Math. Z. 250, 7–22 (2005) **Mathematische Zeitschrift**

DOI: 10.1007/s00209-004-0714-4

Subextension of plurisubharmonic functions with weak singularities

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Received: 17 March 2004 / Published online: 25 February 2005 –

1. Introduction

E. Bedford and D. Burns ([Be-Bu]) and later U. Cegrell ([Ce 1]) proved around 1978 that any smooth bounded domain satisfying certain boundary conditions is a domain of existence of a plurisubharmonic function.

However since plurisubharmonic functions occur in complex analysis through inequalities, it is more natural to ask for the subextension problem.

El Mir gave in 1980 an example of a plurisubharmonic function on the unit bidisc for which the restriction to any smaller bidisc admits no subextension to the whole space (see [El]). He also proved that, after attenuatting the singularities of a given plurisubharmonic function by composition with a suitable convex increasing function, it is possible to obtain a global subextension.

Later Alexander and Taylor gave in 1984 a generalization of this result with a more effective and simple proof (see [Al-Ta]).

On the other hand, Fornaess and Sibony pointed out in 1987 that for a ring domain in $\mathbb{C}^2$, there exists a plurisubharmonic function which admits no subextension inside the hole (see [Fo-Sib]).

Finally E. Bedford and B.A. Taylor proved in 1988 that any smoothly bounded domain in $\mathbb{C}^n$ is a domain of existence of a smooth plurisubharmonic function (see [Be-Ta 3]).

Recently, the first and the last authors proved that plurisubharmonic functions with uniformly bounded Monge-Ampère mass on a bounded hyperconvex domain always admit a plurisubharmonic subextension to any larger hyperconvex domain (see [Ce-Ze]).

⋆ partially supported by the Swedish Research Council contact no 621-2002-5308.
⋆⋆ partially supported by KBN grant 1 P03A 03727.

Here we want to prove several results showing that plurisubharmonic functions with various bounds on their Monge-Ampère masses on a bounded hyperconvex domain always admit global plurisubharmonic subextension with logarithmic growth at infinity.

Let us describe more precisely the content of the article.

In section 2 we recall basic definitions concerning the Cegrell class $\mathcal{E}(\Omega)$ of plurisubharmonic functions of locally uniformly bounded Monge-Ampère masses on a hyperconvex domain $\Omega \Subset \mathbb{C}^n$ and its subclasses of plurisubharmonic functions of finite energy. Then we give a characterization in terms of some capacity of functions from the class $\mathcal{E}(\Omega)$ which provides several examples of functions in this class.

In section 3, we give almost sharp estimates on the size of sublevel sets of plurisubharmonic functions in various subclasses of the class $\mathcal{E}(\Omega)$.

In section 4, we give a generalization of Alexander-Taylor's subextension theorem which implies, using results from section 3, that plurisubharmonic functions of finite energy in the sense of Cegrell admit a global subextention with logarithmic growth of arbitrary small logarithmic type.

Finally in section 5, using recent results from the theory of Monge-Ampère equation on compact Kähler manifolds due to the second author, we prove two results about global subextension of plurisubharmonic functions of uniformly bounded masses on a hyperconvex domain by plurisubharmonic function with logarithmic growth on $\mathbb{C}^n$ with a well defined global Monge-Ampère measure in some cases.

A part of this research was done during the visit of the second author in Laboratoire Emile Picard of the University Paul Sabatier (Toulouse) in January 2004. He would like to thank the staff of the Laboratoire for its hospitality and stimulating discussions.

2. Plurisubharmonic functions with locally uniformly bounded Monge-Ampère masses

Let us first recall some definitions from ([Ce 2, Ce 3]). We use the notation $d^c = \frac{i}{2\pi}(\bar\partial - \partial)$. Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain. We denote by $\mathcal{E}_0(\Omega)$ the set of negative and bounded plurisubharmonic functions φ on Ω which tend to zero at the boundary and satisfy $\int_\Omega (dd^c\varphi)^n < +\infty$. Then for each $p > 0$ define $\mathcal{E}_p(\Omega)$ to be the class of plurisubharmonic functions φ on Ω such that there exists a decreasing sequence of plurisubharmonic functions (φ_j) from the class $\mathcal{E}_0(\Omega)$ which converges to φ such that

$$\sup_j \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n < +\infty. \tag{2.1}$$

If moreover the sequence (φ_j) can be chosen so that $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$, then we say that $\varphi \in \mathcal{F}_p(\Omega)$.

Let us denote by $\mathcal{F}(\Omega)$ the set of all $\varphi \in PSH(\Omega)$ such that there exists a sequence (φ_j) of plurisubharmonic function in $\mathcal{E}_0(\Omega)$ such that $\varphi_j \searrow \varphi$ and $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$. We also need the subclass $\mathcal{F}^a(\Omega)$ of functions from $\mathcal{F}(\Omega)$ whose Monge-Ampère measures put no mass on pluripolar subsets of Ω.

By Cegrell [Ce 2] we have $\mathcal{E}_0(\Omega) \subset \mathcal{F}_p(\Omega) \subset \mathcal{E}_p(\Omega)$; $\mathcal{F}^a(\Omega) \subset \mathcal{F}(\Omega)$, for any $p > 0$. It is also proved in ([Ce 2, Ce 3]) that the complex Monge-Ampère operator is well defined for a function $\varphi \in \mathcal{F}(\Omega)$ as the weak*-limit of the sequence of measures $(dd^c\varphi_j)^n$, where (φ_j) is any decreasing sequence of plurisubharmonic functions from the class $\mathcal{E}_0(\Omega)$ which converges to φ and satisfying the required conditions in the definition.

Finally we denote by $\mathcal{E}(\Omega)$ the set of plurisubharmonic functions which are locally in $\mathcal{F}(\Omega)$. Then the complex Monge-Ampère operator is well defined on the class $\mathcal{E}(\Omega)$ (see [Ce 3]).

The following result will be useful for later refrence; it has been proved in [Ce 3] for $p \geq 1$.

Lemma 2.1. *Let $\varphi \in \mathcal{E}_p(\Omega)$, if $p > 0$ and $\varphi \in \mathcal{F}(\Omega)$ if $p = 0$. Then the pluricomplex $p-$energy of φ defined by the following formula*

$$e_p(\varphi) := \int_\Omega (-\varphi)^p (dd^c\varphi)^n \tag{2.2}$$

is finite and there exists a sequence (φ_j) of plurisubharmonic functions in $\mathcal{E}_0(\Omega)$ such that $\varphi_j \searrow \varphi$ and $\lim_{j\to+\infty} \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n = \int_\Omega (-\varphi)^p (dd^c\varphi)^n$.

Such a sequence will be called a $p-$admissible sequence decreasing to φ.

Proof. Observe that for any sequence (φ_j) of plurisubharmonic functions in $\mathcal{E}_0$, decreasing to φ, with the condition (2.1), the sequence $(-\varphi_j)^p$ is an increasing sequence of lower semi-continuous functions on Ω converging to $(-\varphi)^p$ and the sequence of measures $(dd^c\varphi_j)^n$ converges weakly to $(dd^c\varphi)^n$ on Ω. Therefore it follows that $\int_\Omega (-\varphi)^p (dd^c\varphi)^n \leq \liminf_{j\to+\infty} \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n$, which proves that $e_p(\varphi) < +\infty$. For $p = 0$, the result of the lemma follows from the definition. Now fix $p > 0$. Then, since for the given function $\varphi \in \mathcal{E}_p$ the measure $(dd^c\varphi)^n$ puts no mass on pluripolar sets, it follows from Cegrell's decomposition theorem [Ce 2] that there exists $\psi_0 \in \mathcal{E}_0(\Omega)$ and $0 \leq f \in L^1(\Omega; (dd^c\psi_0)^n)$ such that $(dd^c\varphi)^n = f(dd^c\psi_0)^n$. By Kolodziej's theorem (see [Ko 1, Ce 2]), for any integer $j \geq 1$ there exists $\varphi_j \in \mathcal{E}_0(\Omega)$ such that $(dd^c\varphi_j)^n = \min\{f, j\} \cdot (dd^c\psi_0)^n$. Now by the comparison principle [Be-Ta 1], we see that the sequence (φ_j) decreases to a plurisubharmonic function ψ on Ω such that $\psi \in \mathcal{E}_p(\Omega)$ and $(dd^c\psi)^n = (dd^c\varphi)^n$. By the comparison principle it follows that $\varphi = \psi$ on Ω. Now by the monotone convergence theorem we obtain $\int_\Omega (-\varphi)^p (dd^c\varphi)^n = \lim_{j\to+\infty} \int_\Omega (-\varphi_j)^p \min(f, j)(dd^c\psi_0)^n = \lim_{j\to+\infty} \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n$, which proves the lemma. □

Now let us give a quantitative characterization of the class $\mathcal{E}(\Omega)$ in terms of some capacity introduced by Bedford (see [Be]).

Given $\varphi \in PSH^-(\Omega)$ and $K \subset \Omega$ a borelean set, following [Be] we define the following positive "$\varphi-$capacity"

$$C_\varphi(K; \Omega) := \sup\{\int_K (dd^c\psi)^n \ ; \ \psi \in PSH^-(\Omega) \cap L^\infty(\Omega), \varphi \leq \psi \leq 0\}.$$

Consider also the corresponding extremal function

$$\tilde{\varphi}_K = \sup\{u \in PSH^-(\Omega); u \leq \varphi \text{ q.e. on } \quad K\}. \tag{2.3}$$

where $u \leq \varphi$ q.e. (quasi-everywhere) on K means outside a pluripolar subset of K.

Then we get the following characterization of functions from the class $\mathcal{E}(\Omega)$.

Proposition 2.2. *Let $\varphi \in PSH^-(\Omega)$. Then $\varphi \in \mathcal{E}(\Omega)$ if and only if for any compact set $K \subset \Omega$ we have $C_\varphi(K) < +\infty$. Moreover, if $\varphi \in \mathcal{E}(\Omega)$, then for any borelean set $K \Subset \Omega$, $\tilde{\varphi}_K \in \mathcal{F}(\Omega)$, $\tilde{\varphi}_K = \varphi$ q.e. on K and*

$$C_\varphi(K^\circ) \leq \int_\Omega (dd^c \tilde{\varphi}_K)^n \leq C_\varphi(\overline{K}; \Omega). \tag{2.4}$$

Proof. From the definition of $\tilde{\varphi}_K$, it follows that $\tilde{\varphi}_K$ is plurisubharmonic on Ω, maximal on $\Omega \setminus \overline{K}$ and satisfies the inequality $\varphi \leq \tilde{\varphi}_K \leq 0$ on Ω.
Now assume that $\varphi \in \mathcal{E}(\Omega)$. Then it follows from ([Ce 2, Ce 3]) that $\tilde{\varphi}_K = \sup\{\varphi, \tilde{\varphi}_K\} \in \mathcal{E}(\Omega)$ and $\int_\Omega (dd^c \tilde{\varphi}_K)^n = \int_{\overline{K}} (dd^c \tilde{\varphi}_K)^n < +\infty$.

Let us first prove that $\tilde{\varphi}_K \in \mathcal{F}(\Omega)$. Indeed, take a decreasing sequence $(\varphi_j)_{j\geq 0}$ from $\mathcal{E}_0(\Omega)$ converging to φ on a neighbourhood of $\overline{K}$ such that $\sup_j \int_\Omega (dd^c \varphi_j)^n < +\infty$. Then for each $j \in \mathbb{N}$ define the function $\tilde{\varphi}_j$ by the formula (2.3) with φ replaced by φ_j. Then $(\tilde{\varphi}_j)$ is a decreasing sequence of plurisubharmonic functions from the class $\mathcal{E}_0(\Omega)$ such that $\varphi_j \leq \tilde{\varphi}_j$ on Ω and $\tilde{\varphi}_j = \varphi_j$ q.e. on K. Therefore $(\tilde{\varphi}_j)$ converges to a plurisubharmonic function ψ such that $\tilde{\varphi}_K \leq \psi$ on Ω and $\psi = \varphi$ q.e. on K. Thus $\psi = \tilde{\varphi}_K$. Since $\varphi_j \leq \tilde{\varphi}_j$ on Ω and these functions belong to $\mathcal{E}_0(\Omega)$, it follows that $\int_\Omega (dd^c \tilde{\varphi}_j)^n \leq \int_\Omega (dd^c \varphi_j)^n$ for any $j \geq 0$. Thus the sequence $(\tilde{\varphi}_j)_{j\geq 0}$ decreases to $\tilde{\varphi}_K$ and $\sup_j \int_\Omega (dd^c \tilde{\varphi}_j)^n \leq \sup_j \int_\Omega (dd^c \varphi_j)^n < +\infty$, which proves that $\tilde{\varphi}_K \in \mathcal{F}(\Omega)$. Then since $\varphi \leq \tilde{\varphi}_j$, it follows that $\int_\Omega (dd^c \tilde{\varphi}_j)^n = \int_K (dd^c \tilde{\varphi}_j)^n \leq C_\varphi(K)$. By Cegrell (see [Ce 3]), the sequence of measures $(dd^c \tilde{\varphi}_j)$ converges to the measure $(dd^c \tilde{\varphi})^n$, thus $\int_\Omega (dd^c \tilde{\varphi}_K)^n \leq \liminf_j \int_\Omega (dd^c \tilde{\varphi}_j)^n \leq C_\varphi(\overline{K})$, which proves the second inequality in (2.4).
Now let $\psi \in \mathcal{E}_0(\Omega)$ be chosen so that $\varphi \leq \psi$ on Ω and set $\psi_j := \sup\{\psi, \varphi_j\}$ on Ω. Then $\varphi_j \leq \psi_j$ and then $\tilde{\varphi}_j \leq \psi_j$ q.e. on K. Since functions from $\mathcal{E}_0(\Omega)$ put no mass on pluripolar sets, it follows from Demailly's inequality ([De]) that

$$\begin{aligned}\int_K (dd^c \psi_j)^n &\leq \int_K (dd^c \sup\{\psi_j, \tilde{\varphi}_j\})^n \\ &\leq \int_\Omega (dd^c \sup\{\psi_j, \tilde{\varphi}_j\})^n \\ &\leq \int_\Omega (dd^c \tilde{\varphi}_j)^n = \int_{\overline{K}} (dd^c \tilde{\varphi}_j)^n,\end{aligned}$$

where the last inequality follows from the comparison principle for functions in $\mathcal{E}_0(\Omega)$.

Therefore by the convergence theorem (see [Ce 2, Ce 3]) we have

$$C_\varphi(K^\circ; \Omega) \leq \int_\Omega (dd^c \tilde{\varphi}_K)^n < +\infty,$$

which proves the first inequality in (2.4) as well as the necessary condition of the theorem.

Now assume that the condition on the capacity C_φ is satisfied. Then consider a decreasing sequence of plurisubharmonic functions (φ_j) from the class $\mathcal{E}_0(\Omega)$ converging to φ. Take any subset $K \Subset \Omega$ and define $\tilde{\varphi}_j := (\tilde{\varphi}_j)_K$ for $j \in \mathbb{N}^*$. Then $\tilde{\varphi}_j \in \mathcal{E}_0(\Omega)$ and $\sup_j \int_\Omega (dd^c \tilde{\varphi}_j)^n = \sup_j \int_{\overline{K}} (dd^c \tilde{\varphi}_j)^n \leq C_\varphi(\overline{K}; \Omega) < +\infty$. We know from the first part that $(\tilde{\varphi}_j)$ decreases to $\tilde{\varphi}_K = \varphi$ q.e. on K. Therefore we have proved that $\varphi \in \mathcal{E}(\Omega)$. Moreover by the convergence theorem we obtain the second inequality in (2.4). □

Bedford considered the following class (see [Be]). Let $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ be a monotone decreasing function such that

$$\int_1^{+\infty} \frac{\theta(t)}{t} dt < +\infty \tag{2.5}$$

and the function $t \longmapsto -(-t\ \theta(-t))^{1/n}$ is monotone increasing and convex on $]-\infty, 0[$. Then define $\mathcal{B}(\Omega)$ to be the class of negative function $\psi \in PSH(\Omega)$ such that for any $z_0 \in \Omega$ there exists a neighbourhood ω of z_0, a negative plurisubharmonic function v on ω and a decreasing function θ satisfying (2.5) such that $-(-v\theta(-v))^{1/n} \leq \psi$ on ω.

From the last result we can deduce the following one which provides concrete examples of functions from the class $\mathcal{E}(\Omega)$. This result has been also obtained by the first author (see [Ce 4]).

Proposition 2.3. *For any hyperconvex domain $\Omega \Subset \mathbb{C}^n$, we have $\mathcal{B}(\Omega) \subset \mathcal{E}(\Omega)$. In particular, for any negative plurisubharmonic function u on Ω and any $0 < \alpha < 1/n$, $-(-u)^\alpha \in \mathcal{E}(\Omega)$.*

Proof. Bedford has proved that for any function $\psi \in \mathcal{B}(\Omega)$, the condition $C_\psi(K; \Omega) < +\infty$ holds for any $K \Subset \Omega$ (see [Be]). Therefore the inclusion $\mathcal{B}(\Omega) \subset \mathcal{E}(\Omega)$ follows from the last proposition. Since $\alpha < 1/n$, we clearly have $-(-u)^\alpha = -(-u\theta(-u))^{1/n}$, where $\theta(t) = t^{n\alpha-1}$ which satisfies the condition (2.5). Therefore $-(-u)^\alpha \in \mathcal{B}(\Omega) \subset \mathcal{E}(\Omega)$. □

3. Capacity of sublevel sets of plurisubharmonic functions in subclasses of $\mathcal{E}(\Omega)$

Now we prove the following capacity estimate of the sublevel sets of plurisubharmonic functions of finite energy.

Proposition 3.1. *Let $\varphi \in \mathcal{E}_p(\Omega)$ if $p > 0$ and $\varphi \in \mathcal{F}(\Omega)$ if $p = 0$. Then the following estimate*

$$\mathrm{Cap}(\{z \in \Omega\ ;\ \varphi(z) < -s\}; \Omega) \leq c_{n,p} \cdot e_p(\varphi) \cdot s^{-n-p}, \forall s > 0, \tag{3.1}$$

holds, where $c_{n,p} > 0$ is an absolute constant and $e_p(\varphi)$ is the $p-$energy of φ.

Proof. 1) Assume first that $\varphi \in \mathcal{E}_0(\Omega)$ and define $\Omega(\varphi, s) := \{z \in \Omega; \varphi(z) < -s\}$ for each $s > 0$. Let $K \subset \Omega(\varphi, s)$ a fixed pluriregular subset and h_K the $(-1, 0)-$extremal function of the condenser (K, Ω). Then $h_K \in \mathcal{E}_0(\Omega)$ and since $-\varphi/s \geq 1$ on K, we have

$$\begin{aligned} \operatorname{Cap}(K; \Omega) = \int_K (dd^c h_K)^n &\leq \int_K \left(\frac{-\varphi}{s}\right)^{n+p} (dd^c h_K)^n \qquad (3.2) \\ &\leq \frac{1}{s^{n+p}} \int_\Omega (-\varphi)^{n+p} (dd^c h_K)^n \\ &= \frac{c_{n,p}}{s^{n+p}} \int_\Omega (-\varphi)^p (dd^c \varphi)^n, \end{aligned}$$

where the last inequality follows by integration by parts (see [Ce 3, Bl]). Then we deduce that

$$\operatorname{Cap}\big(\Omega(\varphi, s); \Omega\big) \leq \frac{c_{n,p}}{s^{n+p}} \int_\Omega (-\varphi)^p (dd^c \varphi)^n$$

for each $\varphi \in \mathcal{E}_0(\Omega)$.
2) Let $\varphi \in \mathcal{E}_p(\Omega)$ and (φ_j) a $p-$admissible sequence of plurisubharmonic functions from the class $\mathcal{E}_0(\Omega)$ decreasing to φ. Then applying the estimate (3.1) to each function $\varphi_j \in \mathcal{E}_0(\Omega)$ we obtain the following estimate

$$\operatorname{Cap}\big(\Omega(\varphi_j, s); \Omega\big) \leq \frac{c_{n,p}}{s^{n+p}} \int_\Omega (-\varphi_j)^p (dd^c \varphi_j)^n, \forall j.$$

Now by Lemma 2.1, there exists a decreasing sequence of plurisubharmonic functions (φ_j) from the class $\mathcal{E}_0(\Omega)$ which converges to φ and satisfies the condition $e_p(\varphi) = \lim_j \int_\Omega (-\varphi_j)^p (dd^c \varphi_j)^n$. Then applying the last estimate to this function we get the following estimate

$$\operatorname{Cap}\big(\Omega(\varphi, s); \Omega\big) \leq c_{n,p} \frac{e_p(\varphi)}{s^{n+p}}.$$

This proves our proposition. □

Let us prove a converse to the last result which shows that the estimates (3.1) are almost sharp.

Proposition 3.2. *Let* $\varphi \in \mathcal{E}(\Omega)$ *a function such that there exists an open subset* $\omega \subset \Omega$, *a constant* $A > 0$ *and a real number* $q > n$ *such that*

$$\operatorname{Cap}(\{z \in \omega\,;\ \varphi(z) < -s\}; \Omega) \leq C s^{-q}, \forall s > 0.$$

Then $\tilde{\varphi}_\omega \in \mathcal{E}_p(\Omega)$ *for any real number* p *with* $0 < p < q - n$.

Proof. First we claim that for any plurisubharmonic function $u \in \mathcal{E}(\Omega)$ and any Borel set $B \subset \Omega$, we have

$$\int_B (dd^c u)^n \leq (\sup_B |u|)^n \operatorname{Cap}(B; \Omega), \qquad (3.3)$$

provided that $\sup_B |u| < +\infty$.

To prove this estimate, set $M := \sup_B |u|$ and define the function $v = \sup\{u/M, -1\}$ on Ω. Then $v \in PSH(\Omega)$, $-1 \le v \le 0$ and $v = u/M$ on B. Therefore from Demailly's inequality ([De]), it follows that $M^{-n} \int_B (dd^c u)^n \le \int_B (dd^c v)^n \le \mathrm{Cap}(B; \Omega)$, which proves our claim.

Now define the following sets $B_j := \{z \in \omega \,;\, -2^{j+1} \le \varphi(z) < -2^j\}$ for $j \ge 0$. Then it follows from (3.3) that

$$\int_{B_j} (-\varphi)^p (dd^c \varphi)^n \le (\sup_{A_j} |\varphi|)^{n+p} \mathrm{Cap}(B_j; \Omega) \le C 2^{j(n+p-q)}$$

and then $\int_\omega (-\varphi)^p (dd^c \varphi)^n \le C \sum_{j \ge 0} 2^{(n+p-q)j} < +\infty$, since $p < q - n$. Therefore we have $\int_\Omega (-\tilde{\varphi}_\omega)^p (dd^c \tilde{\varphi}_\omega)^n < +\infty$. □

As a consequence we state the following result which improves the result of Proposition 2.3 and provides examples of functions in the classes $\mathcal{E}_p(\Omega)$.

Corollary 3.3. *Let $u \in PSH^-(\Omega)$, α a real number such that $0 < \alpha < 1/n$ and $\varphi = \varphi_\alpha := -(-u)^\alpha$. Then for any Borel subset $\omega \Subset \Omega$ $\tilde{\varphi}_\omega \in \mathcal{E}_p(\Omega)$ for any real number p such that $0 < p < 1/\alpha - n$,*

Proof. Indeed it is easy to check that $\mathrm{Cap}(\{z \in \omega \,;\, \varphi(z) < -s\}; \Omega) \le A s^{-1/\alpha}$, for any $s > 0$. Thus the result follows from the last one. □

It is possible to characterize the class $\mathcal{F}^a(\Omega)$ by means of the behaviour of the capacity of sublevel sets.

Proposition 3.4. *Let $\varphi \in \mathcal{F}(\Omega)$. Then the following properties are equivalent*

(i) $\varphi \in \mathcal{F}^a(\Omega)$,
(ii) $\int_{\{\varphi = -\infty\}} (dd^c \varphi)^n = 0$,
(iii) $\lim_{s \to +\infty} s^n \mathrm{cap}(\{\varphi < -s\}; \Omega) = 0$.

Proof. Take a sequence (φ_j) of continuous functions from $\mathcal{E}_0(\Omega)$ which decreases to φ and satisfies $\sup_j \int_\Omega (dd^c \varphi_j)^n < +\infty$.

Define the open sets $\Omega_j(s) := \{\varphi_j < -s\}$, $\Omega(s) := \{\varphi < -s\}$ and the functions

$$a_j(s) := \mathrm{cap}(\{\varphi_j < -s\}; \Omega), \quad a(s) := \mathrm{cap}(\{\varphi < -s\}; \Omega),$$

and

$$b_j(s) := \int_{\Omega_j(s)} (dd^c \varphi_j)^n, \quad b(s) := \int_{\Omega(s)} (dd^c \varphi)^n.$$

We claim that for $j \in \mathbb{N}$ and $s > 0$,

$$s^n a_j(2s) \le b_j(s) \le s^n a_j(s). \tag{3.4}$$

Indeed take any function $u \in PSH(\Omega)$ with $-1 \le u \le 0$. Then $\Omega_j(2s) \subset \{\varphi_j/s < u - 1\} \subset \Omega_j(s) \Subset \Omega$. By the comparison principle, we get

$$\int_{\Omega_j(2s)} (dd^c u)^n \le \int_{\{\varphi_j/s < u-1\}} s^{-n} (dd^c \varphi_j)^n \le \int_{\Omega_j(s)} s^{-n} (dd^c \varphi_j)^n = s^{-n} b_j(s).$$

Taking the supremum over all u's, we obtain the first inequality of (3.4).

To obtain the second inequality, observe that for $0 < s < t$, $\sup\{\varphi_j, -t\} = \varphi_j$ on the open set $\{\varphi_j > -t\}$ which is a neighbourhood of $\partial\Omega_j(s)$ and then $\int_{\Omega_j(s)} (dd^c \sup\{\varphi_j, -t\})^n = \int_{\Omega_j(s)} (dd^c \varphi_j)^n$. Therefore

$$a_j(s) \geq t^{-n} \int_{\Omega_j(s)} (dd^c \sup\{\varphi_j, -t\})^n = t^{-n} \int_{\Omega_j(s)} (dd^c \varphi_j)^n,$$

which proves the required inequality since $t > s$ is arbitrarily close to s. Taking the limit in (3.4) when $j \to +\infty$, we obtain

$$s^n a(2s) \leq b(s^-) \text{ and } b(s) \leq s^n a(s), \ \forall s > 0. \tag{3.5}$$

where $b(s^-) := \lim_{\varepsilon \to 0^+} b(s - \varepsilon)$.

From the estimates (3.5), it follows that the conditions (ii) and (iii) are equivalent. Moreover it is clear that (i) implies (ii). So it is enough to prove that (iii) implies (i). Indeed, assume that that $\lim_{s \to +\infty} s^n a(s) = 0$ and take a pluripolar subset K of Ω. It follows from ([De, Ce-Ko]) that

$$\int_{K \setminus \Omega(s)} (dd^c \varphi)^n \leq \int_{K \setminus \Omega(s)} (dd^c \sup\{\varphi, -s\})^n \leq s^n \mathrm{cap}(K; \Omega) = 0.$$

Moreover, by (3.5), we have

$$\int_{K \cap \Omega(s)} (dd^c \varphi)^n \leq b(s) \leq s^n a(s).$$

Therefore $\int_K (dd^c \varphi)^n = 0$ and then $\varphi \in \mathcal{F}^a(\Omega)$. □

4. Global subextension of plurisubharmonic functions with weak singularities

Here we want to prove a general subextension theorem for a class of plurisubharmonic functions of weak singularities, generalizing a theorem by El Mir ([El]) and also by Alexander and Taylor (see [Al-Ta, De]). Then we will apply our result to derive theorems on subextension of plurisubharmonic functions of finite energy.

To sate our results we need to introduce the usual Lelong classes of plurisubharmonic functions.

$$\mathcal{L}_\gamma(\mathbb{C}^n) := \{u \in PSH(\mathbb{C}^n); \limsup_{r \to +\infty} \frac{\max_{|z|=r} u(z)}{\log r} \leq \gamma\}, \gamma > 0. \tag{4.1}$$

When $\gamma = 1$ we write $\mathcal{L}(\mathbb{C}^n) = \mathcal{L}_1(\mathbb{C}^n)$.

Theorem 4.1. *Let $\varphi \in PSH^-(\Omega)$ and $\omega \subset \Omega$ an open subset. Define the function* $\chi(s) = \chi_\varphi(s, \omega) := \mathrm{Cap}(\{z \in \omega \,;\, \varphi(z) < -s\}; \Omega)$. *Assume that the following integral condition*

$$\int_1^{+\infty} \chi(s)^{1/n} ds < +\infty \tag{4.2}$$

holds. Then for any $\varepsilon > 0$, there exists a function $U_\varepsilon \in \mathcal{L}_\varepsilon(\mathbb{C}^n)$ such that $U_\varepsilon \leq \varphi$ on ω.

In particular $\nu_\varphi(a) = 0$ for any $a \in \omega$.

Proof. We use the same construction as in ([Al-Ta]). Indeed let us denote by

$$M(s) := \max_{\overline{\Omega}} V_s = -\log T_{\overline{\Omega}}\big(\omega(\varphi; s)\big), \ s > 0,$$

where V_s is the $\mathcal{L}-$extremal function of the open set $\omega(\varphi; s) := \{z \in \omega \ ; \ \varphi(z) < -s\}$. Then by Alexander-Taylor's inequality ([Al-Ta]), we deduce the following

$$M(s) \geq \chi(s)^{-1/n}, \ \ \forall s > 0, \tag{4.3}$$

where $\chi(s) = \chi_\varphi(s; \omega)$ for $s > 0$. Now define the following function

$$w_s(z) = w(z, s) := V_s(z) - M(s), \ (z, s) \in \mathbb{C}^n \times \mathbb{R}^+. \tag{4.4}$$

We claim that this function satisfies the following properties
(i) $w_s \in \mathcal{L}$ and $w_s(z) \leq a + \log^+ |z|, \ \forall z \in \mathbb{C}^n, \ \forall s > 0,$
(ii) $w_s(z) = -M(s), \ \forall z \in \omega(\varphi, s), \forall s > 0,$
(iii) $\max_{\overline{\Omega}} w_s = 0$ and $\int_\Omega w_s(z) d\lambda(z) \geq -b, \ \forall s > 0$, and the Lebesque measure.
where $a, b > 0$ are absolute constants and λ the Lebesque measure.

Assume for the moment that all the above properties are satisfied and observe that for any fixed $z \in \mathbb{C}^n$, the function $s \longmapsto w_s(z)$ is a function of bounded variation (equal to the difference of two monotone functions) and upper bounded on $\mathbb{R}^+$, by condition (i). Therefore we can define the following function

$$v_c(z) := \int_c^{+\infty} w(z, s) \chi(s)^{1/n} ds, \ z \in \mathbb{C}^n,$$

for each $c > 0$. From condition (i), it follows that

$$v_c(z) \leq a \cdot \eta_c + \eta_c \cdot \log^+ |z|, \ \forall z \in \mathbb{C}^n, \tag{4.5}$$

where $\eta_c := \int_c^{+\infty} \chi(s)^{1/n} ds$. Now from (iii) it follows that

$$\int_\Omega v_c(z) d\lambda(z) \geq -b \cdot \ \eta_c \tag{4.6}$$

Then it follows from (4.6) and (4.5) that v_c is plurisubharmonic on $\mathbb{C}^n$. Now fix $t > c \geq 0$ and $z \in \omega(\varphi, t)$. Then by (ii), for any $s < t$, we have $\varphi(z) < -t$ and $w_s(z) = -M(s)$. Therefore, since $w_s \leq 0$ on Ω, we get from (4.3) the following estimate

$$v_c(z) \leq \int_c^t w_s(z) \chi(s)^{1/n} ds \leq (-t + c).$$

This means that $v_c(z) \leq \varphi(z) + c$ if $\varphi(z) < -c$. But if $\varphi(z) \geq -c$, this inequality is clearly satisfied, since $v_c(z) \leq 0$ for any $z \in \Omega$. Define $u_c(z) := v_c(z) - c$, for $z \in \mathbb{C}^n$. Then it is clear that $u_c \leq \varphi$ on ω. Moreover, from (4.5), it follows that

$$u_c(z) \leq a \cdot \eta_c - c + \eta_c \cdot \log^+ |z|, \ \forall z \in \mathbb{C}^n, \tag{4.7}$$

Now given $\varepsilon > 0$, we can choose $c = c(\varepsilon) > 0$ such that $\eta(c) < \varepsilon$, then the corresponding function $U_\varepsilon := u_{c(\varepsilon)}$ satisfies the conclusions of the theorem with $\gamma(\varepsilon) := a \cdot \eta_{c(\varepsilon)} - c(\varepsilon)$.

Now it remains to prove that our function (4.4) satisfies the properties (i), (ii) and (iii). By definition $w_s \in \mathcal{L}$ and $\max_{\overline{\Omega}} w_s = 0$. Then $w_s \leq V_{\overline{\Omega}}$ on $\mathbb{C}^n$ for any $s > 0$, which proves (i), since $V_{\overline{\Omega}} \in \mathcal{L}$. The condition (ii) is trivial since $V_s = V_{\omega(\varphi,s)} = 0$ on $\omega(\varphi, s)$. The condition (iii) is related to an inequality by Alexander (see [Al], [Sic 2], [De]) and can be proved easily as follows. Observe that the normalized subclass $\dot{\mathcal{L}}_{\overline{\Omega}} := \{w \in \mathcal{L}\ ;\ \max_{\overline{\Omega}} = 0\}$ is a (relatively) compact subset of $\mathcal{L}$ for the L^1_{loc}−topology (see [Ze]) and the functional $w \longmapsto \int_\Omega w(z) d\lambda(z)$ is continuous on $\mathcal{L}$. Therefore it is bounded on $\dot{\mathcal{L}}_{\overline{\Omega}}$, which proves the condition (iii). □

Now from our result we can deduce the Alexander-Taylor's subextension theorem.

Let $h : \mathbb{R}^- \longrightarrow \mathbb{R}^-$ be an increasing convex function such that

$$\int_1^{+\infty} \frac{-h(-t)}{t^{1+1/n}} dt < +\infty. \tag{4.8}$$

Then we obtain the following result.

Corollary 4.2. *Let $u \in PSH^-(\Omega)$ and $h : \mathbb{R}^- \longrightarrow \mathbb{R}^-$ be an increasing convex function satisfying the condition (4.8). Then for any subdomain $\omega \Subset \Omega$, for any $\varepsilon > 0$, there exists a function $U_\varepsilon \in \mathcal{L}_\varepsilon(\mathbb{C}^n)$ such that $U_\varepsilon \leq h(u)$ on ω.*

Proof. Let $g : \mathbb{R}^- \longrightarrow \mathbb{R}^-$ be the inverse function of h. Then $\omega(h(u); s) = \omega(u; g(-s))$ for any $s > 0$. Now use the usual capacity estimate for u to conclude that

$$\mathrm{Cap}(\omega(h(u); s); \Omega) \leq \frac{A}{-g(-s)}, \forall s > 0. \tag{4.9}$$

Now observe that the condition (4.8) on h implies that $\int_1^{+\infty} (-g(-s))^{-1/n} ds < +\infty$. Therefore from the estimate (4.9), it follows that the condition (4.2) is satisfied for the function $h(u)$ and then the corollary follows from the last theorem. □

Now using capacity estimates from section 3 and the last theorem, we easily see that functions from the classes $\mathcal{E}_p(\Omega)$, with $p > 0$, have global subextension of arbitrarily small logarithmic growth at infinity.

Corollary 4.3. *Let $\varphi \in \mathcal{E}_p(\Omega)$, with $p > 0$. Then for any $\varepsilon > 0$, there exists a function $U_\varepsilon \in \mathcal{L}_\varepsilon(\mathbb{C}^n)$ such that $U_\varepsilon \leq \varphi$ on Ω.*

Proof. From the estimates (3.1) of Proposition 3.1, it follows that the condition (4.2) of Theorem 4.1 is satisfied with $\omega = \Omega$, which implies our result. □

5. Global subextension of psh functions with uniformly bounded Monge-Ampère masses

As we pointed out in the introduction, on any smoothly bounded domain in $\mathbb{C}^2$ there is a smooth plurisubharmonic function which admits no subextention to any larger domain (see [Be-Ta 3]). In contrast to this negative result, the first and the third author proved that for any hyperconvex domain $\Omega \Subset \mathbb{C}^n$, functions from the class $\mathcal{F}(\Omega)$ always admit a subextension to any larger bounded hyperconvex domain (see [Ce-Ze]).

Here we want to prove that such functions have a global subextension which is plurisubharmonic of logarithmic growth on $\mathbb{C}^n$.

Besides the Lelong classes $\mathcal{L}_\gamma(\mathbb{C}^n)$ defined in section 4, we also need the following class.

$$\mathcal{L}_\gamma^+(\mathbb{C}^n) := \{u \in PSH(\mathbb{C}^n);\ \sup_{z\in\mathbb{C}^n} |u(z) - \gamma \log^+ |z|| < +\infty\}.$$

Now we can state our main result.

Theorem 5.1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded hyperconvex domain and $\varphi \in \mathcal{F}(\Omega)$. Then there exists a plurisubharmonic function $u \in \mathcal{L}_\gamma(\mathbb{C}^n)$, with $\gamma^n := \int_\Omega (dd^c\varphi)^n$ such that* $\max_{\overline{\Omega}} u = 0$ *and $u \le \varphi$ on Ω.*

Proof. 1) Assume first that $\varphi \in \mathcal{E}_0(\Omega)$ and define the following Borel measure $\mu := \mathbf{1}_\Omega (dd^c\varphi)^n$. Fix a ball $\mathbb{B} \subset \mathbb{C}^n$ such that $\overline{\Omega} \subset \mathbb{B}$. Then in general there is no bounded plurisubharmonic function v on $\mathbb{B}$ such that $(dd^c v)^n \ge \mu$ on $\mathbb{B}$ (see the example below). We will approximate the measure μ by measures for which such bounded plurisubharmonic functions exist. Indeed, since μ puts no mass on pluripolar sets, by ([Ce 1]) there exists $\psi \in \mathcal{E}_0(\mathbb{B})$ and $f \in L^1(\mathbb{B}, \mu)$ such that $\mu = f \cdot (dd^c\psi)^n$ on $\mathbb{B}$. Then consider the sequence of measures $\mu_k := \mathbf{1}_\Omega \inf\{f, k\}(dd^c\psi)^n$, $k \in \mathbb{N}$ with compact support in $\mathbb{B}$.

Fix an integer $k \ge 1$. Since $\mu_k \le (dd^c\psi_k)^n$ on $\mathbb{B}$, where $\psi_k := k^{1/n}\psi \in \mathcal{E}_0(\mathbb{B})$, it follows from ([Ko 2]) that there exists $u_k \in \mathcal{L}^+_{\gamma_k}(\mathbb{C}^n)$ such that $(dd^c u_k)^n = \mu_k$ on $\mathbb{C}^n$, where $\gamma_k^n := \mu_k(\mathbb{B})$. We can normalize u_k so that $\max_{\overline{\Omega}} u_k = 0$. We can also find $g_k \in \mathcal{E}_0(\Omega)$ such that $(dd^c g_k)^n = \mu_k$ on Ω. Then from the comparison principle, we have $u_k \le g_k$ on Ω and since the sequence of measures (μ_k) is increasing, the sequence of plurisubharmonic functions (g_k) decreases to φ on Ω. By Hartogs lemma, $u := (\limsup_{k\to+\infty} u_k)^*$ is plurisubharmonic on $\mathbb{C}^n$ and $\max_{\overline{\Omega}} u = 0$. It is clear that $u \le \varphi$ on Ω and $u \in \mathcal{L}_\gamma(\mathbb{C}^n)$, where $\gamma^n := \int_{\overline{\Omega}} (dd^c u)^n$, since $\gamma_k \le \gamma, \forall k \in \mathbb{N}$.

2) Assume now that $\varphi \in \mathcal{F}(\Omega)$. By Lemma 2.1, there exists a decreasing sequence (φ_j) of functions from the class $\mathcal{E}_0(\Omega)$ which converges to φ on Ω and $\int_\Omega (dd^c\varphi)^n = \lim_j \int_\Omega (dd^c\varphi_j)^n$. Let us define $\gamma > 0$ so that $\gamma^n := \int_\Omega (dd^c\varphi)^n < +\infty$ and fix $j \in \mathbb{N}$. Then by the first case there exists $u_j \in \mathcal{L}^+_{\gamma_j}(\mathbb{C}^n)$ such that $\max_{\overline{\Omega}} u_j = 0$ and $u_j \le \varphi_j$ on Ω, where $\gamma_j^n = \int_\Omega (dd^c\varphi_j)^n$.

Again the function $u := (\limsup_{j\to+\infty} u_j)^* \in \mathcal{L}_\gamma(\mathbb{C}^n)$ and satisfies the inequality $u \le \varphi$ on Ω and by Hartogs' lemma we have $\max_{\overline{\Omega}} u = 0$. □

From this result we get the following one.

Corollary 5.2. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded hyperconvex domain and $\varphi \in \mathcal{E}(\Omega)$. Then for any open set $\omega \Subset \Omega$, there exists a function $u \in \mathcal{L}_\gamma(\mathbb{C}^n)$, where $\gamma > 0$ such that $u \le \varphi$ on ω.*

Proof. This result follows from the last theorem applied to the function $\tilde{\varphi}_\omega$, which belongs to $\mathcal{F}(\Omega)$ and satisfies $\tilde{\varphi}_\omega = \varphi$ on ω by Proposition 2.2. □

It's possible to get a control on the Monge-Ampère measure of the subextension in some cases as the following result shows. Recall that $\mathcal{F}^a(\Omega$ is the set of plurisubharmonic functions $\varphi \in \mathcal{F}(\Omega)$ such that $(dd^c\varphi)^n$ puts no mass on pluripolar subsets of Ω.

Theorem 5.3. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded hyperconvex domain and $\psi \in \mathcal{F}^a(\Omega)$ with $\int_\Omega (dd^c\psi)^n = 1$. Then there exists $u \in \mathcal{L}(\mathbb{C}^n)$ such that $u \le \psi$ on Ω and $(dd^c u)^n = \mathbf{1}_\Omega (dd^c\psi)^n$ on $\mathbb{C}^n$. Here $(dd^c u)^n$ is the unique measure with the property that for any sequence $v_j \in \mathcal{L}^+(\mathbb{C}^n)$ decreasing to u we have $(dd^c v_j)^n \to (dd^c u)^n$ weakly on $\mathbb{C}^n$.*

Proof. Take a hyperconvex domain Ω' containing $\overline{\Omega}$. Then by [Ce 3], there exists a plurisubharmonic function $\varphi \in \mathcal{F}^a(\Omega')$ such that $\varphi \le \psi$ on Ω and $(dd^c\varphi)^n = \mathbf{1}_\Omega (dd^c\psi)^n =: \mu$ as Borel measures on Ω'. Then $\mu(U_j) \to 0$ as $j \to \infty$ when $U_j = \{z \in \Omega'; \varphi(z) < -j\}$. Set $\mu_j = \mu - \mu_{|U_j}$ and observe that $(dd^c \sup\{\varphi, -j\})^n \ge \mu_j$ on Ω' (see [De], [Ce-Ko]). Then by [Ko 1] there exist $\varphi_j \in \mathcal{E}_0(\Omega')$ such that $(dd^c\varphi_j)^n = \mu_j$. Define

$$\alpha_j = \frac{1}{\mu_j(\Omega')}$$

and observe that $\alpha_j \ge 1$. By [Ko 2] there exist $u_j \in \mathcal{L}^+(\mathbb{C}^n)$ with $\sup_{\Omega'} u_j = 0, u_j \le \varphi_j$ and $(dd^c u_j)^n = \alpha_j \mu_j$.

Set

$$v_j = (\sup_{k \ge j} u_k)^*.$$

Then $v_j \ge u_j$ and $\psi \ge \varphi \ge u := \lim v_j \in \mathcal{L}$. Observe that for a fixed $j \in \mathbb{N}$, the sequence

$$\tilde{v}_{j,k} := \sup\{u_\ell; \ j \le \ell \le k\}, k \ge j$$

is an increasing sequence of plurisubharmonic functions in $\mathcal{L}^+(\mathbb{C}^n)$ which converges a.e. on $\mathbb{C}^n$ to v_j. Since $(dd^c u_\ell)^n \ge \mu$ on $\mathbb{C}^n \setminus U_j$, for any $\ell \ge j$, it follows from ([De], [Ce-Ko]) that $(dd^c v_{j,k})^n \ge \mu$ on $\mathbb{C}^n \setminus U_j$. By the convergence theorem [Be-Ta 1], it follows that

$$(dd^c v_j)^n \ge \mu \text{ on } \mathbb{C}^n \setminus U_j$$

and for $M > 0$

$$(dd^c \max(v_j, -M))^n \ge \mu \quad \text{on} \quad \mathbb{C}^n \setminus (U_j \cup V_M), \quad V_M := \{u < -M\}.$$

Since by the convergence theorem [Be-Ta 1],

$$\lim_{j\to\infty} (dd^c \max(v_j, -M))^n = (dd^c \max(u, -M))^n$$

we obtain

$$(dd^c \max(u, -M))^n = \lim_{j\to\infty} (dd^c \max(v_j, -M))^n \ge \mu \text{ on } \mathbb{C}^n \setminus V_M.$$

Therefore

$$\lim_{M\to\infty}(dd^c \max(u,-M))^n \geq \mu$$

on $\mathbb{C}^n$ and since the integrals of both measures are equal, the measures themselves are equal. Hence

$$\mu_M = (dd^c \max(u,-M))^n \to \mu.$$

Take a sequence (w_j) of continuous functions in $\mathcal{L}^+(\mathbb{C}^n)$ decreasing to u. We have to prove that $(dd^c w_j)^n \to d\mu$. It is no loss of generality to assume that $w_j > w_{j+1}$ for all j.

Set for $j \in \mathbb{N}$,

$$\nu_j = (dd^c w_j)^n,$$

and

$$\nu_{j,M} = (dd^c \max(w_j,-M))^n.$$

Fix $t > 1$. Then for $E_j = \{w_j > u + (t-1)\} \cap \{u \geq -M+1\}$ we have

$$\int_{E_j} \mu \to 0. \tag{5.1}$$

(since E_j decrease to $\emptyset$.) Since the set $\{w_j < -M\}$ is relatively compact and the sequence w_j is strictly monotone one can find k_0 so big that for $k > k_0$, $v_k < w_j$ on this set. Note that if $w_j(z) < -M$ then $w_j(z) + M > v_k(z) + M > t(v_k(z) + M)$. Hence, by the comparison principle

$$\begin{aligned}\int_{\{w_j<-M\}}(dd^c w_j)^n &\leq \int_{\{t(v_k+M)<w_j+M\}}(dd^c w_j)^n \\ &\leq t^n \int_{\{t(v_k+M)<w_j+M\}}(dd^c v_k)^n \leq t^n \int_{\{u<-M+1\}\cup E_j}(dd^c v_k)^n.\end{aligned}$$

Then, by (5.1)

$$\limsup_{j\to\infty}\int_{\{w_j<-M\}}(dd^c w_j)^n \leq \liminf_{k\to\infty}\int_{\{u<-M+1\}}(dd^c v_k)^n =: \epsilon(M).$$

From this estimate and the fact that $\nu_{j,M} = \nu_j$ on $\{w_j > -M\}$ we conclude that the total variation

$$||\nu_{j,M} - \nu_j|| \leq 2\epsilon(M/2), \quad j \geq j(M).$$

We claim that $\epsilon(M) \to 0$ as $M \to \infty$. Indeed, since $\int_{\mathbb{C}^n}(dd^c v_k)^n = 1 = \mu(\mathbb{C}^n)$ and $(dd^c v_k)^n \geq \mu$ on $\Omega \setminus U_k \supset \Omega \setminus (U_k \cup V_{M-1})$, it follows that $\int_{U_k\cup V_{M-1}}(dd^c v_k)^n \leq \mu(U_k \cup V_{M-1})$. Now since $\mu(U_k \cup V_{M-1}) \leq \mu(U_k) + \mu(V_{M-1})$ and the measure μ puts no mass on pluripolar sets, it follow that each of these terms tends to 0 and then so does $\epsilon(M) \leq \mu(V_{M-1})$.

Therefore for a test function χ we can make the first and the third term on the right in the formula

$$\int \chi(\nu_j - \mu) = \int \chi(\nu_j - \nu_{j,M}) + \int \chi(\nu_{j,M} - \mu_M) + \int \chi(\mu_M - \mu)$$

arbitrarily small by taking M large enough and $j \geq j(M)$. The middle term goes to zero as $j \to \infty$ by the convergence theorem. Therefore the left hand side tends to zero. □

Remark. 1) Observe that the measure $(dd^c u)^n$ was defined globally. It would be interesing if we can show that it is can be defined locally. It would be also interesting to know if the last theorem is true for $\psi \in \mathcal{F}(\Omega)$.

2) We can define a "canonical" subextention

$$u = \sup\{v \in \mathcal{L} : v \leq \psi\}.$$

Roughly speaking it should have Monge-Ampère mass supported on the set $\{u = \psi\}$.

We will come back to these questions in a subsequent paper.

Example. We give an example of a bounded subharmonic function v on the unit disc $D \subset \mathbb{C}$ such that $\int_D dd^c v < +\infty$ and there is no bounded subharmonic function u on an open neighbourhood D' of $\overline{D}$ such that $dd^c u = \mathbf{1}_D dd^c v$ as measures on D'.

Indeed let (a_j) be a discrete sequence of points in the unit disc $D \subset \mathbb{C}$ which converges to 1. For $j \in \mathbb{N}$, define $v_j(z) := \sup\{g_D(z, a_j), -1\}$ for $z \in D$, where $g_D(z, a_j)$ is the Green function of D with pole at a_j. Let (ε_j) be a sequence of positive numbers such that $\sum_j \varepsilon_j = 1$. Then $v := \sum_j \varepsilon_j v_j$ is a bounded subharmonic function on D such that $-1 \leq v \leq 0$ on D and $\int_D dd^c v < +\infty$.

It is easy to see that $dd^c v_j$ converges weakly to δ_1 as measures on $\mathbb{C}$ since $\int_D dd^c v_j = 1$ for any $j \in \mathbb{N}$ and for $j \in \mathbb{N}$ large enough $dd^c v_j$ puts no mass outside any arbitrary neighbourhood of 1. Then it follows that $\limsup_{j \to +\infty} \int_{\overline{D}} \log|1 - z| dd^c v_j = -\infty$. Therefore, taking a subsequence if necessary, we can assume that the poles and the weights are choosen so that $\sum_j \varepsilon_j \int_{\overline{D}} \log|1 - z| dd^c v_j = -\infty$.

Now if u is a subharmonic function on a disc D' containing $\overline{D}$, such that $dd^c u = \mathbf{1}_D dd^c v$ on D' then by the Riesz decomposition we have $u(1) = c + \int_{\overline{D}} \log|1 - z| dd^c u = c + \sum_j \varepsilon_j \int_{\overline{D}} \log|1 - z| dd^c v_j = -\infty$ by construction.

Remark. As observed by El Mir ([El]), there exists a plurisubharmonic function u on some open subset $\Omega \subset \mathbb{C}^2$ which has no subextension to any larger domain. The main obstruction is the fact that the polar set of u in Ω may contain a non trivial analytic set, which does not extend as an analytic set in a larger domain. This analytic stucture comes from the fact that superlevel sets of Lelong numbers of φ defined by

$$A(\varphi; c) := \{a \in \Omega \,;\, \nu(\varphi, a) \geq c\}, c > 0,$$

are analytic sets by Siu's theorem (cf. [Siu]). Indeed, let u be a plurisubharmonic function on some open subset $\omega \subset \Omega$. Assume that there exists a function $U \in PSH(\Omega)$ such that $U \leq u$ on ω. Then $A(u; c) \subset A(U; c) \cap \omega$. Hence the subextension problem is closely related to the propagation of singularities of plurisubharmonic functions. Observe that for all the functions which were considered in our theorems the sets $A(\varphi; c)$ are finite so that obviously there is no analytic obstruction to subextension.

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