Math. Z. 248, 423–443 (2004) **Mathematische Zeitschrift**

DOI: 10.1007/s00209-004-0663-y

On a periodic Schrödinger equation with nonlocal superlinear part

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Received: 6 June 2003; in final form: 14 November 2003 /
Published online: 27 April 2004 –

Abstract. We consider the Choquard-Pekar equation

$$-\Delta u + Vu = (W * u^2)u \qquad u \in H^1(\mathbb{R}^3)$$

and focus on the case of periodic potential V. For a large class of even functions W we show existence and multiplicity of solutions. Essentially the conditions are that 0 is not in the spectrum of the linear part $-\Delta + V$ and that W does not change sign. Our results carry over to more general nonlinear terms in arbitrary space dimension $N \geq 2$.

Mathematics Subject Classification (2000): 35Q55, 35Q40, 35J10, 35J20, 35J60, 46N50, 49J35, 81V70

1 Introduction

We consider the problem

$$-\Delta u + Vu = (W * u^2)u \qquad u \in H^1(\mathbb{R}^3) \tag{P}$$

where V and W are real functions on $\mathbb{R}^3$, W is even, and u assumes real values. Here, for two functions u, v on $\mathbb{R}^3$, $u * v$ denotes convolution of u and v. Let us define

$$\Psi(u) = \frac{1}{4}\int_{\mathbb{R}^3} (W * u^2)u^2\, dx$$

for $u \in H^1(\mathbb{R}^3)$. Finding weak solutions of (P) is equivalent to finding critical points of the energy functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2)\, dx - \Psi(u)$$

defined on $H^1(\mathbb{R}^3)$.

This type of problem is often referred to as *Choquard-Pekar equation* when $W \geq 0$. It comes up as an approximation to Hartree-Fock theory of a Plasma or in the Hartree theory of bosonic systems (cf. [3, 10, 11]). The case $W \leq 0$ appears as a *Hartree equation* for the Helium atom.

Associated with (P) is the eigenvalue problem

$$-\Delta u + Vu - (W * u^2)u = \lambda u \qquad u \in H^1(\mathbb{R}^3) \tag{EP}$$

that is usually called *Choquard equation* if $W \geq 0$. Here one is interested in solutions with prescribed L^2-norm $|u|_2^2 = M$, $\lambda \in \mathbb{R}$ being a free parameter. Solutions are the critical points of the energy Φ restricted to the L^2-sphere

$$S_M = \{ u \in H^1(\mathbb{R}^3) \mid |u|_2^2 = M \} .$$

For physical reasons let us call V the *exterior potential* and W the *potential of particle interaction*. In the sequel we speak of the *radial* case if V and W are radial functions and existence of radial solutions is investigated. The *periodic* case refers to V being periodic and nonconstant. Moreover, we assume for the whole discussion that W does not change sign.

Both problems have been investigated in the nonperiodic case by many authors, cf. [6, 13–15, 18, 19, 21, 25, 27] and the references therein. Here relative compactness of Palais-Smale (PS) sequences of Φ or of the restriction of Φ to S_M is achieved by exploiting radial symmetry and Strauss' Lemma [24, 28], or the fact that the spectrum of $L = -\Delta + V$ is discrete at the bottom.

In contrast, the compactness issue in the periodic case is much more difficult to handle due to the invariance of (P) and (EP) under the action of the noncompact group $\mathbb{Z}^N$ induced by translation by integer values in the coordinate directions. Minimizers for Φ over S_M have been constructed in the periodic case in [2, 8]. Additional difficulties are encountered when considering excited states, i.e. solutions of (EP) at higher energy levels, or solutions of (EP) with λ in a gap of the spectrum of L.

Even though problem (EP) seems to be more relevant in physics, we concentrate on problem (P). Our assumptions are that V is periodic and that W does not change sign. We believe that the techniques we develop will be useful in studying (EP) as well.

To summarize our results, let us introduce the following notion: Two elements $u, v \in H^1(\mathbb{R}^3)$ are called *geometrically distinct* if u is not contained in the orbit of v under the action of $\mathbb{Z}^N$. The elements of a subset of $H^1(\mathbb{R}^3)$ are called geometrically distinct if they are pairwise geometrically distinct.

In the case of periodic $V > 0$ (the positive definite case) with $W \geq 0$, the existence of *one* nontrivial solution is relatively easy to prove. One can obtain a (PS)-sequence with the Mountain Pass Theorem. Invariance of Φ with respect to the action of $\mathbb{Z}^N$ and weak sequential continuity of Φ' then yield existence. We prove existence of *infinitely* many geometrically distinct solutions for (P) using a theorem of Bartsch and Ding. A multiplicity result for periodic Schrödinger equations was known before only for *local* nonlinear terms, and it was achieved by a multibump construction in [9]. The method of proof used in the latter reference does not apply to the nonlocal problem (P).

The main novelty in our proof is a lemma about decomposition of Φ along (PS)-sequences (cf. Lemma 4.5 below). To show this we prove a variant of Brezis-Lieb's Lemma that should be of independent interest since little regularity is assumed. Results about decomposition were known before in this generality only for local right hand sides in (P), see [9] for example. Nevertheless, partial results about decomposition for nonlocal functionals are already present in [7, 8].

Now we turn to the case of a periodic exterior potential V that changes sign. Here it may happen that the Schrödinger operator L, which has purely continuous spectrum that consists of a union of closed intervals, has essential spectrum below 0. As a consequence the quadratic part of Φ is strongly indefinite and one needs subtle arguments to construct (PS)-sequences. In contrast to the positive definite case, mere existence of *one* solution is hard to prove. This was first achived in [7], assuming that 0 is in a gap of the spectrum of L and that $W(x) = 1/|x|$. The proof makes substantial use of the specific form of Ψ. In fact, consider the symmetric bilinear form sending functions u, v to

$$I(u, v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|} u(y) v(x) \, dy \, dx \; . \tag{1.1}$$

Since the Fourier transform of $1/|x|$ is known to be positive, I is positive definite on an appropriate function space. From this it follows that Ψ is convex, a fact that lies at the heart of the proof in [7]. Moreover, positive definiteness of I is used there to show boundedness of (PS)-sequences. The proof extends to more general W that have nonnegative Fourier transform, but no general criterion is known to decide whether this is the case for a particular choice of W.

For physical reasons it is desirable to treat potentials W without being restricted by the assumption on the Fourier transform of W. Indeed, in work of Fröhlich, Tsai and Yau [10, 11] on the Hartree equation for the thermodynamic limit of systems of non-relativistic bosons, the authors propose to model particle interaction with a potential W that behaves as

$$W(x) \sim \frac{1}{|x|^6} + \frac{C}{|x|} \tag{1.2}$$

for $|x|$ large (see also the discussion in [3]). Here the first term describes van der Waals, the second gravitational attraction between atoms. Near 0 this function must be modified in an appropriate way to be able to work in a variational setting. It is not at all clear how to do this modification such that the Fourier transform of W is nonnegative. Therefore we take a different approach to show existence of solutions to (P) in the periodic and indefinite case, applying generalized linking theorems of Kryszewski-Szulkin and Bartsch-Ding. No convexity of Ψ is required, and we prove boundedness of (PS)-sequences by using a Cauchy-Schwarz type inequality for the bilinear form associated with W as in (1.1), see condition (W_3) below. In [1] we give conditions on W that imply (W_3), allowing for a lot of freedom in choosing the regularization of W described above. Hence we prove the existence of infinitely many geometrically distinct solutions also in this case.

Our method of proof carries over to arbitrary space dimension $N \geq 2$, replacing u^2 by $f(u)$ and u by $f'(u)$ on the right hand side of (P), with suitable growth restrictions on f. Moreover, no radial symmetry of W is assumed, and we treat the cases of $W \geq 0$ and $W \leq 0$, i.e. attractive and repulsive particle interaction.

The organization of the paper is as follows: The next section contains a precise formulation of our results and a discussion of the conditions on W and f. Section 3 deals with mapping properties and regularity of Ψ. It is split into two subsections for simplicity to account for the possibility of W and f being sums of functions with different growth rates. Finally in Sect. 4 we show how to apply the abstract critical point theorems in this setting.

1.1 General notation

We set $E = H^1(\mathbb{R}^N)$, $E^* = H^{-1}(\mathbb{R}^N)$ (the dual space of E). Denote by $\|u\|_E$ the standard norm for $u \in E$. For any measure space Ω and $u \in L^p(\Omega)$ let $|u|_{p,\Omega}$ be the corresponding norm, and set $|u|_p = |u|_{p,\mathbb{R}^N}$.

If X is a metric space, A is a point or a subset of X, and $\rho > 0$, then we set

$$\begin{aligned} U_\rho(A, X) &= \{x \in X \mid \operatorname{dist}_X(x, A) < \rho\} \\ B_\rho(A, X) &= \{x \in X \mid \operatorname{dist}_X(x, A) \leq \rho\} \\ S_\rho(A, X) &= \{x \in X \mid \operatorname{dist}_X(x, A) = \rho\}\,. \end{aligned}$$

When there is no confusion possible we sometimes omit the X-dependency. If $(X, \|\cdot\|)$ is a normed vector space and $A = 0$, we often write $U_\rho X$ instead of $U_\rho(0, X)$, and so forth.

Acknowledgements. The author wishes to thank A. Pankov for suggesting the nonlocal problem, and for many helpful discussions concerning [10, 11] and [23]. Moreover the author thanks T. Bartsch for communicating the theorem used to obtain the multiplicity result.

2 Main results

To be more explicit, consider the following problems:

$$-\Delta u + Vu = (W * f(u))f'(u) \qquad u \in H^1(\mathbb{R}^N) \tag{P$_+$}$$

and

$$-\Delta u + Vu = -(W * f(u))f'(u) \qquad u \in H^1(\mathbb{R}^N)\,. \tag{P$_-$}$$

We define as usual the critical Sobolev exponent $2^* = \infty$ for $N = 2$ and $2^* = 2N/(N-2)$ for $N \geq 3$ and consider the following conditions:

(V_1) $V \in L^\infty(\mathbb{R}^N, \mathbb{R})$, and V is 1-periodic in x_i for $i = 1, 2, \ldots, N$.
(V_2^1) $\sigma(-\Delta + V) \subseteq (0, \infty)$.
(V_2^2) $0 \notin \sigma(-\Delta + V)$ and $\sigma(-\Delta + V) \cap (-\infty, 0) \neq \varnothing$.
(W_1) There are $1 \leq r_1 \leq r_2 < \infty$ such that $W \in L^{r_1}(\mathbb{R}^N) + L^{r_2}(\mathbb{R}^N)$, and W is an even function.
(W_2) $W \geq 0$, and on a neighborhood of 0 we have $W > 0$.

(W_3) There is $C \geq 0$ such that for all nonnegative $\varphi, \psi \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} (W * \varphi)\psi \, dx \leq C \sqrt{\int_{\mathbb{R}^N} (W * \varphi)\varphi \, dx \quad \int_{\mathbb{R}^N} (W * \psi)\psi \, dx} \,. \tag{2.1}$$

(F_1) $f \in C^1(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, and there are $C > 0$ and $p_1, p_2 > 1$ with $2 - 1/r_2 < p_1 \leq p_2 < (2 - 1/r_1)2^*/2$ such that for all $u \in \mathbb{R}$

$$|f'(u)| \leq C(|u|^{p_1 - 1} + |u|^{p_2 - 1}) \,.$$

(F_2) There is $\theta > 2$ such that for all $u \in \mathbb{R} \setminus \{0\}$

$$2f'(u)u \geq \theta f(u) > 0 \,.$$

(F_3) f is an even function.

We can now state for the positive definite case

Theorem 2.1. *If* (V_1)*,* (V_2^1)*,* (W_1)*,* (W_2)*,* (F_1) *and* (F_2) *are satisfied, then* (P_+) *has a nontrivial weak solution. Problem* (P_-) *admits no nontrivial solution. If additionally* (F_3) *holds, then there are infinitely many geometrically distinct weak solutions for* (P_+).

For the strongly indefinite case we have

Theorem 2.2. *If* (V_1)*,* (V_2^2)*,* (W_1)*,* (W_2)*,* (W_3)*,* (F_1) *and* (F_2) *are satisfied, then both* (P_+) *and* (P_-) *have a nontrivial weak solution. If additionally* (F_3) *holds, then there are infinitely many geometrically distinct weak solutions for both of these problems.*

Some comments on the conditions given above are in order. First, for $N = 3$ we have $2^* = 6$, so that for any $1 \leq r_1 \leq r_2 < \infty$ and for $f(u) = u^2$ (F_1)–(F_3) are satisfied with $p_1 = p_2 = 2$ and $\theta = 4$. Therefore our results apply to the special case of (P).

If $r_1 < N/4$ we must require that $r_2 < r_1(N-2)/(N-4r_1)$ for (F_1) to be meaningful. A general model for f is the function $|u|^{p_1} + |u|^{p_2}$ with suitable exponents p_1 and p_2. It satisfies all requirements (using $\theta = 2p_1$). To see that the condition on p_1, p_2 is quite natural, suppose that $N \geq 3$, $W \in L^r$ for some $r \in [1, \infty]$ and $f(u) = |u|^p$ for some $p > 0$. By Young's theorem on convolutions

$$\int_{\mathbb{R}^N} (W * f(u)) f(u) \, dx$$

is well defined if $f(u) \in L^s$ for $s \geq 1$ defined by

$$\frac{1}{r} + \frac{2}{s} = 2 \,.$$

Since $u \in H^1(\mathbb{R}^N)$ we must therefore require that $sp \in [2, 2^*]$ and hence

$$\frac{2}{s} = 2 - \frac{1}{r} \leq p \leq \frac{2^*}{s} = \frac{2^*}{2}\left(2 - \frac{1}{r}\right) \,.$$

Moreover, for the concentration compactness arguments to work, here we need strict inequalities. For the same reason we need $r < \infty$, while in the radial case $r = \infty$ is allowed. In that case compactness is achieved by a different means, as mentioned in the introduction.

To state criteria for checking (W_3), we introduce some more quantities. For any nonempty $X \subseteq \mathbb{R}^N$ let $\alpha(X)$ denote the least positive integer m such that there is a closed convex set $A \subseteq X$ of dimension N, A being symmetric (i.e. $-A = A$), with the property that X can be covered by m translates of A. If $X = \varnothing$ put $\alpha(X) = 0$. If W is a nonnegative Borel function on $\mathbb{R}^N$ put $X(t) = \{ x \in \mathbb{R}^N \mid W(x) \geq t \}$ for $t \geq 0$. The results in [1] yield that W satisfies (W_3) if

$$\limsup_{t\to 0} \alpha(X(t)) + \limsup_{t\to\infty} \alpha(X(t)) < \infty \,. \tag{2.2}$$

In that paper we also give examples that demonstrate that the class of $W \geq 0$ with (W_3) is larger than the class of $W \geq 0$ with nonnegative Fourier transform. In particular, W need not be radially symmetric.

There is a simpler criterion if $W(x) = h(p(x))$ for some seminorm p on $\mathbb{R}^N$ and some nonnegative Borel function h on $[0, \infty)$. For any $Y \subseteq [0, \infty)$ put $\lambda(Y) = \sup\{ t > 0 \mid [0, t] \subseteq Y \}$ and

$$\beta(Y) = \begin{cases} 0 & Y = \varnothing \\ \infty & \lambda(Y) = -\infty \text{ and } Y \neq \varnothing \\ \sup(Y)/\lambda(Y) & \text{otherwise.} \end{cases}$$

Here we set $\infty/a = \infty$ if $a > 0$, and $\infty/\infty = 1$. Now put $Y(t) = \{ s \in [0, \infty) \mid h(s) \geq t \}$ for $t \geq 0$. By [1] W satisfies (W_3) if

$$\limsup_{t\to 0} \beta(Y(t)) + \limsup_{t\to\infty} \beta(Y(t)) < \infty \,. \tag{2.3}$$

The last statement applies in particular to nonnegative radial decreasing functions W (this case was also studied in [20]). For W as in (1.2) we can thus use a simple regularization near 0 as was mentioned in the introduction.

It is clear that any nontrivial even function $W \geq 0$ that satisfies either (2.2) or (2.3) is positive on a neighborhood of 0, so that (W_2) holds.

3 Regularity properties of the nonlinearity

Here we collect properties of the superquadratic part of Φ. Throughout this section we will assume (W_1) and (F_1). Instead of dealing directly with the different exponents r_1, r_2, p_1, p_2 it seems simpler to first consider the case of just two exponents r and p. This is justified by the splitting of $W = W_1 + W_2$ into a sum of functions belonging to L^{r_1} respectively L^{r_2}. Similarly f can be split: Choose a function $\zeta \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\zeta(t) = 0$ for $|t| \geq 2$, $\zeta(t) = 1$ for $|t| \leq 1$ and $\zeta(t) \in [0, 1]$ for all t. Then set

$$f_1(u) = \int_0^u \zeta(t) f'(t)\, dt \qquad \text{and} \qquad f_2 = f - f_1 \,.$$

Clearly we have

$$|f_1'(u)| \le C|u|^{p_1-1} \qquad \text{and} \qquad |f_2'(u)| \le C|u|^{p_2-1} \tag{3.1}$$

where C only depends on f. Now

$$\int_{\mathbb{R}^N} (W * f(u)) f(u)\, dx$$

can be written as a sum of integrals of the form

$$\int_{\mathbb{R}^N} (U * g(u)) h(u)\, dx \ ,$$

where U stands for W_1 or W_2, and g, h each stand for either f_1 or f_2.

3.1 The Simple Case

In this subsection we assume $U \in L^r(\mathbb{R}^N)$ for some $r \in [1, \infty)$, $g, h \in C^1(\mathbb{R}, \mathbb{R})$, $g(0) = h(0) = 0$, and that there exist $p, q > 1$ and a constant $C > 0$ such that

$$|g'(u)| \le C|u|^{p-1} \qquad \text{and} \qquad |h'(u)| \le C|u|^{q-1} \ .$$

Moreover, for $s = 2r/(2r-1)$ we assume $sp, sq \in [2, 2^*)$.

Lemma 3.1. *Let s' be the conjugate exponent for s, let $t \in [s, \infty)$, and let μ be given by $1/s' + 1/t = 1/\mu$. Then the bilinear map $L^s \times L^t \to L^\mu$, sending (u, v) to $(U * u)v$, is well defined and continuous, with*

$$|(U * u)v|_\mu \le |U * u|_{s'} |v|_t \le |U|_r |u|_s |v|_t \ .$$

If $(u_n) \subseteq L^s$ and $(v_n) \subseteq L^t$ are bounded and either $u_n \to u$ in L^s and $v_n \to v$ in L^t_{loc} or $u_n \to u$ in L^s_{loc} and $v_n \to v$ in L^t, then $(U * u_n)v_n \to (U * u)v$ in L^μ.

Proof. If $u \in L^s$ and $v \in L^t$, by Young's Convolution Theorem $U * u$ is in $L^{s'}$ since $1/r + 1/s = 1 + 1/s'$, and

$$|U * u|_{s'} \le |U|_r |u|_s \ .$$

From $t \ge s$ we obtain $\mu \ge 1$. Hölder's inequality then yields the continuity of the bilinear map $(u, v) \mapsto (U * u)v$.

Now let (u_n) and (v_n) be given as in the statement of this lemma. In the case that $u_n \to u$ in L^s we can assume $v_n \to 0$ in L^t_{loc}, and, since (v_n) is bounded, it suffices to show that

$$(U * u)v_n \to 0 \qquad \text{in } L^\mu. \tag{3.2}$$

Let $\varepsilon > 0$. Since $s' < \infty$ there is $R > 0$ such that

$$|U * u|_{s', \mathbb{R}^N \setminus B_R} \le \varepsilon \ .$$

We have

$$\begin{aligned}\int_{\mathbb{R}^N} |(U*u)v_n|^\mu\,dx &= \int_{B_R} |(U*u)v_n|^\mu\,dx + \int_{\mathbb{R}^N\setminus B_R} |(U*u)v_n|^\mu\,dx \\ &\le |U*u|_{s'}^\mu |v_n|_{t,B_R}^\mu + |U*u|_{s',\mathbb{R}^N\setminus B_R}^\mu |v_n|_t^\mu \\ &\le C_1 |v_n|_{t,B_R}^\mu + C_2\varepsilon^\mu\ .\end{aligned}$$

Letting $n\to\infty$ and then $\varepsilon\to 0$ (3.2) follows.

In the case that $v_n\to v$ in L^t, again we can assume that $u_n\to 0$ in L^s_{loc}, and it suffices to show

$$(U*u_n)v\to 0 \qquad \text{in } L^\mu \tag{3.3}$$

since $U*u_n$ is bounded in $L^{s'}$. We claim that

$$U*u_n\to 0 \qquad \text{in } L^{s'}_{\mathrm{loc}}. \tag{3.4}$$

Fix $R_1>0$. For any $\varepsilon>0$ there is $R_2>0$ such that

$$|U|_{r,\mathbb{R}^N\setminus B_{R_2}}\le\varepsilon\ .$$

Put $U_1=\chi_{B_{R_2}}U$ and $U_2=U-U_1$ (here $\chi_{B_{R_2}}$ denotes the characteristic function of B_{R_2}). We have

$$\begin{aligned}|U_1*u_n|_{s',B_{R_1}}^{s'} &\le \int_{B_{R_1}}\left(\int_{\mathbb{R}^N}|U_1(x-y)u_n(y)|\,dy\right)^{s'}dx \\ &= \int_{B_{R_1}}\left(\int_{B_{R_1+R_2}}|U_1(x-y)u_n(y)|\,dy\right)^{s'}dx \\ &\le |U_1|_r^{s'}|u_n|_{s,B_{R_1+R_2}}^{s'}\ .\end{aligned}$$

The last inequality follows from [22, Thm. 3.1], a generalized form of Young's Theorem on convolutions. It follows that

$$\begin{aligned}|U*u_n|_{s',B_{R_1}} &\le |U_1*u_n|_{s',B_{R_1}} + |U_2*u_n|_{s',B_{R_1}} \\ &\le |U_1|_r|u_n|_{s,B_{R_1+R_2}} + |U_2|_r|u_n|_s \\ &\le |U_1|_r|u_n|_{s,B_{R_1+R_2}} + C\varepsilon\ .\end{aligned}$$

Letting $n\to\infty$ and then $\varepsilon\to 0$ we have proved (3.4) since R_1 was arbitrary. Now (3.3) follows from (3.4) as for the first case. □

The following is a variant of Brezis-Lieb's lemma, as already mentioned in the introduction.

Lemma 3.2. *Suppose that $u_n\rightharpoonup v$ in E. Then, after extraction of a subsequence, there is a sequence $(v_n)\subseteq E$ with $v_n\to v$ in E, such that for any $t\ge 1$, $\mu>0$ with $t\mu\in[2,2^*)$ and any continuous $f:\mathbb{R}\to\mathbb{R}$ with*

$$|f(u)|\le C|u|^\mu$$

for some $C>0$ we have

$$f(u_n)-f(u_n-v_n)\to f(v) \qquad \text{in } L^t.$$

Proof. Define functions $Q_n : [0,\infty) \to [0,\infty)$ by

$$Q_n(R) = \int_{B_R} (|\nabla u_n|^2 + u_n^2)\,dx\ .$$

Then the Q_n are uniformly bounded and nondecreasing. There is a subsequence converging almost everywhere to a bounded nondecreasing function Q (cf. [16]). It is easy, extracting another subsequence, to build a sequence $R_n \to \infty$ such that for any $\varepsilon > 0$ there is $R > 0$, arbitrarily large, with

$$\limsup_{n\to\infty}(Q_n(R_n) - Q_n(R)) \le \varepsilon$$

or, stated differently,

$$\limsup_{n\to\infty} \int_{B_{R_n}\setminus B_R} (|\nabla u_n|^2 + u_n^2)\,dx \le \varepsilon\ . \tag{3.5}$$

Here all balls B are taken to have center at 0. Fix a smooth function $\eta : [0,\infty) \to [0,1]$ with $\eta(t) = 1$ for $|t| \le 1$ and $\eta(t) = 0$ for $|t| \ge 2$. Put $v_n(x) = \eta(2|x|/R_n)v(x)$ for $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$.

Given f as in the statement of this lemma, fix $\varepsilon > 0$ and choose $R > 0$ such that (3.5) holds and such that

$$\int_{\mathbb{R}^N\setminus B_R} (|\nabla v|^2 + v^2)\,dx \le \varepsilon\ .$$

Now $u_n \to v$ in $L^{t\mu}(B_R)$ by the compactness of Sobolev embeddings, so that by continuity of the Nemyckii operator induced by f on $L^{t\mu}$ we have

$$\lim_{n\to\infty} \int_{B_R} |f(u_n) - f(u_n - v_n) - f(v_n)|^t\,dx$$
$$= \lim_{n\to\infty} \int_{B_R} |f(u_n) - f(u_n - v) - f(v)|^t\,dx = 0\ .$$

As $n \to \infty$ there is a uniform constant for the continuous embeddings $H^1(B_{R_n} \setminus B_R) \to L^{t\mu}(B_{R_n} \setminus B_R)$. It follows that

$$\limsup_{n\to\infty} |u_n|_{t\mu, B_{R_n}\setminus B_R} \le C\sqrt{\varepsilon}$$
$$\limsup_{n\to\infty} |v_n|_{t\mu, B_{R_n}\setminus B_R} \le |v|_{t\mu,\mathbb{R}^N\setminus B_R} \le C\sqrt{\varepsilon}\ .$$

From this we obtain

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |f(u_n) - f(u_n - v_n) - f(v_n)|^t\,dx$$
$$= \limsup_{n\to\infty} \int_{B_{R_n}\setminus B_R} |f(u_n) - f(u_n - v_n) - f(v_n)|^t\,dx$$

$$
\begin{aligned}
&\le C \limsup_{n\to\infty} \int_{B_{R_n}\setminus B_R} (|u_n|^\mu + |u_n - v_n|^\mu + |v_n|^\mu)^t \,dx \\
&= C \limsup_{n\to\infty} \big| |u_n|^\mu + |u_n - v_n|^\mu + |v_n|^\mu \big|_t^t \\
&\le C \limsup_{n\to\infty} (|u_n|_{t\mu}^\mu + |u_n - v_n|_{t\mu}^\mu + |v_n|_{t\mu}^\mu)^t \\
&\le C \limsup_{n\to\infty} (2\varepsilon^{\mu/2} + (|u_n|_{t\mu} + |v_n|_{t\mu})^\mu)^t \\
&\le C\varepsilon^{t\mu/2} .
\end{aligned}
$$

Here the $L^{t\mu}$ and L^t norms in rows 2–4 counted from the bottom are taken with respect to $B_{R_n} \setminus B_R$, and we have used that $t\mu \ge 1$ and $t \ge 1$. Letting ε tend to 0 we find that

$$
f(u_n) - f(u_n - v_n) - f(v_n) \to 0 \qquad \text{in } L^t .
$$

By noting that $v_n \to v$ in E and thus $f(v_n) \to f(v)$ in L^t we finish the proof. □

Remark 3.3. The preceding lemma can easily be extended to the case of an open subset $\Omega \subseteq \mathbb{R}^N$. Here all is needed is that $\Omega \cap B_R(0)$ satisfies a uniform cone condition for large R, so that we have uniform constants from the Sobolev embeddings. Also the case of f depending on $x \in \mathbb{R}^N$ can be treated with the same proof.

Consider $F\colon E \to \mathbb{R}$ and $G\colon E \to E^*$ given by

$$
\begin{aligned}
F(u) &= \int_{\mathbb{R}^N} (U * g(u))h(u)\,dx \\
G(u)[v] &= \int_{\mathbb{R}^N} (U * g(u))h'(u)v\,dx
\end{aligned}
$$

for $u, v \in E$.

Lemma 3.4. *The maps F and G are well defined and continuous. For $u, v \in E$ we have*

$$
\begin{aligned}
|F(u)| &\le |U|_r |u|_{sp}^p |u|_{sq}^q \\
\|G(u)\|_{E^*} &\le C|U|_r |u|_{sp}^p |u|_{sq}^{q-1} .
\end{aligned}
$$

G is weakly sequentially continuous. If $u_n \rightharpoonup v$ in E there is (after extraction of a subsequence) a sequence $v_n \to v$ in E, independent of g and h, such that

$$
\begin{aligned}
F(u_n) - F(u_n - v_n) &\to F(v) && \text{in } \mathbb{R} \\
G(u_n) - G(u_n - v_n) &\to G(v) && \text{in } E^*.
\end{aligned}
$$

Proof. We have continuous Nemyckii operators $L^{sp} \to L^s$, $L^{sq} \to L^s$, and $L^{sq} \to L^{sq/(q-1)}$ induced by g, h, and h' respectively. Thus the inequality for F follows from Lemma 3.1 with $t = s$ and $\mu = 1$. Continuity of F is then a consequence of continuous Sobolev embeddings $E \to L^{sp}$ and $E \to L^{sq}$. The inequality for and continuity of G follows from Lemma 3.1 with $t = sq/(q-1)$ and $\mu = (sq)'$ (the conjugate exponent for sq), and from the continuous embedding $L^{(sq)'} \to E^*$.

If $u_n \rightharpoonup v$ in E, then $u_n \to v$ in L^{sp}_{loc} and in L^{sq}_{loc}, by the compactness of Sobolev embeddings. Thus

$$\begin{aligned} g(u_n) &\to g(v) && \text{in } L^s_{\mathrm{loc}} \\ h(u_n) &\to h(v) && \text{in } L^s_{\mathrm{loc}} \\ h'(u_n) &\to h'(v) && \text{in } L^{sq/(q-1)}_{\mathrm{loc}}\ , \end{aligned} \tag{3.6}$$

and these sequences are bounded. Clearly (as in the proof of Lemma 3.1) for any $w \in E$ we have $h'(u_n)w \to h'(v)w$ in L^s, so that again by Lemma 3.1 with $t = s$ and $\mu = 1$ $G(u_n)[w] \to G(v)[w]$ in $\mathbb{R}$. Therefore G is weakly sequentially continuous.

By Lemma 3.2 we can, for a subsequence of (u_n), build v_n, independent of g and h, such that $v_n \to v$ in E, $u_n - v_n \rightharpoonup 0$ in E, and (as above)

$$\begin{aligned} g(u_n - v_n) &\to 0 && \text{in } L^s_{\mathrm{loc}} \\ h(u_n - v_n) &\to 0 && \text{in } L^s_{\mathrm{loc}} \\ h'(u_n - v_n) &\to 0 && \text{in } L^{sq/(q-1)}_{\mathrm{loc}} \\ g(u_n) - g(u_n - v_n) &\to g(v) && \text{in } L^s \\ h(u_n) - h(u_n - v_n) &\to h(v) && \text{in } L^s \\ h'(u_n) - h'(u_n - v_n) &\to h(v) && \text{in } L^{sq/(q-1)}\ . \end{aligned}$$

Using this, Lemma 3.1, (3.6), and bilinearity, the last two claims follow easily. □

3.2 The Combined Case

Let us denote

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (W * f(u))f(u)\,dx$$

for $u \in E$. We consider the splitting of W and f discussed above. This yields a splitting of Ψ into a sum of at most six terms. We set $s_i = 2r_i/(2r_i - 1)$ for $i = 1, 2$. From (F_1) it follows that

$$s_i p_j \in (2, 2^*) \tag{3.7}$$

for $i, j \in \{1, 2\}$, so that we can apply the results of Section 3.1.

Lemma 3.5. *Ψ is a C^1-functional where Ψ and Ψ' map bounded sets into bounded sets. Ψ is weakly sequentially lower semicontinuous and Ψ' is weakly sequentially continuous. If $u_n \rightharpoonup v$ in E, there exists (after extraction of a subsequence) a sequence $v_n \to v$ in E such that*

$$\begin{aligned} \Psi(u_n) - \Psi(u_n - v_n) &\to \Psi(v) && \textit{in } \mathbb{R} \\ \Psi'(u_n) - \Psi'(u_n - v_n) &\to \Psi'(v) && \textit{in } E^*. \end{aligned}$$

Proof. By Lemma 3.4 Ψ is well defined and continuous. Let $u_n \rightharpoonup u$ in E. We can assume (after extraction of a subsequence) that $u_n \to u$ pointwise a.e. Since $W, f \geq 0$ Fatou's Lemma yields

$$\Psi(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \lim_{n\to\infty} W(x-y) f(u_n(y)) f(u_n(x))\, dy\, dx \leq \liminf_{n\to\infty} \Psi(u_n) \;.$$

Thus Ψ is weakly sequentially lower semicontinuous.

Consider the map $G\colon E \to E^*$ given by

$$G(u)[v] = \int_{\mathbb{R}^N} (W * f(u)) f'(u) v\, dx$$

for $u, v \in E$. G is well defined, continuous and weakly sequentially continuous by Lemma 3.4. We show that for $u, h \in E$

$$\Psi(u+h) - \Psi(u) = \int_0^1 G(u+sh)[h]\, ds \;. \tag{3.8}$$

Clearly from this and the continuity of G it follows that Ψ is differentiable everywhere and $\Psi' = G$. To show (3.8) recall that W is even. We calculate

$$\begin{aligned}
2 \int_0^1 & G(u+sh)[h]\, ds \\
&= 2 \int_0^1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Big[W(x-y) f(u(y) + sh(y)) \\
&\qquad\qquad\qquad \times f'(u(x) + sh(x)) h(x) \Big]\, dy\, dx\, ds \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W(x-y) \int_0^1 \Big[f'(u(y) + sh(y)) h(y) f(u(x) + sh(x)) \\
&\qquad\qquad + f(u(y) + sh(y)) f'(u(x) + sh(x)) h(x) \Big]\, ds\, dy\, dx \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W(x-y) \Big[f(u(y) + h(y)) f(u(x) + h(x)) \\
&\qquad\qquad\qquad - f(u(y)) f(u(x)) \Big]\, dy\, dx \\
&= 2(\Psi(u+h) - \Psi(u)) \;.
\end{aligned}$$

The integrand in the second row is easily seen to be in $L^1([0,1] \times \mathbb{R}^N \times \mathbb{R}^N)$ by using the splitting of W and f, and the estimates in Section 3.1. This allows us to change the order of integration and (3.8) is proved. The remaining properties of Ψ are clear from Lemma 3.4. □

Lemma 3.6. *If* (W_2) *and* (F_2) *hold, then for all* $u \in E \setminus \{0\}$ *we have*

$$\Psi'(u)[u] \geq \theta \Psi(u) > 0 \;.$$

If in addition (W_3) *holds, then for all* $u \in E$ *we have*

$$\|\Psi'(u)\|_{E^*} \le C(\sqrt{\Psi'(u)[u]} + \Psi'(u)[u]) \ .$$

Proof. From (F_2) and $W, f \ge 0$ it follows that $\Psi'(u)[u] \ge \theta\Psi(u)$ for all $u \in E$. If $u \ne 0$ then also $\Psi(u) > 0$ since $W > 0$ on a neighborhood of 0.

For the proof of the second assertion consider again the splitting of $f = f_1 + f_2$. Let p_1' and p_2' be the conjugate exponents for p_1 and p_2 respectively. From (3.1) we obtain

$$\begin{aligned}|f_1'(u)|^{p_1'} &\le Cf'(u)u\\ |f_2'(u)|^{p_2'} &\le Cf'(u)u \ .\end{aligned}$$

Using this, (F_2), (W_3), and Hölder's inequality we can compute for any $u, v \in E$

$$\begin{aligned}
&\int_{\mathbb{R}^N} (W * f(u))|f_1'(u)v|\,dx\\
&\qquad\le \left(\int (W * f(u))|f_1'(u)|^{p_1'}\right)^{\frac{1}{p_1'}} \left(\int (W * f(u))|v|^{p_1}\right)^{\frac{1}{p_1}}\\
&\qquad\le C\left(\int (W * f(u))f'(u)u\right)^{\frac{1}{p_1'}} \left(\int (W * f(u))|v|^{p_1}\right)^{\frac{1}{p_1}}\\
&\qquad\le C\left(\int (W * f(u))f'(u)u\right)^{\frac{1}{p_1'}} \left(\int (W * f(u))f(u)\right)^{\frac{1}{2p_1}}\\
&\qquad\qquad\qquad\qquad\times\left(\int (W * |v|^{p_1})|v|^{p_1}\right)^{\frac{1}{2p_1}}\\
&\qquad\le C\left(\int (W * f(u))f'(u)u\right)^{\frac{1}{p_1'}} \left(\int (W * f(u))f'(u)u\right)^{\frac{1}{2p_1}}\\
&\qquad\qquad\qquad\qquad\times\left(\int (W * |v|^{p_1})|v|^{p_1}\right)^{\frac{1}{2p_1}}\\
&\qquad\le C(\Psi'(u)[u])^{\frac{1}{p_1'}+\frac{1}{2p_1}}\|v\|_E
\end{aligned}$$

and a similar estimate for f_2 in place of f_1. This, together with

$$|\Psi'(u)[v]| \le \int_{\mathbb{R}^N} (W * f(u))|f_1'(u)v|\,dx + \int_{\mathbb{R}^N} (W * f(u))|f_2'(u)v|\,dx$$

and $1/p_i' + 1/(2p_i) \in (1/2, 1)$ for $i = 1, 2$ yields the desired inequality. □

4 Abstract critical point theory

In this section we assume (V_1), (W_1), (W_2), (F_1) and (F_2) throughout. We also assume that $0 \notin \sigma(-\Delta + V)$.

By Lemma 3.5 the functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2)\,dx - \Psi(u)$$

is of class C^1. Weak solutions of (P$_+$) correspond to critical points of Φ. We have a splitting $E = E^- \oplus E^+$ with orthogonal projections P^- and P^+ corresponding to the decomposition of $\sigma(-\Delta + V)$ in the negative and positive part. Let us define a new norm $\|\cdot\|$ on E by setting

$$\|u^+\|^2 = \int_{\mathbb{R}^N} |\nabla u^+|^2 + V|u^+|^2\,dx$$

$$\|u^-\|^2 = -\int_{\mathbb{R}^N} |\nabla u^-|^2 + V|u^-|^2\,dx$$

where $u^\pm = P^\pm u$. Since $0 \notin \sigma(-\Delta + V)$ the norms $\|\cdot\|$ and $\|\cdot\|_E$ are equivalent. The norm $\|\cdot\|$ is induced by a scalar product $\langle\cdot,\cdot\rangle$, and the projections $P^\pm$ are orthogonal with respect to this new scalar product. For these statements see for example [26]. Note that if (V$^1_2$) holds we have $E^- = \{0\}$ and $\|u^+\| = \|u\|$. Let $\|\cdot\|$ also denote the induced norm on E^*. Now we can write

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Psi(u)\ .$$

4.1 The Geometry of Φ

Lemma 4.1. *There is* $\rho > 0$ *such that* $\inf \Phi(S_\rho E^+) > 0$.

Proof. Suppose that $z \in E^+$ with $\|z\| \le 1$. Using Lemma 3.4 we see that

$$\Phi(z) = \frac{1}{2}\|z\|^2 - \Psi(z) \ge \frac{1}{2}\|z\|^2 - C\|z\|^{2p_1}$$

where $2p_1 > 2$, and the claim follows if we choose ρ small enough. □

Lemma 4.2. *Let* Z *be a finite dimensional subspace of* E^+*. Then* $\Phi(u) \to -\infty$ *as* $\|u\| \to \infty$ *in* $E^- \oplus Z$*.*

Proof. For any $u \in E$ with $\|u\| \ge 1$ and for any $t > 0$ put $g(t) = \Psi(tu/\|u\|) > 0$. By Lemma 3.6 we have

$$\frac{g'(t)}{g(t)} \ge \frac{\theta}{t}$$

for $t > 0$. Integrating this expression over $[1, \|u\|]$ we find

$$\Psi(u) \ge \Psi(u/\|u\|)\|u\|^\theta\ . \tag{4.1}$$

Choose $\beta \in (0,1)$ and set $\gamma = \sin(\arctan\beta) \in (0,1)$. Consider the set

$$K = \{\, u \in E \mid u^+ \in Z, \|u^+\| \ge \gamma, \|u\| = 1\,\}\ .$$

If $Z = \{0\}$ the claim follows from $\Psi \geq 0$. If $\dim Z \geq 1$ there is $(u_n) \subseteq K$ with $\lim_{n\to\infty} \Psi(u_n) = \inf \Psi(K) =: \delta \geq 0$. Since K is bounded we may assume that $u_n \rightharpoonup u \in E$ such that $u_n^+ \to u^+$ in Z. Clearly $\|u^+\| \geq \gamma$ and $u \neq 0$. Now Ψ is weakly sequentially lower semicontinuous. By Lemma 3.6 therefore $\delta \geq \Psi(u) > 0$.

Let $u \in E^- \oplus Z$ satisfy $\|u\| \geq 1$ and let us distinguish two cases: If $\|u^+\|/\|u^-\| \geq \beta$ we have

$$\frac{\|u^+\|}{\|u\|} = \sin\left(\arctan \frac{\|u^+\|}{\|u^-\|}\right) \geq \gamma$$

and therefore $u/\|u\| \in K$. In view of (4.1) and the definition of δ we obtain $\Psi(u) \geq \delta\|u\|^\theta$ and

$$\Phi(u) \leq \frac{1}{2}\|u\|^2 - \delta\|u\|^\theta \ .$$

If $\|u^+\|/\|u^-\| \leq \beta$ we have

$$\Phi(u) \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) \leq -\frac{1-\beta^2}{2(1+\beta^2)}\|u\|^2 \ . \tag{4.2}$$

For $\|u\|$ large we find in either case that (4.2) is satisfied, and the claim is proved since $\beta^2 < 1$. □

Let $\mathcal{K}$ be the set of critical points of Φ.

Lemma 4.3. *If either* (V_2^1) *or* (W_3) *holds, then there is* $\alpha > 0$ *such that for any* $u \in \mathcal{K} \setminus \{0\}$ *we have* $\Phi(u) \geq \alpha$.

Proof. First we show that $\|\cdot\|$ is bounded away from 0 on $\mathcal{K} \setminus \{0\}$. Let $u \in E \setminus \{0\}$ with $\Phi'(u) = 0$. If $\|u\| \leq 1$, using Lemma 3.4 we find

$$\begin{aligned} \|u^+\|^2 &= \Psi'(u)[u^+] \leq C\|u\|^{2p_1-1}\|u^+\| \\ \|u^-\|^2 &= -\Psi'(u)[u^-] \leq C\|u\|^{2p_1-1}\|u^-\| \end{aligned}$$

and therefore

$$\|u\| \leq C\|u\|^{2p_1-1}$$

where $2p_1 - 1 > 1$. This shows that $\|u\| \geq C > 0$ for some independent constant C.

Next, from Lemma 3.6 we see that

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\Phi'(u)[u] + \frac{1}{2}\Psi'(u)[u] - \Psi(u) \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\Psi'(u)[u] \ . \end{aligned}$$

In the case of (V_2^1) we also have $\|u\|^2 = \Psi'(u)[u]$ and thus $\|u\| \leq C\sqrt{\Phi(u)}$ for some independent C.

In the case of (W_3) we argue as follows: If $\Psi'(u)[u] \geq 1$ we have an independent positive lower bound for $\Phi(u)$. If $\Psi'(u)[u] \leq 1$, by Lemma 3.6 it follows that

$$\|\Psi'(u)\| \leq C\sqrt{\Psi'(u)[u]} \leq C\sqrt{\Phi(u)}\ ,$$

leading to

$$\begin{aligned}\|u^+\|^2 &= \Psi'(u)[u^+] \leq C\sqrt{\Phi(u)}\|u^+\| \\ \|u^-\|^2 &= -\Psi'(u)[u^-] \leq C\sqrt{\Phi(u)}\|u^-\|\ .\end{aligned}$$

Again it follows that $\|u\| \leq C\sqrt{\Phi(u)}$. In either case $\Phi(u) \geq C > 0$ for some independent C since $\|u\|$ is bounded away from 0 on $\mathcal{K} \setminus \{0\}$ as shown above. $\square$

4.2 Palais-Smale-Sequences

Lemma 4.4. *Assume* (V^1_2) *or* (W_3)*. If* $(u_n) \subseteq E$ *is a* $(\mathrm{PS})_c$*-sequence for* Φ*, then* $c \geq 0$ *and* (u_n) *is bounded.*

Proof. Suppose that $(u_n) \subseteq E$ with $\Phi(u_n) \leq C$ and $\|\Phi'(u_n)\| \leq \frac{1}{n}$. From

$$\begin{aligned}\Phi(u_n) &= \frac{1}{2}\Phi'(u_n)[u_n] + \frac{1}{2}\Psi'(u_n)[u_n] - \Psi(u_n) \\ &\geq -\frac{\|u_n\|}{2n} + \left(\frac{1}{2} - \frac{1}{\theta}\right)\Psi'(u_n)[u_n]\end{aligned} \tag{4.3}$$

we obtain

$$\Psi'(u_n)[u_n] \leq C\left(1 + \frac{\|u_n\|}{n}\right)\ . \tag{4.4}$$

If (V^1_2) holds then $\Psi'(u_n)[u_n] = \|u_n\|^2 + O(1/n)\|u_n\|$, and (4.4) yields $\|u_n\|^2 \leq C(1 + \|u_n\|/n)$. Consequently $\|u_n\|$ must be bounded.

If (W_3) holds, by Lemma 3.6

$$\|\Psi'(u_n)\| \leq C(1 + \Psi'(u_n)[u_n])\ ,$$

and together with (4.4)

$$\|\Psi'(u_n)\| \leq C\left(1 + \frac{\|u_n\|}{n}\right)\ .$$

Therefore

$$\begin{aligned}\|u_n^+\|^2 &= \Phi'(u_n)[u_n^+] + \Psi'(u_n)[u_n^+] \leq C\left(1 + \frac{\|u_n\|}{n}\right)\|u_n^+\| \\ \|u_n^-\|^2 &= -\Phi'(u_n)[u_n^-] - \Psi'(u_n)[u_n^-] \leq C\left(1 + \frac{\|u_n\|}{n}\right)\|u_n^-\|\ .\end{aligned}$$

We conclude that $\|u_n\| \leq C(1 + \|u_n\|/n)$ and that $\|u_n\|$ must be bounded. In either case, from (4.3) and Lemma 3.6 we find that also $c \geq 0$. $\square$

Consider the action of $\mathbb{Z}^N$ on E given as follows: If $m \in \mathbb{Z}^N$ and $u \in E$ set $(\tau_m u)(x) = u(x-m)$. From (V_1) it follows that $\|\cdot\|$ is invariant under this action, and the same holds for Φ.

Lemma 4.5. *Assume* (V_2^1) *or* (W_3). *For* $c \in \mathbb{R}$ *let* $(u_n) \subseteq E$ *be a* $(PS)_c$*-sequence for* Φ. *Then either* $c = 0$ *and* $u_n \to 0$ *or* $c \geq \alpha$ *and there are* $k \in \mathbb{N}$, $k \leq [c/\alpha]$, *and for each* $1 \leq i \leq k$ *a sequence* $(m_{i,n})_n \subseteq \mathbb{Z}^N$ *and a function* $v_i \in E \setminus \{0\}$ *such that, after extraction of a subsequence of* (u_n),

$$\left\| u_n - \sum_{i=1}^{k} \tau_{m_{i,n}} v_i \right\| \to 0$$
$$\Phi\left(\sum_{i=1}^{k} \tau_{m_{i,n}} v_i\right) \to \sum_{i=1}^{k} \Phi(v_i) = c$$
$$|m_{i,n} - m_{j,n}| \to \infty \qquad \text{for } i \neq j$$
$$\Phi'(v_i) = 0 \qquad \text{for all } i.$$

Proof. By Lemma 4.4 (u_n) is bounded in E. If

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} |u_n|_{2,B_R(x)} = 0 \tag{4.5}$$

for some $R > 0$ then by the well known Lemma I.1 in [17] $u_n \to 0$ in L^p for $p \in (2, 2^*)$. Using the splittings of W and f as in Sect. 3, from Lemma 3.6, (3.7), and Lemma 3.4 it follows that $\|\Psi'(u_n)\| \to 0$, and it is easily seen from $\|\Phi'(u_n)\| \to 0$ that then also $\|u_n\| \to 0$ and thus $c = 0$.

If, on the other hand, (4.5) does not hold, extracting a subsequence there are $R, \beta > 0$ and a sequence $(x_n) \subseteq \mathbb{R}^N$ such that $|u_n|_{2,B_R(x_n)} \geq \beta$. Substituting R by $R + \sqrt{N}/2$ we can choose a sequence $(m_{1,n}) \subseteq \mathbb{Z}^N$ such that $|u_n|_{2,B_R(m_{1,n})} \geq \beta$. Then $\tau_{-m_{1,n}} u_n \rightharpoonup v_1 \in E \setminus \{0\}$ for a subsequence. From weak sequential continuity and invariance of Φ under the action of $\mathbb{Z}^N$ we obtain that $\Phi'(v_1) = 0$. Moreover

$$\begin{aligned}
&\lim_{n\to\infty} (\|u_n^\pm\|^2 - \|u_n^\pm - \tau_{m_{1,n}} v_1^\pm\|^2) \\
&\qquad = \lim_{n\to\infty} (\|\tau_{-m_{1,n}} u_n^\pm\|^2 - \|\tau_{-m_{1,n}} u_n^\pm - v_1^\pm\|^2) \\
&\qquad = \lim_{n\to\infty} 2\langle \tau_{-m_{1,n}} u_n^\pm, v_1^\pm \rangle - \|v_1^\pm\|^2 \\
&\qquad = \|v_1^\pm\|^2 .
\end{aligned}$$

Here we have used that $\tau_{m_{1,n}}$ commutes with the projections $P^\pm$. Extracting subsequences as we go along, by Lemma 3.5 and the last calculation there is a sequence $v_{1,n} \to v_1$ in E such that

$$\Phi(\tau_{-m_{1,n}} u_n) - \Phi(\tau_{-m_{1,n}} u_n - v_{1,n}) \to \Phi(v_1)$$
$$\Phi'(\tau_{-m_{1,n}} u_n) - \Phi'(\tau_{-m_{1,n}} u_n - v_{1,n}) \to \Phi'(v_1) = 0$$

and thus, setting $u_{2,n} = u_n - \tau_{m_{1,n}} v_{1,n}$

$$\Phi(u_{2,n}) \to c - \Phi(v_1)$$
$$\Phi'(u_{2,n}) \to 0$$

as $n \to \infty$. By Lemma 4.3 and Lemma 4.4 $c \ge \Phi(v_1) \ge \alpha$. We can repeat this process for $(u_{2,n})$. After at most $k \le [c/\alpha]$ iterations we find $u_{k+1,n} = u_n - \sum_{i=1}^{k} \tau_{m_{i,n}} v_{i,n} \to 0$ as $n \to \infty$. Here we can replace $v_{i,n}$ by v_i. Also we see that $\sum_{i=1}^{k} \Phi(v_i) = c$. Noting that (u_n) is bounded and that Φ' maps bounded sets into bounded sets, clearly

$$\Phi(u_n) - \Phi\left(\sum_{i=1}^{k} \tau_{m_{i,n}} v_i\right) \to 0 \,.$$

To show the remaining assertion, assume that $|m_{i,n} - m_{j,n}|$ is bounded as $n \to \infty$ for some $1 \le i < j \le k$. We can assume that $|m_{i,n} - m_{l,n}| \to \infty$ for any $i < l < j$. Suppose that (u_n) is the final extracted subsequence. Put $m_n^* = m_{i,n} - m_{j,n}$. By construction $\tau_{-m_{i,n}} u_{j,n} \rightharpoonup 0$ and thus $\tau_{m_n^*} \tau_{-m_{i,n}} u_{j,n} \rightharpoonup 0$. But we also have $\tau_{-m_{j,n}} u_{j,n} \rightharpoonup v_j$ and $\tau_{m_n^*} \tau_{-m_{i,n}} = \tau_{-m_{j,n}}$, leading to $v_j = 0$. Contradiction. □

4.3 Proof of the Main Theorems

Now we can prove Theorem 2.1 and Theorem 2.2. If (V_2^1) or (W_3) is satisfied, fix $z \in E^+$ with $\|z\| = 1$. By Lemma 4.2 there is $r > \rho$ such that $\Phi(u) \le 0$ for all $u \in E^- \oplus [z]$ with $\|u\| \ge r$. Here $[z]$ denotes the span of $\{z\}$. Consider

$$M = \{\, y + tz \mid y \in E^-,\ \|y + tz\| \le r,\ t \ge 0 \,\}$$

and let M_0 be the boundary of M in $E^- \oplus [z]$. Then $\sup \Phi(M) < \infty$ by Lemma 3.5 since M is bounded, and $\sup \Phi(M_0) \le 0 < \inf \Phi(S_\rho E^+)$ from the choice of r, since $\Phi \le 0$ on E^-, and by Lemma 4.1. In view of Lemma 3.5 and [28, Cor. 6.11] we can apply the theorem of Kryszewski and Szulkin (cf. [28, Thm. 6.10] or [12]) to obtain a $(\mathrm{PS})_c$-sequence $(u_n) \subseteq E$ for Φ, with $c > 0$. For $E^- = \{0\}$ this is of course the same as constructing a (PS)-sequence from the Mountain Pass Theorem. By Lemma 4.5 there exists a nontrivial weak solution for (P_+).

The proof of the multiplicity results for (P_+) follows the proof of [4, Thm. 1.2]. It rests on [5, Thm. 5.2]. For the convenience of the reader we state the latter theorem here.

Let us write E_w^- for the subspace E^- with the weak topology. Set $\Phi_a^b = \{u \in E \mid a \le \Phi(u) \le b\}$. Given an interval $I \subset \mathbb{R}$, call a set $\mathcal{A} \subset E$ a $(\mathrm{PS})_I$-*attractor* if for any $(\mathrm{PS})_c$-sequence (u_n) with $c \in I$, and any $\varepsilon, \delta > 0$ one has $u_n \in U_\varepsilon(\mathcal{A} \cap \Phi_{c-\delta}^{c+\delta})$ provided n is large enough. Consider the following hypotheses on Φ:

- (Φ_1) $\Phi \in C^1(E, \mathbb{R})$ is even and $\Phi(0) = 0$.
- (Φ_2) There exist $\kappa, \rho > 0$ such that $\Phi(z) \ge \kappa$ for every $z \in E^+$ with $\|z\| = \rho$.
- (Φ_3) There exists a strictly increasing sequence of finite-dimensional subspaces $Z_n \subset E^+$ such that $\sup \Phi(E_n) < \infty$ where $E_n := E^- \oplus Z_n$, and an increasing sequence of real numbers $r_n > 0$ with $\Phi(E_n \setminus B_{r_n}) < \inf \Phi(B_\rho)$.

(Φ_4) $\Phi(u) \to -\infty$ as $\|u^-\| \to \infty$ and $\|u^+\|$ bounded.
(Φ_5) $\Phi' \colon E_w^- \oplus E^+ \to E_w^*$ is sequentially continuous, and $\Phi \colon E_w^- \oplus E^+ \to \mathbb{R}$ is sequentially upper semicontinuous.
(Φ_6) For any compact interval $I \subset (0, \infty)$ there exists a $(\mathrm{PS})_I$-attractor $\mathcal{A}$ such that $\inf\{ \|u^+ - v^+\| \mid u, v \in \mathcal{A}, u^+ \neq v^+ \} > 0$.

Theorem 4.6 (Bartsch-Ding, 1999). *If* Φ *satisfies* (Φ_1)–(Φ_6) *then there exists an unbounded sequence* (c_n) *of positive critical values.*

Now we assume that either (V_2^1) or (W_3) holds and that (F_3) is satisfied. Let $\mathcal{F}$ consist of arbitrarily chosen representatives of the orbits in $\mathcal{K}$ under the action of $\mathbb{Z}^N$. By the evenness of Φ we can also assume that $\mathcal{F} = -\mathcal{F}$. Suppose that there are only finitely many geometrically distinct solutions of (P_+) or, equivalently, that $\mathcal{F}$ is finite. To reach a contradiction we want to apply Theorem 4.6 and have to show that hypotheses (Φ_1)–(Φ_6) are satisfied for Φ. From (F_3) it follows that Φ is even and thus (Φ_1). (Φ_2) is stated in Lemma 4.1. (Φ_3) follows from Lemma 3.5 and Lemma 4.2. Condition (Φ_4) holds since $\Psi \geq 0$.

The embedding $E_w^- \oplus E^+ \hookrightarrow E_w$ is sequentially continuous. Therefore, by Lemma 3.5, Ψ' is sequentially continuous on $E_w^- \oplus E^+$, and the same holds for Φ'. For the same reason Ψ is sequentially lower semicontinuous on $E_w^- \oplus E^+$. Moreover $\|\cdot\|$ is sequentially lower semicontinuous on E_w^-. These facts together give (Φ_5).

Given any compact interval $I \subseteq (0, \infty)$ with $d = \max I$ we set $k = [d/\alpha]$ and

$$[\mathcal{F}, k] = \left\{ \sum_{i=1}^{j} \tau_{m_i} v_i \;\middle|\; 1 \leq j \leq k, m_i \in \mathbb{Z}^N, v_i \in \mathcal{F} \right\} .$$

By Lemma 4.5 $[\mathcal{F}, k]$ is a $(\mathrm{PS})_I$-attractor. Since the projections $P^{\pm}$ commute with the action of $\mathbb{Z}^N$ on E, it is clear from [9, Prop. 2.57] that (Φ_6) is also satisfied. We reach a contradiction, because now Theorem 4.6 provides us with infinitely many geometrically distinct solutions.

It remains to prove the assertions pertaining to problem (P_-). Consider the functional

$$\Phi_-(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \Psi(u) .$$

Critical points of Φ_- are in correspondence with solutions to (P_-). If (V_2^1) is satisfied, for any critical point u of Φ_- we have

$$\|u\|^2 = -\Psi'(u)[u] \leq 0$$

by Lemma 3.6, so there is no nontrivial solution in this case.

Note that we have nowhere used that $\sigma(-\Delta + V)$ is bounded below. So if (W_3) and (V_2^2) hold, for our discussion the subspaces E^- and E^+, both being infinite dimensional separable Hilbert spaces, are equivalent. By this we mean that we can apply the arguments from the existence proofs above to the functional Φ_- by interchanging the roles of E^- and E^+. The proof of the theorems is complete.

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