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Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces

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Abstract. Using vertex algebra techniques, we determine a set of generators for the cohomology ring of the Hilbert schemes of points on an arbitrary smooth projective surface over the field of complex numbers.

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1. Introduction

The Hilbert scheme $X^{[n]}$ of points on a smooth projective surface X is a desingularization of the n-th symmetric product of X (see [Fog]). An element ξ in $X^{[n]}$ is a length-n 0-dimensional closed subscheme of X. Recently, there are two surprising discoveries, mainly due to the work of Göttsche [Go1], Nakajima [Na1,Na2] and Grojnowski [Gro], that the sum of the cohomology groups $\mathbb{H}_n = H^*(X^{[n]})$ with $\mathbb{Q}$-coefficients of the Hilbert schemes $X^{[n]}$ for $n \geq 0$ have relationships with modular forms on the one hand and with representations of infinite dimensional Heisenberg algebras on the other hand (see aslo the work of Vafa and Witten [V-W] for connections with string theory). These results have been used by Lehn [Leh] to investigate the relation between the Heisenberg algebra structure and the cup product structure of $\mathbb{H}_n$. In particular, Lehn constructed the Virasoro algebra

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in a geometric fashion and studied certain tautological sheaves over $X^{[n]}$. Some other recent work on Hilbert schemes includes [dCM, EGL, Go2, Hai, K-T, LQZ, Wan].

In this paper, by using vertex algebra techniques (see [Bor, FLM, Kac]) and generalizing the work of Nakajima, Grojnowski and Lehn [Na1, Gro, Na2, Leh], we study the cohomology ring structure of the Hilbert schemes $X^{[n]}$. We determine the ring generators of $H^*(X^{[n]})$ for an arbitrary smooth projective surface X over the field of complex numbers. In particular, we recover the result of Ellingsrud and Strømme [ES2] for $X = \mathbb{P}^2$. More precisely, we find a set of $(n \cdot \dim H^*(X))$ generators for the cohomology ring $\mathbb{H}_n$, and interpret the relations among these generators in terms of certain operators in $\mathrm{End}(\mathbb{H})$ where $\mathbb{H} = \bigoplus_{n \geq 0} \mathbb{H}_n$. Our results also clearly indicate that there are deep interplays between the geometry of Hilbert schemes and vertex algebra structures which go beyond the Heisenberg and Virasoro algebras.

To state our result, we establish some notations and refer the details to Definition 5.1. Let $\mathcal{Z}_n$ be the universal codimension-2 subscheme of $X^{[n]} \times X$, and p_1 and p_2 be the projections of $X^{[n]} \times X$ to $X^{[n]}$ and X respectively. For $\gamma \in H^s(X)$ and $n \geq 0$, let $G_i(\gamma, n)$ be the $H^{s+2i}(X^{[n]})$-component of

$$G(\gamma, n) \stackrel{\text{def}}{=} p_{1*}(\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n}) \cdot p_2^* \mathrm{td}(X) \cdot p_2^* \gamma) \in \mathbb{H}_n \tag{1.1}$$

(we refer to the Conventions at the end of this section for the conventions used in the paper). For $\gamma \in H^*(X)$ and $i \in \mathbb{Z}$, define an operator $\mathfrak{G}_i(\gamma) \in \mathrm{End}(\mathbb{H})$ which acts on the component $\mathbb{H}_n = H^*(X^{[n]})$ by the cup product by the class $G_i(\gamma, n)$.

Theorem 1.2. *Let X be a smooth projective surface over the field of complex numbers. For $n \geq 1$, the cohomology ring $\mathbb{H}_n = H^*(X^{[n]})$ is generated by*

$$G_i(\gamma, n) = \mathfrak{G}_i(\gamma)(1_{X^{[n]}})$$

where $0 \leq i < n$ and γ runs over a linear basis of $H^*(X)$. Moreover, the relations among these generators are precisely the relations among the restrictions $\mathfrak{G}_i(\gamma)|_{\mathbb{H}_n}$ of the corresponding operators $\mathfrak{G}_i(\gamma)$ to $\mathbb{H}_n$.

The above Theorem and its proof are inspired mainly by two sources. The first one is Lehn's approach of determining the cohomology ring structure of $(\mathbb{C}^2)^{[n]}$ by using vertex operator techniques (the cohomology ring structure of $(\mathbb{C}^2)^{[n]}$ has been first obtained by Ellingsrud and Strømme [ES2]). Lehn's approach is very instrumental and valuable to us. Our first result here is that although it is difficult to describe completely the operators $\mathfrak{G}_i(\gamma)$ as vertex operators or differential operators, we are able to determine the leading terms of the operators $\mathfrak{G}_i(\gamma)$ as the degree-0 components of some explicit vertex operators (see the paragraph preceding Theorem 4.12 for the definition of the leading term). These vertex operators are natural generalization of the Virasoro operators $\mathfrak{L}_n(\alpha)$ considered by Lehn [Leh]. Such descriptions of the leading terms allow us to use induction to derive our Theorem above. As a byproduct, we also show that the commutator between the operator $\mathfrak{G}_i(\gamma)$ and the Heisenberg generator $\mathfrak{q}_n(\alpha)$ depends only on the cup product $\gamma\alpha$, which we refer to as the *transfer property*. We remark that such a transfer property seems to be a general phenomenon among this type of commutation relations.

The second one is the work of Ellingsrud and Strømme [ES1,ES2] (see also the work of Beauville, Fantechi, Göttsche, Yoshioka and Markman [Bea,F-G, Yos,Mar] on the cohomology ring structures of other moduli spaces of sheaves). In [ES2], Ellingsrud and Strømme proved that for $X = \mathbb{P}^2$, the cohomology ring $\mathbb{H}_n = H^*(X^{[n]})$ is generated by the Chern classes of the tautological rank-n bundles

$$p_{1*}(\mathcal{O}_{\mathcal{Z}_n} \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(-j)) \tag{1.3}$$

with $j = 1, 2, 3$. Equivalently, this says that $\mathbb{H}_n$ is generated by the $H^{2i}(X^{[n]})$-components with $0 \le i \le n$ of the Chern characters of the three bundles in (1.3). By the Grothendieck-Riemann-Roch Theorem [Har], we have

$$\mathrm{ch}(p_{1*}(\mathcal{O}_{\mathcal{Z}_n} \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(-j))) = p_{1*}(\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n} \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(-j)) \cdot p_2^* \,\mathrm{td}(X))$$

for $j = 1, 2, 3$. Note that $\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n} \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(-j)) = \mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n}) \cdot p_2^* \,\mathrm{ch}(\mathcal{O}_{\mathbb{P}^2}(-j)) = \mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n}) \cdot p_2^*([X] - j[\ell] + j^2[x]/2)$ where ℓ (respectively, x) stands for a line (respectively, a point) in $X = \mathbb{P}^2$. So the result of Ellingsrud and Strømme says that the cohomology ring $\mathbb{H}_n$ is generated by the $H^{2i}(X^{[n]})$-components of

$$p_{1*}(\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n}) \cdot p_2^* \,\mathrm{td}(X) \cdot p_2^*\gamma) \in \mathbb{H}_n$$

where $0 \le i \le n$ and $\gamma = [X], [\ell], [x]$. Indeed, this is our motivation for (1.1).

We may also put our present work in a different perspective. From the pioneering work [Na1,Gro,Leh], it is clear that there are deep connections between the geometry of Hilbert schemes and vertex operators. However, the structure of a vertex algebra [Bor,FLM,Kac] is far more richer than the appearance of the Heisenberg and Virasoro vertex operators which are of conformal weight one and two respectively. Thus a natural question here, which is easy to post but difficult to answer, is to understand the full symmetry of vertex algebras in terms of the geometry of Hilbert schemes. Our present work provides a strong evidence that the vertex operators of higher conformal weights afford nice geometric interpretations. In a work in progress, we will further clarify precise connections between vertex algebras and the geometry of Hilbert schemes.

The paper is organized as follows. In Sect. 2, we recall constructions and results of Nakajima, Grojnowski and Lehn. In Sect. 3, we study the relation between Lehn's boundary operator and the Heisenberg generators, and introduce the transfer property. In Sect. 4, we prove that the leading terms for certain linear operators of geometric significance are the degree-0 components of some vertex operators. Finally, we prove our main results in Sect. 5.

Conventions: Throughout the paper, all cohomology groups are in $\mathbb{Q}$-coefficients. The cup product between two cohomology classes α and β is denoted by $\alpha \cdot \beta$ or simply by $\alpha\beta$. For a continuous map $p : Y_1 \to Y_2$ between two smooth compact manifolds and for $\alpha_1 \in H^*(Y_1)$, the push-forward $p_*(\alpha_1)$ is defined by

$$p_*(\alpha_1) = \mathrm{PD}^{-1} p_*(\mathrm{PD}(\alpha_1))$$

where PD stands for the Poincaré duality. We make no distinction between an algebraic cycle and its corresponding cohomology class so that intersections among algebraic cycles correspond to cup products among the corresponding cohomology classes. For instance, for two algebraic cycles $[a]$ and $[b]$ on a smooth projective variety Y, it is understood that $[a] \cdot [b] \in H^*(Y)$.

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2. Results of Nakajima, Grojnowski and Lehn

In this section, we shall fix some notations, and recall some results of Nakajima, Grojnowski and Lehn [Na1,Gro,Leh]. For convenience, we also review certain basic facts for the Hilbert scheme of points in a smooth projective surface.

Let X be a smooth projective surface over $\mathbb{C}$, and $X^{[n]}$ be the Hilbert scheme of points in X. An element in the Hilbert scheme $X^{[n]}$ is represented by a length-n 0-dimensional closed subscheme ξ of X, which sometimes is called a length-n 0-cycle. For $\xi \in X^{[n]}$, let I_ξ and $\mathcal{O}_\xi$ be the corresponding sheaf of ideals and structure sheaf respectively. For a point $x \in X$, let ξ_x be the component of ξ supported at x and $I_{\xi,x} \subset \mathcal{O}_{X,x}$ be the stalk of I_ξ at x. It is known from [Fog] that $X^{[n]}$ is smooth. In $X^{[n]} \times X$, we have the universal codimension-2 subscheme:

$$\mathcal{Z}_n = \{(\xi, x) \subset X^{[n]} \times X \mid x \in \mathrm{Supp}\,(\xi)\} \subset X^{[n]} \times X.$$

In general, for $n > m > 0$, we have the closed subscheme

$$X^{[n,m]} = \{(\xi, \eta) \in X^{[n]} \times X^{[m]} \mid \xi \supset \eta\}$$

of $X^{[n]} \times X^{[m]}$. Let ψ and ϕ be the natural maps from $X^{[n,m]}$ to $X^{[n]}$ and $X^{[m]}$ respectively. Then, $X^{[n,m]}$ parametrizes the two flat families $\psi_X^{-1}(\mathcal{Z}_n) \supset \phi_X^{-1}(\mathcal{Z}_m)$ over $X^{[n,m]} \times X$ where $\psi_X = \psi \times \mathrm{Id}_X : X^{[n,m]} \times X \to X^{[n]} \times X$ and

$\phi_X = \phi \times \mathrm{Id}_X : X^{[n,m]} \times X \to X^{[m]} \times X$. In addition, there exists an exact sequence

$$0 \to \mathcal{I}_{n,m} \to \psi_X^* \mathcal{O}_{\mathcal{Z}_n} \to \phi_X^* \mathcal{O}_{\mathcal{Z}_m} \to 0. \tag{2.1}$$

When $m = n-1$, there is a morphism $\rho : X^{[n,n-1]} \to X$ which maps a point $(\xi, \eta) \in X^{[n,n-1]}$ to the support of (I_η / I_ξ). So we have the standard diagram

$$\begin{array}{ccccc} X & \stackrel{\rho}{\leftarrow} & X^{[n,n-1]} & \stackrel{\psi}{\rightarrow} & X^{[n]} \\ & & \downarrow \phi & & \\ & & X^{[n-1]}. & & \end{array} \tag{2.2}$$

It is known from [Che,Tik,ES3] that $X^{[n,n-1]}$ is irreducible, smooth and of dimension $2n$. In fact, $X^{[n,n-1]}$ is isomorphic to the blowup of $X^{[n-1]} \times X$ along $\mathcal{Z}_{n-1}$. Let E_n be the exceptional divisor in $X^{[n,n-1]}$. Then, we have

$$E_n = \{(\xi, \eta) \in X^{[n,n-1]} \mid \mathrm{Supp}(\xi) = \mathrm{Supp}(\eta)\}.$$

Moreover, (2.1) can be simplified to the exact sequence (see p.193 in [Leh]):

$$0 \to \rho_X^* \mathcal{O}_{\Delta_X} \otimes p_1^* \mathcal{O}_{X^{[n,n-1]}}(-E_n) \to \psi_X^* \mathcal{O}_{\mathcal{Z}_n} \to \phi_X^* \mathcal{O}_{\mathcal{Z}_{n-1}} \to 0 \tag{2.3}$$

where Δ_X is the diagonal in $X \times X$, and p_1 is the projection of $X^{[n,n-1]} \times X$ to $X^{[n,n-1]}$. Finally, we let $X^n = \underbrace{X \times \cdots \times X}_{n}$ be the n-th Cartesian product, and

$$X^{[n_1],\ldots,[n_k]} = X^{[n_1]} \times \cdots \times X^{[n_k]}. \tag{2.4}$$

We formulate below various notations and definitions which will be used later.

Definition 2.5. (i) Let $\mathbb{H} = \bigoplus_{n,i \geq 0} \mathbb{H}^{n,i}$ denote the double graded vector space with components $\mathbb{H}^{n,i} \stackrel{\text{def}}{=} H^i(X^{[n]})$, and $\mathbb{H}_n \stackrel{\text{def}}{=} H^*(X^{[n]}) \stackrel{\text{def}}{=} \bigoplus_{i=0}^{4n} H^i(X^{[n]})$. The element 1 in $H^0(X^{[0]}) = \mathbb{Q}$ is called the *vacuum vector* and denoted by $|0\rangle$;

(ii) A linear operator $\mathfrak{f} \in \mathrm{End}(\mathbb{H})$ is *homogeneous of bi-degree* (ℓ, m) if

$$\mathfrak{f}(\mathbb{H}^{n,i}) \subset \mathbb{H}^{n+\ell, i+m}. \tag{2.6}$$

Furthermore, $\mathfrak{f} \in \mathrm{End}(\mathbb{H})$ is *even* (respectively, *odd*) if m is even (respectively, odd).

(iii) For two homogeneous linear operators $\mathfrak{f}$ and $\mathfrak{g} \in \mathrm{End}(\mathbb{H})$ of bi-degrees (ℓ, m) and (ℓ_1, m_1) respectively, define the *Lie superalgebra bracket* $[\mathfrak{f}, \mathfrak{g}]$ by

$$[\mathfrak{f}, \mathfrak{g}] = \mathfrak{f}\mathfrak{g} - (-1)^{mm_1} \mathfrak{g}\mathfrak{f}. \tag{2.7}$$

A non-degenerate super-symmetric bilinear form $(,)$ on $\mathbb{H}$ is induced from the standard one on $\mathbb{H}_n = H^*(X^{[n]})$. For a homogeneous linear operator $\mathfrak{f} \in \text{End}(\mathbb{H})$ of bi-degree (ℓ, m), we can define its *adjoint* $\mathfrak{f}^\dagger \in \text{End}(\mathbb{H})$ by

$$(\mathfrak{f}(\alpha), \beta) = (-1)^{m \cdot |\alpha|} \cdot (\alpha, \mathfrak{f}^\dagger(\beta))$$

where $|\alpha| = s$ for $\alpha \in H^s(X^{[n]})$. Note that the bi-degree of $\mathfrak{f}^\dagger$ is $(-\ell, m - 4\ell)$. Also,

$$(\mathfrak{f}\mathfrak{g})^\dagger = (-1)^{mm_1} \cdot \mathfrak{g}^\dagger \mathfrak{f}^\dagger \qquad \text{and} \qquad [\mathfrak{f}, \mathfrak{g}]^\dagger = -[\mathfrak{f}^\dagger, \mathfrak{g}^\dagger] \tag{2.8}$$

where $\mathfrak{g} \in \text{End}(\mathbb{H})$ is another homogeneous linear operator of bi-degree (ℓ_1, m_1).

Next, we collect from [Na1,Leh] the definitions of the closed subset $Q^{[n+\ell,n]}$ in $X^{[n+\ell]} \times X \times X^{[n]}$, the Heisenberg generator $\mathfrak{q}_n$, the Virasoro generator $\mathfrak{L}_n$, the boundary operator $\mathfrak{d}$, and the derivative $\mathfrak{f}'$ of a linear operator $\mathfrak{f} \in \text{End}(\mathbb{H})$.

Definition 2.9. (see [Na1,Leh]) (i) For $n \geq 0$, define $Q^{[n,n]} = \emptyset$. For $n \geq 0$ and $\ell > 0$, define $Q^{[n+\ell,n]} \subset X^{[n+\ell]} \times X \times X^{[n]}$ to be the closed subset

$$\{(\xi, x, \eta) \in X^{[n+\ell]} \times X \times X^{[n]} \mid \xi \supset \eta \ \text{ and } \ \text{Supp}(I_\eta/I_\xi) = \{x\}\}; \tag{2.10}$$

(ii) For $n \in \mathbb{Z}$, define linear maps $\mathfrak{q}_n : H^*(X) \to \text{End}(\mathbb{H})$ as follows. When $n \geq 0$, the linear operator $\mathfrak{q}_n(\alpha) \in \text{End}(\mathbb{H})$ with $\alpha \in H^*(X)$ is defined by

$$\mathfrak{q}_n(\alpha)(a) = \tilde{p}_{1*}([Q^{[m+n,m]}] \cdot \tilde{\rho}^*\alpha \cdot \tilde{p}_2^* a) \tag{2.11}$$

for all $a \in \mathbb{H}_m = H^*(X^{[m]})$, where $[Q^{[m+n,m]}]$ is (the cohomology class corresponding to) the algebraic cycle associated to $Q^{[m+n,m]}$, and $\tilde{p}_1, \tilde{\rho}, \tilde{p}_2$ are the projections of $X^{[m+n]} \times X \times X^{[m]}$ to $X^{[m+n]}, X, X^{[m]}$ respectively. When $n < 0$, define the operator $\mathfrak{q}_n(\alpha) \in \text{End}(\mathbb{H})$ with $\alpha \in H^*(X)$ by

$$\mathfrak{q}_n(\alpha) = (-1)^n \cdot \mathfrak{q}_{-n}(\alpha)^\dagger; \tag{2.12}$$

(iii) For $n \in \mathbb{Z}$, define linear maps $\mathfrak{L}_n : H^*(X) \to \text{End}(\mathbb{H})$ by putting

$$\mathfrak{L}_n = \begin{cases} \frac{1}{2} \cdot \sum_{m \in \mathbb{Z}} \mathfrak{q}_m \mathfrak{q}_{n-m} \tau_{2*}, & \text{if } n \neq 0 \\ \sum_{m > 0} \mathfrak{q}_m \mathfrak{q}_{-m} \tau_{2*}, & \text{if } n = 0 \end{cases} \tag{2.13}$$

where $\tau_{2*} : H^*(X) \to H^*(X^2)$ is the linear map induced by the diagonal embedding $\tau_2 : X \to X^2$, and the operator $\mathfrak{q}_m \mathfrak{q}_\ell \tau_{2*}(\alpha)$ stands for

$$\sum_j \mathfrak{q}_m(\alpha_{j,1}) \mathfrak{q}_\ell(\alpha_{j,2})$$

when $\tau_{2*}\alpha = \sum_j \alpha_{j,1} \otimes \alpha_{j,2}$ via the Künneth decomposition of $H^*(X^2)$;

(iv) Define the linear operator $\mathfrak{d} \in \mathrm{End}(\mathbb{H})$ by

$$\mathfrak{d} = \bigoplus_n c_1(p_{1*}\mathcal{O}_{\mathcal{Z}_n}) = \bigoplus_n (-[\partial X^{[n]}]/2) \tag{2.14}$$

where p_1 is the projection of $X^{[n]} \times X$ to $X^{[n]}$, $\partial X^{[n]}$ is the boundary of $X^{[n]}$ consisting of all $\xi \in X^{[n]}$ with $|\operatorname{Supp}(\xi)| < n$, and the first Chern class $c_1(p_{1*}\mathcal{O}_{\mathcal{Z}_n})$ of the rank-n bundle $p_{1*}\mathcal{O}_{\mathcal{Z}_n}$ acts on $\mathbb{H}_n = H^*(X^{[n]})$ by the cup product.

(v) For a linear operator $\mathfrak{f} \in \mathrm{End}(\mathbb{H})$, define its *derivative* $\mathfrak{f}'$ by

$$\mathfrak{f}' \stackrel{\text{def}}{=} \mathrm{ad}(\mathfrak{d})\mathfrak{f} \stackrel{\text{def}}{=} [\mathfrak{d}, \mathfrak{f}]. \tag{2.15}$$

The higher derivative $\mathfrak{f}^{(k)}$ of $\mathfrak{f}$ is defined inductively by $\mathfrak{f}^{(k)} = [\mathfrak{d}, \mathfrak{f}^{(k-1)}]$.

We remark that the definition of the Virasoro generator $\mathfrak{L}_n$ will be generalized in Definition 4.3 (ii) below. Also, $\mathfrak{q}_n(\alpha)$, $\mathfrak{L}_n(\alpha)$, and $\mathfrak{d}$ are homogeneous of bidegrees $(n, 2n - 2 + |\alpha|)$, $(n, 2n + |\alpha|)$, and $(0, 2)$ respectively.

Finally, we recall from [Na1,Leh] the formulas for the derivative $\mathfrak{q}_n'(\alpha)$ as well as the commutation relations among the Heisenberg generators $\mathfrak{q}_n(\alpha)$ and the Virasoro generator $\mathfrak{L}_m(\beta)$. These formulas will be used frequently in the sequel.

Theorem 2.16. *Let K_X and $c_2(X)$ be the canonical divisor and the second Chern class of X respectively. Let $n, m \in \mathbb{Z}$ and $\alpha, \beta \in H^*(X)$. Then,*

(i) $[\mathfrak{q}_n(\alpha), \mathfrak{q}_m(\beta)] = n \cdot \delta_{n+m} \cdot \int_X (\alpha\beta) \cdot \mathrm{Id}_{\mathbb{H}}$;

(ii) $[\mathfrak{L}_n(\alpha), \mathfrak{q}_m(\beta)] = -m \cdot \mathfrak{q}_{n+m}(\alpha\beta)$;

(iii) $[\mathfrak{L}_n(\alpha), \mathfrak{L}_m(\beta)] = (n - m) \cdot \mathfrak{L}_{n+m}(\alpha\beta) - \frac{n^3-n}{12} \cdot \delta_{n+m} \cdot \int_X (c_2(X)\alpha\beta) \cdot \mathrm{Id}_{\mathbb{H}}$;

(iv) $\mathfrak{q}_n'(\alpha) = n \cdot \mathfrak{L}_n(\alpha) + \frac{n(|n|-1)}{2} \mathfrak{q}_n(K_X\alpha)$;

(v) $[\mathfrak{q}_n'(\alpha), \mathfrak{q}_m(\beta)] = -nm \cdot \left\{ \mathfrak{q}_{n+m}(\alpha\beta) + \frac{|n|-1}{2} \cdot \delta_{n+m} \cdot \int_X (K_X\alpha\beta) \cdot \mathrm{Id}_{\mathbb{H}} \right\}$.

We notice that Theorem 2.16 (i) was proved by Nakajima [Na1] subject to some universal nonzero constant, which was determined subsequently in [ES3]. The other four formulas in Theorem 2.16 were obtained by Lehn [Leh]. Moreover, as observed by Nakajima and Grojnowski in [Na1,Gro], $\mathbb{H}$ is an irreducible representation of the Heisenberg algebra generated by the $\mathfrak{q}_i(\alpha)$'s with $|0\rangle \in H^0(X^{[0]})$ being the highest weight vector. In particular, a linear basis of $\mathbb{H}$ is given by

$$\mathfrak{q}_{i_1}(\alpha_1)\mathfrak{q}_{i_2}(\alpha_2) \cdots \mathfrak{q}_{i_k}(\alpha_k)|0\rangle,$$

where $k \geq 0$, $i_1 \geq i_2 \geq \cdots \geq i_k > 0$, and each of the cohomology classes $\alpha_1, \alpha_2, \ldots, \alpha_k$ runs over a fixed linear basis of $H^*(X) = \bigoplus_{i=0}^4 H^i(X)$.

3. The higher derivatives of $\mathfrak{q}_n(\alpha)$

In this section, we study the higher derivatives $\mathfrak{q}_n^{(k)}(\alpha)$ of the Heisenberg generator $\mathfrak{q}_n(\alpha)$ by computing their commutators with other Heisenberg generators $\mathfrak{q}_m(\beta)$. In addition, for two series of operators $\mathfrak{A}(\alpha), \mathfrak{B}(\beta) \in \mathrm{End}(\mathbb{H})$ depending linearly on the cohomology classes $\alpha, \beta \in H^*(X)$, we introduce the transfer property for the commutators $[\mathfrak{A}(\alpha), \mathfrak{B}(\beta)]$, i.e., the property that

$$[\mathfrak{A}(\alpha), \mathfrak{B}(\beta)] = [\mathfrak{A}(1_X), \mathfrak{B}(\alpha\beta)] = [\mathfrak{A}(\alpha\beta), \mathfrak{B}(1_X)].$$

Then, we prove that the commutators $[\mathfrak{q}_n^{(k)}(\alpha), \mathfrak{q}_m(\beta)]$ satisfy the transfer property.

Lemma 3.1. *Let $k \geq 0, n_0, \ldots, n_k \in \mathbb{Z}$, and $\alpha_0, \ldots, \alpha_k \in H^*(X)$. Then,*

$$\begin{aligned} &[[\ldots[\mathfrak{q}_{n_0}^{(k)}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_k}(\alpha_k)] \\ &= a \cdot \mathfrak{q}_{n_0+\ldots+n_k}(\alpha_0 \ldots \alpha_k) + b \cdot \int_X (K_X \alpha_0 \ldots \alpha_k) \cdot \mathrm{Id}_{\mathbb{H}} \end{aligned} \tag{3.2}$$

where a and b are constants depending only on $k, n_0, \ldots, n_k$.

Proof. For simplicity, denote the left-hand-side of (3.2) by $\mathcal{C}(k; n_0, \alpha_0; \ldots; n_k, \alpha_k)$. Note that (3.2) is trivially true for $k = 0$. By Theorem 2.16 (v), (3.2) is true for $k = 1$. In the following, by assuming that (3.2) holds for some positive integer k, we shall prove that (3.2) also holds for $(k + 1)$. Recall the Jacobi identity

$$[[\mathfrak{f}_1, \mathfrak{f}_2], \mathfrak{f}_3] = (-1)^{m_2 m_3}[[\mathfrak{f}_1, \mathfrak{f}_3], \mathfrak{f}_2] + [\mathfrak{f}_1, [\mathfrak{f}_2, \mathfrak{f}_3]] \tag{3.3}$$

if the bi-degree of $\mathfrak{f}_i \in \mathrm{End}(\mathbb{H})$ is (ℓ_i, m_i). We have by the Jacobi identity that

$$\begin{aligned} &\mathcal{C}(k+1; n_0, \alpha_0; \ldots; n_{k+1}, \alpha_{k+1}) \\ &= [[\ldots[[\mathfrak{q}_{n_0}^{(k+1)}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \mathfrak{q}_{n_2}(\alpha_2)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\ &= [[\ldots[\{[\mathfrak{q}_{n_0}^{(k)}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)]' - [\mathfrak{q}_{n_0}^{(k)}(\alpha_0), \mathfrak{q}_{n_1}'(\alpha_1)]\}, \mathfrak{q}_{n_2}(\alpha_2)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\ &= [[\ldots[[\mathfrak{q}_{n_0}^{(k)}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)]', \mathfrak{q}_{n_2}(\alpha_2)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\ &\quad -[[\ldots[[\mathfrak{q}_{n_0}^{(k)}(\alpha_0), \mathfrak{q}_{n_1}'(\alpha_1)], \mathfrak{q}_{n_2}(\alpha_2)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})]. \end{aligned} \tag{3.4}$$

Repeating the above process, we conclude that

$$\begin{aligned} &\mathcal{C}(k+1; n_0, \alpha_0; \ldots; n_{k+1}, \alpha_{k+1}) \\ &= [\mathcal{C}(k; n_0, \alpha_0; \ldots; n_k, \alpha_k), \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})]' - \sum_{i=1}^{k+1} [\ldots[\mathfrak{q}_{n_0}^{(k)}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \ldots \\ &\qquad \ldots], q_{n_{i-1}}(\alpha_{i-1})], q_{n_i}'(\alpha_i)], q_{n_{i+1}}(\alpha_{i+1})], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})]. \end{aligned} \tag{3.5}$$

By the induction hypothesis on k and Theorem 2.16 (i), we get

$$[\mathcal{C}(k; n_0, \alpha_0; \ldots; n_k, \alpha_k), \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})]' = 0. \tag{3.6}$$

Using the Jacobi identity (3.3) again, we see that

$$
\begin{aligned}
&[\dots[\mathfrak{q}^{(k)}_{n_0}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \dots], q_{n_{i-1}}(\alpha_{i-1})], q'_{n_i}(\alpha_i)], \\
&\qquad q_{n_{i+1}}(\alpha_{i+1})], \dots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\
&= (-1)^{\sum\limits_{\ell=i+1}^{k+1} |\alpha_\ell||\alpha_i|} [\mathcal{C}(k; n_0, \alpha_0; \dots ; n_{i-1}, \alpha_{i-1}; n_{i+1}, \alpha_{i+1}; \dots ; \\
&\qquad n_{k+1}, \alpha_{k+1}), q'_{n_i}(\alpha_i)] \\
&\quad + [\dots[\mathfrak{q}^{(k)}_{n_0}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \dots], q_{n_{i-1}}(\alpha_{i-1})], [q'_{n_i}(\alpha_i), q_{n_{i+1}}(\alpha_{i+1})]], \\
&\qquad q_{n_{i+2}}(\alpha_{i+2})], \dots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\
&\quad + \sum_{j=i+2}^{k+1} (-1)^{\sum_{\ell=i+1}^{j-1} |\alpha_\ell||\alpha_i|} \cdot [\dots[\mathfrak{q}^{(k)}_{n_0}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \dots], \\
&\qquad q_{n_{i-1}}(\alpha_{i-1})], q_{n_{i+1}}(\alpha_{i+1})], \dots], q_{n_{j-1}}(\alpha_{j-1})], [q'_{n_i}(\alpha_i), q_{n_j}(\alpha_j)]], \\
&\qquad q_{n_{j+1}}(\alpha_{j+1})], \dots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\
&= (-1)^{\sum_{\ell=i+1}^{k+1} |\alpha_\ell||\alpha_i|} [\mathcal{C}(k; n_0, \alpha_0; \dots ; \\
&\qquad n_{i-1}, \alpha_{i-1}; n_{i+1}, \alpha_{i+1}; \dots ; n_{k+1}, \alpha_{k+1}), q'_{n_i}(\alpha_i)] \\
&\quad + (-n_i n_{i+1}) \cdot \mathcal{C}(k; n_0, \alpha_0; \dots ; n_{i-1}, \alpha_{i-1}; n_i + n_{i+1}, \alpha_i\alpha_{i+1}; \\
&\qquad n_{i+2}, \alpha_{i+2}; \dots ; n_{k+1}, \alpha_{k+1}) \\
&\quad + \sum_{j=i+2}^{k+1} (-1)^{\sum_{\ell=i+1}^{j-1} |\alpha_\ell||\alpha_i|} \cdot (-n_i n_j) \cdot \mathcal{C}(k; n_0, \alpha_0; \dots ; \\
&\qquad n_{i-1}, \alpha_{i-1}; n_{i+1}, \alpha_{i+1}; \dots ; n_{j-1}, \alpha_{j-1}; \\
&\qquad n_i + n_j, \alpha_i\alpha_j; n_{j+1}, \alpha_{j+1}; \dots ; n_{k+1}, \alpha_{k+1})
\end{aligned}
$$

where we have used Theorem 2.16 (v) in the last step. By induction hypothesis,

$$
\begin{aligned}
&[\dots[\mathfrak{q}^{(k)}_{n_0}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \dots], q_{n_{i-1}}(\alpha_{i-1})], q'_{n_i}(\alpha_i)], q_{n_{i+1}}(\alpha_{i+1})], \dots], \\
&\qquad \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\
&= (-1)^{\sum_{\ell=i+1}^{k+1} |\alpha_\ell||\alpha_i|} \cdot [a_i \cdot \mathfrak{q}_{\sum_{\ell=0}^{k+1} n_\ell - n_i}(\alpha_0 \dots \alpha_{i-1}\alpha_{i+1} \dots \alpha_{k+1}), q'_{n_i}(\alpha_i)] \\
&\quad + \sum_{j=i+1}^{k+1} (-n_i n_j) \Big\{ a_{i,j} \mathfrak{q}_{n_0+\dots+n_{k+1}}(\alpha_0 \dots \alpha_{k+1}) + b_{i,j} \\
&\quad \cdot \int_X (K_X \alpha_0 \dots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}} \Big\} \\
&= a_i \cdot \left(\sum_{\ell=0}^{k+1} n_\ell - n_i \right) n_i \Big\{ \mathfrak{q}_{n_0+\dots+n_{k+1}}(\alpha_0 \dots \alpha_{k+1}) + b_i \\
&\quad \cdot \int_X (K_X \alpha_0 \dots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}} \Big\}
\end{aligned}
$$

$$+\sum_{j=i+1}^{k+1}(-n_in_j)\Big\{a_{i,j}\mathfrak{q}_{n_0+\ldots+n_{k+1}}(\alpha_0\ldots\alpha_{k+1})+b_{i,j}$$
$$\cdot\int_X(K_X\alpha_0\ldots\alpha_{k+1})\cdot\mathrm{Id}_{\mathbb{H}}\Big\}$$

where all the numbers $a_i, b_i, a_{i,j}, b_{i,j}$ depend only on $k, n_0, \ldots, n_k$. Combining the above with (3.5) and (3.6), we obtain (3.2). □

Lemma 3.7. *Let $k \geq 0$, $n_0, \ldots, n_k \in \mathbb{Z}$, and $\alpha_0, \ldots, \alpha_k \in H^*(X)$. Then,*

$$[[\ldots[\mathfrak{q}^{(k)}_{n_0}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_k}(\alpha_k)]$$
$$=\big((-1)^k\cdot k!\cdot n_0^k\cdot n_1\ldots n_k\big)\cdot\mathfrak{q}_{n_0+\ldots+n_k}(\alpha_0\ldots\alpha_k)+b\cdot\int_X(K_X\alpha_0\ldots\alpha_k)\cdot\mathrm{Id}_{\mathbb{H}}$$

where b is a constant depending only on $k, n_0, \ldots, n_k$.

Proof. For simplicity, denote the constant a in (3.2) by $a(k; n_0, \ldots, n_k)$. In view of (3.2), it remains to show that $a(k; n_0, \ldots, n_k) = (-1)^k \cdot k! \cdot n_0^k \cdot n_1 \ldots n_k$.

This is trivially true for $k = 0$. By Theorem 2.16 (v), this is true for $k = 1$. In the following, assuming $a(k; n_0, \ldots, n_k) = (-1)^k \cdot k! \cdot n_0^k \cdot n_1 \ldots n_k$, we shall prove

$$a(k+1; n_0, \ldots, n_{k+1}) = (-1)^{k+1}\cdot(k+1)!\cdot n_0^{k+1}\cdot n_1\ldots n_{k+1}. \tag{3.8}$$

Indeed, we see from the proof of Lemma 3.1 that $a(k+1; n_0, \ldots, n_{k+1})$ is equal to

$$-\sum_{i=1}^{k+1}\Big\{a(k; n_0, \ldots, n_{i-1}, n_{i+1}, \ldots, n_{k+1})\cdot\left(\sum_{\ell=0}^{k+1}n_\ell-n_i\right)n_i$$
$$+\sum_{j=i+1}^{k+1}(-n_in_j)\cdot a(k; n_0, \ldots, n_{i-1}, n_{i+1}, \ldots, n_{j-1}, n_i+n_j, n_{j+1}, \ldots, n_{k+1})\Big\}.$$

So by using the induction hypothesis, we conclude that

$$\begin{aligned}&a(k+1; n_0, \ldots, n_{k+1})\\&=-\sum_{i=1}^{k+1}\{(-1)^k\cdot k!\cdot n_0^k\cdot n_1\ldots n_{i-1}n_{i+1}\ldots n_{k+1}\cdot\left(\sum_{\ell=0}^{k+1}n_\ell-n_i\right)n_i\\&\quad+\sum_{j=i+1}^{k+1}(-n_in_j)\cdot(-1)^k\cdot k!\cdot n_0^k\cdot n_1\ldots n_{i-1}n_{i+1}\ldots n_{j-1}(n_i+n_j)n_{j+1}\ldots n_{k+1}\}\\&=(-1)^{k+1}\cdot(k+1)!\cdot n_0^{k+1}\cdot n_1\ldots n_{k+1}.\end{aligned}$$

□

Lemma 3.9. *Let $k \geq 0$, $n_0, \ldots, n_{k+1} \in \mathbb{Z}$, and $\alpha_0, \ldots, \alpha_{k+1} \in H^*(X)$. Then,*

$$\begin{aligned}&[\ldots[\mathfrak{q}_{n_0}^{(k)}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\ &= k! \cdot n_0^k \cdot \prod_{\ell=1}^{k+1}(-n_\ell) \cdot \delta_{n_0+\ldots+n_{k+1}} \cdot \int_X (\alpha_0 \ldots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}}\,.\end{aligned}$$

Proof. Follows from Lemma 3.7 and Theorem 2.16 (i). □

Proposition 3.10. *Let $\mathfrak{f} \in \mathrm{End}(\mathbb{H})$ be an operator of bi-degree $(0, i)$ with $\mathfrak{f}' = 0$ and $\mathfrak{f}^\dagger = \mathfrak{f}$. Assume that $[\mathfrak{f}, \mathfrak{q}_1(\alpha)] = \sum_{j=0}^k c_j \mathfrak{q}_1^{(j)}(\lambda_j \alpha)$ for every $\alpha \in H^*(X)$, where $k \geq 0$, $c_k = 1/k!$, and $k, c_j \in \mathbb{Q}$, $\lambda_j \in H^{i-2j}(X)$ depend only on $\mathfrak{f}$. Then,*

$$\begin{aligned}&[[\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)], \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\ &= -\textstyle\prod_{\ell=0}^{k+1}(-n_\ell) \cdot \delta_{n_0+\ldots+n_{k+1}} \cdot \int_X (\lambda_k \alpha_0 \ldots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}}\end{aligned} \tag{3.11}$$

for all $n_0, \ldots, n_{k+1} \in \mathbb{Z}$ and all $\alpha_0, \ldots, \alpha_{k+1} \in H^*(X)$.

Proof. (i) First, we show that (3.11) holds for $n_0 \geq 0$. We shall use induction on n_0. When $n_0 = 0$, (3.11) is true since $\mathfrak{q}_0(\alpha_0) = 0$ for any α_0. When $n_0 = 1$, we see from the assumptions on $\mathfrak{f}$ that $[\mathfrak{f}, \mathfrak{q}_1(\alpha_0)] = \sum_{j=0}^k c_j \mathfrak{q}_1^{(j)}(\lambda_j \alpha_0)$. So

$$\begin{aligned}&[[\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)], \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\ &= \textstyle\sum_{j=0}^k c_j [[\ldots[\mathfrak{q}_1^{(j)}(\lambda_j \alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})].\end{aligned} \tag{3.12}$$

By Lemma 3.9, $[\ldots[\mathfrak{q}_1^{(j)}(\lambda_j \alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] = 0$ if $0 \leq j < k$. Also,

$$\begin{aligned}&[\ldots[\mathfrak{q}_1^{(k)}(\lambda_k \alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\ &= -k! \cdot \textstyle\prod_{\ell=0}^{k+1}(-n_\ell) \cdot \delta_{n_0+\ldots+n_{k+1}} \cdot \int_X (\lambda_k \alpha_0 \ldots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}}\end{aligned} \tag{3.13}$$

by Lemma 3.9 again. It follows from (3.12) and (3.13) that (3.11) holds for $n_0 = 1$.

Next, assuming that (3.11) is true for some positive integer n_0, we shall prove that (3.11) still holds if n_0 is replaced by $(n_0 + 1)$. Note from Theorem 2.16 (v) that $\mathfrak{q}_{n_0+1}(\alpha_0) = -1/n_0 \cdot [\mathfrak{q}_1'(1_X), \mathfrak{q}_{n_0}(\alpha_0)]$. Thus, we obtain

$$\begin{aligned}[\mathfrak{f}, \mathfrak{q}_{n_0+1}(\alpha_0)] &= -\frac{1}{n_0} \cdot [\mathfrak{f}, [\mathfrak{q}_1'(1_X), \mathfrak{q}_{n_0}(\alpha_0)]] \\ &= -\frac{1}{n_0} \cdot \{[[\mathfrak{f}, \mathfrak{q}_1'(1_X)], \mathfrak{q}_{n_0}(\alpha_0)] + [\mathfrak{q}_1'(1_X), [\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)]]\}.\end{aligned}$$

Since $\mathfrak{f}' = 0$, $[\mathfrak{f}, \mathfrak{q}_1'(1_X)] = [\mathfrak{f}, \mathfrak{q}_1(1_X)]' = \sum_{j=0}^{k} c_j \mathfrak{q}_1^{(j+1)}(\lambda_j)$. Therefore, we have

$$[\mathfrak{f}, \mathfrak{q}_{n_0+1}(\alpha_0)] = -\frac{1}{n_0}\left\{\sum_{j=0}^{k} c_j [\mathfrak{q}_1^{(j+1)}(\lambda_j), \mathfrak{q}_{n_0}(\alpha_0)] - [[\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)], \mathfrak{q}_1'(1_X)]\right\}. \tag{3.14}$$

Note that $[[\ldots[\mathfrak{q}_1^{(j+1)}(\lambda_j), \mathfrak{q}_{n_0}(\alpha_0)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})]$ is equal to 0 if $0 \le j < k$, and $(k+1)! \cdot \prod_{\ell=0}^{k+1}(-n_\ell) \cdot \delta_{1+n_0+\ldots+n_{k+1}} \cdot \int_X (\lambda_k \alpha_0 \ldots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}}$ if $j = k$. So

$$\begin{aligned}
&[[\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0+1}(\alpha_0)], \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\
&= -\frac{1}{n_0}\Bigg\{(-1)^k (k+1) \cdot \prod_{\ell=0}^{k+1} n_\ell \cdot \delta_{1+n_0+\ldots+n_{k+1}} \cdot \int_X (\lambda_k \alpha_0 \ldots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}} \\
&\qquad - [\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)], \mathfrak{q}_1'(1_X)], \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})]\Bigg\} \\
&= -\frac{1}{n_0}\Bigg\{(-1)^k (k+1) \cdot \prod_{\ell=0}^{k+1} n_\ell \cdot \delta_{1+n_0+\ldots+n_{k+1}} \cdot \int_X (\lambda_k \alpha_0 \ldots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}} \\
&\qquad - [\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)], \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})], \mathfrak{q}_1'(1_X)] \\
&\qquad - \sum_{j=1}^{k+1} [\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)], \ldots], \mathfrak{q}_{n_{j-1}}(\alpha_{j-1})], [q_1'(1_X), \mathfrak{q}_{n_j}(\alpha_j)]], \\
&\qquad\qquad \mathfrak{q}_{n_{j+1}}(\alpha_{j+1})], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})]\Bigg\}.
\end{aligned}$$

By induction hypothesis, $[\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)], \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})], \mathfrak{q}_1'(1_X)] = 0$. By Theorem 2.16 (v) and induction hypothesis, we get

$$\begin{aligned}
&[\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)], \ldots], \mathfrak{q}_{n_{j-1}}(\alpha_{j-1})], [q_1'(1_X), \mathfrak{q}_{n_j}(\alpha_j)]], \mathfrak{q}_{n_{j+1}}(\alpha_{j+1})], \ldots \\
&\qquad \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\
&= (-n_j) \cdot [\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)], \ldots], \mathfrak{q}_{n_{j-1}}(\alpha_{j-1})], \mathfrak{q}_{n_j+1}(\alpha_j)], \mathfrak{q}_{n_{j+1}}(\alpha_{j+1})], \ldots \\
&\qquad \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\
&= (1+n_j) \cdot \prod_{\ell=0}^{k+1}(-n_\ell) \cdot \delta_{1+n_0+\ldots+n_{k+1}} \cdot \int_X (\lambda_k \alpha_0 \ldots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}}.
\end{aligned}$$

It follows that $[[\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0+1}(\alpha_0)], \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})]$ is equal to

$$-(-(1+n_0)) \prod_{\ell=1}^{k+1}(-n_\ell) \cdot \delta_{1+n_0+\ldots+n_{k+1}} \cdot \int_X (\lambda_k \alpha_0 \ldots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}}, \tag{3.15}$$

i.e., (3.11) is still true if n_0 is replaced by $(n_0 + 1)$.

(ii) Now we show that (3.11) holds for $n_0 < 0$. Note that $\mathfrak{q}_{-n}(\alpha) = (-1)^n \mathfrak{q}_n(\alpha)^\dagger$, $(\mathfrak{g}_1\mathfrak{g}_2)^\dagger = (-1)^{m_1 m_2} \cdot (\mathfrak{g}_2)^\dagger (\mathfrak{g}_1)^\dagger$ and $[\mathfrak{g}_1, \mathfrak{g}_2]^\dagger = -[(\mathfrak{g}_1)^\dagger, (\mathfrak{g}_2)^\dagger]$ for $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathrm{End}(\mathbb{H})$ of bi-degrees $(\ell_1, m_1), (\ell_2, m_2)$ respectively. Since $\mathfrak{f}^\dagger = \mathfrak{f}$ by assumption,

$$\begin{aligned} &[\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)], \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\ &= (-1)^{(1+n_0)+\ldots+(1+n_{k+1})} \cdot [\ldots[[\mathfrak{f}, \mathfrak{q}_{-n_0}(\alpha_0)], \mathfrak{q}_{-n_1}(\alpha_1)], \ldots], \mathfrak{q}_{-n_{k+1}}(\alpha_{k+1})]^\dagger. \end{aligned}$$

By what we have proved in (i) for the positive integer $(-n_0)$, we have

$$\begin{aligned} &[\ldots[[\mathfrak{f}, \mathfrak{q}_{n_0}(\alpha_0)], \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] \\ &= (-1)^{n_0+\ldots+n_{k+1}+k} \cdot \left(-\prod_{\ell=0}^{k+1} n_\ell \cdot \delta_{-n_0-\ldots-n_{k+1}} \cdot \int_X (\lambda_k \alpha_0 \ldots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}} \right)^\dagger \\ &= -\prod_{\ell=0}^{k+1} (-n_\ell) \cdot \delta_{n_0+\ldots+n_{k+1}} \cdot \int_X (\lambda_k \alpha_0 \ldots \alpha_{k+1}) \cdot \mathrm{Id}_{\mathbb{H}}\,. \end{aligned}$$

□

Next, we shall define the transfer property for certain commutators, and verify that the commutator $[\mathfrak{q}_{n_0}^{(k)}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)]$ satisfies the transfer property.

Definition 3.16. Let $\mathfrak{A}(\alpha), \mathfrak{B}(\beta) \in \mathrm{End}(\mathbb{H})$ be two series of operators depending linearly on $\alpha, \beta \in H^*(X)$. Then, the commutator $[\mathfrak{A}(\alpha), \mathfrak{B}(\beta)]$ satisfies *the transfer property* if $[\mathfrak{A}(\alpha), \mathfrak{B}(\beta)] = [\mathfrak{A}(1_X), \mathfrak{B}(\alpha\beta)] = [\mathfrak{A}(\alpha\beta), \mathfrak{B}(1_X)]$ for all α, β.

Proposition 3.17. *Let k be a nonnegative integer. Let $n_0, n_1 \in \mathbb{Z}$, and $\alpha_0, \alpha_1 \in H^*(X)$. Then, the commutator $[\mathfrak{q}_{n_0}^{(k)}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)]$ satisfies the transfer property:*

$$[\mathfrak{q}_{n_0}^{(k)}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)] = [\mathfrak{q}_{n_0}^{(k)}(1_X), \mathfrak{q}_{n_1}(\alpha_0\alpha_1)] = [\mathfrak{q}_{n_0}^{(k)}(\alpha_0\alpha_1), \mathfrak{q}_{n_1}(1_X)]. \tag{3.18}$$

Proof. By Theorem 2.16 (i) and (v), (3.18) is true for $k = 0, 1$. In the following, we assume that $k \geq 2$. In addition, we may assume that $(\alpha_0, \alpha_1) \neq (1_X, 1_X)$. Let $\mathfrak{e}$ be the difference of any of the two commutators in (3.18). By Lemma 3.1, we have

$$[\ldots[\mathfrak{e}, \mathfrak{q}_{n_2}(\alpha_2)], \ldots], \mathfrak{q}_{n_{k-1}}(\alpha_{k-1})], \mathfrak{q}_{n_k}(\alpha_k)] = 0 \tag{3.19}$$

for all $n_2, \ldots, n_k \in \mathbb{Z}$ and all $\alpha_2, \ldots, \alpha_k \in H^*(X)$. Since $\mathbb{H}$ is irreducible (see the last paragraph in Sect. 2), we see from Schur's lemma that

$$\mathfrak{e}_{k-1} \stackrel{\mathrm{def}}{=} [\ldots[\mathfrak{e}, \mathfrak{q}_{n_2}(\alpha_2)], \ldots], \mathfrak{q}_{n_{k-1}}(\alpha_{k-1})]$$

must be a scalar multiple of the identity operator. Now, the bi-degree of $\mathfrak{e}_{k-1}$ is

$$\left(\sum_{j=0}^{k-1} n_j, 2k + \sum_{j=0}^{k-1} (2n_j - 2 + |\alpha_j|) \right).$$

So when $\sum_{j=0}^{k-1} n_j \neq 0$, the bi-degree of $\mathfrak{e}_{k-1}$ is nontrivial. When $\sum_{j=0}^{k-1} n_j = 0$, the bi-degree of $\mathfrak{e}_{k-1}$ is $(0, \sum_{j=0}^{k-1} |\alpha_j|)$ which again is nontrivial since we have assumed $(\alpha_0, \alpha_1) \neq (1_X, 1_X)$. This forces $\mathfrak{e}_{k-1} = 0$, i.e., we have

$$[\dots[\mathfrak{e}, \mathfrak{q}_{n_2}(\alpha_2)], \dots], \mathfrak{q}_{n_{k-2}}(\alpha_{k-2})], \mathfrak{q}_{n_{k-1}}(\alpha_{k-1})] = 0$$

for all the integers $n_2, \dots, n_{k-1} \in \mathbb{Z}$ and all the cohomology classes $\alpha_2, \dots, \alpha_{k-1} \in H^*(X)$. Repeating the above process, we conclude that $\mathfrak{e} = 0$. □

4. The operators W_n^k as the leading terms

In this section, we shall define certain operators $W_n^k(\alpha)$ and determine their commutation relations with the Heisenberg generator $\mathfrak{q}_m(\beta)$. Our main result (see Theorem 4.12 below) says that if $\mathfrak{f} \in \mathrm{End}(\mathbb{H})$ satisfies the same conditions as in Proposition 3.10, then the leading term of $\mathfrak{f}$ is equal to $-W_0^{k+2}(\lambda_k)$.

First, we recall the normally ordered product $: \cdot :$ from [Bor, FLM, Kac]. Let

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{n-\Delta}$$

be a vertex operator of conformal weight Δ, that is, a generating function in a formal variable z with $a_{(n)} \in \mathrm{End}(\mathbb{H})$ of bi-degree $(n, *)$. Put

$$a_+(z) = \sum_{n>0} a_{(n)} z^{n-\Delta} \qquad \text{and} \qquad a_-(z) = \sum_{n \le 0} a_{(n)} z^{n-\Delta}$$

(note that our sign convention on vertex operators throughout this paper differs from the standard one used in the vertex algebra literature [Bor, FLM, Kac]). If $b(z)$ is another vertex operator, we define a new vertex operator, which is called *the normally ordered product* of $a(z)$ and $b(z)$, to be:

$$: a(z)b(z) := a_+(z)b(z) + (-1)^{ab} b(z)a_-(z) \tag{4.1}$$

where $(-1)^{ab}$ is -1 if both $a(z)$ and $b(z)$ are odd and 1 otherwise. Inductively we can define the normally ordered product of k vertex operators from right to left by

$$: a_1(z)a_2(z)\cdots a_k(z) :=: a_1(z)(: a_2(z)\cdots a_k(z) :) : . \tag{4.2}$$

Definition 4.3. (i) For $\alpha \in H^*(X)$, we define a vertex operator $\alpha(z)$ by putting

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \mathfrak{q}_n(\alpha) z^{n-1}; \tag{4.4}$$

(ii) Let $k \geq 1, n \in \mathbb{Z}$, and $\alpha \in H^*(X)$. Let $\tau_{k*} : H^*(X) \to H^*(X^k)$ be the linear map induced by the diagonal embedding $\tau_k : X \to X^k$, and

$$\tau_{k*}\alpha = \sum_j \alpha_{j,1} \otimes \ldots \otimes \alpha_{j,k}$$

via the Künneth decomposition of $H^*(X^k)$. We define the operator $W_n^k(\alpha) \in \mathrm{End}(\mathbb{H})$ to be the coefficient of z^{n-k} in the vertex operator

$$\frac{1}{k!} \cdot (\tau_{k*}\alpha)(z) \overset{\text{def}}{=} \frac{1}{k!} \cdot \sum_j : \alpha_{j,1}(z) \cdots \alpha_{j,k}(z) : . \tag{4.5}$$

Note that $W_n^k(\alpha)$ is a homogeneous linear operator of bi-degree

$$(n, 2n + 2k - 4 + |\alpha|).$$

Also, $W_n^1(\alpha)$ coincides with the Heisenberg generator $\mathfrak{q}_n(\alpha)$, and $W_n^2(\alpha)$ coincides with the Virasoro generator $\mathfrak{L}_n(\alpha)$. The next lemma generalizes Theorem 2.16 (ii) and indicates that the commutator $[W_n^k(\alpha), \mathfrak{q}_m(\beta)]$ satisfies the transfer property.

Lemma 4.6. *Let $k \geq 2$. Let $n, m \in \mathbb{Z}$ and $\alpha, \beta \in H^*(X)$. Then,*

$$[W_n^k(\alpha), \mathfrak{q}_m(\beta)] = (-m) \cdot W_{n+m}^{k-1}(\alpha\beta). \tag{4.7}$$

Proof. First of all, we rewrite the commutation relation Theorem 2.16 (i) as

$$[\alpha(z), \beta(w)] = \int_X (\alpha\beta) \cdot \sum_{n\in\mathbb{Z}} (n \cdot z^{n-1} w^{-n-1}).$$

Assume that $\tau_{k*}\alpha = \sum_j \alpha_{j,1} \otimes \ldots \otimes \alpha_{j,k} \in H^*(X^k)$. Then, we have

$$\begin{aligned}
&[(\tau_{k*}\alpha)(z), \mathfrak{q}_m(\beta)] \\
&= \mathrm{Res}_{w=0}\, w^{-m} [(\tau_{k*}\alpha)(z), \beta(w)] \\
&= \mathrm{Res}_{w=0}\, w^{-m} \textstyle\sum_j [: \alpha_{j,1}(z) \cdots \alpha_{j,k}(z) :, \beta(w)] \\
&= \mathrm{Res}_{w=0}\, w^{-m} \textstyle\sum_{s=1}^k \sum_j : \alpha_{j,1}(z) \cdots \widehat{\alpha}_{j,s}(z) \cdots \alpha_{j,k}(z) : \cdot \\
&\quad \cdot (-1)^{|\beta| \sum_{\ell=s+1}^k |\alpha_{j,\ell}|} \textstyle\int_X \alpha_{j,s}\beta \sum_{n\in\mathbb{Z}} n z^{n-1} w^{-n-1} \\
&= (-m z^{-m-1}) \cdot \textstyle\sum_{s=1}^k \sum_j : \alpha_{j,1}(z) \cdots \widehat{\alpha}_{j,s}(z) \cdots \alpha_{j,k}(z) : \cdot \\
&\quad \cdot (-1)^{|\beta| \sum_{\ell=s+1}^k |\alpha_{j,\ell}|} \textstyle\int_X \alpha_{j,s}\beta.
\end{aligned} \tag{4.8}$$

To simplify (4.8), we fix s satisfying $1 \le s \le k$. Let p_s (respectively, π_s) be the projection of X^k to the s-th factor (respectively, to the product of the remaining $(k-1)$ factors). So we have a commutative diagram of morphisms:

$$\begin{array}{ccccc} X & \xrightarrow{\tau_k} & X^k & \xrightarrow{p_s} & X \\ & \searrow^{\tau_{k-1}} & \downarrow \pi_s & & \\ & & X^{k-1} & & \end{array}$$

Now, by the projection formula, we conclude that

$$\begin{aligned} &(\tau_{k-1})_*(\alpha\beta) = (\pi_s \circ \tau_k)_*(\alpha\beta) \\ &= \pi_{s*}(\tau_{k*}(\alpha \cdot \tau_k^* p_s^* \beta)) = \pi_{s*}(\tau_{k*}\alpha \cdot p_s^* \beta) \\ &= \sum_j \alpha_{j,1} \otimes \cdots \otimes \widehat{\alpha}_{j,s} \otimes \cdots \otimes \alpha_{j,k} \cdot (-1)^{|\beta| \sum_{\ell=s+1}^k |\alpha_{j,\ell}|} \int_X \alpha_{j,s}\beta. \end{aligned}$$

Combining this with (4.8), we conclude that

$$[(\tau_{k*}\alpha)(z), \mathfrak{q}_m(\beta)] = (-mz^{-m-1}) \cdot k \cdot ((\tau_{k-1})_*(\alpha\beta))(z). \tag{4.9}$$

Now comparing the coefficients of z^{n-k} on both sides of (4.9), we obtain (4.7).

$\square$

Proposition 4.10. *Let $k \ge 1, n_0, \ldots, n_k \in \mathbb{Z}$, and $\alpha_0, \ldots, \alpha_k \in H^*(X)$. Then,*

(i) $[[\ldots[W^k_{n_0}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k-1}}(\alpha_{k-1})]$ *is equal to*

$$\prod_{\ell=1}^{k-1}(-n_\ell) \cdot \mathfrak{q}_{n_0+\ldots+n_{k-1}}(\alpha_0 \ldots \alpha_{k-1});$$

(ii) $[[\ldots[W^k_{n_0}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_k}(\alpha_k)]$ *is equal to*

$$\prod_{\ell=1}^{k}(-n_\ell) \cdot \delta_{n_0+\ldots+n_k} \cdot \int_X (\alpha_0 \ldots \alpha_k) \cdot \mathrm{Id}_{\mathbb{H}}.$$

Proof. Applying Lemma 4.6 repeatedly, we see that

$$\begin{aligned} &[\ldots[W^k_{n_0}(\alpha_0), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k-1}}(\alpha_{k-1})] \\ &= \prod_{\ell=1}^{k-1}(-n_\ell) \cdot W^1_{n_0+\ldots+n_{k-1}}(\alpha_0 \ldots \alpha_{k-1}) \\ &= \prod_{\ell=1}^{k-1}(-n_\ell) \cdot \mathfrak{q}_{n_0+\ldots+n_{k-1}}(\alpha_0 \ldots \alpha_{k-1}). \end{aligned}$$

This proves (i). Now (ii) follows from (i) and Theorem 2.16 (i). □

Let $\mathfrak{f} \in \mathrm{End}(\mathbb{H})$ be a linear operator. We write the operator $\mathfrak{f}$ as a (possibly infinite) linear combination of finite products of Heisenberg generators:

$$\mathfrak{q}_{m_1}(\beta_1) \cdots \mathfrak{q}_{m_i}(\beta_i) \tag{4.11}$$

where those $\mathfrak{q}_{m_j}(\beta_j)$ with $m_j < 0$ are put to the right. In other words, we regard $\mathfrak{f}$ as an element in the completion of the universal enveloping algebra of the Heisenberg algebra. We can always do so because $\mathbb{H}$ is irreducible as a representation of the Heisenberg algebra. Assume that the lengths i of all the product terms (4.11) appearing in $\mathfrak{f}$ have a common upper bound (this is the case for all the operators $\mathfrak{f}$ considered below). Then we define the *leading term* of $\mathfrak{f}$ to be the sum of those products (4.11) in $\mathfrak{f}$ such that i is the largest. For example, each product in the operator $W_n^k(\alpha)$ has k factors of $\mathfrak{q}$'s by Definition 4.3 (ii). So the leading term of $W_n^k(\alpha)$ is itself. Our next result says that if $\mathfrak{f} \in \mathrm{End}(\mathbb{H})$ satisfies the same assumptions as in Proposition 3.10, then the leading term of $\mathfrak{f}$ is $-W_0^{k+2}(\lambda_k)$.

Theorem 4.12. *Let $\mathfrak{f} \in \mathrm{End}(\mathbb{H})$ be of bi-degree $(0, i)$ with $\mathfrak{f}' = 0$ and $\mathfrak{f}^\dagger = \mathfrak{f}$. Assume $[\mathfrak{f}, \mathfrak{q}_1(\alpha)] = \sum_{j=0}^k c_j \mathfrak{q}_1^{(j)}(\lambda_j \alpha)$ for every $\alpha \in H^*(X)$, where $k \geq 0$, $c_k = 1/k!$, and $k, c_j \in \mathbb{Q}$, $\lambda_j \in H^{i-2j}(X)$ depend only on $\mathfrak{f}$. Put $\epsilon(\mathfrak{f}) = \mathfrak{f} + W_0^{k+2}(\lambda_k)$. Then,*

(i) *for all $n_1, \ldots, n_{k+2} \in \mathbb{Z}$ and all $\alpha_1, \ldots, \alpha_{k+2} \in H^*(X)$,*

$$[\ldots[\epsilon(\mathfrak{f}), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+2}}(\alpha_{k+2})] = 0;$$

(ii) *for all $n_1, \ldots, n_{k+1} \in \mathbb{Z}$ with $\sum_{j=1}^{k+1} n_j \neq 0$ and all $\alpha_1, \ldots, \alpha_{k+1} \in H^*(X)$,*

$$[\ldots[\epsilon(\mathfrak{f}), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] = 0;$$

(iii) *the leading term of $\mathfrak{f}$ is $-W_0^{k+2}(\lambda_k)$.*

Proof. (i) Follows from Proposition 3.10 and Proposition 4.10 (ii).

(ii) Denote $[\ldots[\epsilon(\mathfrak{f}), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})]$ by $\mathfrak{g}$. Then, we see from (i) that $[\mathfrak{g}, \mathfrak{q}_{n_{k+2}}(\alpha_{k+2})] = 0$ for all $n_{k+2} \in \mathbb{Z}$ and all $\alpha_{k+2} \in H^*(X)$. Since $\mathbb{H}$ is irreducible, $\mathfrak{g}$ must be a scalar. Now, the bi-degree of $\mathfrak{g}$ is equal to

$$\left(\sum_{j=1}^{k+1} n_j, i + \sum_{j=1}^{k+1} (2n_j - 2 + |\alpha_j|) \right)$$

which is nontrivial since $\sum_{j=1}^{k+1} n_j \neq 0$. Therefore, $\mathfrak{g} = 0$.

(iii) As in (4.11), we write the operator $\mathfrak{f}$ as a (possibly infinite) linear combination of finite products of the Heisenberg generators:

$$\mathfrak{q}_{m_1}(\beta_1) \cdots \mathfrak{q}_{m_i}(\beta_i) \tag{4.13}$$

where those $\mathfrak{q}_{m_j}(\beta_j)$ with $m_j < 0$ are put to the right. Note that

$$\mathfrak{f} = \epsilon(\mathfrak{f}) - W_0^{k+2}(\lambda_k).$$

Now (i) says that $\epsilon(\mathfrak{f})$ is a (possibly infinite) linear combination of finite products of at most $(k+1)$ Heisenberg generators. Also, from the discussion in the paragraph preceding Theorem 4.12, we see that each product in the operator $W_0^{k+2}(\lambda_k)$ has $(k+2)$ factors of $\mathfrak{q}$'s. Therefore, the leading term of $\mathfrak{f}$ is $-W_0^{k+2}(\lambda_k)$. □

5. The cohomology ring structure

In this section, we study the cohomology ring structure of $\mathbb{H}_n = H^*(X^{[n]})$. In particular, we find the ring generators of $H^*(X^{[n]})$. The basic idea is to introduce certain operator $\mathfrak{G}(\gamma) \in \text{End}(\mathbb{H})$ for $\gamma \in H^*(X)$, which generalizes the considerations in [Leh]. We show that $\mathfrak{G}(\gamma)$ satisfies the assumptions in Theorem 4.12. So the leading term of $\mathfrak{G}(\gamma)$ is related to the operator $W_n^k(\alpha)$ from Definition 4.3 (ii). Then an inductive procedure proves our main result.

Definition 5.1. (i) For $\gamma \in H^*(X)$ and $n \geq 0$, define

$$G(\gamma, n) = p_{1*}(\text{ch}(\mathcal{O}_{\mathcal{Z}_n}) \cdot p_2^* \text{td}(X) \cdot p_2^* \gamma) \in H^*(X^{[n]}) \tag{5.2}$$

where $\text{ch}(\mathcal{O}_{\mathcal{Z}_n})$ is the Chern character of the sheaf $\mathcal{O}_{\mathcal{Z}_n}$, $\text{td}(X)$ is the Todd class of X, and p_1, p_2 are the natural projections of $X^{[n]} \times X$ to $X^{[n]}, X$ respectively. We define the linear operator $\mathfrak{G}(\gamma) \in \text{End}(\mathbb{H})$ by putting

$$\mathfrak{G}(\gamma) = \bigoplus_{n \geq 0} G(\gamma, n)$$

where $G(\gamma, n)$ acts on $\mathbb{H}_n = H^*(X^{[n]})$ by the cup product;

(ii) For $i \in \mathbb{Z}$ and $\gamma \in H^s(X)$, define $G_i(\gamma, n)$ to be the component of $G(\gamma, n)$ in $H^{s+2i}(X^{[n]})$. We define the operator $\mathfrak{G}_i(\gamma) \in \text{End}(\mathbb{H})$ by

$$\mathfrak{G}_i(\gamma) = \bigoplus_{n \geq 0} G_i(\gamma, n)$$

where again $G_i(\gamma, n)$ acts on $\mathbb{H}_n = H^*(X^{[n]})$ by the cup product.

Notice that $\mathfrak{G}_i(\gamma) \in \text{End}(\mathbb{H})$ is homogeneous of bi-degree $(0, |\gamma| + 2i)$. Moreover,

$$\mathfrak{G}_i(\gamma)' = 0 \qquad \text{and} \qquad \mathfrak{G}_i(\gamma)^\dagger = \mathfrak{G}_i(\gamma). \tag{5.3}$$

By comparing the degrees on both sides of (5.2), we see that

$$G(\gamma,n)=\sum_{i\in\mathbb{Z}}G_i(\gamma,n).$$

Therefore, by the definition of the operators $\mathfrak{G}(\gamma)$ and $\mathfrak{G}_i(\gamma)$, we have

$$\mathfrak{G}(\gamma)=\sum_{i\in\mathbb{Z}}\mathfrak{G}_i(\gamma). \tag{5.4}$$

When $\gamma = 1_X$, we have $G(1_X,n) = p_{1*}(\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n})\cdot p_2^*\,\mathrm{td}(X))$. So we see from the Grothendieck-Riemann-Roch Theorem [Har] that

$$G(1_X,n)=\mathrm{ch}(p_{1*}\mathcal{O}_{\mathcal{Z}_n}) \tag{5.5}$$

where we have made no distinction between an algebraic cycle and its corresponding cohomology class. In particular, we have the following two formulas:

$$\mathfrak{G}_0(1_X)=\bigoplus_n G_0(1_X,n)=\bigoplus_n(n\cdot\mathrm{Id}_{X^{[n]}})=\mathfrak{L}_0(-1_X), \tag{5.6}$$

$$\mathfrak{G}_1(1_X)=\bigoplus_n G_1(1_X,n)=\bigoplus_n c_1(p_{1*}\mathcal{O}_{\mathcal{Z}_n})=\mathfrak{d}. \tag{5.7}$$

Our next lemma and its proof are parallel to the Theorem 4.2 and its proof in [Leh]. Together with (5.3), this lemma enables us to apply Theorem 4.12.

Lemma 5.8. *Let $\gamma,\alpha\in H^*(X)$. Then, we have*

$$[\mathfrak{G}(\gamma),\mathfrak{q}_1(\alpha)]=\exp\,(\mathrm{ad}(\mathfrak{d}))(\mathfrak{q}_1(\gamma\alpha));$$

or equivalently, using the component $\mathfrak{G}_k(\gamma)$ of $\mathfrak{G}(\gamma)$ (see (5.4)), we obtain

$$[\mathfrak{G}_k(\gamma),\mathfrak{q}_1(\alpha)]=\frac{1}{k!}\cdot\mathfrak{q}_1^{(k)}(\gamma\alpha). \tag{5.9}$$

Proof. Recall the standard diagram (2.2) and the exact sequence (2.3). We have

$$\psi_X^*\,\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n})-\phi_X^*\,\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_{n-1}})=\rho_X^*\,\mathrm{ch}(\mathcal{O}_{\Delta_X})\cdot p_1^*\,\exp\,(-[E_n]) \tag{5.10}$$

where we have used the fact that $\mathrm{ch}(\rho_X^*\mathcal{O}_{\Delta_X}\otimes p_1^*\mathcal{O}_{X^{[n,n-1]}}(-E_n))=\rho_X^*\,\mathrm{ch}(\mathcal{O}_{\Delta_X})\cdot p_1^*\,\mathrm{ch}(\mathcal{O}_{X^{[n,n-1]}}(-E_n))=\rho_X^*\,\mathrm{ch}(\mathcal{O}_{\Delta_X})\cdot p_1^*\,\exp\,(-[E_n])$.

Claim. $\psi^*G(\gamma,n)=\phi^*G(\gamma,n-1)+\rho^*\gamma\cdot\,\exp\,(-[E_n])$.

Proof. For $i\geq 1$, let $p_{i,1}$ and $p_{i,2}$ be the projections of $X^{[i]}\times X$ to $X^{[i]}$ and X respectively. From the definition of $G(\gamma,n)$, we see that

$$\begin{aligned}\psi^*G(\gamma,n)&=\psi^*(p_{n,1})_*(\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n})\cdot(p_{n,2})^*\,\mathrm{td}(X)\cdot(p_{n,2})^*\gamma)\\&=p_{1*}\psi_X^*(\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n})\cdot(p_{n,2})^*\,\mathrm{td}(X)\cdot(p_{n,2})^*\gamma)\\&=p_{1*}(\psi_X^*\,\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n})\cdot p_2^*\,\mathrm{td}(X)\cdot p_2^*\gamma)\end{aligned}$$

where p_2 is the projection of $X^{[n,n-1]} \times X$ to X. Similarly,

$$\phi^* G(\gamma, n-1) = p_{1*}(\phi_X^* \operatorname{ch}(\mathcal{O}_{\mathcal{Z}_{n-1}}) \cdot p_2^* \operatorname{td}(X) \cdot p_2^* \gamma).$$

So combining these with the formula (5.10) above, we conclude that

$$\begin{aligned}
&\psi^* G(\gamma, n) - \phi^* G(\gamma, n-1) \\
&= p_{1*}((\psi_X^* \operatorname{ch}(\mathcal{O}_{\mathcal{Z}_n}) - \phi_X^* \operatorname{ch}(\mathcal{O}_{\mathcal{Z}_{n-1}})) \cdot p_2^* \operatorname{td}(X) \cdot p_2^* \gamma) \\
&= p_{1*}(\rho_X^* \operatorname{ch}(\mathcal{O}_{\Delta_X}) \cdot p_1^* \exp(-[E_n]) \cdot p_2^* \operatorname{td}(X) \cdot p_2^* \gamma) \\
&= p_{1*}(\rho_X^* \operatorname{ch}(\mathcal{O}_{\Delta_X}) \cdot p_2^* \operatorname{td}(X) \cdot p_2^* \gamma) \cdot \exp(-[E_n]) \\
&= p_{1*}\rho_X^* (\operatorname{ch}(\mathcal{O}_{\Delta_X}) \cdot (p_{1,2})^* \operatorname{td}(X) \cdot (p_{1,2})^* \gamma) \cdot \exp(-[E_n]) \\
&= \rho^* (p_{1,1})_* (\operatorname{ch}(\mathcal{O}_{\Delta_X}) \cdot (p_{1,2})^* \operatorname{td}(X) \cdot (p_{1,2})^* \gamma) \cdot \exp(-[E_n]).
\end{aligned}$$

Thus, it remains to show that over $X \times X$, we have

$$(p_{1,1})_* (\operatorname{ch}(\mathcal{O}_{\Delta_X}) \cdot (p_{1,2})^* \operatorname{td}(X) \cdot (p_{1,2})^* \gamma) = \gamma.$$

Indeed, applying the Grothendieck-Riemann-Roch Theorem to the diagonal embedding $\tau_2 : X \to X \times X$, we get $\operatorname{ch}(\mathcal{O}_{\Delta_X}) \cdot (p_{1,2})^* \operatorname{td}(X) = \tau_{2*}[X]$. So

$$\begin{aligned}
&(p_{1,1})_* (\operatorname{ch}(\mathcal{O}_{\Delta_X}) \cdot (p_{1,2})^* \operatorname{td}(X) \cdot (p_{1,2})^* \gamma) \\
&= (p_{1,1})_* (\tau_{2*}[X] \cdot (p_{1,2})^* \gamma) = \gamma.
\end{aligned}$$ □

We continue the proof of the lemma. By (2.11), for any $a \in H^*(X^{[n-1]})$,

$$\mathfrak{q}_1(\alpha)(a) = \tilde{p}_{1*}([Q^{[n,n-1]}] \cdot \tilde{\rho}^* \alpha \cdot \tilde{p}_2^* a).$$

Define $\iota : X^{[n,n-1]} \to X^{[n]} \times X \times X^{[n-1]}$ by putting $\iota(\xi, \eta) = (\xi, \rho(\xi, \eta), \eta)$. Then, ι is an embedding. Moreover, $\iota_*[X^{[n,n-1]}] = [Q^{[n,n-1]}]$. Thus, we have

$$\begin{aligned}
\mathfrak{q}_1(\alpha)(a) &= \tilde{p}_{1*}(\iota_*[X^{[n,n-1]}] \cdot \tilde{\rho}^* \alpha \cdot \tilde{p}_2^* a) \\
&= \tilde{p}_{1*}\iota_*([X^{[n,n-1]}] \cdot (\iota^* \circ \tilde{\rho}^*)\alpha \cdot (\iota^* \circ \tilde{p}_2^*) a) \\
&= \psi_*([X^{[n,n-1]}] \cdot \rho^* \alpha \cdot \phi^* a).
\end{aligned}$$

Combining this with the above Claim, we conclude that

$$\begin{aligned}
\mathfrak{G}(\gamma)\mathfrak{q}_1(\alpha)(a) &= G(\gamma, n) \cdot \psi_*([X^{[n,n-1]}] \cdot \rho^* \alpha \cdot \phi^* a) \\
&= \psi_*(\psi^* G(\gamma, n) \cdot [X^{[n,n-1]}] \cdot \rho^* \alpha \cdot \phi^* a) \\
&= \psi_*(\phi^* G(\gamma, n-1) \cdot [X^{[n,n-1]}] \cdot \rho^* \alpha \cdot \phi^* a) \\
&\quad + \psi_*(\rho^* \gamma \cdot \exp(-[E_n]) \cdot [X^{[n,n-1]}] \cdot \rho^* \alpha \cdot \phi^* a) \\
&= \psi_*([X^{[n,n-1]}] \cdot \phi^* G(\gamma, n-1) \rho^* \alpha \cdot \phi^* a) \\
&\quad + \psi_*((\exp(-[E_n]) \cdot [X^{[n,n-1]}]) \cdot \rho^*(\gamma\alpha) \cdot \phi^* a).
\end{aligned}$$

Thus, $[\mathfrak{G}(\gamma), \mathfrak{q}_1(\alpha)](a) = \psi_*((\exp(-[E_n]) \cdot [X^{[n,n-1]}]) \cdot \rho^*(\gamma\alpha) \cdot \phi^* a)$ which is equal to $\exp(\operatorname{ad}(\mathfrak{d}))(\mathfrak{q}_1(\gamma\alpha))(a)$ by the Lemma 3.9 in [Leh]. □

Proposition 5.11. *Assume that $n \in \mathbb{Z}$ and $\gamma, \alpha \in H^*(X)$. Then, the commutator $[\mathfrak{G}(\gamma), \mathfrak{q}_n(\alpha)]$ satisfies the transfer property, i.e., we have*

$$[\mathfrak{G}(\gamma), \mathfrak{q}_n(\alpha)] = [\mathfrak{G}(1_X), \mathfrak{q}_n(\gamma\alpha)] = [\mathfrak{G}(\gamma\alpha), \mathfrak{q}_n(1_X)]. \tag{5.12}$$

Proof. Since the operator $\mathfrak{G}(\gamma)$ is self-adjoint, we see from (2.8) and (2.12) that we need only to prove (5.12) for positive integers n. In the following, we shall use induction on these positive integers n. By Lemma 5.8, (5.12) is true for $n = 1$. Assume that (5.12) is true for n. Since $[\mathfrak{G}(\gamma), \mathfrak{q}_1(1_X)] = \exp(\mathrm{ad}(\mathfrak{d}))(\mathfrak{q}_1(\gamma))$, the same proof to (3.14) (replacing the operator $\mathfrak{f}$ there by $\mathfrak{G}(\gamma)$) yields

$$[\mathfrak{G}(\gamma), \mathfrak{q}_{n+1}(\alpha)] = -\frac{1}{n}\left\{\sum_{j=0}^{+\infty}\frac{1}{j!}\cdot[\mathfrak{q}_1^{(j+1)}(\gamma), \mathfrak{q}_n(\alpha)] - [[\mathfrak{G}(\gamma), \mathfrak{q}_n(\alpha)], \mathfrak{q}_1'(1_X)]\right\}.$$

Now the transfer property of $[\mathfrak{G}(\gamma), \mathfrak{q}_{n+1}(\alpha)]$ follows from the induction hypothesis and the transfer property of $[\mathfrak{q}_1^{(j+1)}(\gamma), \mathfrak{q}_n(\alpha)]$ in Proposition 3.17. □

Theorem 5.13. *Let $k \in \mathbb{Z}$ be nonnegative, $\gamma \in H^*(X)$ be a nonzero cohomology class, and $\epsilon(\mathfrak{G}_k(\gamma)) = \mathfrak{G}_k(\gamma) + W_0^{k+2}(\gamma)$. Then,*

(i) *for all $n_1, \ldots, n_{k+2} \in \mathbb{Z}$ and all $\alpha_1, \ldots, \alpha_{k+2} \in H^*(X)$,*

$$[\ldots[[\epsilon(\mathfrak{G}_k(\gamma)), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+2}}(\alpha_{k+2})] = 0;$$

(ii) *for all $n_1, \ldots, n_{k+1} \in \mathbb{Z}$ with $\sum_{j=1}^{k+1} n_j \neq 0$ and all $\alpha_1, \ldots, \alpha_{k+1} \in H^*(X)$,*

$$[\ldots[[\epsilon(\mathfrak{G}_k(\gamma)), \mathfrak{q}_{n_1}(\alpha_1)], \ldots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})] = 0;$$

(iii) *the leading term of $\mathfrak{G}_k(\gamma)$ is $-W_0^{k+2}(\gamma)$;*
(iv) *$\mathfrak{G}_0(\gamma) = -W_0^2(\gamma)$;*
(v) *$\mathfrak{G}_1(\gamma) = -W_0^3(\gamma)$ if K_X is numerically trivial. In particular,*

$$\mathfrak{G}_1(1_X) = \mathfrak{d} = -W_0^3(1_X). \tag{5.14}$$

Proof. The first three statements follow from (5.3), (5.9) and Theorem 4.12.

To prove (iv), recall from the definitions that $W_0^2(\gamma) = \mathfrak{L}_0(\gamma)$. By (5.6), we may assume that γ is not a scalar multiple of 1_X. So $|\gamma| > 0$. Now,

$$[\mathfrak{G}_0(\gamma), \mathfrak{q}_n(\alpha)] = [\mathfrak{G}_0(1_X), \mathfrak{q}_n(\gamma\alpha)] = [-\mathfrak{L}_0(1_X), \mathfrak{q}_n(\gamma\alpha)] \tag{5.15}$$

by (5.12) and (5.6). In view of Theorem 2.16 (ii), we have

$$[\mathfrak{G}_0(\gamma) + W_0^2(\gamma), \mathfrak{q}_n(\alpha)] = [-\mathfrak{L}_0(1_X), \mathfrak{q}_n(\gamma\alpha)] + [\mathfrak{L}_0(\gamma), \mathfrak{q}_n(\alpha)] = 0$$

for all $n \in \mathbb{Z}$ and $\alpha \in H^*(X)$. Since $\mathbb{H}$ is irreducible, $\mathfrak{G}_0(\gamma) + W_0^2(\gamma)$ must be a scalar multiple of the identity operator, which has to be zero since the bi-degree of $(\mathfrak{G}_0(\gamma) + W_0^2(\gamma))$ is $(0, |\gamma|)$. So we have $\mathfrak{G}_0(\gamma) = -W_0^2(\gamma)$.

Finally, to prove (v), we notice from Lemma 4.6 that

$$[W_0^3(\gamma), \mathfrak{q}_n(\alpha)] = -nW_n^2(\gamma\alpha) = -n\mathfrak{L}_n(\gamma\alpha). \tag{5.16}$$

Next, we have $\mathfrak{G}_1(1_X) = \mathfrak{d}$ in view of (5.7). By (5.12) and Theorem 2.16 (iv),

$$[\mathfrak{G}_1(\gamma), \mathfrak{q}_n(\alpha)] = [\mathfrak{G}_1(1_X), \mathfrak{q}_n(\gamma\alpha)] = \mathfrak{q}_n'(\gamma\alpha) = n\mathfrak{L}_n(\gamma\alpha). \tag{5.17}$$

Combining this with (5.16), we see that $[\mathfrak{G}_1(\gamma) + W_0^3(\gamma), \mathfrak{q}_n(\alpha)] = 0$ for all $n \in \mathbb{Z}$ and $\alpha \in H^*(X)$. As argued in the proof of (iv), we have $\mathfrak{G}_1(\gamma) = -W_0^3(\gamma)$. $\square$

We remark that a result parallel to Theorem 5.13 (v) has been proved by Frenkel and Wang [F-W] within the framework of wreath products (compare p.205 of [Leh]). Also, $-W_0^k(\gamma)$ is the degree-0 component of the vertex operator in (4.5).

Definition 5.18. Fix a positive integer n. Define $\mathbb{H}_n'$ to be the subring of $\mathbb{H}_n = H^*(X^{[n]})$ generated by the following cohomology classes:

$$G_i(\gamma, n) = \mathfrak{G}_i(\gamma)(1_{X^{[n]}}) \tag{5.19}$$

where $0 \le i < n$ and γ runs over a linear basis of $H^*(X)$.

Note that the subring $\mathbb{H}_n'$ is generated by $(n \cdot \dim H^*(X))$ elements. Our goal is to show that $\mathbb{H}_n' = \mathbb{H}_n$, i.e., the cohomology ring $\mathbb{H}_n = H^*(X^{[n]})$ is generated by those $(n \cdot \dim H^*(X))$ classes $G_i(\gamma, n)$ in (5.19). We shall use induction on the reverse lexicographic order $\prec$ of all the partitions $\mu = (\mu_1, \mu_2, \ldots)$ of n, where μ_i denotes the number of parts in the partition μ equal to i (compare with the proof of Theorem 4.10 in [Leh]). Under this ordering, the partition $(n, 0, ...)$ is the smallest. Our induction goes as follows. First of all, we prove that

$$\prod_{j=1}^{n} \mathfrak{q}_1(\alpha_j)|0\rangle \in \mathbb{H}_n' \tag{5.20}$$

for all $\alpha_j \in H^*(X)$ in Lemma 5.23 below. Then, by assuming that

$$\prod_{i\ge1}\prod_{j=1}^{\mu_i'} \mathfrak{q}_i(\beta_{i,j})|0\rangle \in \mathbb{H}_n' \tag{5.21}$$

for all $\beta_{i,j} \in H^*(X)$ and all $\mu' = (\mu_1', \mu_2', \ldots) \prec \mu = (\mu_1, \mu_2, \ldots)$, we prove that

$$\prod_{i\ge1}\prod_{j=1}^{\mu_i} \mathfrak{q}_i(\alpha_{i,j})|0\rangle \in \mathbb{H}_n' \tag{5.22}$$

for all $\alpha_{i,j} \in H^*(X)$. Now we begin with the statement and proof of Lemma 5.23.

Lemma 5.23. *For all cohomology classes $\alpha_j \in H^*(X)$, we have*

$$\prod_{j=1}^{n} \mathfrak{q}_1(\alpha_j)|0\rangle \in \mathbb{H}'_n. \tag{5.24}$$

Proof. We shall use induction on k to show that for all $\alpha_1, \dots, \alpha_k \in H^*(X)$,

$$\prod_{j=1}^{k} \mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(1_X)^{n-k}|0\rangle \in \mathbb{H}'_n. \tag{5.25}$$

First of all, we claim that (5.25) is true for $k = 0$. Indeed, in view of (5.6),

$$\mathfrak{q}_1(1_X)^n|0\rangle = n! \cdot 1_{X^{[n]}} = (n-1)! \cdot G_0(1_X, n) \in \mathbb{H}'_n.$$

Next, assuming that (5.25) is true for some k with $0 \le k < n$, we shall verify that (5.25) holds as well if k is replaced by $(k+1)$. We may assume that α_{k+1} is homogeneous and $|\alpha_{k+1}| = s$. By (5.9), we have $[\mathfrak{G}_0(\alpha_{k+1}), \mathfrak{q}_1(\alpha)] = \mathfrak{q}_1(\alpha_{k+1}\alpha)$. Taking the cup product of (5.25) with $G_0(\alpha_{k+1}, n) \in \mathbb{H}'_n$, we obtain

$$\begin{aligned}
\mathbb{H}'_n \ &\ni G_0(\alpha_{k+1}, n) \cdot \left(\prod_{j=1}^{k} \mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(1_X)^{n-k}|0\rangle\right) \\
&= \mathfrak{G}_0(\alpha_{k+1}) \prod_{j=1}^{k} \mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(1_X)^{n-k}|0\rangle \\
&= \sum_{i=1}^{k} \pm \mathfrak{q}_1(\alpha_1)\cdots\mathfrak{q}_1(\alpha_{i-1})\mathfrak{q}_1(\alpha_{k+1}\alpha_i)\mathfrak{q}_1(\alpha_{i+1})\cdots\mathfrak{q}_1(\alpha_k)\mathfrak{q}_1(1_X)^{n-k}|0\rangle \\
&\quad + (-1)^{s\sum_{j=1}^{k}|\alpha_j|}(n-k) \cdot \prod_{j=1}^{k} \mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(\alpha_{k+1})\mathfrak{q}_1(1_X)^{n-(k+1)}|0\rangle \\
&\equiv (-1)^{s\sum_{j=1}^{k}|\alpha_j|}(n-k) \cdot \prod_{j=1}^{k+1} \mathfrak{q}_1(\alpha_j)\mathfrak{q}_1(1_X)^{n-(k+1)}|0\rangle \quad (\text{mod } \mathbb{H}'_n)
\end{aligned}$$

where we have used the induction hypothesis in the first and last steps. So (5.25) holds if k is replaced by $(k+1)$. This completes the proof of (5.24). □

Lemma 5.26. *Fix a, b with $1 \le a \le b$. Let $\mathfrak{g} \in \mathrm{End}(\mathbb{H})$ be of bi-degree (ℓ, s), and*

$$A = \mathfrak{q}_{m_1}(\beta_1)\cdots\mathfrak{q}_{m_b}(\beta_b)|0\rangle.$$

Then, $\mathfrak{g}(A)$ *is equal to the sum of the following two terms:*

$$\sum_{i=0}^{a-1} \sum_{\sigma_i} \pm \prod_{\ell \in \sigma_i^0} \mathfrak{q}_{m_\ell}(\beta_\ell)[[\cdots[\mathfrak{g}, \mathfrak{q}_{m_{\sigma_i(1)}}(\beta_{\sigma_i(1)})], \cdots], \mathfrak{q}_{m_{\sigma_i(i)}}(\beta_{\sigma_i(i)})]|0\rangle \tag{5.27}$$

and

$$\sum_{\sigma_a} (-1)^{\sum_{k=0}^{a-1}(s+\sum_{\ell=1}^{k}|\beta_{\sigma_a(\ell)}|)\sum_{\sigma_a(k)<j<\sigma_a(k+1)}|\beta_j|} \prod_{\ell \in \sigma_a^1} \mathfrak{q}_{m_\ell}(\beta_\ell) \cdot$$

$$\cdot [[\cdots[\mathfrak{g}, \mathfrak{q}_{m_{\sigma_a(1)}}(\beta_{\sigma_a(1)})], \cdots], \mathfrak{q}_{m_{\sigma_a(a)}}(\beta_{\sigma_a(a)})] \prod_{\ell \in \sigma_a^2} \mathfrak{q}_{m_\ell}(\beta_\ell)|0\rangle \tag{5.28}$$

where for each fixed i *with* $0 \le i \le a$*,* σ_i *runs over all the maps*

$$\{1, \ldots, i\} \to \{1, \ldots, b\}$$

satisfying $\sigma_i(1) < \cdots < \sigma_i(i)$*. Moreover,* $\sigma_i^0 = \{\ell \mid 1 \le \ell \le b, \ell \ne \sigma_i(1), \ldots, \sigma_i(i)\}$*,* $\sigma_a^1 = \{\ell \mid 1 \le \ell < \sigma_a(a), \ell \ne \sigma_a(1), \ldots, \sigma_a(a)\}$*, and* $\sigma_a^2 = \{\ell \mid \sigma_a(a) < \ell \le b\}$*.*

Proof. Note that for all i with $0 \le i < a$ and for all the above σ_i, we move

$$[\cdots[\mathfrak{g}, \mathfrak{q}_{m_{\sigma_i(1)}}(\beta_{\sigma_i(1)})], \cdots], \mathfrak{q}_{m_{\sigma_i(i)}}(\beta_{\sigma_i(i)})]$$

all the way to the right. This produces (5.27). In doing so, we obtain (5.28) by repeatedly applying the elementary fact that

$$\mathfrak{g}_1\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_2] + (-1)^{s_1 s_2}\mathfrak{g}_2\mathfrak{g}_1 \tag{5.29}$$

for two operators $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathrm{End}(\mathbb{H})$ of bi-degrees $(\ell_1, s_1), (\ell_2, s_2)$ respectively. □

Theorem 5.30. *For* $n \ge 1$*, the cohomology ring* $\mathbb{H}_n = H^*(X^{[n]})$ *is generated by*

$$G_i(\gamma, n) = \mathfrak{G}_i(\gamma)(1_{X^{[n]}})$$

where $0 \le i < n$ *and* γ *runs over a linear basis of* $H^*(X)$*. Moreover, the relations among these generators are precisely the relations among the restrictions* $\mathfrak{G}_i(\gamma)|_{\mathbb{H}_n}$ *of the corresponding operators* $\mathfrak{G}_i(\gamma)$ *to* $\mathbb{H}_n$*.*

Proof. Note that the second statement follows from the fact that the operators $\mathfrak{G}_i(\gamma)|_{\mathbb{H}_n}$ are defined in terms of the cup products by the cohomology classes $G_i(\gamma, n)$. In the following, we prove the first statement.

Let $\mathbb{H}_n'$ be defined as in Definition 5.18. We want to show that $\mathbb{H}_n' = \mathbb{H}_n$. As indicated in the paragraph following Definition 5.18, we use induction on the reverse lexicographic order $\prec$ of all the partitions $\mu = (\mu_1, \mu_2, \ldots)$ of n. By Lemma 5.23, (5.20) is true. In the following, assuming that (5.21) is true for all

$\beta_{i,j} \in H^*(X)$ and all $\mu' = (\mu'_1, \mu'_2, \dots) \prec \mu = (\mu_1, \mu_2, \dots)$, we shall prove (5.22).

Since $\mu = (\mu_1, \mu_2, \dots) \neq (n, 0, \dots)$, we let a be the smallest index such that $a > 1$ and $\mu_a \geq 1$ (i.e., $\mu = (\mu_1, 0, \dots, 0, \mu_a, \mu_{a+1}, \dots)$). Also, some of the classes $\alpha_{1,j}$ in (5.22) might be (scalar multiples of) 1_X. Without loss of generality, we may assume that $\alpha_{1,1} = \dots = \alpha_{1,r-a} = 1_X$ for some $r \geq a$ and that $|\alpha_{1,j}| > 0$ for all j satisfying $(r-a) < j \leq \mu_1$. Then (5.22) can be rewritten as

$$\mathfrak{q}_1(1_X)^{r-a} \prod_{j=(r-a+1)}^{\mu_1} \mathfrak{q}_1(\alpha_{1,j})\mathfrak{q}_a(\alpha_{a,1}) \prod_{j=2}^{\mu_a} \mathfrak{q}_a(\alpha_{a,j}) \prod_{i>a} \prod_{j=1}^{\mu_i} \mathfrak{q}_i(\alpha_{i,j})|0\rangle \in \mathbb{H}'_n. \tag{5.31}$$

Put $\mu' = (a + \mu_1, 0, \dots, 0, \mu_a - 1, \mu_{a+1}, \dots)$ and

$$\begin{aligned} A &\stackrel{\text{def}}{=} \mathfrak{q}_1(1_X)^r \textstyle\prod_{j=(r-a+1)}^{\mu_1} \mathfrak{q}_1(\alpha_{1,j}) \prod_{j=2}^{\mu_a} \mathfrak{q}_a(\alpha_{a,j}) \prod_{i>a} \prod_{j=1}^{\mu_i} \mathfrak{q}_i(\alpha_{i,j})|0\rangle \\ &\stackrel{\text{def}}{=} \mathfrak{q}_{m_1}(\beta_1) \cdots \mathfrak{q}_{m_b}(\beta_b)|0\rangle \end{aligned} \tag{5.32}$$

where $b = (a + \mu_1) + (\mu_a - 1) + \mu_{a+1} + \dots$ is the length of μ'. We have

$$m_1 = \dots = m_{a+\mu_1} = 1, m_i > 1 \text{ for } (a + \mu_1) < i \leq b, \tag{5.33}$$

$$\beta_1 = \dots = \beta_r = 1_X, |\beta_i| > 0 \text{ for } r < i \leq (a + \mu_1). \tag{5.34}$$

Since A corresponds to the partition μ' and $\mu' \prec \mu$, $A \in \mathbb{H}'_n$ by induction.

Claim. (5.31) is true as long as $|\alpha_{a,1}| = 4$.

Proof. Note that $a \leq n$. So we see from the definition of $\mathbb{H}'_n$ that $G_{a-1}(\alpha_{a,1}, n) \in \mathbb{H}'_n$. Taking the cup product of $A \in \mathbb{H}'_n$ with $G_{a-1}(\alpha_{a,1}, n) \in \mathbb{H}'_n$ yields

$$\mathbb{H}'_n \ni G_{a-1}(\alpha_{a,1}, n) \cdot A = \mathfrak{G}_{a-1}(\alpha_{a,1})(A).$$

Put $\mathfrak{e} = \epsilon(\mathfrak{G}_{a-1}(\alpha_{a,1})) = \mathfrak{G}_{a-1}(\alpha_{a,1}) + W_0^{a+1}(\alpha_{a,1})$. Then,

$$\mathbb{H}'_n \ni \mathfrak{G}_{a-1}(\alpha_{a,1})(A) = -W_0^{a+1}(\alpha_{a,1})(A) + \mathfrak{e}(A). \tag{5.35}$$

Applying Lemma 5.26 to the operator $\mathfrak{e}$, we see that $\mathfrak{e}(A)$ consists of two parts (5.27) and (5.28). By (5.33), the number of $\mathfrak{q}_1$'s in every nonvanishing term of (5.27) is at least $(a + \mu_1) - (a - 1) = \mu_1 + 1 > \mu_1$. So every nonvanishing term in (5.27) corresponds to some partition $\mu' \prec \mu$. By induction hypothesis, (5.27) is contained in $\mathbb{H}'_n$. By Theorem 5.13 (ii) (replacing the integer k there by $(a - 1)$), we see that (5.28) is 0. In summary, $\mathfrak{e}(A) \in \mathbb{H}'_n$. By (5.35), we obtain

$$-W_0^{a+1}(\alpha_{a,1})(A) \in \mathbb{H}'_n. \tag{5.36}$$

Now we apply Lemma 5.26 to the operator $-W_0^{a+1}(\alpha_{a,1})$. So $-W_0^{a+1}(\alpha_{a,1})(A)$ consists of two parts (5.27) and (5.28). Again, (5.27) is contained in $\mathbb{H}_n'$. Let $N(\sigma_a)$ be the number of $\mathfrak{q}_1$'s in a nonvanishing term in (5.28) corresponding to σ_a. By (5.33), $N(\sigma_a) \geq (a+\mu_1) - a = \mu_1$. If $N(\sigma_a) > \mu_1$, then by induction hypothesis, this nonvanishing term in (5.28) corresponding to σ_a is contained in $\mathbb{H}_n'$. Also, $N(\sigma_a) = \mu_1$ if and only if $m_{\sigma_a(1)} = \ldots = m_{\sigma_a(a)} = 1$, i.e.,

$$1 \leq \sigma_a(1) < \ldots < \sigma_a(a) \leq (a+\mu_1)$$

by (5.33). So this nonvanishing term in (5.28) is of the form:

$$(-1)^{\sum_{k=0}^{a-1}(4+\sum_{\ell=1}^{k}|\beta_{\sigma_a(\ell)}|)\sum_{\sigma_a(k)<j<\sigma_a(k+1)}|\beta_j|} \cdot \prod_{\ell\in\sigma_a^1} \mathfrak{q}_1(\beta_\ell)\cdot$$

$$\cdot[[\cdots[-W_0^{a+1}(\alpha_{a,1}), \mathfrak{q}_1(\beta_{\sigma_a(1)})], \cdots], \mathfrak{q}_1(\beta_{\sigma_a(a)})] \prod_{\ell\in\sigma_a^2} \mathfrak{q}_{m_\ell}(\beta_\ell)|0\rangle$$

which can be simplified to the following by Proposition 4.10 (i):

$$(-1)^{\sum_{k=0}^{a-1}(4+\sum_{\ell=1}^{k}|\beta_{\sigma_a(\ell)}|)\sum_{\sigma_a(k)<j<\sigma_a(k+1)}|\beta_j|} \cdot \prod_{\ell\in\sigma_a^1} \mathfrak{q}_1(\beta_\ell)\cdot$$

$$\cdot(-1)^{a+1}\cdot \mathfrak{q}_a(\alpha_{a,1}\beta_{\sigma_a(1)}\cdots\beta_{\sigma_a(a)}) \prod_{\ell\in\sigma_a^2} \mathfrak{q}_{m_\ell}(\beta_\ell)|0\rangle. \qquad (5.37)$$

Since $\alpha_{a,1} \in H^4(X)$, the term (5.37) being nonzero forces

$$|\beta_{\sigma_a(1)}| = \ldots = |\beta_{\sigma_a(a)}| = 0. \qquad (5.38)$$

Since $1 \leq \sigma_a(1) < \ldots < \sigma_a(a) \leq (a+\mu_1)$, we see from (5.34) that $1 \leq \sigma_a(1) < \ldots < \sigma_a(a) \leq r$. So (5.37) can be further simplified to

$$\begin{aligned}
&(-1)^{a+1}\cdot \textstyle\prod_{\ell\in\sigma_a^1} \mathfrak{q}_1(1_X)\mathfrak{q}_a(\alpha_{a,1}) \prod_{\ell>\sigma_a(a)} \mathfrak{q}_{m_\ell}(\beta_\ell)|0\rangle \\
&= (-1)^{a+1}\cdot \mathfrak{q}_1(1_X)^{|\sigma_a^1|}\mathfrak{q}_a(\alpha_{a,1})\mathfrak{q}_1(1_X)^{r-a-|\sigma_a^1|}\cdot \\
&\quad \cdot \textstyle\prod_{j=(r-a+1)}^{\mu_1} \mathfrak{q}_1(\alpha_{1,j}) \prod_{j=2}^{\mu_a} \mathfrak{q}_a(\alpha_{a,j}) \prod_{i>a}\prod_{j=1}^{\mu_i} \mathfrak{q}_i(\alpha_{i,j})|0\rangle \qquad (5.39)\\
&= (-1)^{a+1}\cdot \mathfrak{q}_1(1_X)^{r-a} \textstyle\prod_{j=(r-a+1)}^{\mu_1} \mathfrak{q}_1(\alpha_{1,j})\cdot \\
&\quad \cdot\mathfrak{q}_a(\alpha_{a,1}) \textstyle\prod_{j=2}^{\mu_a} \mathfrak{q}_a(\alpha_{a,j}) \prod_{i>a}\prod_{j=1}^{\mu_i} \mathfrak{q}_i(\alpha_{i,j})|0\rangle.
\end{aligned}$$

Now there are exactly $\binom{r}{a}$ such terms in (5.28). Therefore, by (5.36), we have

$$\binom{r}{a}\cdot(-1)^{a+1}\cdot\prod_{j=1}^{\mu_1}\mathfrak{q}_1(\alpha_{1,j})\mathfrak{q}_a(\alpha_{a,1})\prod_{j=2}^{\mu_a}\mathfrak{q}_a(\alpha_{a,j})\prod_{i>a}\prod_{j=1}^{\mu_i}\mathfrak{q}_i(\alpha_{i,j})|0\rangle \in \mathbb{H}_n'.$$

It follows that (5.31) is true as long as $|\alpha_{a,1}| = 4$. □

We continue the proof of Theorem 5.30. In view of the above Claim, it remains to prove that if there is an integer s such that $0 \le s \le 3$ and (5.31) is true as long as $|\alpha_{a,1}| \ge (s+1)$, then (5.31) holds as well if $|\alpha_{a,1}| = s$. In view of Theorem 5.13, we put $\tilde{\mathfrak{e}} = \epsilon(\mathfrak{G}_{a-1}(\alpha_{a,1})) = \mathfrak{G}_{a-1}(\alpha_{a,1}) + W_0^{a+1}(\alpha_{a,1})$. Then,

$$\mathbb{H}'_n \ni \mathfrak{G}_{a-1}(\alpha_{a,1})(A) = -W_0^{a+1}(\alpha_{a,1})(A) + \tilde{\mathfrak{e}}(A). \tag{5.40}$$

As in the proof of the above Claim, we see from Lemma 5.26 that $\mathbb{H}'_n$ contains

$$\sum_{\sigma_a} (-1)^{\sum_{k=0}^{a-1}(s+\sum_{\ell=1}^{k}|\beta_{\sigma_a(\ell)}|)\sum_{\sigma_a(k)<j<\sigma_a(k+1)}|\beta_j|} \cdot \prod_{\ell\in\sigma_a^1} \mathfrak{q}_1(\beta_\ell)\cdot$$

$$\cdot(-1)^{a+1}\cdot \mathfrak{q}_a(\alpha_{a,1}\beta_{\sigma_a(1)}\cdots\beta_{\sigma_a(a)}) \prod_{\ell\in\sigma_a^2} \mathfrak{q}_{m_\ell}(\beta_\ell)|0\rangle \tag{5.41}$$

where σ_a runs over all the maps $\{1,\ldots,a\} \to \{1,\ldots,b\}$ with $\sigma_a(1) < \cdots < \sigma_a(a)$, and satisfies $m_{\sigma_a(1)} = \ldots = m_{\sigma_a(a)} = 1$. So by (5.33), we have $\sigma_a(1) < \cdots < \sigma_a(a) \le (a+\mu_1)$. Note that every nonvanishing term in (5.41) corresponds to the partition μ. Moreover, if $|\beta_{\sigma_a(1)}|+\cdots+|\beta_{\sigma_a(a)}| > 0$, then $|\alpha_{a,1}|+|\beta_{\sigma_a(1)}|+\cdots+|\beta_{\sigma_a(a)}| \ge (s+1)$; so this nonvanishing term in (5.41) is already contained in $\mathbb{H}'_n$ by our assumption. Therefore, the subring $\mathbb{H}'_n$ contains

$$\sum_{\sigma_a} (-1)^{\sum_{k=0}^{a-1}(s+\sum_{\ell=1}^{k}|\beta_{\sigma_a(\ell)}|)\sum_{\sigma_a(k)<j<\sigma_a(k+1)}|\beta_j|} \cdot \prod_{\ell\in\sigma_a^1} \mathfrak{q}_1(\beta_\ell)\cdot$$

$$\cdot(-1)^{a+1}\cdot \mathfrak{q}_a(\alpha_{a,1}\beta_{\sigma_a(1)}\cdots\beta_{\sigma_a(a)}) \prod_{\ell\in\sigma_a^2} \mathfrak{q}_{m_\ell}(\beta_\ell)|0\rangle \tag{5.42}$$

where σ_a runs over all the maps $\{1,\ldots,a\} \to \{1,\ldots,b\}$ with $\sigma_a(1) < \cdots < \sigma_a(a) \le (a+\mu_1)$, and satisfies $|\beta_{\sigma_a(1)}| = \cdots = |\beta_{\sigma_a(a)}| = 0$. So we have $\sigma_a(1) < \cdots < \sigma_a(a) \le r$ by (5.34). Now as in the last paragraph (starting from the line below (5.37)) in the proof of the above Claim, we obtain

$$\binom{r}{a}\cdot(-1)^{a+1}\cdot\prod_{j=1}^{\mu_1}\mathfrak{q}_1(\alpha_{1,j})\mathfrak{q}_a(\alpha_{a,1})\prod_{j=2}^{\mu_a}\mathfrak{q}_a(\alpha_{a,j})\prod_{i>a}\prod_{j=1}^{\mu_i}\mathfrak{q}_i(\alpha_{i,j})|0\rangle \in \mathbb{H}'_n.$$

So (5.31) holds if $|\alpha_{a,1}| = s$. This completes the proof of Theorem 5.30. □

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