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Spectral multipliers for sub-Laplacians on amenable Lie groups with exponential volume growth

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Introduction

Let G be a Lie group and Δ a left invariant sub-Laplacian on G. If $L^2(G)$ denotes the space of square integrable functions with respect to the right invariant Haar measure on G, then Δ is a selfadjoint operator on $L^2(G)$. Therefore every bounded Borel function f on $\mathbb{R}$ induces a continuous operator $f(\Delta)$ on $L^2(G)$.

It is now natural to ask, under which additional conditions on f the operator $f(\Delta)$ is necessarily bounded on $L^p(G)$, $p \neq 2$. In this case we call f an *L^p-multiplier for* Δ. For more background information and various multiplier theorems we refer to [1], [3], [2], [5], [10], [8] and the literature mentioned therein.

Here we focus our attention on amenable groups with exponential volume growth and continuous functions f with compact support. Our aim is to show for a reasonably large class of Lie groups and sub-Laplacians that a certain degree of differentiability of f is sufficient for $f(\Delta)$ to extend to a bounded operator on $L^p(G)$, i. e. that Δ has *differentiable L^p-functional calculus*.

That this is not true on any group with exponential growth (in contrast to the situation on Lie groups with polynomial growth, cf. [1]), was shown by M. Christ and D. Müller in [2]. They gave examples of sub-Laplacians Δ on solvable Lie groups, which are for any $p \neq 2$ of *holomorphic L^p-type*, i. e., there exists some non-isolated point λ in the L^2-spectrum of Δ and an open complex neighbourhood U of λ in $\mathbb{C}$ such that every continuous L^p-multiplier, which vanishes at infinity, extends holomorphically to U. (More recent articles dealing with this topic are [8] and [7].)

Therefore it is interesting to study new classes of groups and sub-Laplacians, to find out whether they admit differentiable L^p-functional calculus or not.

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In Part I of this article we consider compact extensions of a class of solvable Lie groups. We modify and extend some methods, which were used in [6], to show that any sub-Laplacian on these groups possesses differentiable L^p-functional calculus for each $p \in [1, \infty]$.

In Part II we turn to some semidirect products of the 3-dimensional Heisenberg group $\mathbb{H}_1$ and the real line and study distinguished sub-Laplacians thereon. In fact, up to some exceptional cases, the groups and operators here are treated in Part I as well, namely when K is chosen to be the trivial compact group $\{1_K\}$. But in Part II we use different methods (introduced in [5] and again employed in [10]) to derive differentiable L^p-functional calculus. From the quantitative point of view our results here are better than the results about these special sub-Laplacians in Part I.

Part I: Compact extensions of solvable groups

Preliminaries

Let $\mathfrak{n}$ be a real m-dimensional nilpotent Lie algebra, and let N be $\mathfrak{n}$, endowed with the Campbell-Hausdorff multiplication. Then, up to isomorphism, N is the uniquely determined connected and simply connected nilpotent Lie group whose Lie algebra is $\mathfrak{n}$. Although the exponential map $\exp_N$ of N is in fact the identity on $\mathfrak{n}$, we will use the notation $\exp_N$ to make a clear distinction between the levels of Lie group and Lie algebra.

Let D be a derivation on $\mathfrak{n}$ with eigenvalues $\lambda_i, i = 1, \ldots, q$, whose real parts ρ_i are all strictly positive (or all strictly negative). We define ρ to be the real part ρ_i, which has the smallest absolute value. The trace of D will be denoted by Q. If D is diagonalizable over the field of complex numbers, we shall say that D is *semisimple*.

Let $\theta : \mathbb{R} \to \mathrm{Aut}(\mathfrak{n}) = \mathrm{Aut(N)}$ be the group homomorphism defined by $\theta(s) = e^{sD}$. Thus we can consider the semidirect product $H := N \rtimes_\theta \mathbb{R}$.

Furthermore, let K be a connected compact Lie group and $\gamma : K \to \mathrm{Aut(H)}$ a group homomorphism such that the mapping

$$H \times K \to H\,, (h, k) \mapsto \gamma(k)h$$

is analytic. The group of main interest in this section is $G := H \rtimes_\gamma K$.

The measures we would like to consider, are the Lebesgue measures on N and $\mathbb{R}$, dn and dr, as well as the biinvariant Haar measure dk on K. For simplicity we may assume $dk(K) = 1$. The Lie algebra of the group K is denoted by $\mathfrak{k}$.

A right invariant Haar measure on H is given by $d^r h := dn\,dr$, and the measure $d^l h := e^{-rQ}\,dn\,dr$ is left invariant. We shall denote by μ the modular factor $\mu(n,r) := e^{rQ}$. It is easy to verify that $d^r g := dn\,dr\,dk$ is a right invariant Haar measure on G. As K is compact, the modular function m on G is given by $m(n,r,k) = \mu(n,r)$. Hence the left invariant Haar measure on G is of the form $d^l g := e^{-rQ}\,dn\,dr\,dk$. For $p \in [1, \infty]$ let $L^p(G) := L^p(G, d^r g)$.

Since the modular functions μ and m are not trivial, the groups H and G are of exponential volume growth (cf. [11], §IX.1).

Let $\mathcal{Y}_1, \dots, \mathcal{Y}_p$ be left invariant vector fields on G, which generate the Lie algebra $\mathfrak{g}$ of G. We are interested in the sub-Laplacian

$$\Delta = -\sum_{j=1}^{p} \mathcal{Y}_j^2 \tag{1}$$

and its heat kernel $(\phi_z)_{z \in \mathcal{H}_r}$, where $\mathcal{H}_r$ denotes the right open complex halfplane. The heat kernel is defined by $e^{-z\Delta} f = f * \phi_z$ for all $f \in C_c^\infty(G)$.

Results

In the situation described above the following two theorems hold:

Theorem 1. *For any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for each $s \in \mathbb{R}$*

$$\|\phi_{1+is}\|_{L^1(G)} \leq C_\varepsilon (1 + |s|)^{\frac{Q}{2\rho} + 2 + \varepsilon} . \tag{2}$$

If D is semisimple, there exists a $C_0 > 0$ such that for all $s \in \mathbb{R}$

$$\|\phi_{1+is}\|_{L^1(G)} \leq C_0 (1 + |s|)^{\frac{Q}{2\rho} + 2} . \tag{3}$$

Theorem 2. *Let $f \in C_c(\mathbb{R})$, $\kappa > \frac{Q}{2\rho} + \frac{5}{2}$ and $p \in [1, \infty]$. If f lies in the Sobolev space $H^\kappa(\mathbb{R})$, the operator $f(\Delta)$ extends to a bounded endomorphism on $L^p(G)$, given by convolution from the right with the $L^1(G)$-function*

$$k_f := \frac{1}{2\pi} \int_{\mathbb{R}} (f \cdot \exp)^\wedge(\xi) \phi_{1-i\xi} \, d\xi .$$

Theorem 2 follows directly from Theorem 1: With Theorem 1 and the Cauchy-Schwartz inequality it is easily verified, that for $f \in H^\kappa(\mathbb{R})$ the function k_f is integrable on G. The Fourier inversion formula implies for all $\varphi \in L^p \cap L^2(G)$

$$f(\Delta)\varphi = \frac{1}{2\pi} \int_{\mathbb{R}} (f \cdot \exp)^\wedge(\xi) e^{-(1-i\xi)\Delta} \varphi \, d\xi ,$$

so we obtain $f(\Delta)\varphi = \varphi * k_f$ and $\|f(\Delta)\varphi\|_{L^p(G)} \leq \|k_f\|_{L^1(G)} \|\varphi\|_{L^p(G)}$.

Remark 3. Consider the special case, where K is the trivial group $\{1_K\}$, N a stratified group and $\mathbb{R}$ is acting on N by the natural dilations. More precisely, let $V_i, i = 1, \dots, q$, be vector spaces with $\mathfrak{n} = V_1 \oplus \cdots \oplus V_q$ and $[V_i, V_j] = V_{i+j}$ (with the convention $V_l = \{0\}$ for $l > q$), and let $Dv_j = jv_j$ for each $v_j \in V_j$.

In this situation Theorem 1 and 2 were proved in [6] (with slight restrictions on the form of the considered sub-Laplacians). In the Section *Improvements and open problems* of that article W. Hebisch mentioned that his results can be extended to any semidirect product H, defined as in our preceding section. So in the case $K = \{1_K\}$ our proof of Theorem 1 serves as a rigorous verification of the statement made by W. Hebisch.

When K and γ are non-trivial our results are new.

Proof of Theorem 1

If all ρ_i are strictly negative, the mapping $\tau(n,r,k) = (n,-r,k)$ is a group isomorphism between G and $\tilde{G} := (N \rtimes_{\tilde{\theta}} \mathbb{R}) \rtimes_{\tilde{\gamma}} K$ with $\tilde{\theta}(r) = e^{r(-D)}$ and $\tilde{\gamma} = \tau \circ \gamma \circ \tau^{-1}$. The operator $\tilde{\Delta} := d\tau(\Delta)$ is a sub–Laplacian on $\tilde{G}$ and its heat kernel is given by $\tilde{\phi}_z = \phi_z \circ \tau$, which implies $\|\phi_z\|_{L^1(G)} = \|\tilde{\phi}_z\|_{L^1(\tilde{G})}$.

So we just have to prove the case, where all real parts ρ_i, $i = 1, \ldots, q$, are strictly positive. We shall reduce the L^1-estimate of the heat kernel to a weighted L^2-estimate in Proposition 6. But previously, we have to define a reasonable weight function w and to prove two preparatory lemmas.

Let $|\cdot|_D$ be a *homogeneous norm on N with respect to D*, i. e., a continuous mapping $|\cdot|_D : N \to [0,\infty[$, which is smooth away from the origin and which fulfils the conditions $|x|_D = 0$ iff $x = 0$, $|-x|_D = |x|_D$ and $|e^{sD}x|_D = e^s|x|_D$. (Such a homogeneous norm exists iff all the ρ_i, $i = 1, \ldots, q$, are strictly positive; see e. g. [4], §2.5.) F_s shall denote the compact smooth surface $\{n \in N : |n|_D = e^s\}$. The weight function $w : G \to [0,\infty[$ is defined by $w(n,r,k) = |n|_D^Q$.

We consider a left invariant Riemannian metric d on G. Let 1_G be the unit element in G. Then we define $d(g)$ to be $d(1_G, g)$ for any $g \in G$.

Lemma 4. *There is a constant $C > 0$ such that for all $g = (n,r,k) \in G$*

$$|n|_D \le Ce^{Cd(g)} \quad \textit{and} \quad |r| \le C(1 + d(g)) .$$

Proof. Let B_r denote the Riemannian ball in G with centre 1_G and radius r. As its closure $\overline{B}_r$ is compact, there are q, $p \in \mathbb{N}$ with

$$\overline{B}_1 \subset (B_q \cap H) \times K \quad \text{and} \quad \gamma(K)(\overline{B}_q \cap H) \subset B_p \cap H .$$

If $g_0 = (h_0, k_0)$ is in B_j, there exist $g_i = (h_i, k_i) \in B_1$, $i = 1, \ldots, j$, with

$$g_0 = g_1 \cdot \ldots \cdot g_j = (h_1 \cdot \gamma(k_1)h_2 \cdot \ldots \cdot \gamma(k_1 \cdot \ldots \cdot k_{j-1})h_j, k_1 \cdot \ldots \cdot k_j) .$$

Thus $h_0 = (n_0, r_0)$ is in $(B_p \cap H)^j$, and we can find $h_i' = (n_i, r_i) \in B_p \cap H$, $i = 1, \ldots, j$, with

$$h_0 = h_1' \cdot \ldots \cdot h_j' = (n_1 \cdot e^{r_1 D} n_2 \cdot \ldots \cdot e^{(r_1 + \ldots + r_{j-1})D} n_j, r_1 + \ldots + r_j) .$$

There is a $C > 0$ with $\overline{B}_p \cap H \subset \{(n,r) : |n|_D \le C, |r| \le C\}$, which implies $|r_0| \le Cj$. For all $n', m' \in N$ we have $|n' \cdot m'|_D \le M \cdot \max\{|n'|_D, |m'|_D\}$, where $M := \max\{|n \cdot m|_D : |n|_D, |m|_D \le 1\}$. Hence

$$|n_0|_D \le M^{j-1} \max\{|n_1|_D, \ldots, e^{r_1 + \ldots + r_{j-1}}|n_j|_D\} \le CM^{j-1}e^{C(j-1)} .$$

□

Lemma 5. *There exists a $C > 0$ such that for each $R > 0$*

$$\int_{d(g)<R} (1 + w(g))^{-1} d^r g \le C(1 + R)^2 . \tag{4}$$

Proof. Lemma 4 ensures the existence of a constant $C > 0$ independent of $g = (n, r, k)$ such that $|n|_D \leq Ce^{Cd(g)}$ and $2|r| \leq C(1 + d(g))$ are satisfied. That implies

$$\int_{d(g)<R} \frac{d^r g}{1 + w(g)} \leq C(1 + R) \int_{|n|_D \leq Ce^{CR}} \frac{dn}{1 + |n|_D^Q} .$$

For the sake of simplicity we confine our analysis to the situation, where F_0 can be parametrized up to a set with surface measure zero by one chart $\varphi : U \to \mathbb{R}^m = \mathfrak{n}$. Here U is an open subset in $\mathbb{R}^{m-1}$. Let $R' := (CR + \ln(C))/Q$. The mapping $e^{sD} \circ \varphi$ is a parametrization of F_s and

$$\Phi : U \times]-\infty, R'[\to \{n \in N \setminus \{0\} : |n|_D < e^{R'Q}\} , (u, s) \mapsto e^{sD}(\varphi(u))$$

is a diffeomorphism onto the range of Φ with Jacobian determinant

$$e^{sQ} |\det(\partial_u \varphi(u), D(\varphi(u)))| .$$

With a suitable $C_0 > 0$ we obtain

$$\begin{aligned} \int_{|n|_D \leq e^{R'Q}} \frac{dn}{1 + |n|_D^Q} &= \int_{-\infty}^{R'} \Big(\int_U \frac{e^{sQ} |\det(\partial_u \varphi(u), D(\varphi(u)))|}{1 + e^{sQ}} \, du \Big) \, ds \\ &= C_0 \ln(1 + e^{R'Q}) . \end{aligned}$$

□

Proposition 6. *There exists a $C > 0$ such that for every $s \in \mathbb{R}$*

$$\|\phi_{1+is}\|_{L^1(G)} \leq C(1 + |s|)^2 (1 + \|w^{1/2} \phi_{1+is}\|) . \tag{5}$$

Here and in the sequel, $\| \cdot \|$ shall denote the norm on $L^2(G) = L^2(G, d^r g)$.

Because of inequality 4, the argument from [5], p. 160 (or [6], p. 438 – 439) can be used to prove Proposition 6. Consequently, in order to prove Theorem 1 it suffices to verify the following proposition:

Proposition 7. *For given $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for each $s \in \mathbb{R}$*

$$\|w^{1/2} \phi_{1+is}\| \leq C_\varepsilon (1 + |s|)^{\frac{Q}{2\rho} + \varepsilon} . \tag{6}$$

If D is semisimple, then there exists a $C_0 > 0$, independent of s, with

$$\|w^{1/2} \phi_{1+is}\| \leq C_0 (1 + |s|)^{\frac{Q}{2\rho}} . \tag{7}$$

For the proof of Proposition 7 it is useful to consider a distinguished basis of the Lie algebra $\mathfrak{g}$. Let $\{\mathcal{X}_1, ..., \mathcal{X}_m\}$ be a basis of $\mathfrak{n}$, $\mathcal{X}_0 = (0, 1, 0) \in \mathfrak{n} \times \mathbb{R} \times \mathfrak{k}$ and $\{\mathcal{X}_{-1}, ..., \mathcal{X}_{-n}\}$ a basis of $\mathfrak{k}$. These Lie algebra elements induce left invariant vector fields on G by

$$\mathcal{X}_j f(g) = \frac{d}{dt} f(g \cdot \exp_G(t\mathcal{X}_j))|_{t=0}\,, \quad j = -n, ..., 0, ..., m\,,$$

which we will identify with the Lie algebra elements themselves. If $j = 0, ..., m$, we can also consider left invariant vector fields $\mathcal{X}_j^H$ on H, defined by

$$\mathcal{X}_j^H \varphi(h) = \frac{d}{dt} \varphi(h \cdot \exp_H(t\mathcal{X}_j))|_{t=0}\,.$$

Analogously, we define for $j = 1, ..., m$ and a function ψ on N

$$\mathcal{X}_j^N \psi(n) = \frac{d}{dt} \psi(n \cdot \exp_N(t\mathcal{X}_j))|_{t=0}\,.$$

Then the vector fields $\mathcal{X}_j^H$, $j = 0, ..., m$, on H are given by

$$\mathcal{X}_0^H = \partial_r \quad \text{and} \quad \mathcal{X}_i^H = (e^{rD}\mathcal{X}_i)^N \quad \text{for all } i = 1, ..., m. \tag{8}$$

For a given $k \in K$ let now $\tilde{\gamma}(k)$ be the uniquely determined linear mapping, which ensures commutativity in the diagram below:

$$\begin{array}{ccc} H & \xrightarrow{\gamma(k)} & H \\ {\scriptstyle \exp_H}\uparrow & & \uparrow{\scriptstyle \exp_H} \\ \mathfrak{h} & \xrightarrow[\tilde{\gamma}(k)]{} & \mathfrak{h} \end{array}$$

(A maybe more common notation for $\tilde{\gamma}(k)$ would be $d\gamma(k)$.) If $\tilde{\gamma}(k)$ will be represented as a matrix in the sequel, this is always meant with respect to the basis $\{\mathcal{X}_0, ..., \mathcal{X}_m\}$ of $\mathfrak{h}$. With this convention we obtain for each $j \in \{0, ..., m\}$ and $n \in N \setminus \{0\}$, $r \in \mathbb{R}$, $k \in K$

$$(\mathcal{X}_j w)(n, r, k) = \sum_{i=1}^{m} \tilde{\gamma}(k)_{i,j} [(e^{rD}\mathcal{X}_i)^N w](n)\,. \tag{9}$$

The following statement can be calculated directly by transforming D into complex Jordan normal form: There are functions $P_{i,l,\nu} : \mathbb{R} \to \mathbb{C}$, $i, l = 1, ..., m$; $\nu = 1, ..., q$, and constants $C, \mu > 0$ independent of $s \in \mathbb{R}$, satisfying

$$e^{sD}\mathcal{X}_i = \sum_{l=1}^{m} \Big(\sum_{\nu=1}^{q} e^{s\rho_\nu} P_{i,l,\nu}(s) \Big) \mathcal{X}_l \quad \textit{and} \quad |P_{i,l,\nu}(s)| \le C(1 + |s|^{\mu})\,. \tag{10}$$

If D is semisimple, each $P_{i,l,\nu}$ can be chosen as a bounded function.

Now we are going to establish a proposition, which is essential for our approach to the proof of Proposition 7:

Proposition 8. *For any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for all $i \in \{1, ..., p\}$, $n \in N \setminus \{0\}$, $r \in \mathbb{R}$ and $k \in K$ the following inequality holds:*

$$|\mathcal{Y}_i w|(n, r, k) \le C_\varepsilon \sum_{\nu=1}^{q} \left\{ e^{r(\rho_\nu+\varepsilon)} w(n)^{\frac{Q-\rho_\nu-\varepsilon}{Q}} + e^{r(\rho_\nu-\varepsilon)} w(n)^{\frac{Q-\rho_\nu+\varepsilon}{Q}} \right\}. \tag{11}$$

If D is semisimple, one can estimate

$$|\mathcal{Y}_i w|(n, r, k) \le C_0 \sum_{\nu=1}^{q} e^{r\rho_\nu} w(n)^{\frac{Q-\rho_\nu}{Q}} , \tag{12}$$

with a constant $C_0 > 0$ independent of i, n, r and k.

Proof. For $i = -n, ..., -1$ the functions $\mathcal{X}_i w$ are identically zero, as w does not depend on the variable k.

Using formula (10), we get for $n \in F_0$ and $i = \{1, ..., m\}$

$$\begin{aligned}
[(e^{rD}\mathcal{X}_i)^N w](e^{sD}n) &= e^{sQ} \frac{d}{dt} w(n \cdot \exp_N(te^{(r-s)D}\mathcal{X}_i))|_{t=0} \\
&= e^{sQ}(e^{(r-s)D}\mathcal{X}_i)^N w(n) = \sum_{\nu=1}^{q} e^{sQ+(r-s)\rho_\nu} \sum_{l=1}^{m} P_{i,l,\nu}(r-s)(\mathcal{X}_l^N w)(n) .
\end{aligned}$$

Hence there exists for any $\varepsilon > 0$ a constant $c_\varepsilon > 0$, fulfilling

$$|(e^{rD}\mathcal{X}_i)^N w|(e^{sD}n) \le c_\varepsilon \sum_{\nu=1}^{q} e^{sQ+(r-s)\rho_\nu+|r-s|\varepsilon} \sum_{l=1}^{m} |\mathcal{X}_l^N w|(n) . \tag{13}$$

Now define $C := c_\varepsilon \max\{\sum_{l=1}^{N} |(\mathcal{X}_l^N w)|(n) : n \in F_0\}$. Because of $w|_{F_0} \equiv 1$, there holds for $n \in F_0$, $r \in \mathbb{R}$

$$\begin{aligned}
&|(e^{rD}\mathcal{X}_i)^N w|(e^{sD}n) \\
&\quad \le Ce^{sQ} \sum_{\nu=1}^{q} \left\{ e^{(r-s)(\rho_\nu+\varepsilon)} w(n)^{\frac{Q-\rho_\nu-\varepsilon}{Q}} + e^{(r-s)(\rho_\nu-\varepsilon)} w(n)^{\frac{Q-\rho_\nu+\varepsilon}{Q}} \right\} \\
&\quad = C \sum_{\nu=1}^{q} \left\{ e^{r(\rho_\nu+\varepsilon)} w(e^{sD}n)^{\frac{Q-\rho_\nu-\varepsilon}{Q}} + e^{r(\rho_\nu-\varepsilon)} w(e^{sD}n)^{\frac{Q-\rho_\nu+\varepsilon}{Q}} \right\} . \qquad (14)
\end{aligned}$$

We have $N \setminus \{0\} = \{e^{sD}F_0 : s \in \mathbb{R}\}$. Furthermore, the functions $\tilde{\gamma}_{i,j}$ are bounded on K and the $\mathcal{Y}_i$s are linear combinations of the $\mathcal{X}_j s$. Thus formulae (9) and (14) imply inequality (11). If D is semisimple, (13) can be simplified to

$$|(e^{rD}\mathcal{X}_i)^N w|(e^{sD}n) \le c_0 \sum_{\nu=1}^{q} e^{sQ+(r-s)\rho_\nu} \sum_{l=1}^{m} |\mathcal{X}_l^N w|(n)$$

with a suitable constant $c_0 > 0$. Therefore inequality (12) follows. □

To simplify the proof of Proposition 7, we state two preliminary lemmas:

Lemma 9. *For any $\delta > 0$ there exists a $C > 0$ such that for each $j \in \{1, ..., p\}$ and each $z \in \mathbb{C}$ with $\Re(z) \geq \delta$ the inequality $\|e^{r\frac{Q}{2}} \mathcal{Y}_j \phi_z\| \leq C$ holds; here $e^{r\frac{Q}{2}}$ denotes the function $(n, r, k) \mapsto e^{r\frac{Q}{2}}$.*

Proof. Let $z \in \mathbb{C}$ with $\Re(z) \geq \delta$. If $\mathcal{Y}_j$ has the form $\mathcal{Y}_j = \sum_{i=-n}^{m} a_{i,j} \mathcal{X}_i$ with $a_{i,j} \in \mathbb{R}$, then, by using the notation $\eta_j := \sum_{i=0}^{m} a_{i,j} \tilde{\gamma}_{0,i}$, we get

$$\langle \Delta\phi_z, e^{rQ}\phi_z \rangle = \sum_{j=1}^{p} (\|e^{r\frac{Q}{2}} \mathcal{Y}_j \phi_z\|^2 + \langle \mathcal{Y}_j \phi_z, \eta_j Q e^{rQ} \phi_z \rangle) .$$

The Cauchy-Schwarz inequality and the fact that ϕ_z solves the homogeneous heat equation with respect to Δ imply

$$\begin{aligned} \sum_{j=1}^{p} \|e^{r\frac{Q}{2}} \mathcal{Y}_j \phi_z\|^2 &\leq |\langle e^{r\frac{Q}{2}} \Delta\phi_z, e^{r\frac{Q}{2}} \phi_z \rangle| + \sum_{j=1}^{p} |\eta_j|_\infty Q \langle |e^{r\frac{Q}{2}} \mathcal{Y}_j \phi_z|, |e^{r\frac{Q}{2}} \phi_z| \rangle \\ &\leq \|e^{r\frac{Q}{2}} \partial_z \phi_z\| \|e^{r\frac{Q}{2}} \phi_z\| + \sum_{j=1}^{p} \frac{1}{2} \big(\|e^{r\frac{Q}{2}} \mathcal{Y}_j \phi_z\|^2 + |\eta_j|_\infty^2 Q^2 \|e^{r\frac{Q}{2}} \phi_z\|^2 \big) . \end{aligned}$$

From $(e^{-z\Delta})^* = e^{-z^*\Delta}$ follows $\phi_z(g^{-1}) = m(g)\phi_z(g)$. As the modular function m is given by e^{rQ}, we get $\|e^{r\frac{Q}{2}} \phi_z\| = \|\phi_z\|$. Since $\phi_z = e^{-(z-\delta)\Delta}\phi_\delta$, $\|e^{r\frac{Q}{2}} \phi_z\| \leq \|\phi_\delta\|$ holds. By using Cauchy's formula, it is easy to verify that

$$\|e^{r\frac{Q}{2}} \partial_z \phi_z\| \leq \frac{2}{\delta} \sup\{\|e^{r\frac{Q}{2}} \phi_\zeta\| : |z - \zeta| < \tfrac{\delta}{2}\} .$$

□

Lemma 10. *Let $j \in \{1, ..., p\}$, $\delta > 0$, $\eta > 0$ and $\tilde{C} > 0$.*

(i) If $Q \geq 2\eta$, there exists a $C > 0$ such that for $\alpha > 0$ and z with $\Re(z) \geq \delta$

$$\langle |\mathcal{Y}_j \phi_z|, e^{r\eta} w^{\frac{Q-\eta}{Q}} |\phi_z| \rangle \leq \frac{\alpha}{\tilde{C}} \|w^{1/2} \mathcal{Y}_j \phi_z\|^2 + \frac{C}{\alpha} \|w^{1/2} \phi_z\|^{\frac{2Q-4\eta}{Q}} . \quad (15)$$

(ii) If $Q < 2\eta < 2Q$, there exists a $C > 0$ with

$$\langle |\mathcal{Y}_j \phi_z|, e^{r\eta} w^{\frac{Q-\eta}{Q}} |\phi_z| \rangle \leq \frac{\alpha}{\tilde{C}} \|w^{1/2} \mathcal{Y}_j \phi_z\|^2 + C \alpha^{\frac{\eta-Q}{\eta}} \quad (16)$$

for all $\alpha > 0$ and z with $\Re(z) \geq \delta$.

Proof. Consider $\alpha, \tilde{C}, \eta, \delta > 0$. Let $j \in \{1, ..., p\}$ and z with $\Re(z) \geq \delta$.

(i) $Q \geq 2\eta$ implies

$$\langle |\mathcal{Y}_j\phi_z|, e^{r\eta} w^{\frac{Q-\eta}{Q}} |\phi_z|\rangle \leq \frac{\alpha}{\tilde{C}} \|w^{1/2}\mathcal{Y}_j\phi_z\|^2 + \frac{\tilde{C}}{\alpha} \|e^{r\eta} w^{\frac{Q-2\eta}{2Q}} \phi_z\|^2 .$$

By using Hölder's inequality, one gets

$$\|e^{r\eta} w^{\frac{Q-2\eta}{2Q}} \phi_z\| \leq \|e^{r\frac{Q}{2}} \phi_z\|^{\frac{2\eta}{Q}} \|w^{1/2}\phi_z\|^{\frac{Q-2\eta}{Q}} \leq \|\phi_\delta\|^{\frac{2\eta}{Q}} \|w^{1/2}\phi_z\|^{\frac{Q-2\eta}{Q}} .$$

(ii) Let $Q < 2\eta < 2Q$. By using Hölder's inequality with exponents $p = Q/2(Q-\eta)$ and $p' = Q/(2\eta - Q)$, we can estimate

$$\begin{aligned}\langle |\mathcal{Y}_j\phi_z|, e^{r\eta} w^{\frac{Q-\eta}{Q}} |\phi_z|\rangle &\leq \|e^{r\frac{Q}{2}}\phi_z\| \|e^{r(\eta-\frac{Q}{2})} w^{\frac{Q-\eta}{Q}} \mathcal{Y}_j\phi_z\| \\ &\leq \|\phi_\delta\| \|e^{r(\eta-\frac{Q}{2})} w^{\frac{Q-\eta}{Q}} \mathcal{Y}_j\phi_z\| \leq \|\phi_\delta\| \|w^{1/2}\mathcal{Y}_j\phi_z\|^{\frac{2(Q-\eta)}{Q}} \|e^{\frac{rQ}{2}} \mathcal{Y}_j\phi_z\|^{\frac{2\eta-Q}{Q}} .\end{aligned}$$

As there exists a constant $C > 0$ with $\|e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z\| \leq C$ (cf. Lemma 9) and as the inequality $ab \leq a^r + b^{r'}$, $r' = \frac{r}{r-1}$ holds for all $a, b > 0$ and $r > 1$, we obtain inequality (16) with $r = Q/(Q-\eta)$ and a suitable $C > 0$. □

Proof of Proposition 7. Lemma 4 states the existence of a $C > 0$ with $w(g) \leq Ce^{Cd(g)}$, and $(e^{-z\Delta})_{\Re(z)>0}$ is a holomorphic semigroup of operators on each weighted L^2-space $L^2(G, e^{sd(g)}\, d^r g)$, $s \in \mathbb{R}$ (cf. [5], Lemma 1.2). Hence $z \mapsto w^{1/2}\phi_z$ is a holomorphic mapping from the right open complex halfplane into $L^2(G)$. Therefore there exists a $C > 0$ such that $\|w^{1/2}\phi_{1+is}\| \leq C$ for each $s \in [0, 1]$. Since $\phi_{1-is} = (\phi_{1+is})^*$, we have to consider only the case where $s \geq 1$. For $0 < \alpha \leq 1$ we define

$$\psi_\alpha(s) := \|w^{1/2}\phi_{\frac{1}{2}+(i+\alpha)s}\|^2 .$$

Further we define $z := \frac{1}{2} + (i+\alpha)s$. Using this notation, we obtain

$$\begin{aligned}\partial_s\psi_\alpha(s) &= 2\Re\langle (i+\alpha)\partial_z\phi_z, w\phi_z\rangle = -2\Re\big((i+\alpha)\langle \Delta\phi_z, w\phi_z\rangle\big) \\ &\leq 2\sum_{j=1}^{p} \big(-\alpha\|w^{1/2}\mathcal{Y}_j\phi_z\|^2 + 2\langle |\mathcal{Y}_j\phi_z|, |\mathcal{Y}_j w||\phi_z|\rangle\big) . \qquad (17)\end{aligned}$$

Case (1): $\dim \mathfrak{n} = 1$. Here, D is given by the 1×1-matrix (Q). Estimate (12) leads us to the inequality $|\mathcal{Y}_j w|(n, r, k) \leq C_0 e^{rQ}$ for $j \in \{1, ..., p\}$. Hence

$$\langle |\mathcal{Y}_j\phi_z|, |\mathcal{Y}_j w||\phi_z|\rangle \leq C_0 \|e^{r\frac{Q}{2}}\mathcal{Y}_j\phi_z\| \|e^{r\frac{Q}{2}}\phi_z\| \leq C ,$$

with $C > 0$ independent of s, α (cf. Lemma 9). Thus there exists a $C > 0$ such that $\partial_s\psi_\alpha(s) \leq C$ for all s and α. That implies (with $\alpha := \frac{1}{2s}$)

$$\|w^{1/2}\phi_{1+is}\| \leq C\sqrt{1+|s|} \quad \text{for } s \geq 1.$$

Case (2): $\dim \mathfrak{n} \geq 2$. For $j = 1, ..., p$ and $\varepsilon > 0$ we get from Proposition 8

$$\begin{aligned} &\langle |\mathcal{Y}_j \phi_z|, |\mathcal{Y}_j w||\phi_z|\rangle \\ &\leq C_\varepsilon \sum_{\nu=1}^{q} \langle |\mathcal{Y}_j \phi_z|, (e^{r(\rho_\nu+\varepsilon)} w^{\frac{Q-(\rho_\nu+\varepsilon)}{Q}} + e^{r(\rho_\nu-\varepsilon)} w^{\frac{Q-(\rho_\nu-\varepsilon)}{Q}})|\phi_z|\rangle . \end{aligned} \tag{18}$$

(If D is semisimple, we can exchange ε by 0 in (18) and in the rest of this proof.) For every $\varepsilon < \rho$, $\rho_\nu + \varepsilon$ fulfills $Q > \rho_\nu + \varepsilon$ for any $\nu \in \{1, ..., q\}$. According to (17), (18), (15) and (16) (with $\tilde{C} := 4qC_\varepsilon$), $\partial_s \psi_\alpha$ is majorized by a sum over $\eta \in \{\rho_\nu \pm \varepsilon : \nu = 1, ..., q\}$, consisting of terms of the form

$$\frac{C}{\alpha} \|w^{1/2}\phi_z\|^{\frac{2Q-4\eta}{Q}} = \frac{C}{\alpha} \psi_\alpha^{\frac{Q-2\eta}{Q}} \qquad \text{for } Q \geq 2\eta$$

and

$$C\alpha^{\frac{\eta-Q}{\eta}} \leq \frac{C}{\alpha} \qquad \text{for } Q < 2\eta < 2Q.$$

Hence, there exists a $C > 0$ such that for all $\alpha \in]0, 1]$, $s \geq 1$ the function ψ_α is majorized by the solution u of the initial value problem

$$u'(s) = \frac{C}{\alpha}(1 + u(s))^{\frac{Q-2(\rho-\varepsilon)}{Q}} , \quad u(1) = \psi_\alpha(1) ,$$

which is given by

$$u(s) = \Big(\frac{2(\rho - \varepsilon)C}{Q\alpha}(s - 1) + (1 + \psi_\alpha(1))^{\frac{2(\rho-\varepsilon)}{Q}}\Big)^{\frac{Q}{2(\rho-\varepsilon)}} - 1 .$$

Hence, for $\alpha = \frac{1}{2s}$ there exists a constant $c_\varepsilon > 0$, independent of s, with

$$\|w^{1/2}\phi_{1+is}\| = \sqrt{\psi_\alpha(s)} \leq c_\varepsilon(1 + |s|)^{\frac{Q}{2(\rho-\varepsilon)}} .$$

□

Part II: Semidirect products of the Heisenberg group $\mathbb{H}_1$ and the real axis

Motivation

Noteworthy about Theorem 1 and 2 is, that the exponents in (2), (3) and the exponent κ in Theorem 2 tend to infinity with the ratio Q/ρ. There are indications that this phenomenon might be a consequence of our method of proof and does not reflect any underlying mathematical reality.

In [10] e. g., groups of the form $H = \mathbb{R}^2 \rtimes_\theta \mathbb{R}$ with $\theta(t) = e^{tD}$, D any 2×2-matrix, were studied and for distinguished sub-Laplacians and their heat kernels the estimate

$$\|\phi_{1+i\xi}\|_{L^1(H)} \leq C(1 + |\xi|)^5 \tag{19}$$

was proven. (In some cases the estimates are better; the method which handled the most delicate case had been introduced in [5].) So the exponent of (19) is bounded, regardless of the action e^{tD}.

If we consider semidirect products $\mathbb{H}_1 \rtimes_\theta \mathbb{R}$ of the 3-dimensional Heisenberg group with the real axis, we can of course not expect such a result. The article [2] shows that not even all of this semidirect products admit differentiable L^p-functional calculus.

But if we confine ourselves to group homomorphisms θ, which are induced by derivations D in diagonal form with non-negative entries (or non-positive entries), we are able to derive an estimate like (19) with exponent 6 for all θ by transferring the methods from [5] and [10] to our situation.

Preliminaries

The *Heisenberg group* $\mathbb{H}_1$ is the set $\mathbb{R}^3$ endowed with the multiplication

$$(x, y, u)(x', y', u') = \left(x + x', y + y', u + u' + \frac{1}{2}(xy' - yx')\right).$$

The Lie algebra of $\mathbb{H}_1$ is the *Heisenberg algebra* $\mathfrak{h}_1$.

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \geq 0$, and let D be the derivation on $\mathfrak{h}_1$ defined by $D(p, q, t) = (\alpha p, \beta q, (\alpha + \beta)t)$. The trace Q of D is then equal to $2(\alpha + \beta)$. Here our object of interest is the group $G := \mathbb{H}_1 \rtimes_\theta \mathbb{R}$, where $\theta(r) = e^{rD}$.

As in Part I the right invariant Haar measure $d^r g$ is simply the Lebesgue measure $dx\,dy\,du\,dr$ on $\mathbb{R}^4$ and the modular function is $m(g) = m(x, y, u, r) = e^{2(\alpha+\beta)r}$.

The left invariant vector fields on G, induced by the Lie algebra elements $\mathcal{X}_1 := (1, 0, 0, 0)$, $\mathcal{X}_2 := (0, 1, 0, 0)$, $\mathcal{X}_3 := (0, 0, 1, 0)$ and $\mathcal{X}_0 := (0, 0, 0, 1)$ from $\mathfrak{g} = \mathfrak{h}_1 \times \mathbb{R}$, are explicitly given by

$$\begin{aligned} \mathcal{X}_1 &= e^{\alpha r}\left(\partial_x - \frac{1}{2} y \partial_u\right), & \mathcal{X}_2 &= e^{\beta r}\left(\partial_y + \frac{1}{2} x \partial_u\right), \\ \mathcal{X}_3 &= e^{(\alpha+\beta) r} \partial_u\,, & \mathcal{X}_0 &= \partial_r\,. \end{aligned}$$

The operator $\Delta_S := -\sum_{j=0}^{2} \mathcal{X}_j^2$ is a sub-Laplacian and $\Delta_L := -\sum_{j=0}^{3} \mathcal{X}_j^2$ a full Laplacian on G. Let ϕ_t^S and ϕ_t^L denote the heat kernels of Δ_S and Δ_L, respectively. Further let $J(\Delta_S) := \{0, 1, 2\}$ and $J(\Delta_L) := \{0, 1, 2, 3\}$. In the sequel Δ shall denote the sub-Laplacian Δ_S as well as the full Laplacian Δ_L, and $(\phi_t)_{t>0}$ shall denote the heat kernel of Δ.

Results

Theorem 11. *There exists a $C > 0$ such that for each $\xi \in \mathbb{R}$ the inequality*

$$\|\phi_{1+i\xi}\|_{L^1(G)} \leq C(1 + |\xi|)^\kappa \tag{20}$$

holds; hereby we have

$$\kappa = \begin{cases} \frac{\alpha+\beta}{\min\{\alpha,\beta\}} + 2 & \textit{if } \frac{\alpha}{\beta} \in [\frac{1}{3}, 3], \\ 6 & \textit{otherwise.} \end{cases} \tag{21}$$

Like Theorem 1 in the first part, Theorem 11 implies a multiplier theorem:

Theorem 12. *Let $p \in [1,\infty]$, $\varepsilon > 0$ and κ as in* (21). *Then each $f \in C_c \cap H^{\kappa+\frac{1}{2}+\varepsilon}(\mathbb{R})$ is an L^P-multiplier for Δ.*

Remark 13. (a) Extending the results of Theorem 11 and Theorem 12 to compact extensions $G \rtimes_\gamma K$ of G and sub-Laplacians $\Delta_K + d\gamma(\Delta)$, Δ_K a sub-Laplacian on K, is more or less trivial:
If $\alpha \neq \beta$, it can be shown that any homomorphism $\gamma : K \to \mathrm{Aut(G)}$ has to be trivial, i. e., $\gamma(k)$ is the identity on G for any $k \in K$. (One can e. g. calculate the entries of the matrix $\tilde{\gamma} = d\gamma$ successively.) But then our sub-Laplacian is of the form $\Delta_K + \Delta$ and its heat kernel is given by $\phi_z^K \otimes \phi_z$, ϕ_z^K the heat kernel of Δ_K.
If $\alpha = \beta \neq 0$, the extension of the results is contained in Part I.
If $\alpha = 0 = \beta$, then $G \rtimes_\gamma K$ has polynomial growth, so we refer to [1].
(b) If $\phi_z = \phi_z^S$ is the heat kernel of the sub-Laplacian and if $\alpha = \beta \neq 0$, Inequality (20) and Theorem 12 hold even with $\kappa = 3/2$ (cf. [9] or [4]).

Outline of the proof of Theorem 11

The general strategy for proving Theorem 11 is the same as in the proof of Theorem 1. That is, we want to reduce the L^1-estimate of the heat kernel to a weighted L^2-estimate. But this time we utilize a weight function w, which is independent of the action θ: We define $w : G \to \mathbb{R}$ by

$$w(x, y, u, r) = (1 + |x|)(1 + |y|)(1 + |u|)\,.$$

Then there exists a $C > 0$ such that for any $R > 0$

$$\int_{d(g)\leq R} w(g)^{-1}\, d^r g \leq C(1 + R)^4\,.$$

Again, by using the same argument as in [5], we are able to find a constant $C > 0$ such that for each $\xi \in \mathbb{R}$

$$\|\phi_{1+i\xi}\|_{L^1(G)} \leq C(1 + |\xi|)^4(1 + \|w^{1/2}\phi_{1+i\xi}\|)\,. \tag{22}$$

Therefore we are interested in estimating the term $|\partial_\xi \|w^{1/2}\phi_{1+i\xi}\|^2|$. We will do this step by step, beginning with weights of low order in x, y, u (like $|x|^{1/2}$ and $|y|^{1/2}$) and using the estimations of these terms to estimate higher order terms in x, y and u. We start with the analogue of Lemma 9:

Lemma 14. *For any $\delta > 0$ there exists a $C > 0$ such that for each $j \in J(\Delta)$ and each $z \in \mathbb{C}$ with $\Re(z) \geq \delta$ the inequality $\|e^{(\alpha+\beta)r}\mathcal{X}_j\phi_z\| \leq C$ holds.*

Lemma 14 can be verified easily by mimicking the proof of Lemma 9.

Lemma 15. *For any compact set $K \subset]0,\infty[$ there exists a $C > 0$ with*

$$\| \, |x|^{1/2} \phi_{\rho+i\xi} \| + \| \, |y|^{1/2} \phi_{\rho+i\xi} \| \le C(1+|\xi|)^{1/2} \tag{23}$$

and

$$\| \, |x|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \| + \| \, |y|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi} \| \le C(1+|\xi|)^{1/2} \tag{24}$$

for all $\rho \in K$, $\xi \in \mathbb{R}$ and all $j \in J(\Delta)$.

Proof. By using the notation $z = \rho + i\xi$ we get

$$\langle \Delta\phi_z, |y|\phi_z \rangle = \sum_{j \in J(\Delta)} \| \, |y|^{1/2} \mathcal{X}_j \phi_z \|^2 + \langle \mathcal{X}_2 \phi_z, e^{\beta r} \operatorname{sgn}(y) \phi_z \rangle \, .$$

On the one hand, from this and Lemma 14 there follows

$$\begin{aligned} |\partial_\xi \| \, |y|^{1/2} \phi_z \|^2| &= 2|\Re(i \langle \partial_z \phi_z, |y| \phi_z \rangle)| = 2|\Im \langle \Delta\phi_z, |y|\phi_z \rangle| \\ &\le 2|\langle \mathcal{X}_2 \phi_z, e^{\beta r} \operatorname{sgn}(y) \phi_z \rangle| \le 2\|\phi_z\| \, \|e^{\beta r} \mathcal{X}_2 \phi_z\| \le C \, . \end{aligned}$$

We obtain $\| \, |y|^{1/2} \phi_z \|^2 \le C(1+|\xi|)$, because the mapping $K \ni \rho \mapsto \| \, |y|^{1/2} \phi_\rho \|^2$ is in particular continuous and thus bounded.

On the other hand, it follows from $(\partial_z + \Delta)\phi_z = 0$ and Cauchy's formula that

$$\sum_{j \in J(\Delta)} \| \, |y|^{1/2} \mathcal{X}_j \phi_z \|^2 \le C + \| \, |y|^{1/2} \phi_z \| \, \| \, |y|^{1/2} \partial_z \phi_z \| \le C(1+|\xi|) \, .$$

The rest of the statement can be obtained analogously. □

Lemma 16. *For any compact set $K \subset]0,\infty[$ one can choose a $C > 0$ in such a way that for each $\rho \in K$, $\xi \in \mathbb{R}$ and each $j \in J(\Delta)$*

$$\|e^{(\frac{\alpha}{2}+\beta) r} |x|^{1/2} \phi_{\rho+i\xi}\| + \|e^{(\alpha+\frac{\beta}{2}) r} |y|^{1/2} \phi_{\rho+i\xi}\| \le C(1+|\xi|)^{1/2} \tag{25}$$

and

$$\|e^{(\frac{\alpha}{2}+\beta) r} |x|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi}\| + \|e^{(\alpha+\frac{\beta}{2}) r} |y|^{1/2} \mathcal{X}_j \phi_{\rho+i\xi}\| \le C(1+|\xi|)^{1/2} \, . \tag{26}$$

Proof. Let again $z := \rho + i\xi$. With the notation $\gamma(r) = \exp((\alpha + \frac{\beta}{2})r)$ we get $\|\gamma |y|^{1/2} \phi_{\rho+i\xi}\| = \| |y|^{1/2} \phi_{\rho+i\xi}\|$, because of $\phi_z(g^{-1}) = m(g)\phi_z(g)$. Further

$$\begin{aligned} \langle \Delta\phi_z, \gamma^2 |y| \phi_z \rangle = \sum_{j \in J(\Delta)} &\|\gamma |y|^{1/2} \mathcal{X}_j \phi_z\|^2 + \langle \mathcal{X}_2 \phi_z, m \operatorname{sgn}(y) \phi_z \rangle \\ &+ \langle \mathcal{X}_0 \phi_z, (2\alpha+\beta)\gamma^2 |y| \phi_z \rangle \, . \end{aligned}$$

Here the absolute value of the last term can be majorized by

$$\frac{1}{2}\|\gamma |y|^{1/2} \mathcal{X}_0 \phi_z\|^2 + \frac{(2\alpha+\beta)^2}{2} \|\gamma |y|^{1/2} \phi_z\|^2 \, .$$

Hence

$$\sum_{j\in J(\Delta)} \|\gamma|y|^{1/2}\mathcal{X}_j\phi_z\|^2 \le 2\|\gamma|y|^{1/2}\phi_z\|\,\|\gamma|y|^{1/2}\partial_z\phi_z\| + 2\|\sqrt{m}\phi_z\|\,\|\sqrt{m}\mathcal{X}_2\phi_z\| + (2\alpha+\beta)^2\|\gamma|y|^{1/2}\phi_z\|^2 \le C(1+|\xi|)\,.$$

□

Lemma 17. *For any compact set $K \subset]0,\infty[$ there exists a $C > 0$ with*

$$\|\,|u|^{1/2}\phi_{\rho+i\xi}\| + \|\,|u|^{1/2}\mathcal{X}_j\phi_{\rho+i\xi}\| \le C(1+|\xi|) \tag{27}$$

for every $\rho \in K$, $\xi \in \mathbb{R}$ and $j \in J(\Delta)$.

Proof. We consider just the case $\Delta = \Delta_L$. (The proof for the heat kernel of the sub-Laplacian Δ_S is contained in the proof for the Laplacian Δ_L – one has just to ignore all terms in which $\mathcal{X}_3$ occurs.)

With $z = \rho + i\xi$ we get

$$\langle \Delta_L\phi_z^L, |u|\phi_z^L\rangle = \sum_{j=0}^{3}\|\,|u|^{1/2}\mathcal{X}_j\phi_z^L\|^2 - \frac{1}{2}\langle \mathcal{X}_1\phi_z^L, e^{\alpha r}y\,\mathrm{sgn}(u)\phi_z^L\rangle + \frac{1}{2}\langle \mathcal{X}_2\phi_z^L, e^{\beta r}x\,\mathrm{sgn}(u)\phi_z^L\rangle + \langle \mathcal{X}_3\phi_z^L, e^{(\alpha+\beta)r}\,\mathrm{sgn}(u)\phi_z^L\rangle\,.$$

By proceeding as in the proof of (23) we obtain $|\partial_\xi\|\,|u|^{1/2}\phi_z^L\|^2| \le C(1+|\xi|)$, and from this follows $\|\,|u|^{1/2}\phi_{\rho+i\xi}^L\|^2 \le C(1+|\xi|)^2$. As in the proof of (24) one derives eventually $\sum_{j=0}^3\|\,|u|^{1/2}\mathcal{X}_j\phi_z^L\|^2 \le C(1+|\xi|)^2$. □

In Lemma 16 we got the same upper bound $C(1+|\xi|^{1/2})$ for the terms $\|e^{(\alpha+\frac{\beta}{2})r}|y|^{1/2}\phi_{\rho+i\xi}\|$ and $\|e^{(\alpha+\frac{\beta}{2})r}|y|^{1/2}\mathcal{X}_j\phi_{\rho+i\xi}\|$ (which we shall call *related terms of* $\||y|^{1/2}\phi_{\rho+i\xi}\|$) as for the terms $\||y|^{1/2}\phi_{\rho+i\xi}\|$ and $\||y|^{1/2}\mathcal{X}_j\phi_{\rho+i\xi}\|$ in Lemma 15. Similarly, we get from Lemma 17 the estimate

$$\|e^{\frac{\alpha+\beta}{2}r}|u|^{1/2}\phi_{\rho+i\xi}\| + \|e^{\frac{\alpha+\beta}{2}r}|u|^{1/2}\mathcal{X}_j\phi_{\rho+i\xi}\| \le C(1+|\xi|) \tag{28}$$

for the sum of the related terms of $\||u|^{1/2}\phi_{\rho+i\xi}\|$. The last inequality holds again for any compact $K \subset]0,\infty[$, $\rho \in K$, $\xi \in \mathbb{R}$, $j \in J$ and the constant C depends just on K.

By using the same notation and just the same techniques that we have used so far, we obtain

$$\|x\phi_{\rho+i\xi}\| + \|y\phi_{\rho+i\xi}\| \le C(1+|\xi|) \tag{29}$$

and this upper bound holds also for the related terms $\|e^{\beta r}x\phi_{\rho+i\xi}\|$, $\|e^{\alpha r}y\phi_{\rho+i\xi}\|$, $\|e^{\beta r}x\mathcal{X}_j\phi_{\rho+i\xi}\|$ and $\|e^{\alpha r}y\mathcal{X}_j\phi_{\rho+i\xi}\|$.

After this it is easy to derive that $\| |xu|^{1/2}\phi_{\rho+i\xi}\|$, $\| |yu|^{1/2}\phi_{\rho+i\xi}\|$ and the related terms are bounded by $C(1+|\xi|)^{3/2}$.

In a similar way one establishes

$$\|x|y|^{1/2}\phi_{\rho+i\xi}\| + \| |x|^{1/2} y\phi_{\rho+i\xi}\| \leq C(1+|\xi|)^{3/2} \tag{30}$$

(and this bound holds also for the related terms). With this bunch of estimates we are able to verify $\|w^{1/2}\phi_{1+i\xi}\| \leq C(1+|\xi|)^2$. Therefore, with regard to inequality (22) and Theorem 1, Theorem 11 holds.

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