

Cohomological invariants and R -triviality of adjoint classical groups

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Abstract Using a cohomological obstruction, we construct examples of absolutely simple adjoint classical groups of type 2A_n with $n \equiv 3 \pmod{4}$, C_n or 1D_n with $n \equiv 0 \pmod{4}$, which are not R -trivial hence not stably rational.

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1 Introduction

For an algebraic group G defined over a field F , let $G(F)/R$ be the group of R -equivalence classes introduced by Manin in [6]. The algebraic group G is called R -trivial if $G(L)/R = 1$ for every field extension L/F . It was established by Colliot-Thélène and Sansuc in [2] (see also [7, Proposition 1]) that the group G is R -trivial if the variety of G is stably rational.

In this paper, we focus on the case where G is an absolutely simple classical group of adjoint type. Adjoint groups of type 1A_n or B_n are easily seen to be rational (see [7, pp. 199, 200]). Voskresenskii and Klyachko [11, Cor. of Th. 8] proved that adjoint groups of type 2A_n are rational if n is even, and Merkurjev [7, Prop. 4] showed that adjoint groups of type C_n are stably rational for n odd. On the other hand, Merkurjev also produced in [7] examples of adjoint groups of type 2A_3 ($= {}^2D_3$) and of type 2D_n for any $n \geq 4$ which are not R -trivial, hence not stably rational. Examples of adjoint groups of type 1D_4 which are not R -trivial were constructed by Gille in [3].

The goal of the present paper is to construct examples of adjoint groups of type 2A_n with $n \equiv 3 \pmod{4}$ and of adjoint groups of type C_n or 1D_n with $n \equiv 0 \pmod{4}$ which are not R -trivial. Our constructions are based on Merkurjev's computation in [7] of the group of R -equivalence classes of adjoint classical groups, which we

now recall briefly. According to Weil (see [4, §26]), every absolutely simple classical group of adjoint type over a field F of characteristic different from 2 can be obtained as the connected component of the identity in the automorphism group of a central¹ simple algebra with involution (A, σ) over F . Let $\mathbf{Sim}(A, \sigma)$ be the algebraic group of similitudes of (A, σ) , defined (as a group scheme) by

$$\mathbf{Sim}(A, \sigma)(E) = \{u \in A \otimes_F E \mid (\sigma \otimes \text{Id})(u)u \in E^\times\}$$

for every commutative F -algebra E , and let $\mathbf{PSim}(A, \sigma)$ be the group of projective similitudes,

$$\mathbf{PSim}(A, \sigma) = \mathbf{Sim}(A, \sigma)/R_{K/F}(\mathbf{G}_{m,K})$$

where K is the center of A . The connected component of the identity in these groups is denoted by $\mathbf{Sim}^+(A, \sigma)$ and $\mathbf{PSim}^+(A, \sigma)$ respectively. We let $\text{Sim}(A, \sigma)$, $\text{PSim}(A, \sigma)$, $\text{Sim}^+(A, \sigma)$ and $\text{PSim}^+(A, \sigma)$ denote the corresponding groups of F -rational points:

$$\text{Sim}(A, \sigma) = \mathbf{Sim}(A, \sigma)(F), \quad \text{PSim}(A, \sigma) = \mathbf{PSim}(A, \sigma)(F), \quad \text{etc.}$$

The group $\mathbf{PSim}^+(A, \sigma)$ is canonically isomorphic (under the map which carries every similitude g to the induced inner automorphism $\text{Int}(g)$) to the connected component of the identity in the automorphism group of (A, σ) . To describe the group of R -equivalence classes of $\text{PSim}^+(A, \sigma)$, consider the homomorphism

$$\mu: \mathbf{Sim}(A, \sigma) \rightarrow \mathbf{G}_m$$

which carries every similitude to its multiplier

$$\mu(g) = \sigma(g)g.$$

Let $G^+(A, \sigma) = \mu(\text{Sim}^+(A, \sigma)) \subset F^\times$ and $NK^\times = \mu(K^\times) \subset F^\times$ (so $NK^\times = F^{\times 2}$ if $K = F$). Let also $\text{Hyp}(A, \sigma)$ be the subgroup of F^\times generated by the norms of the finite extensions L of F such that (A, σ) becomes hyperbolic after scalar extension to L . In [7, Theorem 1], Merkurjev shows that the multiplier map μ induces a canonical isomorphism

$$\mathbf{PSim}^+(A, \sigma)/R \simeq G^+(A, \sigma)/(NK^\times \cdot \text{Hyp}(A, \sigma)). \tag{1}$$

For any positive integer d , let $H^d(F, \mu_2)$ be the degree d cohomology group of the absolute Galois group of F with coefficients $\mu_2 = \{\pm 1\}$. In Section 4 we consider the case where σ is of the first kind. If it is orthogonal, we assume further that its discriminant is trivial. Assuming the index of A divides $\frac{1}{2} \deg A$, we construct a homomorphism

¹We use the same terminology as in [4]. In particular, the center of A is F if σ is of the first kind; it is a quadratic étale extension of F if σ is of the second kind.

$$\Theta_1: \text{PSim}^+(A, \sigma)/R \rightarrow H^4(F, \mu_2),$$

and give examples where this homomorphism is nonzero, hence

$$\text{PSim}^+(A, \sigma)/R \neq 1.$$

Similarly, if σ is of the second kind and the exponent of A divides $\frac{1}{2} \deg A$, we construct in Section 5 a homomorphism

$$\Theta_2: \text{PSim}^+(A, \sigma)/R \rightarrow H^3(F, \mu_2)$$

and show that this map is nonzero in certain cases. In all the examples where we show $\Theta_1 \neq 0$ or $\Theta_2 \neq 0$, the algebra with involution has the form $(A, \sigma) = (B, \rho) \otimes (C, \tau)$ where ρ is an orthogonal involution which admits improper similitudes.

Throughout the paper, the characteristic of the base field F is different from 2.

2 Improper similitudes

Let (A, σ) be a central simple F -algebra with orthogonal involution of degree $n = 2m$. The group of similitudes $\mathbf{Sim}(A, \sigma)$ is not connected. Its connected component of the identity $\mathbf{Sim}^+(A, \sigma)$ is defined by the equation

$$\text{Nrd}_A(g) = \mu(g)^m,$$

where Nrd_A is the reduced norm. As mentioned in the introduction, we denote by $\text{Sim}(A, \sigma)$ and $\text{Sim}^+(A, \sigma)$ the group of F -rational points

$$\text{Sim}(A, \sigma) = \mathbf{Sim}(A, \sigma)(F), \quad \text{Sim}^+(A, \sigma) = \mathbf{Sim}^+(A, \sigma)(F).$$

The elements in $\text{Sim}^+(A, \sigma)$ are called *proper similitudes*, and those in the non-trivial coset

$$\text{Sim}^-(A, \sigma) = \{g \in \text{Sim}(A, \sigma) \mid \text{Nrd}_A(g) = -\mu(g)^m\}$$

are called *improper similitudes*.

For example, if $m = 1$ (i.e. A is a quaternion algebra), then every orthogonal involution has the form $\sigma = \text{Int}(q) \circ \gamma$, where γ is the canonical involution, q is an invertible pure quaternion and $\text{Int}(q)$ is the inner automorphism induced by q , mapping $x \in A$ to qxq^{-1} . It is easily checked that

$$\text{Sim}^+(A, \sigma) = F(q)^\times \quad \text{and} \quad \text{Sim}^-(A, \sigma) = q'F(q)^\times,$$

where q' is a unit which anticommutes with q . Therefore, $\text{Sim}^-(A, \sigma) \neq \emptyset$.

If $m > 1$, the existence of improper similitudes is an important restriction on A and σ , since it implies that A is split by the quadratic étale F -algebra $F[\sqrt{\text{disc } \sigma}]$, where $\text{disc } \sigma$ is the discriminant of σ , see [9, Theorem A] or [4, (13.38)]. In particular, the index of A satisfies $\text{ind } A \leq 2$, i.e. A is Brauer-equivalent to a quaternion algebra. Moreover, if m is even, then $-1 \in \text{Nrd}_A(A)$, see [9, Corollary 1.13]. There is no other restriction on A , as the following proposition shows.

Proposition 1. *Let H be an arbitrary quaternion F -algebra and let m be an arbitrary integer. If m is even, assume $-1 \in \text{Nrd}_H(H^\times)$. Then the algebra $M_m(H)$ carries an orthogonal involution which admits improper similitudes.*

Proof. Suppose first m is odd. Let i, j be elements in a standard quaternion basis of H . We set

$$\sigma = t \otimes (\text{Int}(i) \circ \gamma) \quad \text{on } M_m(H) = M_m(F) \otimes_F H,$$

where t is the transpose map on $M_m(F)$ and γ is the canonical involution on H . It is readily verified that $1 \otimes j$ is an improper similitude of σ .

Suppose next m is even, and $q \in H$ satisfies $\text{Nrd}_H(q) = -1$. We pick a quaternion basis $1, i, j, k = ij$ such that i commutes with q , and set

$$\sigma = \text{Int} \text{diag}(j, i, \dots, i) \circ (t \otimes \gamma) \quad \text{and} \quad g = \text{diag}(j, qj, \dots, qj).$$

Again, computation shows that g is an improper similitude of σ . □

Necessary and sufficient conditions for the existence of improper similitudes for a given involution σ are not known if $m \geq 4$. For $m = 2$ (resp. $m = 3$), Corollary (15.9) (resp. (15.26)) in [4] shows that $\text{Sim}^-(A, \sigma) \neq \emptyset$ if and only if the Clifford algebra $C(A, \sigma)$ has outer automorphisms (resp. outer automorphisms which commute with its canonical involution). (For $m = 2$ another equivalent condition is that A is split by the center of $C(A, \sigma)$, see [4, (15.11)] or [9, Prop. 1.15].) We use this fact to prove the following result:

Proposition 2. *Let (A, σ) be a central simple F -algebra with orthogonal involution of degree 4. Assume that A is not split and $\text{disc } \sigma \neq 1$. Then there exists a field extension L/F such that A_L is not split and $\text{Sim}^-(A_L, \sigma_L) \neq \emptyset$.*

Proof. By hypothesis, $F(\sqrt{\text{disc } \sigma})$ is a quadratic field extension of F . We denote it by K for simplicity and let ι be its nontrivial F -automorphism. The Clifford algebra $C = C(A, \sigma)$ is a quaternion K -algebra. Let X be the Severi-Brauer variety of $C \otimes_K {}^t C$ and let L be the function field of its Weil transfer:

$$L = F(R_{K/F}(X)).$$

Then $(C \otimes_K {}^t C) \otimes_K KL$ splits, so C_{KL} is isomorphic to ${}^t C_{KL}$, which means that C_{KL} has outer automorphisms. By [4, (15.9)], it follows that $\text{Sim}^-(A_L, \sigma_L) \neq \emptyset$.

On the other hand, by [9, Corollary 2.12], the kernel of the scalar extension map $\text{Br}(F) \rightarrow \text{Br}(L)$ is generated by the corestriction of $C \otimes_K {}^t C$. Since this corestriction is trivial, A_L is not split. □

3 Trace forms

In this section, A is a central simple F -algebra of even degree with an involution σ of the first kind. We consider the quadratic forms T_A and T_σ on A defined by

$$T_A(x) = \text{Tr}_A(x^2), \quad T_\sigma(x) = \text{Tr}_A(\sigma(x)x) \quad \text{for } x \in A,$$

where Trd_A is the reduced trace on A . We denote by T_σ^+ (resp. T_σ^-) the restriction of T_σ to the space $\text{Sym}(\sigma)$ of symmetric elements (resp. to the space $\text{Skew}(\sigma)$ of skew-symmetric elements), so that

$$T_A = T_\sigma^+ \perp -T_\sigma^- \quad \text{and} \quad T_\sigma = T_\sigma^+ \perp T_\sigma^-. \tag{2}$$

Recall that if σ is orthogonal the (signed) discriminant $\text{disc } T_\sigma^+$ is equal to the discriminant $\text{disc } \sigma$ up to a factor which depends only on the degree of A , see for instance [4, (11.5)]. In the following, we denote by $I^n F$ the n -th power of the fundamental ideal IF of the Witt ring WF .

Lemma 3. *Let σ, σ_0 be two involutions of the first kind on A .*

- *If σ and σ_0 are both symplectic, then $T_\sigma^+ - T_{\sigma_0}^+ \in I^3 F$.*
- *If σ and σ_0 are both orthogonal, then $\text{disc}(T_\sigma^+ - T_{\sigma_0}^+) = \text{disc } \sigma \text{ disc } \sigma_0$. Moreover, if $\text{disc } \sigma = \text{disc } \sigma_0$, then $T_\sigma^+ - T_{\sigma_0}^+ \in I^3 F$.*

Proof. The symplectic case has been considered in [1, Theorem 4]. For the rest of the proof, we assume that σ and σ_0 are both orthogonal. By [4, (11.5)], there is a factor $c \in F^\times$ such that

$$\text{disc } T_\sigma^+ = c \text{ disc } \sigma \quad \text{and} \quad \text{disc } T_{\sigma_0}^+ = c \text{ disc } \sigma_0,$$

hence

$$\text{disc}(T_\sigma^+ - T_{\sigma_0}^+) = \text{disc } T_\sigma^+ \text{ disc } T_{\sigma_0}^+ = \text{disc } \sigma \text{ disc } \sigma_0.$$

To complete the proof, observe that the Witt-Clifford invariant $e_2(T_\sigma^+)$ (or, equivalently, the Hasse invariant $w_2(T_\sigma^+)$) depends only on $\text{disc } \sigma$ and on the Brauer class of A , as was shown by Quéguiner [10, p. 307]. Therefore, if $\text{disc } \sigma = \text{disc } \sigma_0$, then $e_2(T_\sigma^+) = e_2(T_{\sigma_0}^+)$, hence $T_\sigma^+ - T_{\sigma_0}^+ \in I^3 F$ by a theorem of Merkurjev. \square

We next compute the Arason invariant $e_3(T_\sigma^+ - T_{\sigma_0}^+) \in H^3(F, \mu_2)$ in the special case where σ and σ_0 decompose. We use the following notation: $[A] \in H^2(F, \mu_2)$ is the cohomology class corresponding to the Brauer class of A under the canonical isomorphism $H^2(F, \mu_2) = {}_2\text{Br}(F)$. For $a \in F^\times$ we denote by (a) the cohomology class corresponding to the square class of a under the canonical isomorphism $H^1(F, \mu_2) = F^\times / F^{\times 2}$.

Lemma 4. *Suppose $A = B \otimes_F C$ for some central simple F -algebras B, C of even degree. Let ρ and ρ_0 be orthogonal involutions on B and let τ be an involution of the first kind on C . Let also $\sigma = \rho \otimes \tau$ and $\sigma_0 = \rho_0 \otimes \tau$.*

If τ (hence also σ and σ_0) is symplectic, then

$$e_3(T_\sigma^+ - T_{\sigma_0}^+) = \begin{cases} 0 & \text{if } \deg C \equiv 0 \pmod{4}, \\ (\text{disc } \rho \text{ disc } \rho_0) \cup [C] & \text{if } \deg C \equiv 2 \pmod{4}. \end{cases}$$

If τ (hence also σ and σ_0) is orthogonal, then

$$e_3(T_\sigma^+ - T_{\sigma_0}^+) = \begin{cases} (\text{disc } \rho \text{ disc } \rho_0) \cup (\text{disc } \tau) \cup (-1) & \text{if } \deg C \equiv 0 \pmod{4}, \\ (\text{disc } \rho \text{ disc } \rho_0) \cup ((\text{disc } \tau) \cup (-1) + [C]) & \text{if } \deg C \equiv 2 \pmod{4}. \end{cases}$$

Proof. The decomposition

$$\text{Sym}(\sigma) = (\text{Sym}(\rho) \otimes \text{Sym}(\tau)) \oplus (\text{Skew}(\rho) \otimes \text{Skew}(\tau))$$

yields

$$T_\sigma^+ = T_\rho^+ T_\tau^+ + T_\rho^- T_\tau^- \quad \text{in } WF.$$

Since $T_B = T_\rho^+ - T_\rho^-$ we may eliminate T_ρ^- in the equation above to obtain

$$T_\sigma^+ = T_\rho^+ T_\tau^+ + (T_\rho^+ - T_B) T_\tau^-.$$

Similarly,

$$T_{\sigma_0}^+ = T_{\rho_0}^+ T_\tau^+ + (T_{\rho_0}^+ - T_B) T_\tau^-$$

and subtracting the two equalities yields

$$T_\sigma^+ - T_{\sigma_0}^+ = (T_\rho^+ - T_{\rho_0}^+) T_\tau^+ + (T_\rho^+ - T_{\rho_0}^+) T_\tau^- = (T_\rho^+ - T_{\rho_0}^+) T_\tau.$$

Since $\deg C$ is even, we have $T_\tau \in I^2 F$ (see [4, (11.5)]), hence

$$e_3(T_\sigma^+ - T_{\sigma_0}^+) = (\text{disc}(T_\rho^+ - T_{\rho_0}^+) \cup e_2(T_\tau)) \quad \text{in } H^3(F, \mu_2).$$

By Lemma 3 we have

$$\text{disc}(T_\rho^+ - T_{\rho_0}^+) = \text{disc } \rho \text{ disc } \rho_0.$$

The computation of $e_2(T_\tau)$ in [10, Theorem 1] or [5] completes the proof. □

Remark 5. If σ and σ_0 are symplectic, the Arason invariant $e_3(T_\sigma^+ - T_{\sigma_0}^+)$ is the discriminant $\Delta_{\sigma_0}(\sigma)$ investigated in [1].

4 Involutions of the first kind

In this section, A is a central simple F -algebra of even degree, and σ is an involution of the first kind on A . We assume $\text{ind } A$ divides $\frac{1}{2} \deg A$, i.e. $A \simeq M_2(A_0)$ for some central simple F -algebra A_0 , so that A carries a hyperbolic involution σ_0 of the same type as σ . If σ is orthogonal, we assume $\text{disc } \sigma = 1$ ($= \text{disc } \sigma_0$), so that in all cases $T_\sigma^+ - T_{\sigma_0}^+ \in I^3 F$, by Lemma 3.

Proposition 6. *The map $\theta_1: \text{Sim}(A, \sigma) \rightarrow H^4(F, \mu_2)$ defined by*

$$\theta_1(g) = (\mu(g)) \cup e_3(T_\sigma^+ - T_{\sigma_0}^+)$$

induces a homomorphism

$$\Theta_1: \text{PSim}^+(A, \sigma)/R \rightarrow H^4(F, \mu_2).$$

Moreover, for all $g \in \text{Sim}(A, \sigma)$, we have

$$\theta_1(g) \cup (-1) = 0 \quad \text{in } H^5(F, \mu_2).$$

Proof. In view of the isomorphism (1), it suffices to show that for every finite field extension L/F such that $(A, \sigma) \otimes_F L$ is hyperbolic and for every $x \in L^\times$,

$$(N_{L/F}(x)) \cup e_3(T_\sigma^+ - T_{\sigma_0}^+) = 0 \quad \text{in } H^4(F, \mu_2).$$

The projection formula yields

$$(N_{L/F}(x)) \cup e_3(T_\sigma^+ - T_{\sigma_0}^+) = \text{cor}_{L/F}((x) \cup e_3(T_\sigma^+ - T_{\sigma_0}^+)_L).$$

Since σ_L is hyperbolic, the involutions σ_L and $(\sigma_0)_L$ are conjugate, hence

$$e_3(T_\sigma^+ - T_{\sigma_0}^+)_L = 0.$$

For the last equality, observe that (2) yields the following equations in WF :

$$T_\sigma + T_A = \langle 1, 1 \rangle T_\sigma^+ \quad \text{and} \quad T_{\sigma_0} + T_A = \langle 1, 1 \rangle T_{\sigma_0}^+,$$

hence

$$T_\sigma - T_{\sigma_0} = \langle 1, 1 \rangle (T_\sigma^+ - T_{\sigma_0}^+).$$

Since σ_0 is hyperbolic, we have $T_{\sigma_0} = 0$. Moreover, for $g \in \text{Sim}(A, \sigma)$ the map $x \mapsto gx$ is a similitude of T_σ with multiplier $\mu(g)$, hence

$$\langle 1, -\mu(g) \rangle T_\sigma = \langle 1, -\mu(g) \rangle \langle 1, 1 \rangle (T_\sigma^+ - T_{\sigma_0}^+) = 0.$$

Since

$$e_5(\langle 1, -\mu(g) \rangle \langle 1, 1 \rangle (T_\sigma^+ - T_{\sigma_0}^+)) = \theta_1(g) \cup (-1),$$

the proposition follows. □

Proposition 7. *Let $(A, \sigma) = (B, \rho) \otimes (C, \tau)$, where B and C are central simple F -algebras of even degree and ρ, τ are involutions of the first kind. Suppose $\text{ind } B$ divides $\frac{1}{2} \deg B$ and ρ is orthogonal. For $g \in \text{Sim}^-(B, \rho)$, we have $g \otimes 1 \in \text{Sim}^+(A, \sigma)$ and*

$$\theta_1(g \otimes 1) = \begin{cases} 0 & \text{if } \deg C \equiv 0 \pmod{4}, \\ [B] \cup [C] & \text{if } \deg C \equiv 2 \pmod{4}. \end{cases}$$

Proof. For $g \in \text{Sim}(B, \rho)$, we have

$$\sigma(g \otimes 1)g \otimes 1 = \rho(g)g = \mu(g)$$

and

$$\text{Nrd}_A(g \otimes 1) = \text{Nrd}_B(g)^{\deg C},$$

so $g \otimes 1 \in \text{Sim}^+(A, \sigma)$.

Since $\text{ind } B$ divides $\frac{1}{2} \deg B$, we may find a hyperbolic orthogonal involution ρ_0 on B , and set $\sigma_0 = \rho_0 \otimes \tau$, a hyperbolic involution on A of the same type as σ .

If τ is symplectic, Lemma 4 yields

$$e_3(T_\sigma^+ - T_{\sigma_0}^+) = \begin{cases} 0 & \text{if } \deg C \equiv 0 \pmod{4}, \\ (\text{disc } \rho) \cup [C] & \text{if } \deg C \equiv 2 \pmod{4}. \end{cases}$$

The proposition follows by taking the cup-product with $(\mu(g))$, since

$$(\mu(g)) \cup (\text{disc } \rho) = [B]$$

by [9, Theorem A] (see also [4, (13.38)]).

Suppose next τ is orthogonal. By Lemma 4,

$$e_3(T_\sigma^+ - T_{\sigma_0}^+) = \begin{cases} (\text{disc } \rho) \cup (\text{disc } \tau) \cup (-1) & \text{if } \deg C \equiv 0 \pmod{4}, \\ (\text{disc } \rho) \cup ((\text{disc } \tau) \cup (-1) + [C]) & \text{if } \deg C \equiv 2 \pmod{4}. \end{cases}$$

Using again the equation $(\mu(g)) \cup (\text{disc } \rho) = [B]$ and taking into account the equation $(-1) \cup [B] = 0$, which follows from [9, Corollary 1.13], we obtain the formula for $\theta_1(g \otimes 1)$. □

Using Proposition 7, it is easy to construct examples where $\theta_1 \neq 0$. For these examples, the map Θ_1 of Proposition 6 is not trivial, hence $\mathbf{PSim}^+(A, \sigma)$ is not R -trivial.

Corollary 8. *Let Q, H be quaternion F -algebras satisfying*

$$(-1) \cup [H] = 0 \text{ in } H^3(F, \mu_2) \quad \text{and} \quad [H] \cup [Q] \neq 0 \text{ in } H^4(F, \mu_2).$$

Let $A = M_{2r}(H) \otimes M_s(Q)$, where r is arbitrary and s is odd. Let ρ be an orthogonal involution on $M_{2r}(H)$ which admits improper similitudes (see Lemma 1), and let τ be any involution of the first kind on $M_s(Q)$. Then $\mathbf{PSim}^+(A, \rho \otimes \tau)$ is not R -trivial.

To obtain explicit examples, we may take for F the field of rational fractions in four indeterminates $F = \mathbb{C}(x_1, y_1, x_2, y_2)$ and set $H = (x_1, y_1)_F, Q = (x_2, y_2)_F$. Note that the degree of A can be any multiple of 8 and that the conditions on Q and H in Corollary 8 imply $\text{ind } A = 4$. Indeed, if there is a quadratic extension of F which splits Q and H , then $[H] \cup [Q]$ is a multiple of $(-1) \cup [H]$.

Other examples can be obtained from Proposition 2.

Corollary 9. *Let (B, ρ) be a central simple algebra of degree 4 and index 2 with orthogonal involution of nontrivial discriminant over a field F_0 . Let $F = F_0(x, y)$ be the field of rational fractions in two indeterminates x, y over F_0 , and let (C, τ) be a central simple F -algebra with involution of the first kind such that*

$$\deg C \equiv 2 \pmod{4} \quad \text{and} \quad [C] = (x) \cup (y) \in H^2(F, \mu_2).$$

Then $\mathbf{PSim}^+(B \otimes C, \rho \otimes \tau)$ is not R -trivial.

Proof. Proposition 2 yields an extension L_0/F_0 such that ρ_{L_0} admits an improper similitude g and B_{L_0} is not split. Set $L = L_0(x, y)$. By Proposition 7,

$$g \otimes 1 \in \mathbf{Sim}^+(B \otimes C, \rho \otimes \tau)(L) \quad \text{and} \quad \theta_1(g \otimes 1) = [B_L] \cup (x) \cup (y).$$

Since $[B_{L_0}] \neq 0$, taking successive residues for the x -adic and the y -adic valuations shows that $\theta_1(g \otimes 1) \neq 0$. Therefore, $\mathbf{PSim}^+(B \otimes C, \rho \otimes \tau)(L)/R \neq 1$, hence $\mathbf{PSim}^+(B \otimes C, \rho \otimes \tau)$ is not R -trivial. □

5 Involutions of the second kind

We assume in this section that (A, σ) is a central simple algebra with unitary involution over F . In this case, the group of similitudes is connected,

$$\mathbf{Sim}^+(A, \sigma) = \mathbf{Sim}(A, \sigma) \quad \text{and} \quad \mathbf{PSim}^+(A, \sigma) = \mathbf{PSim}(A, \sigma).$$

We denote by K the center of A and write $K = F[X]/(X^2 - \alpha)$. We assume the degree of A is even, $\deg A = n = 2m$, and denote by $D(A, \sigma)$ the discriminant algebra of (A, σ) (see [4, §10] for a definition).

Lemma 10. *$D(A, \sigma)$ is split if (A, σ) is hyperbolic.*

Proof. The lemma is clear if A is split, for then σ is adjoint to a hyperbolic hermitian form h and $[D(A, \sigma)] = (\alpha) \cup (\text{disc } h)$ by [4, (10.35)]. The general case is reduced to the case where A is split by scalar extension to the field of functions $L = F(R_{K/F}(\text{SB}(A)))$ of the Weil transfer of the Severi-Brauer variety of A . Indeed, $A \otimes_F L$ is split and the scalar extension map $\text{Br}(F) \rightarrow \text{Br}(L)$ is injective by [9, Corollary 2.12]. □

Proposition 11. *Suppose $A^{\otimes m}$ is split. The map*

$$\theta_2: \text{Sim}(A, \sigma) \rightarrow H^3(F, \mu_2)$$

defined by

$$\theta_2(g) = (\mu(g)) \cup [D(A, \sigma)]$$

induces a homomorphism

$$\Theta_2: \text{PSim}(A, \sigma)/R \rightarrow H^3(F, \mu_2).$$

Moreover, for any $g \in \text{Sim}(A, \sigma)$,

$$\theta_2(g) \cup (\alpha) = 0 \quad \text{in } H^4(F, \mu_2).$$

Proof. In view of the isomorphism (1), it suffices to show that for every finite field extension L/F such that $(A, \sigma) \otimes_F L$ is hyperbolic and for every $x \in L^\times$,

$$(N_{L/F}(x)) \cup [D(A, \sigma)] = 0 \quad \text{in } H^3(F, \mu_2),$$

and that for every $\lambda \in K^\times$,

$$(N_{K/F}(\lambda)) \cup [D(A, \sigma)] = 0 \quad \text{in } H^3(F, \mu_2).$$

As in the proof of Proposition 6, we are reduced by the projection formula to proving that $D(A, \sigma)$ is split by K and by every extension L/F such that $(A, \sigma) \otimes L$ is hyperbolic. The latter assertion follows from the lemma. On the other hand, by [4, (10.30)] and by the hypothesis on B we have

$$[D(A, \sigma)_K] = [\lambda^m A] = m[A] = 0.$$

To prove the last part, we use the trace form T_σ defined as in Section 3,

$$T_\sigma(x) = \text{Trd}_A(\sigma(x)x) \quad \text{for } x \in A,$$

and its restrictions T_σ^+, T_σ^- to $\text{Sym}(A, \sigma)$ and $\text{Skew}(A, \sigma)$ respectively. In the case of involutions of unitary type we have

$$T_\sigma = T_\sigma^+ \perp T_\sigma^- = \langle 1, -\alpha \rangle T_\sigma^+.$$

The computation of the Clifford algebra of T_σ^+ in [4, (11.17)] shows that $T_\sigma \in I^3 F$ and

$$e_3(T_\sigma) = (\alpha) \cup [D(A, \sigma)].$$

Now, for $g \in \text{Sim}(A, \sigma)$ the map $x \mapsto gx$ is a similitude of T_σ with multiplier $\mu(g)$, hence $\langle 1, -\mu(g) \rangle T_\sigma = 0$ in WF . Taking the image under e_4 yields

$$0 = (\mu(g)) \cup e_3(T_\sigma) = \theta_2(g) \cup (\alpha). \quad \square$$

Remark 12. 1. If $\text{ind } A$ divides $\frac{1}{2} \text{deg } A$, so that A carries a hyperbolic unitary involution σ_0 , then [4, (11.17)] and Lemma 10 yield

$$[D(A, \sigma)] = e_2(T_\sigma^+ - T_{\sigma_0}^+).$$

This observation underlines the analogy between θ_2 and the map θ_1 of Proposition 6. Note however that no hypothesis on the index of A is required in Proposition 11.

2. For $g \in \text{Sim}(A, \sigma)$, the equation $\theta_2(g) \cup (\alpha) = 0$ implies that $\theta_2(g)$ lies in the image of the corestriction map $\text{cor}_{K/F}: H^3(K, \mu_2) \rightarrow H^3(F, \mu_2)$, by [4, (30.12)]. On the other hand, if the characteristic does not divide m , Corollary 1.18 of [8] yields an explicit element $\xi \in H^3(K, \mu_m^{\otimes 2})$ such that $\text{cor}_{K/F}(\xi) = \theta_2(g)$. In particular, if m is odd it follows that $\theta_2 = 0$.

The following explicit computation yields examples where $\theta_2 \neq 0$.

Proposition 13. *Let ι be the nontrivial automorphism of K/F , and assume*

$$(A, \sigma) = (B, \rho) \otimes_F (K, \iota)$$

for some central simple F -algebra with orthogonal involution (B, ρ) of degree $n = 2m$. Assume m is even. For $g \in \text{Sim}^-(B, \rho)$ we have $g \otimes 1 \in \text{Sim}(A, \sigma)$ and

$$\theta_2(g \otimes 1) = (\alpha) \cup [B].$$

Proof. For $g \in \text{Sim}^-(B, \rho)$,

$$\sigma(g \otimes 1)g \otimes 1 = \rho(g)g = \mu(g),$$

so $g \otimes 1 \in \text{Sim}(A, \sigma)$. By [4, (10.33)], we have

$$[D(A, \sigma)] = m[B] + (\alpha) \cup (\text{disc } \rho).$$

Since m is even, the first term on the right side vanishes. The proposition follows by taking the cup-product with $(\mu(g))$, since $[B] = (\mu(g)) \cup (\text{disc } \rho)$ by [9, Theorem A] (see also [4, (13.38)]). □

Remark 14. If m is odd in Proposition 13, then the definition of θ_2 requires the extra hypothesis that B is split by K . Computation then shows that $\theta_2(g \otimes 1) = 0$ for all $g \in \text{Sim}^-(B, \rho)$, as follows also from Remark 12.2 above.

Corollary 15. *Let r be an arbitrary integer. Let H be a quaternion F -algebra, $\alpha \in F^\times$, $K = F[X]/(X^2 - \alpha)$, and let ι be the nontrivial automorphism of K/F . Assume*

$$(-1) \cup [H] = 0 \text{ in } H^3(F, \mu_2) \quad \text{and} \quad (\alpha) \cup [H] \neq 0 \text{ in } H^3(F, \mu_2).$$

Let ρ be an orthogonal involution on $M_{2r}(H)$ which admits improper similitudes (see Lemma 1). Then $\mathbf{PSim}(M_{2r}(H) \otimes_F K, \rho \otimes \iota)$ is not R-trivial.

As in the previous section (see Corollary 9), alternative examples can be constructed from Proposition 2:

Corollary 16. *Let (B, ρ) be a central simple algebra of degree 4 with orthogonal involution over a field F_0 . Assume B is not split and $\text{disc } \rho \neq 1$. Let $F = F_0(x)$ be the field of rational fractions in one indeterminate over F_0 , let $K = F(\sqrt{x})$ and let ι be the nontrivial automorphism of K/F . The group $\mathbf{PSim}(B \otimes_{F_0} K, \rho \otimes \iota)$ is not R-trivial.*

Note that this corollary also follows from [7, Theorem 3].

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