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Atoms and regularity for measures in a partially defined free convolution semigroup

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Abstract. Consider a Borel probability measure μ on the real line, and denote by $\{\mu_t : t \geq 1\}$ the free additive convolution semigroup defined by Nica and Speicher. We show that the singular part of μ_t is purely atomic and the density of μ_t is locally analytic, provided that $t > 1$. The main ingredient is a global inversion theorem for analytic functions on a half plane.

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1 Introduction

Given two Borel probability measures μ, ν on $\mathbb{R}$, we denote by $\mu \boxplus \nu$ their free additive convolution. We recall briefly the definition of free convolution, and refer to [11] for a systematic exposition of the subject. Denote by $(L(\mathbb{F}_2), \tau)$ the von Neumann algebra of the free group with two generators a, b, endowed with its usual trace τ. We can find elements x, y affiliated with the subalgebra generated by a, b, respectively, and with distributions μ, ν. In other words,

$$\tau(u(x)) = \int_{-\infty}^{\infty} u(t)\, d\mu(t), \quad \tau(u(y)) = \int_{-\infty}^{\infty} u(t)\, d\nu(t)$$

for every bounded Borel function u on the real line. The measure $\mu \boxplus \nu$ is the distribution of $x + y$. It has been shown that the Cauchy transform

$$G_{\mu\boxplus\nu}(z) = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d(\mu \boxplus \nu)(t), \quad \Im z > 0,$$

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is subordinate to G_μ, in the sense that $G_{\mu\boxplus\nu} = G_\mu \circ \omega$ for some analytic self map of the upper half-plane $\mathbb{C}^+ = \{z = x + iy \in \mathbb{C} : y > 0\}$. This was shown under a genericity assumption in [8], extended with combinatorial tools in [4], and then proved again in [9] under more general circumstances.

On the other hand, it was already shown by A. Nica and R. Speicher in [6] that the discrete semigroup

$$\mu_n = \underbrace{\mu \boxplus \mu \boxplus \cdots \boxplus \mu}_{n \text{ times}}, \quad n = 1, 2, \ldots$$

can be embedded in a continuous family $\{\mu_t : t \geq 1\}$ such that $\mu_{s+t} = \mu_s \boxplus \mu_t$. (The existence of μ_t for large values of t was shown in [2] in case μ has compact support.)

By the subordination result mentioned above, there exist analytic selfmaps ω_n of the upper half-plane satisfying $G_{\mu_n} = G_\mu \circ \omega_n$. Our purpose is to extend this subordination result for arbitrary values of $t > 1$. In fact, our proof of subordination does not rely on any of the earlier arguments and it also yields an alternate proof of the existence of μ_t for $t > 1$. We will also use this subordination result in order to show that μ_t has no continuous singular part if $t > 1$, and that the density of its absolutely continuous part is locally analytic. The subordination functions ω_t turn out to be injective, and their existence follows from a global inversion theorem. The inversion theorem essentially follows from the existence of free convolutions; we are not aware of a classical proof.

2 An analytic inversion result and subordination for μ_t

Given a Borel probability measure μ on $\mathbb{R}$, there exist $\varepsilon > 0$, and an analytic function

$$R_\mu : \{z = x + iy : -\varepsilon < x < \varepsilon, -|x| < y < 0\} \to \mathbb{C}^+,$$

such that $G_\mu(\frac{1}{z} + R_\mu(z)) = z$ and $\lim_{y\to 0} yR_\mu(iy) = 0$ (see [1]). (This local inverse $\frac{1}{z} + R_\mu(z)$ of $G_\mu(z)$ is usually denoted $K_\mu(z)$.) The function R_μ is called the R-transform of μ. Its relevance to free convolution arises from the remarkable equation

$$R_{\mu\boxplus\nu} = R_\mu + R_\nu,$$

which is valid in the common domain of the three functions (see [10], [11], [5], and [1] for the original statement and succesive extensions.)

Lemma 2.1. *Consider a Borel probability measure μ on $\mathbb{R}$, and set $H_2(z) = 2z - \frac{1}{G_\mu(z)}$, and $\omega_2(z) = \frac{1}{2}\left[z + \frac{1}{G_{\mu\boxplus\mu}(z)}\right]$, $z \in \mathbb{C}^+$. Then*

(1) *ω_2 is injective on $\mathbb{C}^+$;*
(2) *$\Im\omega_2(z) \geq \Im z$, $z \in \mathbb{C}^+$;*
(3) *$H_2(\omega_2(z)) = z$, $z \in \mathbb{C}^+$; and*
(4) *$G_{\mu\boxplus\mu}(z) = G_\mu(\omega_2(z))$, $z \in \mathbb{C}^+$.*

Proof. Let us note that the function $\omega(z) = \frac{1}{G_{\mu\boxplus\mu}(z)} + R_\mu(G_{\mu\boxplus\mu}(z))$ is defined in $\Gamma_M = \{z \in \mathbb{C}^+ : z = x + iy,\ M < |x| < \frac{y}{2}\}$, provided that M is sufficiently large. Observe that for $z \in \Gamma_M$

$$\begin{aligned}
\omega(z) &= \frac{1}{G_{\mu\boxplus\mu}(z)} + R_\mu(G_{\mu\boxplus\mu}(z)) \\
&= \frac{1}{G_{\mu\boxplus\mu}(z)} + \frac{1}{2} R_{\mu\boxplus\mu}(G_{\mu\boxplus\mu}(z)) \\
&= \frac{1}{G_{\mu\boxplus\mu}(z)} + \frac{1}{2}\left[K_{\mu\boxplus\mu}(G_{\mu\boxplus\mu}(z)) - \frac{1}{G_{\mu\boxplus\mu}(z)}\right] \\
&= \omega_2(z).
\end{aligned}$$

From the definition of R_μ we see that $G_\mu(\omega(z)) = G_{\mu\boxplus\mu}(z)$ for $z \in \Gamma_M$, and we conclude by analytic continuation that (4) holds. Next we calculate

$$\begin{aligned}
H_2(\omega_2(z)) &= 2\omega_2(z) - \frac{1}{G_\mu(\omega_2(z))} \\
&= 2\omega_2(z) - \frac{1}{G_{\mu\boxplus\mu}(z)},
\end{aligned}$$

and (3) follows from the definition of ω_2. Clearly (3) implies (1), and (2) follows because $\Im \frac{1}{G_\mu(z)} \geq \Im z,\ z \in \mathbb{C}^+$, for any probability measure μ (cf. [5] and [1]). □

The preceding observation leads to the following global inversion theorem.

Theorem 2.2. *Let $H : \mathbb{C}^+ \to \mathbb{C}$ be an analytic function satisfying the following two conditions:*

(1) $\Im H(z) < 2\Im z,\ z \in \mathbb{C}^+$, *and*
(2) $\lim_{y\to+\infty} H(iy)/iy = 1$.

Then there exists an analytic function $\omega : \mathbb{C}^+ \to \mathbb{C}^+$ such that $H(\omega(z)) = z,\ z \in \mathbb{C}^+$. Moreover, $\Im\omega(z) \geq \Im z,\ z \in \mathbb{C}^+$, and $\lim_{y\to+\infty} \omega(iy)/iy = 1$.

Proof. Let us define

$$G(z) = \frac{1}{2z - H(z)},\ z \in \mathbb{C}^+,$$

and observe that conditions (1) and (2) translate into $\Im G(z) < 0$ for $z \in \mathbb{C}^+$, and $\lim_{y\to+\infty} iyG(iy) = 1$, respectively. According to [1], Theorem 5.1, these two conditions imply the existence of a Borel probability measure μ on $\mathbb{R}$ such that $G = G_\mu$. Using the notations of the preceding lemma, we have $H(z) = H_2(z) = 2z - \frac{1}{G_\mu(z)}$, and therefore the theorem follows with

$$\omega(z) = \omega_2(z) = \frac{1}{2}\left[z + \frac{1}{G_{\mu\boxplus\mu}(z)}\right].$$

The last assertions of the theorem follow easily from the corresponding properties of H. □

It may be worthwile to note that the hypothesis of Theorem 2.2 can be weakened somewhat.

Proposition 2.3. *Let $t > 1$, and let $H : \mathbb{C}^+ \longrightarrow \mathbb{C}$ be an analytic function such that $\Im H(z) < t\Im z$, $z \in \mathbb{C}^+$, and $\lim_{y\to\infty} \frac{H(iy)}{iy} = 1$. Then $\Im H(z) \leq \Im z$ for $z \in \mathbb{C}^+$.*

Proof. As in the proof of Theorem 2.3 there must exist a probability measure μ such that

$$G_\mu(z) = \frac{t-1}{tz - H(z)}, \quad z \in \mathbb{C}^+.$$

The conclusion follows immediatly from the formula

$$H(z) = tz - (t-1)\frac{1}{G_\mu(z)},$$

since $\Im \frac{1}{G_\mu(z)} \geq \Im z$, $z \in \mathbb{C}^+$. □

We note below some useful properties of the function ω provided by the preceding theorem.

Proposition 2.4. *Let $\omega, H : \mathbb{C}^+ \to \mathbb{C}$ be analytic functions such that $\omega(\mathbb{C}^+) \subset \mathbb{C}^+$, $\Im H(z) \leq \Im z$, and $H(\omega(z)) = z$ for $z \in \mathbb{C}^+$. Then for every $x \in \mathbb{R}$, the limit $\omega(x) = \lim_{z\to x} \omega(z)$ exists in the extended complex plane. Moreover, if $\omega(x) \in \mathbb{C}^+$, there exists $\delta > 0$ such that ω can be continued analytically through the intervals $(x-\delta, x)$ and $(x, x+\delta)$. The limit $\lim_{z\to\infty} \omega(z)$ also exists.*

Proof. First assume that there exists a sequence $z_n \to x$ such that the limit $\lambda = \lim_{n\to\infty} \omega(z_n)$ exists and belongs to $\mathbb{C}^+$. In this case we have $H(\lambda) = x$. Denote by $n \geq 1$ the order of the zero of $H(z) - x$ at $z = \lambda$. We can find analytic functions $\omega_1, \omega_2, \ldots, \omega_n$ defined in a set of the form $\Omega = \{w : 0 < |w - x| < \delta, w \notin x - i\mathbb{R}_+\}$ such that $H(\omega_j(w)) = w$ for $w \in \Omega$ and $j = 1, 2, \ldots, n$. Clearly ω must coincide with one of the functions ω_j on $\Omega \cap \mathbb{C}^+$ and it follows that ω extends continuously to the interval $(x-\delta, x+\delta)$ and the extension is analytic on $(x-\delta, x)$ and $(x, x+\delta)$.

Assume to the contrary that there is no sequence z_n as in the first part of the argument. In other words, if $z_n \to x$ and $\lim_{n\to\infty} \omega(z_n)$ exists, this limit is either infinite or real. Assume now that two sequences $z_n, w_n \in \mathbb{C}^+$ have limit equal to x and the limits $\lim_{n\to\infty} \omega(z_n)$, $\lim_{n\to\infty} \omega(w_n)$ exist and are different. Consider a continuous path $\gamma : (0, 1) \longrightarrow \mathbb{C}^+$ passing through all the points z_n and w_n, and such that $\lim_{t\to 1} \gamma(t) = x$. There exists then an open interval $(\alpha, \beta) \subset \mathbb{R}$ such that for every $s \in (\alpha, \beta)$ there is a sequence $t_n \to 1$ such that $\omega(\gamma(t_n)) \to s$. In fact t_n can be chosen so that $\omega(\gamma(t_n)) \to s$ nontangentially as $n \to \infty$. Since $H(\omega(\gamma(t_n))) = \gamma(t_n)$, we deduce that the nontangential limit $H(s)$ of H at s is equal to x almost everywhere. The F. and M. Riesz theorem shows now that H must be constant, and this is a contradiction. Therefore $\lim_{z\to x} \omega(z)$ exists. The case $x = \infty$ is treated similarily. □

As noted in the introduction, it was shown in [6] that measures μ_t such that $R_{\mu_t} = tR_\mu$ exist for $t \geq 1$. The following theorem provides an alternative approach to this result.

Theorem 2.5. *Consider a Borel probability measure μ on $\mathbb{R}$, and a real number $t \geq 1$.*

(1) *There exists a probability measure μ_t satisfying $R_{\mu_t}(z) = tR_\mu(z)$ for z in the common domain of the two functions.*
(2) *There exists an injective analytic map $\omega_t : \mathbb{C}^+ \to \mathbb{C}^+$ such that $G_{\mu_t}(z) = G_\mu(\omega_t(z))$, for $z \in \mathbb{C}^+$.*
(3) *We have $\omega_t(z) = \frac{1}{t}z + \left(1 - \frac{1}{t}\right)\frac{1}{G_{\mu_t}(z)}$, and $H_t(\omega_t(z)) = z$, where $H_t(z) = tz + (1-t)\frac{1}{G_\mu(z)}$, for $z \in \mathbb{C}^+$.*
(4) *If $t > 1$, the functions ω_t and $\frac{1}{G_{\mu_t}}$ extend continuously to functions from $\overline{\mathbb{C}^+} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$.*

Proof. If $t = 1$, clearly $\mu_1 = \mu$ and $\omega_1(z) = z$ will satisfy the conclusions of the theorem. Assume therefore that $t > 1$. We clearly have

$$\begin{aligned}\Im H_t(z) &= t\Im z - (t-1)\Im\frac{1}{G_\mu(z)} \\ &\leq t\Im z - (t-1)\Im z \\ &= \Im z,\end{aligned}$$

and

$$\lim_{y\to+\infty} H_t(iy)/iy = t - (t-1)\lim_{y\to+\infty}\frac{1}{iyG_\mu(iy)} = 1.$$

Therefore, Theorem 2.2 implies the existence of an analytic function $\omega_t : \mathbb{C}^+ \to \mathbb{C}^+$ satisfying $H_t(\omega_t(z)) = z,\ z \in \mathbb{C}^+$. We also have $\Im\omega_t(z) \geq \Im z$ and $\lim_{y\to+\infty}\omega_t(iy)/iy = 1$. It follows that the function

$$G_t(z) = \frac{t-1}{t\omega_t(z) - z},\ z \in \mathbb{C}^+$$

satisfies the conditions $\Im G_t(z) \leq 0,\ z \in \mathbb{C}^+$, and $\lim_{y\to+\infty} iyG_t(iy) = 1$. These conditions imply the existence of a Borel probability measure μ_t on $\mathbb{R}$ satisfying $G_{\mu_t} = G_t$. Note that the definition of G_t yields the first formula in (3). To prove (2) we observe that

$$G_\mu(z) = \frac{t-1}{tz - H_t(z)},\ z \in \mathbb{C}^+,$$

so that

$$G_\mu(\omega_t(z)) = \frac{t-1}{t\omega_t(z) - z} = G_t(z),\ z \in \mathbb{C}^+.$$

Finally, let us observe that, for z in the domain of definition of R_{μ_t}, we have

$$\begin{aligned}
z &= G_{\mu_t}\left(\frac{1}{z} + R_{\mu_t}(z)\right)\\
&= G_{\mu}\left(\omega_t\left(\frac{1}{z} + R_{\mu_t}(z)\right)\right)\\
&= G_{\mu}\left(\frac{1}{t}\left[\frac{1}{z} + R_{\mu_t}(z)\right] + \left(1 - \frac{1}{t}\right)\frac{1}{G_{\mu_t}(\frac{1}{z} + R_{\mu_t}(z))}\right)\\
&= G_{\mu}\left(\frac{1}{t}\left[\frac{1}{z} + R_{\mu_t}(z)\right] + \left(1 - \frac{1}{t}\right)\frac{1}{z}\right)\\
&= G_{\mu}\left(\frac{1}{z} + \frac{1}{t}R_{\mu_t}(z)\right),
\end{aligned}$$

where we used (2) in the second equality, and (3) in the third equality. We conclude that the function $\rho(z) = \frac{1}{t}R_{\mu_t}(z)$ satisfies $\lim_{y\to 0} y\rho(iy) = 0$ and $G_\mu\left(\frac{1}{z} + \rho(z)\right) = z$. Therefore, $\rho(z) = R_\mu(z)$, which proves (1). Finally, property (4) is a consequence of Proposition 2.4. □

Let us note that the calculation of μ_t involves in principle two function inverses: first we calculate R_μ by inverting G_μ, then we calculate G_{μ_t} by inverting $\frac{1}{z} + tR_\mu(z)$. The preceding result allows us to calculate G_{μ_t} with just one inversion. Thus, we invert H_t to calculate ω_t, and then we find

$$G_{\mu_t}(z) = \frac{t-1}{t\omega_t(z) - z}.$$

We illustrate this in the case of $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$. For this measure,

$$G_\mu(z) = \frac{z}{z^2 - 1}$$

so that

$$H_t(z) = tz - (t-1)\frac{z^2-1}{z} = \frac{z^2 + t - 1}{z}$$

and therefore

$$\omega_t(z) = \frac{z + \sqrt{z^2 - 4(t-1)}}{2},$$

where the square root must be chosen to be positive for large real values of z. After some simple manipulations we obtain

$$G_{\mu_t}(z) = \frac{(2-t)z + t\sqrt{z^2 - 4(t-1)}}{2(z^2 - t^2)}.$$

Note that the function G_{μ_t} has poles at $z = \pm t$ with residue

$$\frac{2 - t + |2 - t|}{4}.$$

This indicates that μ_t has atoms at $\pm t$ provided that $t < 2$. The absolutely continuous part of μ_t is concentrated on $[-2\sqrt{t-1}, 2\sqrt{t-1}]$ and it has density

$$\frac{t\sqrt{4(t-1)-x^2}}{2\pi(t^2-x^2)}, \ x \in [-2\sqrt{t-1}, 2\sqrt{t-1}].$$

This density is bounded for all t, except for $t = 2$ when it equals

$$\frac{1}{\pi\sqrt{4-x^2}}.$$

This behaviour for atoms and the absolutely continuous part is rather general, as seen in the following section.

3 Atoms and regularity for μ_t

It is known (see [3]) that a free convolution $\mu \boxplus \nu$ generally has a finite number of atoms, fewer than either μ or ν. This fact extends to the measures μ_t.

Theorem 3.1. *Let μ be a probability measure on $\mathbb{R}$, and let μ_t be such that $R_{\mu_t} = tR_\mu,\ t > 1$. A number $\alpha \in \mathbb{R}$ is an atom of μ_t if and only if α/t is an atom of μ such that $\mu(\{\alpha/t\}) > 1 - \frac{1}{t}$. In this case,*

$$\mu_t(\{\alpha\}) = t\mu\left(\left\{\frac{\alpha}{t}\right\}\right) - (t-1).$$

If N_t denotes the number of atoms of μ_t, we have $N_t < \frac{t}{t-1}$.

Proof. The estimate on N_t follows immediatly from the inequality $\mu\left(\left\{\frac{\alpha}{t}\right\}\right) > 1-\frac{1}{t}$. Recall the fact that for every Borel probability measure on $\mathbb{R}$ and for every $\alpha \in \mathbb{R}$, we have

$$\mu_t(\{\alpha\}) = \lim_{\substack{z \longrightarrow \alpha \\ \sphericalangle}} (z-\alpha)G_{\mu_t}(z),$$

where the notation $z \underset{\sphericalangle}{\longrightarrow} \alpha$ indicates nontangential convergence (see [3], Lemma 7.1).

Using the notation of Theorem 2.5, we have

$$\omega_t(z) - \frac{\alpha}{t} = \frac{1}{t}(z-\alpha) + \left(1-\frac{1}{t}\right)\frac{1}{G_{\mu_t}(z)}.$$

Since $\frac{1}{G_{\mu_t}(z)}$ converges to zero as z tends to α nontangentially, we conclude that $\omega_t(\alpha) = \alpha/t$. Moreover, since

$$\lim_{\substack{z \longrightarrow \alpha \\ \sphericalangle}} \frac{\omega_t(z) - \frac{\alpha}{t}}{z-\alpha} = \frac{1}{t} + \left(1-\frac{1}{t}\right)\lim_{\substack{z \longrightarrow \alpha \\ \sphericalangle}} \frac{1}{(z-\alpha)G_{\mu_t}(z)} = \frac{1}{t} + \left(1-\frac{1}{t}\right)\frac{1}{\mu_t(\{\alpha\})},$$

we deduce that $\omega_t(z)$ approaches α/t nontangentially as $z \underset{\sphericalangle}{\longrightarrow} \alpha$. Using these facts we deduce that

$$\mu\left(\left\{\frac{\alpha}{t}\right\}\right) = \lim_{\substack{w \longrightarrow \alpha/t \\ \sphericalangle}} \left(w - \frac{\alpha}{t}\right) G_\mu(w) = \lim_{\substack{z \longrightarrow \alpha \\ \sphericalangle}} \left(\omega_t(z) - \frac{\alpha}{t}\right) G_\mu(\omega_t(z))$$
$$= \lim_{\substack{z \longrightarrow \alpha \\ \sphericalangle}} \left(\omega_t(z) - \frac{\alpha}{t}\right) G_{\mu_t}(z)$$
$$= \lim_{\substack{z \longrightarrow \alpha \\ \sphericalangle}} \frac{\omega_t(z) - \alpha/t}{z - \alpha} \mu_t(\{\alpha\})$$
$$= \left(\frac{1}{t} + \left(1 - \frac{1}{t}\right) \frac{1}{\mu_t(\{\alpha\})}\right) \mu_t(\{\alpha\})$$
$$= \frac{1}{t}\mu_t(\{\alpha\}) + 1 - \frac{1}{t}.$$

Thus α/t must be an atom of μ with the required mass.

Conversely, let us assume that α/t is an atom of μ such that $\mu(\{\alpha/t\}) > 1 - \frac{1}{t}$. We will use the realization of μ_t given in [6]. Namely, there exists a W^*-probability space $(\mathcal{M}, \tau)$, a selfadjoint random variable X affiliated with $\mathcal{M}$, and a projection $p \in \mathcal{M}$ such that X and p are free, $\tau(p) = t^{-1}$, X has distribution μ, and $tpXp$ has distribution μ_t in the W^*-probability space $(p\mathcal{M}p, t\tau)$. Since α/t is an atom for μ, we have $Xq = \frac{\alpha}{t}q$ for some projection $q \in \mathcal{M}$ with $\tau(q) = \mu(\{\alpha/t\})$. We conclude that $(tpXp)(p \wedge q) = \alpha(p \wedge q)$, and the projection $p \wedge q$ is not zero because $\tau(p) + \tau(q) > \frac{1}{t} + \left(1 - \frac{1}{t}\right) = 1$. We conclude that α is indeed an atom for μ_t. □

We restate one of the ingredients of the previous proof which will be useful later.

Lemma 3.2. *With the notations of Theorem 3.1, assume that* $t > 1$, $\alpha, \beta \in \mathbb{R}$, *and there exists a sequence* $z_n \in \mathbb{C}^+$ *such that* $z_n \longrightarrow \alpha$ *nontangentially,* $\omega_t(z_n) \longrightarrow \beta$ *nontangentially, and* $\frac{1}{G_{\mu_t}(z_n)} \longrightarrow 0$ *as* $n \to \infty$. *Then* β *is an atom of* μ, $\beta = \alpha/t$, *and*

$$\mu(\{\beta\}) = \frac{1}{t}\mu_t(\{\alpha\}) + 1 - \frac{1}{t}.$$

Proof. The relation

$$\omega_t(z_n) = \frac{1}{t}z_n + \left(1 - \frac{1}{t}\right) \frac{1}{G_{\mu_t}(z_n)}$$

implies immediatly $\beta = \alpha/t$. The fact that β is an atom with the required mass follows now from

$$(\omega(z_n) - \beta)G_\mu(\omega_t(z_n)) = \frac{1}{t}(z_n - \alpha)G_{\mu_t}(z_n) + 1 - \frac{1}{t}.$$

□

Theorem 3.1 indicates that the atomic part of μ_t loses mass in a piecewise linear fashion as t increases. The next result gives some indication of where the lost mass goes.

Proposition 3.3. *With the notation of Theorem 3.1, assume that $t > 1$ and μ_t has atoms $\alpha < \beta$. Then we have $\mu_t((\alpha, \beta)) > 0$.*

Proof. Assume to the contrary that $\mu_t((\alpha, \beta)) = 0$. In this case G_{μ_t} takes real values on the interval (α, β), $G_{\mu_t}(\alpha + 0) = +\infty$, and $G_{\mu_t}(\beta - 0) = -\infty$. We deduce the existence of $x \in (\alpha, \beta)$ satisfying $G_{\mu_t}(x) = 0$, and henceforth $\omega_t(x) = \infty$. On the other hand, $\omega_t(\alpha) = \alpha/t$, $\omega_t(\beta) = \beta/t$, and $\omega_t(\infty) = \infty$. We claim that this combination of values is not possible. Indeed, consider a simple path γ joining α and β in the upper half-plane. Now, $\omega_t(\gamma)$ cuts the range of ω_t into two components, one of which must be bounded. Since $\omega_t(\infty)$ and $\omega_t(x)$ belong to the closures of different components, only one of these limits can be infinite. □

The preceding argument shows in fact that G_{μ_t} cannot change sign on any interval disjoint from the support of μ_t; G_{μ_t} is real, continuous, and decreasing on such intervals.

We will see next that μ_t has no singular spectrum beside the atoms discussed in Theorem 3.1. We will denote by μ_t^{ac} the absolutely continous part of the measure μ_t.

Theorem 3.4. *Let μ be a probability measure on $\mathbb{R}$, and let μ_t satisfy $R_{\mu_t} = tR_\mu$ for some $t > 1$. The measure μ_t has no continuous singular part. Moreover, there exists a closed set $\sigma \subset \mathbb{R}$ with $\mu_t^{ac}(\sigma) = 0$ such that the density $\frac{d\mu_t(x)}{dx}$ is locally analytic for $x \in \mathbb{R} \setminus \sigma$.*

Proof. Assume that μ_t does have a singular continuous part ν. In this case, it is known that

$$\lim_{z \underset{\sphericalangle}{\longrightarrow} a} \frac{1}{|G_{\mu_t}(z)|} = 0$$

for ν-almost every $a \in \mathbb{R}$ and, by Theorem 2.7 of [7], the convergence of $1/G_{\mu_t}(z)$ to zero is not typically tangential. In particular, there exist uncountably many values $a \in \mathbb{R}$ for which there is a sequence $z_n \in \mathbb{C}^+$ converging to a nontangentially, such that $\lim_{n\to\infty} 1/|G_{\mu_t}(z_n)| = 0$ and

$$\sup_{n\in\mathbb{N}} \left| \frac{\Re G_{\mu_t}(z_n)}{\Im G_{\mu_t}(z_n)} \right| < \infty.$$

Using again the notations of Theorem 2.5,

$$\omega_t(z_n) - \frac{a}{t} = \frac{1}{t}(z_n - a) + \left(1 - \frac{1}{t}\right) \frac{1}{G_{\mu_t}(z_n)},$$

and this shows that $\omega_t(z_n) \longrightarrow \frac{a}{t}$ nontangentially as $n \to \infty$. By Lemma 3.2, a/t is an atom of μ. Since we can only have at most countably many atoms, we arrive at a contradiction, showing that indeed $\nu = 0$.

For μ_t^{ac}-almost every $a \in \mathbb{R}$, the nontangential limit $\lim_{z\to a} G_{\mu_t}(z)$ exists, and it has imaginary part equal to $-\pi \frac{d\mu_t}{dx}(a)$. We deduce that the nontangential limit $\lim_{z\to a} \omega_t(z)$ also exists, and it has nonzero imaginary part. Proposition 2.4 implies now that ω_t, and hence G_{μ_t}, extends continuously to some interval $(a-\delta, a+\delta)$ and is analytic on that interval except possibly at a. The set σ can be taken to be the common complement of the intervals $(a-\delta, a+\delta) \setminus \{a\}$ obtained this way. □

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