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Restriction theorems for a surface with negative curvature

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Abstract We prove a bilinear restriction theorem for a surface of negative curvature. This is the analogue of the results of T. Wolff [19] and T. Tao [14], [15] for cones and paraboloids. As a consequence we obtain an almost sharp linear restriction theorem.

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1 Introduction

Let S be a smooth compact hypersurface with boundary in $\mathbf{R}^n$ and let $d\sigma$ denote the Lebesgue measure on S. We say that the (linear) adjoint restriction estimate $\mathbf{R}^*_S(p \to r)$ holds if

$$\|\widehat{f d\sigma}\|_r \leq C\|f\|_p \tag{1}$$

for all test functions supported on S, with a constant C independent of f. The operator

$$\widehat{f d\sigma}(x) = \int_S f(\xi)^{2\pi i x\cdot\xi}\, d\sigma(\xi),$$

can be considered as the adjoint of the operator of restriction of the Fourier transform to S.

E. M. Stein posed this problem in the seventies. The conjecture is that if S has non-vanishing Gaussian curvature, then (1) holds whenever

$$r > 2n/(n-1), \qquad p' \leq \frac{n-1}{n+1} r. \tag{2}$$

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The problem was first studied in dimension 2 by Stein and C. Fefferman (see [7]). The conjecture was proven in dimension 2 by A. Zygmund [20]. In higher dimensions it is still open. The first results were due to P. Tomas [18], P. Sjölin [6], R. S. Strichartz [12] and E. M. Stein [11] who proved that (1) was true for $p = 2$ and r in the range given in (2), $r \geq 2\frac{n+1}{n-1}$. Many years later, J. Bourgain [2,3,5] showed the estimate for some $p \geq 2$, $p' < \frac{n-1}{n+1}r$. There were some improvements, due to A. Moyua, T. Tao, L. Vega and A. Vargas [9], [10], [16],[17] for hypersurfaces of elliptic type. Recently, T. Tao [15] proved the estimate for all $r > 2\frac{n+2}{n}$ and p in the range (2), $p' \leq \frac{n-1}{n+1}r$, for paraboloids.

Given two surfaces S_1, S_2 with measures $d\sigma_1$, and $d\sigma_2$ respectively, we say that the bilinear restriction estimate $\mathbf{R}^*_{S_1,S_2}(p \times p \to q)$ holds if

$$\|\widehat{f_1\, d\sigma_1}\,\widehat{f_2\, d\sigma_2}\|_q \leq C\|f_1\|_p\|f_2\|_p \tag{3}$$

for all test functions f_1, f_2 on S_1 and S_2 respectively.

If (2) were true, then, for all S_1, $S_2 \subset S$ we would have that $\mathbf{R}^*_{S_1,S_2}(p \times p \to q)$ for all $q = r/2$, r in the same range. It was observed that under certain assumptions, a wider range was allowed for (3). Tao, Vega and Vargas [16], [17], proved several results in this direction, for elliptic type hypersurfaces under the hypothesis that S_1 and S_2 are separated compact subsets of S. They also proved that the bilinear restriction estimates (with the separation hypothesis) imply the linear estimates (Theorem 2.2 in in [16]). Finally, Tao [15] proved that if S is a paraboloid (or more generally, a hypersurface of elliptic type), under the same hypothesis on S_1 and S_2, (3) holds also for

$$q > \frac{n+2}{n}, \qquad \frac{n+2}{2q} + \frac{n}{p} < n. \tag{4}$$

It is worth to mention that the analogous problem for cones (case of null curvature) has been solved. The linear theorem in $\mathbf{R}^3$ is due to B. Barcelo, [1]. For the bilinear theorem, there were partial results due to Bourgain [4] and T. Tao and A. Vargas [17]. Finally, T. Wolff [19] and T. Tao [14] gave the optimal estimates.

Here, we want to consider the case of a surface with negative Gaussian curvature. The model surface is the hyperbolic paraboloid, $z = xy$, in $\mathbf{R}^3$. Concerning bilinear restriction estimates, the first remark that we have to make is that the hypothesis on S_1 and S_2 has to be different from the one that we had in the elliptic case. The separation condition is not enough to give a range for (3) wider than (2). The existence of line segments in a hyperbolic paraboloid makes the following example possible:

Remark 1.1. Consider the hyperbolic paraboloid

$$S = \{(\xi, \eta, \tau) /\ \ \tau = \xi\eta\} \subset \mathbf{R}^3.$$

Define the subsets of S

$$S_1 = S \cap \{(\xi, \eta, \tau) /\ \ 1/2 < \xi < 1,\ -1 < \eta < 1\}$$

and

$$S_2 = S \cap \{(\xi, \eta, \tau) / \ -1 < \xi < -1/2, \ -1 < \eta < 1\}.$$

Then, $\mathbf{R}^*_{S_1,S_2}(p \times p \to q)$ is false for any $p' > q$.

To prove this statement, we just consider the sets $A = S \cap \{(\xi, \eta, \tau) / \ 1/2 < \xi < 1, \ |\eta| < \epsilon\}$, $B = S \cap \{(\xi, \eta, \tau) / \ -1 < \xi < -1/2, \ |\eta| < \epsilon\}$, and the functions

$$f_1 = \chi_A, \qquad f_2 = \chi_B.$$

Then, for $i = 1, 2$, and $|x| \leq 1/10$, $|y| \leq 1/(10\epsilon)$, $|z| \leq 1/(10\epsilon)$, we have

$$|\widehat{f_i \, d\sigma_i}(x, y, z)| \geq c\epsilon.$$

Hence,

$$\|\widehat{f_1 \, d\sigma_1}\,\widehat{f_2 \, d\sigma_2}\|_q \geq \epsilon^{2-2/q},$$

while,

$$\|f_i\|_p \sim \epsilon^{1/p}.$$

This proves the remark.

We will state a bilinear restriction theorem with some hypothesis that avoid this type of example. Since the line segments in the hyperbolic paraboloid all lie above the axis parallel lines in the (ξ, η) parameter space, one is lead to the formulation of the separation condition: the two subsets S_1 and S_2 are separated both in the ξ and the η parameter. Under this hypothesis, we can follow the argument due to T. Tao [15] to obtain,

Theorem 1.2. *Consider the surface*

$$S = \{(\xi, \eta, \tau) / \ \tau = \xi\eta, \quad |\xi|, \ |\eta| \leq 1\} \subset \mathbf{R}^3.$$

Consider compact subsets of S, S_1 *and* S_2 *satisfying:*

for all $(\xi_1, \eta_1, \xi_1\eta_1) \in S_1$ *and* $(\xi_2, \eta_2, \xi_2\eta_2) \in S_2$ *we have* $|\xi_1 - \xi_2| \geq 1$ *and* $|\eta_1 - \eta_2| \geq 1$.

Then, $\mathbf{R}^*_{S_1,S_2}(p \times p \to q)$ *holds for any* $q > 5/3$, $\frac{5}{2q} + \frac{3}{p} < 3$.

This separation condition appeared in [17], section 9. It was proven there that, under the assumptions of Theorem 1.2, $\mathbf{R}^*_{S_1,S_2}(2 \times 2 \to 12/7)$ holds.

As in the case of elliptic surfaces, the bilinear theorem will imply the right linear restriction estimate. Since our hypotheses here are different, we can not use directly Theorem 2.2 of [16]. We need to prove a theorem suited for our case, adapting the ideas of [16]. Unfortunately, we lose the endpoint.

Theorem 1.3. *Define*

$$S = \{(\xi, \eta, \tau), / \ \tau = \xi\eta, \quad |\xi|, \ |\eta| \leq 1\} \subset \mathbf{R}^3.$$

Then $\mathbf{R}^*_S(p \to r)$ *holds for* $r > 10/3$ *and* $p' < r/2$.

In section 2 we derive Theorem 1.3 from Theorem 1.2. In section 3 we give the proof of Theorem 1.2.

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After the submission of this paper, the author was informed that Sanghyuk Lee [8] obtained independently the same result, giving also a more general version of Theorem 1.2.

2 Proof of Theorem 1.3

By scaling and translation, Theorem 1.2 implies the following

Proposition 2.1. *Let j, k, m, n, m', n' be natural numbers such that $|m-m'| = 2$, $|n-n'| = 2$, and functions f, g, with supp $f \subset S \cap \{(\xi,\eta,\tau) : m2^{-k} \le \xi \le (m+1)2^{-k},\ n2^{-j} \le \eta \le (n+1)2^{-j}\}$ and supp $g \subset S \cap \{(\xi,\eta,\tau) : m'2^{-k} \le \xi \le (m'+1)2^{-k},\ n'2^{-j} \le \eta \le (n'+1)2^{-j}\}$. Then, for all $q > 5/3$, $\frac{5}{2q} + \frac{3}{\tilde p} < 3$,*

$$\|\widehat{f\,d\sigma}\,\widehat{g\,d\sigma}\|_{L^q} \le C2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\|f\|_{L^{\tilde p}}\|g\|_{L^{\tilde p}},$$

where C is a constant independent of j, k, m, n, f and g.

By interpolation, to obtain Theorem 1.3, it is enough to prove the restricted type estimate,

$$\|\widehat{\chi_\Omega\,d\sigma}\|_{L^{2q}} \le C\|\chi_\Omega\|_{L^p},$$

for all $\Omega \subset S$, $2 > q > 5/3$, and p such that $\frac{1}{p} + \frac{1}{q} < 1$.

For each j, k natural numbers, we decompose S into "rectangles" $\tau_l^{k,j}$ of the form $\{(\xi,\eta,\xi\eta) : m2^{-k} \le \xi \le (m+1)2^{-k},\ n2^{-j} \le \eta \le (n+1)2^{-j}\}$, $l = (m,n) \in \mathbf{Z}\times\mathbf{Z}$. If $\tau_l^{k,j}$ and $\tau_h^{k,j}$ are two rectangles with $l = (m,n)$ and $h = (m',n')$ and $|m-m'| = 2$, $|n-n'| = 2$, we say that these rectangles are *close* and write $\tau_l^{k,j} \sim \tau_h^{k,j}$. For almost every $(x,y), (x',y') \in [-1,1]\times[-1,1]$ there exists a unique pair of close rectangles $\tau_l^{k,j}$, $\tau_h^{k,j}$ containing (x,y) and (x',y') respectively. Thus we have

$$\widehat{\chi_\Omega\,d\sigma}\,\widehat{\chi_\Omega\,d\sigma} = \sum_{k,j}\ \sum_{l,h:\tau_l^{k,j}\sim\tau_h^{k,j}} \widehat{\chi_{\Omega\cap\tau_l^{k,j}}d\sigma}\,\widehat{\chi_{\Omega\cap\tau_h^{k,j}}d\sigma}.$$

Hence,

$$\|\widehat{\chi_\Omega\,d\sigma}\|_{L^{2q}}^2 = \|\widehat{\chi_\Omega\,d\sigma}\,\widehat{\chi_\Omega\,d\sigma}\|_{L^q} \le \sum_{k,j}\Bigg\|\sum_{l,h:\tau_l^{k,j}\sim\tau_h^{k,j}} \widehat{\chi_{\Omega\cap\tau_l^{k,j}}d\sigma}\,\widehat{\chi_{\Omega\cap\tau_h^{k,j}}d\sigma}\Bigg\|_{L^q}.$$

As in [16] this can be majorized by

$$\sum_{k,j}\Bigg(\sum_{l,h:\tau_l^{k,j}\sim\tau_h^{k,j}}\|\widehat{\chi_{\Omega\cap\tau_l^{k,j}}d\sigma}\,\widehat{\chi_{\Omega\cap\tau_h^{k,j}}d\sigma}\|_{L^q}^q\Bigg)^{1/q}.$$

By proposition 2.1, this is less than or equal to

$$\sum_{k,j}2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\Bigg(\sum_{l,h:\tau_l^{k,j}\sim\tau_h^{k,j}}|\Omega\cap\tau_l^{k,j}|^{q/\tilde p}|\Omega\cap\tau_h^{k,j}|^{q/\tilde p}\Bigg)^{1/q}$$

for some $\tilde p$ that we will choose later.

Due to the fact that we have a double sum, in k and j, the rest of the argument has some technical complication. We need to decompose the set Ω in convenient subsets. For $\eta\in[-1,1]$, we define $\Omega_\eta=\{\xi\ :\ (\xi,\eta,\xi\eta)\in\Omega\}$. For each natural number K, set $\Omega(K)=\{(\xi,\eta,\xi\eta)\in\Omega\ :\ 2^{-K}<|\Omega_\eta|\le 2^{-K+1}\}$. For $\Delta\subset\mathbf{R}^3$, denote by $P(\Delta)$ the orthogonal projection onto the second coordinate axis. For each K, denote $J=J(K)$, a natural number so that the length $|P(\Omega_K)|\sim 2^{-J}$. Then,

$$|\Omega(K)|\sim 2^{-K-J}. \tag{5}$$

We write

$$\Omega=\bigcup_{K=0}^{\infty}\Omega(K)$$

and

$$\widehat{\chi_\Omega\,d\sigma}=\sum_{K=0}^{\infty}\widehat{\chi_{\Omega(K)}\,d\sigma}.$$

We are first going to show

$$\|\widehat{\chi_{\Omega(K)}\,d\sigma}\|_{L^{2q}}\le C|\Omega(K)|^{1-1/q}. \tag{6}$$

We fix K and estimate, as above,

$$\begin{aligned}
&\|\widehat{\chi_{\Omega(K)}\,d\sigma}\|_{L^{2q}}^2\\
&\quad=\|\widehat{\chi_{\Omega(K)}\,d\sigma}\,\widehat{\chi_{\Omega(K)}\,d\sigma}\|_{L^q}\\
&\quad\le\sum_{k,j}2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\Bigg(\sum_{l,h:\tau_l^{k,j}\sim\tau_h^{k,j}}|\Omega(K)\cap\tau_l^{k,j}|^{q/\tilde p}|\Omega(K)\cap\tau_h^{k,j}|^{q/\tilde p}\Bigg)^{1/q}.
\end{aligned}$$

We use the fact that for each rectangle $\tau_l^{k,j}$ there are only four rectangles $\tau_h^{k,j}$ such that $\tau_l^{k,j}\sim\tau_h^{k,j}$. We define, for $l=(m,n)$,

$$5\tau_l^{k,j}:=\{(\xi,\eta,\xi\eta):(m-2)2^{-k}\le\xi\le(m+3)2^{-k},\ (n-2)2^{-j}\le\eta\le(n+3)2^{-j}\}.$$

Then, for each l,

$$\sum_{h:\tau_l^{k,j}\sim\tau_h^{k,j}} |\Omega(K)\cap\tau_l^{k,j}|^{q/\tilde p}|\Omega(K)\cap\tau_h^{k,j}|^{q/\tilde p} \le C|\Omega(K)\cap 5\tau_l^{k,j}|^{2q/\tilde p}.$$

Hence,

$$\|\widehat{\chi_{\Omega(K)}\,d\sigma}\|_{L^{2q}}^2 \le C\sum_{k,j} 2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\left(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}|^{2q/\tilde p}\right)^{1/q}. \tag{7}$$

We will take into account that

$$|\Omega(K)\cap 5\tau_l^{k,j}| \le 25\min\{2^{-j-k}, 2^{-k}2^{-J}, 2^{-j}2^{-K}, 2^{-K}2^{-J}\}. \tag{8}$$

To bound (7) we decompose it in four sums: first one for $j \le J(K)$ and $k \ge K$, second $j \ge J(K)$ and $k \le K$, third $j \ge J(K)$ and $k \ge K$, and fourth $j \le J(K)$ and $k \le K$. We begin by

$$S_1 := \sum_{k\ge K}\sum_{j\le J} 2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\left(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}|^{2q/\tilde p}\right)^{1/q}.$$

Note that for $q > 3/2$ and $\tilde p$ close to p_0, $\frac{1}{p_0}+\frac{1}{q}=1$, we have that $2q/\tilde p > 1$. For $j,\ k$ such that, $j \le J$ and $k \ge K$, we have, by (8),

$$\begin{aligned}
&\le C\sum_{k\ge K}\sum_{j\le J} 2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\left(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}|[2^{-J}2^{-k}]^{2q/\tilde p-1}\right)^{1/q}\\
&\le C[2^{-J}]^{2/\tilde p-1/q}\sum_{k\ge K}\sum_{j\le J} 2^{-j(2-\frac{2}{q}-\frac{2}{\tilde p})}2^{-k(2-\frac{3}{q})}\left(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}|\right)^{1/q}.
\end{aligned}$$

We observe that

$$\sum_l |\Omega(K)\cap 5\tau_l^{k,j}| \le C|\Omega(K)|.$$

We can choose $\tilde p$ such that, $(2-\frac{2}{q}-\frac{2}{\tilde p}) < 0$, while $2-\frac{3}{q} > 0$. We sum both series to bound

$$S_1 \le C[2^{-J}]^{2/\tilde p-1/q}|\Omega(K)|^{1/q}2^{-J(2-\frac{2}{q}-\frac{2}{\tilde p})}2^{-K(2-\frac{3}{q})} \le C|\Omega(K)|^{2-2/q},$$

by (5).

We do similarly for the second sum

$$S_2 := \sum_{k\le K}\sum_{j\ge J} 2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\left(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}|^{2q/\tilde p}\right)^{1/q}.$$

In this case, we use the estimate (8) to obtain,

$$\leq C \sum_{k\leq K}\sum_{j\geq J} 2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\Big(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}|2^{(-K-j)(2q/\tilde p-1)}\Big)^{1/q}$$

$$\leq C2^{-K(2/\tilde p-1/q)}\sum_{k\leq K}\sum_{j\geq J} 2^{-k(2-\frac{2}{q}-\frac{2}{\tilde p})}2^{-j(2-\frac{3}{q})}\Big(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}|\Big)^{1/q}.$$

The same reasoning above, gives us

$$S_2 \leq C|\Omega(K)|^{2-2/q}.$$

The third piece of the sum,

$$\begin{aligned} S_3 &:= \sum_{k\leq K}\sum_{j\leq J} 2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\Big(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}|^{2q/\tilde p}\Big)^{1/q}\\ &\leq C \sum_{k\leq K}\sum_{j\leq J} 2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\Big(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}||\Omega(K)|^{2q/\tilde p-1}\Big)^{1/q}.\\ &\leq C \sum_{k\leq K}\sum_{j\leq J} 2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\Big(|\Omega(K)|^{2q/\tilde p}\Big)^{1/q}. \end{aligned}$$

Again, we obtain, $S_3 \leq C|\Omega(K)|^{2-2/q}$.

About the fourth term,

$$\begin{aligned} S_4 &:= \sum_{k\geq K}\sum_{j\geq J} 2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\Big(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}|^{2q/\tilde p}\Big)^{1/q}\\ &\leq C \sum_{k\geq K}\sum_{j\geq J} 2^{-(j+k)(2-\frac{2}{q}-\frac{2}{\tilde p})}\Big(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}|[2^{-(j+k)}]^{2q/\tilde p-1}\Big)^{1/q}\\ &= C \sum_{k\geq K}\sum_{j\geq J} 2^{-(j+k)(2-\frac{3}{q})}\Big(\sum_l |\Omega(K)\cap 5\tau_l^{k,j}|\Big)^{1/q} \leq C|\Omega(K)|^{2-2/q}. \end{aligned}$$

This proves (6).

Once we have this, to be able to sum in K, we have some loss. Using (6)

$$\|\widehat{\chi_\Omega\, d\sigma}\|_{L^q} \leq \sum_K \|\widehat{\chi_{\Omega(K)}\, d\sigma}\|_{L^q} \leq C\sum_K |\Omega(K)|^{1-1/q}.$$

Now, simply notice that $|\Omega(K)| \leq 2^{-K}$, and write,

$$\sum_K |\Omega(K)|^{1-1/q} \leq C\sum_{K\geq 0} |\Omega|^{1-1/q-\epsilon}2^{-K\epsilon} \leq C_\epsilon |\Omega|^{1-1/q-\epsilon},$$

for all $\epsilon > 0$. This finishes the proof.

3 Proof of Theorem 1.2

By interpolation, it suffices to prove that $\mathbf{R}^*_{S_1,S_2}(2\times 2 \to q)$ holds for all $q > 5/3$. Our proof follows the lines of the proof in [15]. The "local part"of the argument is exactly the same. We only need to check the transversality properties which are crucial for the "global part". For the sake of completeness, we will recall the main steps of [15].

By translation, we can assume that

$$S_1 = S \cap \{(\xi,\eta,\tau)/ \;\; -1 < \xi < -1/2, \; -1 < \eta < -1/2\}$$

and

$$S_2 = S \cap \{(\xi,\eta,\tau)/ \;\; 1/2 < \xi < 1, \; 1/2 < \eta < 1\}$$

Denote by $\mathbf{R}^*_{S_1,S_2}(2\times 2 \to q,\alpha)$ the estimate

$$\|\widehat{f_1\,d\sigma_1}\,\widehat{f_2\,d\sigma_2}\|_{L^q(B(p,R))} \le CR^{\alpha}\|f_1\|_2\|f_2\|_2 \tag{9}$$

for all balls of radius R and all functions f_1, f_2 in S_1, S_2. Then, the epsilon removal argument on [13] reduces the proof of the theorem to show that $\mathbf{R}^*_{S_1,S_2}(2\times 2 \to 5/3,\alpha)$ holds for all $\alpha > 0$. Moreover, following Wolff's induction on scale argument we just have to prove:

Proposition 3.1. *There is a constant $C > 0$ such that, if*

$$\mathbf{R}^*_{S_1,S_2}(2\times 2 \to 5/3,\alpha) \tag{10}$$

holds for some $\alpha > 0$, then

$$\mathbf{R}^*_{S_1,S_2}(2\times 2 \to 5/3, \max\{(1-\delta)\alpha, C\delta\}+\epsilon)$$

holds for all $\delta > 0$ and all $0 < \epsilon << 1$.

Fix $R > 0$. To prove this proposition, we decompose f_j, $j = 1,2$ following the notation in Lemma 4.1 in [15]. There the *tubes* T are defined as as the sets of the form

$$T = \{(x,y,t)\; : \; R/2 \le t \le R; \; |(x,y)-(x(T),y(T))-tv(T)| \le R^{1/2}\},$$

where $(x(T),y(T)) \in R^{1/2}\mathbf{Z}^2$ is the *initial position* of T and $v(T) \in R^{-1/2}\mathbf{Z}^2$ is the *velocity*. Then, Tao shows that we can write

$$\widehat{f_j\,d\sigma} = \sum_{T_j} c_{T_j}\phi_{T_j},$$

for some coefficients c_{T_j} satisfying

$$\sum_{T_j} |c_{T_j}|^2 \le \|f_j\|_2^2.$$

and some functions ϕ_{T_j} adapted to tubes T_j, i.e. satisfying

$$|\phi_{T_j}(x, y, t)| \le C_N R^{-1/2}\left(1 + \frac{|(x, y) - (x(T_j), y(T_j)) - tv(T_j)|}{R^{1/2}}\right)^{-N} \tag{11}$$

for all $N > 0$.

Moreover, the functions ϕ_{T_j} are of the form

$$\phi_{T_j} = \widehat{f_{T_j}\, d\sigma} \tag{12}$$

where f_{T_j} is supported in a $R^{-1/2}$–neighborhood of some point $(\xi_j, \eta_j, \xi_j\eta_j) \in S_j$ and $v(T_j) = (-\eta_j, -\xi_j)$. Notice that the axis of the tube T_j is orthogonal to the surface S_j at the point $(\xi_j, \eta_j, \xi_j\eta_j)$. Finally, we also have, for any family of tubes, $\mathbf{T}$,

$$\Big\| \sum_{T_j \in \mathbf{T}} f_{T_j} \Big\|_2^2 \le C\#\mathbf{T} \tag{13}$$

or equivalently

$$\Big\| \sum_{T_j \in \mathbf{T}} \phi_{T_j}(\cdot, t) \Big\|_2^2 \le C\#\mathbf{T} \qquad \text{uniformly in } t. \tag{14}$$

Denote by $\mathbf{T}(S_j)$, $j = 1, 2$, the family of all the *tubes* T, such that $v(T) = (-\eta, -\xi)$ for some $(\xi, \eta, \xi\eta) \in S_j$. A transversality condition holds:

$$\text{If } T_1 \in \mathbf{T}(S_1) \text{ and } T_2 \in \mathbf{T}(S_2),\ \text{ then } T_1 \text{ and } T_2 \text{ are transversal,} \tag{15}$$

meaning this that the angle between their axes is bounded below by a positive absolute constant (independent of the tubes).

Let us go back to the proof of the proposition. We assume that $\mathbf{R}^*_{S_1,S_2}(2 \times 2 \to q, \alpha)$ holds and consider $Q_R \subset \{(x, y, t) : R/2 < t < R\}$ a square of side length $R/2$. It suffices to prove the estimate

$$\|\widehat{f_1\, d\sigma_1}\,\widehat{f_2\, d\sigma_2}\|_{L^{5/3}(Q_R)} \le C_{\epsilon,\delta} R^{\epsilon}\big(R^{(1-\delta)\alpha} + R^{c\delta}\big)\|f_1\|_2\|f_2\|_2. \tag{16}$$

Using the decomposition of f_1 and f_2 and some pigeonholing argument, this can be reduced to

Proposition 3.2.

$$\Big\| \sum_{T_1 \in \mathbf{T}_1} \sum_{T_2 \in \mathbf{T}_2} \phi_{T_1}\phi_{T_2} \Big\|_{L^{5/3}(Q_R)} \le C\big(R^{(1-\delta)\alpha} + R^{c\delta}\big)(\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2} \tag{17}$$

for all collections of tubes $\mathbf{T}_j \subset \mathbf{T}(S_j)$, $j = 1, 2$.

We cover the ball Q_R by balls B of radius $R^{1-\delta}$ with finite overlap. Then

$$\Big\|\sum_{T_1}\sum_{T_2}\phi_{T_1}\phi_{T_2}\Big\|_{L^{5/3}(Q_R)} \le \sum_B \Big\|\sum_{T_1}\sum_{T_2}\phi_{T_1}\phi_{T_2}\Big\|_{L^{5/3}(B)}$$

We will also decompose each ball B in cubes q of side length $R^{1/2}$ with finite overlap.

By a pigeonholing argument we can assume that there are numbers $1 \le \mu_1,\ \mu_2 \le R^{200}$ such that, for all the cubes q that we will consider, we have:

$$\mu_j \le \#\{T_j \in \mathbf{T}_j : T_j \cap R^\delta q \neq \emptyset\} \le 2\mu_j \qquad \text{for } j = 1, 2. \tag{18}$$

Also, we can assume that there is $\lambda_1 > 0$, such that for all tubes T_1,

$$\lambda_1 \le \#\{q \text{ satisfying (18)} :\ \ T_1 \cap R^\delta q \neq \emptyset\} \le 2\lambda_1. \tag{19}$$

We associate tubes in $\mathbf{T}_1 \cup \mathbf{T}_2$ and balls B as in [15]. There is a relation $\sim$, between tubes and balls, such that, for all $T \in \mathbf{T}_1 \cup \mathbf{T}_2$ we have

$$1 \le \#\{B\ :\ T \sim B\} \le CR^{c\delta}, \tag{20}$$

and by (19), for all $T_1 \in \mathbf{T}_1$, $T_1 \sim B$,

$$\#\{q\ :\ q \cap 10B \neq \emptyset,\ T_1 \cap R^\delta q \neq \emptyset\} \ge R^{-c\delta}\lambda_1. \tag{21}$$

For each ball B set $\hat{B}$ as the union of all the cubes q in B satisfying (18). We estimate the local part

$$\sum_B \Big\|\sum_{T_1\sim B}\sum_{T_2\sim B}\phi_{T_1}\phi_{T_2}\Big\|_{L^{5/3}(\hat{B})} \le \sum_B \Big\|\sum_{T_1\sim B}\sum_{T_2\sim B}\phi_{T_1}\phi_{T_2}\Big\|_{L^{5/3}(B)}$$

as in [15], using the induction hypothesis (10) and (14), by

$$\le \sum_B CR^{(1-\delta)\alpha}(\#\{T_1 \in \mathbf{T}_1 :\ T_1 \sim B\})^{1/2}(\#\{T_2 \in \mathbf{T}_2 :\ T_2 \sim B\})^{1/2}.$$

By Cauchy–Schwarz inequality,

$$\le CR^{(1-\delta)\alpha}\left(\sum_B \#\{T_1 \in \mathbf{T}_1 :\ T_1 \sim B\}\right)^{1/2}\left(\sum_B \#\{T_2 \in \mathbf{T}_2 :\ T_2 \sim B\}\right)^{1/2}.$$

Finally, by (20), this is less than or equal to

$$CR^{(1-\delta)\alpha}(\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2}.$$

The geometry of the surface will be important for the estimate of the global part

$$\sum_{T_1\not\sim B}\sum_{T_2\sim B}\phi_{T_1}\phi_{T_2} + \sum_{T_1\not\sim B}\sum_{T_2\not\sim B}\phi_{T_1}\phi_{T_2} + \sum_{T_1\sim B}\sum_{T_2\not\sim B}\phi_{T_1}\phi_{T_2}.$$

We will consider the first term of this sum, the others being similar. We are going to interpolate an L^1 and an L^2 estimate for that sum. For the L^1 estimate, we apply Cauchy–Schwarz inequality

$$\Big\| \sum_{T_1 \not\sim B} \sum_{T_2 \sim B} \phi_{T_1}\phi_{T_2} \Big\|_{L^1(B)} \leq \Big\| \sum_{T_1 \not\sim B} \phi_{T_1} \Big\|_{L^2(B)} \Big\| \sum_{T_2 \sim B} \phi_{T_2} \Big\|_{L^2(B)}$$

and directly estimate the norms using (14) to obtain

$$\leq R(\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2}.$$

The proof will end if we show the estimate

$$\Big\| \sum_{T_1 \not\sim B} \sum_{T_2 \sim B} \phi_{T_1}\phi_{T_2} \Big\|^2_{L^2(\hat{B})} \leq C R^{C\delta} R^{-1/2}(\#\mathbf{T}_1)(\#\mathbf{T}_2). \tag{22}$$

As we said before, we decompose $\hat{B}$ in cubes q of side length $R^{1/2}$, satisfying (18). We estimate the left hand side of (22) by

$$\sum_q \Big\| \sum_{T_1 \not\sim B} \sum_{T_2 \sim B} \phi_{T_1}\phi_{T_2} \Big\|^2_{L^2(q)}.$$

We can assume that we have (18) and (19). Set, for $q \subset 2B$,

$$\mathbf{T}_1(q) = \{T_1 \in \mathbf{T}_1 : \ T_1 \not\sim B, \ T_1 \cap R^\delta q \neq \emptyset\}, \tag{23}$$

and

$$\mathbf{T}_2(q) = \{T_2 \in \mathbf{T}_2 : \ T_2 \cap R^\delta q \neq \emptyset\},$$

We will consider the contribution of the cubes q and tubes T_1, satisfying (18) and (19), such that $T_1 \in \mathbf{T}_1(q)$ and $T_2 \in \mathbf{T}_2(q)$, the remaining being easy to estimate by (11).

Denote $\Lambda_j = \{(\xi, \eta) \ : \ (\xi, \eta, \xi\eta) \in S_j\}$, $j = 1, 2$, the orthogonal projections of S_j onto the $\xi\eta$–plane. For $(\xi_1, \eta_1) \in \Lambda_1$ and $(\xi_2', \eta_2') \in \Lambda_2$, define the set

$$\begin{aligned} &\pi((\xi_1, \eta_1), (\xi_2', \eta_2')) \\ &\quad = \{(\xi_1', \eta_1') \in \Lambda_1 : (\xi_1, \eta_1, \xi_1\eta_1) + (\xi_2, \eta_2, \xi_2\eta_2) \\ &\quad = (\xi_1', \eta_1', \xi_1'\eta_1') + (\xi_2', \eta_2' + \xi_2'\eta_2') \text{ for some } (\xi_2, \eta_2) \in \Lambda_2\}. \end{aligned} \tag{24}$$

It turns out that $\pi((\xi_1, \eta_1), (\xi_2', \eta_2'))$ is contained in a straight line. Actually, if we set $A = \xi_2' - \xi_1$ and $B = \eta_2' - \eta_1$, then, $\pi((\xi_1, \eta_1), (\xi_2', \eta_2'))$ is contained in the straight line that passes through (ξ_1, η_1) and orthogonal to the vector (B, A), i.e.

$$\mathbf{r} = \mathbf{r}((\xi_1, \eta_1), (\xi_2', \eta_2')) := \{(\xi_1', \eta_1') : \ B\xi_1' + A\eta_1' = B\xi_1 + A\eta_1\}. \tag{25}$$

Note also that by the definition of S_1 and S_2, $A \sim 1$, and $B \sim 1$. This implies that for all $(\xi_2, \eta_2) \in \Lambda_2$, $\mathbf{r}((\xi_1, \eta_1), (\xi_2', \eta_2'))$ is transversal to the vector (ξ_2, η_2), and that, the distance from (ξ_2, η_2) to $\mathbf{r}((\xi_1, \eta_1), (\xi_2', \eta_2'))$ is ~ 1. This has an important consequence for us: define the plane

$$\mathbf{P}((\xi_1, \eta_1), (\xi_2', \eta_2'))$$

as the plane containing the point $(0, 0, 1)$ and the straight line $\{(\xi, \eta, 0) : (\xi, \eta) \in \mathbf{r}((\xi_1, \eta_1), (\xi_2', \eta_2'))\}$. Then, for all $(\xi_2, \eta_2) \in \Lambda_2$,

$$(-\xi_2, -\eta_2, 1) \quad \text{and} \quad \mathbf{P}((\xi_1, \eta_1), (\xi_2', \eta_2')) \quad \text{are transversal.} \tag{26}$$

For each cube q we want to estimate

$$\Big\| \sum_{T_1 \in \mathbf{T}_1(q)} \sum_{T_2 \in \mathbf{T}_2(q)} \phi_{T_1} \phi_{T_2} \Big\|_{L^2(q)}^2 .$$

First, note that, by (11) and the fact that the tubes $T_1 \in \mathbf{T}_1$ and $T_2 \in \mathbf{T}_2$ are transversal, we have

$$\|\phi_{T_1} \phi_{T_2}\|_{L^2}^2 \le C R^{-1/2}.$$

Moreover, we can write

$$\Big\| \sum_{T_1 \in \mathbf{T}_1(q)} \sum_{T_2 \in \mathbf{T}_2(q)} \phi_{T_1} \phi_{T_2} \Big\|_2^2 = \sum_{T_1 \in \mathbf{T}_1(q)} \sum_{T_2 \in \mathbf{T}_2(q)} \sum_{T_1' \in \mathbf{T}_1(q)} \sum_{T_2' \in \mathbf{T}_2(q)} \int \phi_{T_1} \phi_{T_2} \overline{\phi_{T_1'} \phi_{T_2'}},$$

and use (12)

$$= \sum_{T_1 \in \mathbf{T}_1(q)} \sum_{T_2 \in \mathbf{T}_2(q)} \sum_{T_1' \in \mathbf{T}_1(q)} \sum_{T_2' \in \mathbf{T}_2(q)} (f_{T_1}\, d\sigma) * (f_{T_2} d\sigma) * (\widetilde{f_{T_1'}\, d\sigma}) * (\widetilde{f_{T_2'}\, d\sigma})(0).$$

(Here we use the notation $\widetilde{g} = g(-\cdot)$.) Now, notice that

$$(f_{T_1}\, d\sigma) * (f_{T_2} d\sigma) * (\widetilde{f_{T_1'}\, ds}) * (\widetilde{f_{T_2'}\, ds})(0)$$

it not zero only if (ξ_1', η_1') belongs to a $R^{-1/2}$–neighborhood of $\pi((\xi_1, \eta_1), (\xi_2', \eta_2'))$ and

$$(\xi_1, \eta_1, \xi_1 \eta_1) + (\xi_2, \eta_2, \xi_2 \eta_2) = (\xi_1', \eta_1', \xi_1' \eta_1') + (\xi_2', \eta_2', \xi_2' \eta_2') + O(R^{-1/2}).$$

For fixed (ξ_1', η_1'), (ξ_1, η_1) and (ξ_2, η_2), there are at most $O(1)$ points (ξ_2', η_2') satisfying that equation.

Denote by $\nu(q)$ the supremum on all the points $(\xi_1, \eta_1) \in \Lambda_1$, $(\xi_2', \eta_2') \in \Lambda_2$ of

$$\#\{T_1' \in \mathbf{T}_1(q) : \; v(T_1') = (-\eta_1', -\xi_1') \quad \text{such that } (\xi_1', \eta_1') \text{ belongs to a}$$
$$R^{-1/2} - \text{neighborhood of } \pi((\xi_1, \eta_1), (\xi_2', \eta_2'))\}$$

Then, the above reasoning shows that

$$\Big\| \sum_{T_1 \in \mathbf{T}_1(q)} \sum_{T_2 \in \mathbf{T}_2(q)} \phi_{T_1} \phi_{T_2} \Big\|_{L^2(q)}^2 \le C R^{-1/2} \nu(q) (\#\mathbf{T}_1(q)) (\#\mathbf{T}_2(q)).$$

To end the proof we need to show that

$$\sum_q \nu(q)(\#\mathbf{T}_1(q))(\#\mathbf{T}_2(q)) \le CR^{C\delta}(\#\mathbf{T}_1)(\#\mathbf{T}_2).$$

By (19)

$$\sum_q \#\mathbf{T}_1(q) \le \lambda_1 \#\mathbf{T}_1$$

and by (18),

$$\#\mathbf{T}_2(q) \le 2\mu_2.$$

Therefore, as in [15] this reduces our problem to show that for each cube q,

$$\nu(q) \le CR^{C\delta}\frac{\#\mathbf{T}_2}{\lambda_1\mu_2}.$$

Fix a cube q_0 and $(\xi_1, \eta_1) \in \Lambda_1$, $(\xi_2', \eta_2') \in \Lambda_2$. Denote

$$\begin{aligned}\mathbf{T}_1' := \{&T_1' \in \mathbf{T}_1(q_0) : v(T_1') = (-\eta_1', -\xi_1'),\ (\xi_1', \eta_1') \text{ belongs to a } R^{-1/2}\\ &-\text{neighborhood of } \pi((\xi_1, \eta_1), (\xi_2', \eta_2'))\}.\end{aligned}$$

We want to prove

$$\#\mathbf{T}_1' \le CR^{C\delta}\frac{\#\mathbf{T}_2}{\lambda_1\mu_2}.$$

By (18) and (21),

$$\begin{aligned}&\#\{(q, T_1', T_2) : T_1' \in \mathbf{T}_1',\ T_1' \cap R^\delta q \ne \emptyset,\ T_2 \cap R^\delta q \ne \emptyset,\ dist(q, q_0) \ge R^{-C\delta}R\}\\ &\ge R^{-C\delta}\lambda_1\mu_2\#\mathbf{T}_1'.\end{aligned}$$

Besides,

$$\begin{aligned}&\#\{(q, T_1', T_2) : T_1' \in \mathbf{T}_1',\ T_1' \cap R^\delta q \ne \emptyset,\ T_2 \cap R^\delta q \ne \emptyset,\ dist(q, q_0) \ge R^{-C\delta}R\}\\ &\le (\#\mathbf{T}_2)\cdot \sup_{T_2\in\mathbf{T}_2} \#\{(q, T_1') : T_1' \in \mathbf{T}_1',\ T_1' \cap R^\delta q \ne \emptyset,\ T_2 \cap R^\delta q \ne \emptyset,\\ &dist(q, q_0) \ge R^{-C\delta}R\}.\end{aligned}$$

Hence, what we need to prove is

Lemma 3.3. *For all $T_2 \in \mathbf{T}_2$*

$$\begin{aligned}&\#\{(q, T_1') : T_1' \in \mathbf{T}_1',\ T_1' \cap R^\delta q \ne \emptyset,\ T_2 \cap R^\delta q \ne \emptyset,\ dist(q, q_0) \ge R^{-C\delta}R\}\\ &\le CR^{C\delta}.\end{aligned}$$

To prove the lemma, note that once T_2 is fixed, if (q, T_1') is as in the lemma, q is contained in a ball of radius $R^{C\delta}R^{1/2}$ determined by $R^{C\delta}T_1' \cap R^{C\delta}T_2$. Hence, we just have to count the tubes T_1' such that

(a) $T_1' \cap R^\delta q_0 \neq \emptyset$,
(b) (ξ_1', η_1') belongs to a $R^{-1/2}$–neighborhood of $\pi((\xi_1, \eta_1), (\xi_2', \eta_2'))$
and for which there is a cube q such that
(c) $T_1' \cap R^\delta q \neq \emptyset$,
(d) $T_2 \cap R^\delta q \neq \emptyset$
(e) $dist(q, q_0) \geq R^{-C\delta} R$.

We observe that the union of all tubes T_1' satisfying (a) and (b) is contained in a $R^{C\delta}$ neighborhood of a plane that passes through q_0 and is parallel to the plane

$$\tilde{P} = \{t(-\eta', -\xi', 1) : \ t \in \mathbf{R}, \ (\xi', \eta') \in \mathbf{r}((\xi_1, \eta_1), (\xi_2', \eta_2'))\}.$$

Note that T_2 crosses that plane transversally, for all $T_2 \in \mathbf{T}_2$. This can be seen as a consequence of (26), since under the orthogonal transformation $L(u, v, w) = (v, u, w)$, $v(T_2)$ goes to $(-\xi_2, -\eta_2, 1)$ and $\tilde{P}$ goes to $\mathbf{P}((\xi_1, \eta_1), (\xi_2', \eta_2'))$. Moreover the sets

$$R^{C\delta} T_1' \cap \{p \in \mathbf{R}^3 : \ dist(p, q_0) \geq R^{-C\delta} R\}$$

have overlap bounded by $R^{C\delta}$. Hence, we conclude that $R^{C\delta} T_2$ intersects at most $R^{C\delta}$ of those $R^{C\delta} T_1'$. This proves the lemma.

Remarks. The proof of Theorem 1.2 can be adapted to other pairs of surfaces, S_1, S_2, with non vanishing curvature. To repeat the proof for general S_1 and S_2, the axis of the *tubes* $T \in \mathbf{T}_j$ have to be normal to S_j, $j = 1, 2$, at some points. For a point $p \in S_j$ denote $N(p)$ a normal vector to S_j at p. To have (15) we need

(A) For all $p_1 \in S_1$ and $p_2 \in S_2$, $N(p_1)$ and $N(p_2)$ are transversal.

For $p_1 \in S_1$ and $p_2' \in S_2$, denote

$$\pi(p_1, p_2') = \{p_1' \in S_1 \ / \ p_1 + p_2 = p_1' + p_2' \quad \text{for some } p_2 \in S_2\}.$$

A second transversality condition is needed in the Proof of Lemma 3.3. The definition of $\tilde{P}$ is replaced by

$$\tilde{P} = \{tN(p) \ : \ t \in \mathbf{R} \ \ p \in \pi(p_1, p_2')\}.$$

The second condition that we need is

(B) For all $p_1 \in S_1$ and $p_2, \ p_2' \in S_2$, $N(p_2)$ is tranversal to $\tilde{P}$.

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