R/\mathcal{P} + A_e^{[q']}) \leq \lambda(R/A_e) \cdot \lambda(R/\mathcal{P} + \mathfrak{m}^{[q']}).$$

Indeed, the left side of the inequality is immediate since $\mathcal{P} + A_{e'}^{[q]} \subset A_{e+e'}$. Set $k = \lambda(R/A_e)$ and write a composition series $A_e = I_o \subset \cdots \subset I_i \subset \cdots \subset I_k = R$. Then $(\mathcal{P} + A_e^{[q']}) \subset \cdots \subset (\mathcal{P} + I_i) \subset \cdots \subset (\mathcal{P} + I_k) = R$. The successive quotients of this filtration are homomorphic images of $R/\mathcal{P} + \mathfrak{m}^{[q]}$. Hence, $\lambda(R/A_{e+e'}) \leq \lambda(R/\mathcal{P} + A_e^{[q']}) \leq k \cdot \lambda(R/\mathcal{P} + \mathfrak{m}^{[q]})$.

By dividing on both sides by $(qq')^j$, with $j = \operatorname{sdim}(R) = \dim(R/\mathcal{P})$, and letting q, q' approach infinity we obtain the inequality stated in ii).

For iii), note that a one-dimensional strongly F-regular ring R is a DVR. Part iv) follows at once from i), ii) and iii) by noting that the Hilbert-Kunz multiplicity of a local regular ring equals one.

For v), $\dim(R/\mathcal{P}) = \dim(R)$ implies that $\mathcal{P}$ is a minimal prime of R. Since $\mathcal{P}$ contains all the minimal primes of R, it follows that $\mathcal{P}$ is, in fact, the only minimal prime of R. But R is reduced, so $\mathcal{P} = 0$. Hence, R is strongly F-regular.

That depth $R \geq \operatorname{sdim} R$ follows from [17], Lemma 2.2. □

Question 4.9. Is is true that, for a local reduced and F-finite ring $(R, \mathfrak{m}, k)$, $\operatorname{sdim}(R) = \dim(R/\mathcal{P})$?

Our results so far seem to indicate an affirmative answer to this question, since we have the equality $\operatorname{sdim}(R) = \dim(R/\mathcal{P})$ if $\dim(R/\mathcal{P}) \in \{0, 1, \dim(R)\}$.

Proposition 4.10. *Let $R = S/I$ be a Stanley-Reisner ring, then $\sum Q = \mathcal{P}$ where the sum runs over all the minimal primes Q of R.*

Proof. By the definition of Stanley-Reisner rings, I is a square-free monomial ideal. In particular, I can be written as an intersection of some prime ideals in S, each of them generated by a string of indeterminates.

Let us assume that the union of all the minimal primes of R lifted to S involves all the indeterminates of S. But then $\sum Q = \mathfrak{m} \subset \mathcal{P} \subset \mathfrak{m}$ and we are done.

In the general case, let us assume that $x_1, \ldots, x_k$ are all the indeterminates that do not appear in any of the minimal primes of R. Then one can write $R = A[x_1, \ldots, x_k]$, with A a Stanley-Reisner ring that satisfies the condition on the minimal primes of the previous paragraph. Hence $\mathfrak{m}_A = \mathcal{P}(A)$. It is easy to show that $\mathcal{P}(R) = \mathcal{P}(A)R$. But then, $\mathcal{P}(R) = \mathfrak{m}_A R$ and this equals the sum of all the minimal primes Q of R as it can be easily checked. □

5. The module case

In what follows, we would like to extend some of our considerations to the case of R-modules. Let M be a finitely generated R-module. For every e, we can define a new R-module on the $\mathbf{Z}$-module M with the R-multiplication given as follows: $r \cdot m = r^{p^e} m$, for every $r \in R$ and $m \in M$. Denote this new module by $M^{(e)}$ and, as before, let us denote $q = p^e$. The reader should note that $M^{(e)} = M$ as abelian groups.

Denote by a_q the maximal rank of a free direct summand of M. Hence, $M^{(e)} = R^{a_q} \oplus M_q$ with M_q an R-module with no free direct summands. Also, for every $m \in M$, denote $\phi_{m,e} : R \to M^{(e)}$ the map that takes r to $r^q m$ for every $r \in R$. This is clearly a map of R-modules.

Definition 5.1. *Let u be the socle generator for $E_R(k)$. For every positive integer e, one can define*

$$A_e(M) = \{m \in M^{(e)} : m \otimes u = 0\}$$

in $M^{(e)} \otimes E_R(k)$. Here $M^{(e)}$ is the R-module just introduced above. It is easy to see that $A_e(M)$ is a submodule of $M = M^{(e)}$ over R with either of the two module structures that can be considered. Let us also define, $\mathcal{P}(M) := \cap_{e \gg 0} A_e(M)$ seen as an R-submodule of M.

Note that the above Definition naturally extends the Definition 3.1.

Also, $\mathfrak{m}^{[q]} M + \mathcal{P}(M) \subset A_e(M)$ for every e: if $r \in \mathfrak{m}$ and $m \in M$, then $r^q m \otimes u = m \otimes ru = m \otimes 0 = 0$.

The following Theorem is similar to the Theorem 2.8.

Theorem 5.2. *With the notations introduced above, one has that*

$$\lambda_R(M/A_e(M)) \cdot q^{\alpha(R)} = a_q.$$

Proof. Regard $A_e(M)$ as an R-submodule of M. Then $(M/A_e(M))^{(e)} = M^{(e)}/A_e(M)$.

The proof is similar to that of Theorem 2.8 and one can show that $a_q = \lambda_R(M^{(e)}/A_e(M))$. But, for every R-module N one has that $\lambda_R(N^{(e)}) = \lambda_R(N) \cdot q^{\alpha(R)}$. Hence, $a_q = \lambda_R(M/A_e(M)) \cdot q^{\alpha(R)}$. □

Theorem 5.3. *Let M be a finitely generated R-module and $r \in R$ and $m \in M$. Then $rm \in \mathcal{P}(M)$ if and only if $r \in \mathcal{P}(R)$ or $m \in \mathcal{P}(M)$.*

Proof. The proof is similar to the that of Theorem 3.3.

Let $r \in \mathcal{P}(R)$ and $m \in M$. We need to show that $rm \in \mathcal{P}(M)$. Consider the map $R^{(e)} \to M^{(e)}$ that sends r' to $r'm$ (seen as an element of M under the usual R-module structure) for every $r' \in R^{(e)}$. Tensor this map with $E_R(k)$ and let u be the socle element of this module. For every $e \gg 0$, $r \otimes u = 0$. This maps further to zero, so $rm \otimes u = 0$ for $e \gg 0$. Hence, $rm \in \mathcal{P}(M)$.

For the converse implication, assume that $rm \in \mathcal{P}(M)$ and that $r \notin \mathcal{P}(R)$ and $m \notin \mathcal{P}(M)$.

Since $r \notin \mathcal{P}(R)$, we have that for $e \gg 0$ the map $\phi_{r,e} : R \to R^{(e)}$ splits over R. Similarly, for $m \notin \mathcal{P}(M)$, we have that $\phi_{m,e'} : R \to M^{(e')}$ splits over R.

Fix $e' \gg 0$ so that $\phi_{m,e'}$ splits over R. Let $q' = p^{e'}$. We will show that $r^{q'}m \notin \mathcal{P}(M)$ which contradicts the fact that $rm \in \mathcal{P}(M)$. This will prove the converse implication.

Take $e \gg 0$, so that $\phi_{r,e}$ splits. Define the map $\psi_{m,e'} : R^{(e)} \to M^{(e+e')}$ by $\psi_{m,e'}(r') = r'^{q'}m$. This is essentially the map $\phi_{m,e'}$ "lifted" to $R^{(e)}$. It is an R-linear map and it splits over R. The composition of the two maps $\phi_{r,e}$ and $\psi_{m,e'}$ defines an R-linear map $R \to M^{(e+e')}$ that sends $1 \to r^{q'}m$. This map splits over R for all $e \gg 0$ (or, equivalently, $e + e' \gg 0$) and hence $r^{q'}m \notin \mathcal{P}(M)$. □

We can define the s-dimension and the Frobenius splitting ratio of an R-module.

Definition 5.4. *The* s-dimension *of* M, $\operatorname{sdim}(M)$, *is the largest integer* k *such that*

$$\liminf_{q\to\infty} \frac{a_q}{q^{k+\alpha(R)}}$$

is not zero. The lower Frobenius splitting ratio *of* M, $r_F(M)$, *equals the value of the limit introduced above, that is* $r_F^-(M) = \liminf_{q\to\infty} \frac{a_q}{q^{\operatorname{sdim}(M)+\alpha(R)}}$. *Similarly, the* upper Frobenius splitting ratio *of* M, $r_F^+(M)$, *equals* $\limsup_{q\to\infty} \frac{a_q}{q^{\operatorname{sdim}(M)+\alpha(R)}}$. *Whenever* $r_F^-(M) = r_F^+(M)$, *we call this number the* Frobenius splitting ratio *of* M.

The following result is an extension of the case when $M = R$.

Theorem 5.5. *Let* M *be an* R*-module. Then* $\operatorname{sdim}(M) \leq \dim(M/\mathcal{P}(M))$. *If* R *is strongly* F*-regular and* M *is an torsion free* R*-module, then* $\operatorname{sdim}(M) = \dim(R) = \dim(M)$.

Proof. From the definition of $M^{(e)}$ one can see that $(\operatorname{Ann}_R(M^{(e)}))^q \subset \operatorname{Ann}_R(M)$, so $\dim(M^{(e)}) = \dim(M)$. Write $M^{(e)} = R^{a_q} \oplus M_q$. If $a_q \neq 0$, for some q, then $\dim(M^{(e)}) = \dim(R)$, because R injects into $M^{(e)}$.

We have seen that $a_q = \lambda_R(M/A_e(M))$, and $\mathfrak{m}^{[q]}M + \mathcal{P}(M) \subset A_e(M)$. So, $a_q \geq \lambda(M/\mathfrak{m}^{[q]}M + \mathcal{P}(M))$. Since, $e_{HK}(\mathfrak{m}, M/\mathcal{P}(M)) > 0$, the first part of the Theorem follows.

If R is strongly F-regular, then $\mathcal{P}(R) = 0$. If M is torsion-free over R, then M injects into R^n for some n. Hence, if $m \otimes u = 0$ in $M^{(e)} \otimes E_R(k)$ for all $e \gg 0$, then the image of m in R^n, say $(r_1, \ldots, r_n)$, belongs to $\mathcal{P}(R)$ component-wise. However, $\mathcal{P}(R) = 0$, and so $\mathcal{P}(M) = 0$. So, $a_q \leq \lambda(M/\mathfrak{m}^{[q]}M)$, and since $e_{HK}(\mathfrak{m}, M) > 0$ we see that $\operatorname{sdim}(M) = \dim(M) = \dim(R)$. □

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