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A priori estimates for a semilinear elliptic system without variational structure and their applications

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1 Introduction

Consider the weakly coupled semilinear elliptic system

$$\begin{array}{l}\Delta u + f(x,u,v) = 0,\\ \Delta v + g(x,u,v) = 0,\end{array} \qquad u,\ v \geq 0 \quad \text{in } \Omega, \tag{I}$$

together with the homogeneous Dirichlet boundary condition

$$u = v = 0 \quad \text{in } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded smooth domain. Recently, there have been significant studies of (I), see [1–4, 6–7, 14–16, 20–23] and the references therein. System (I) arises from studying various nonlinear phenomena, such as pattern formation, population evolution, chemical reaction, etc., where u and v represent concentrations of different species in the process. Naturally positive solutions of (I) is of particular interest.

In this paper, we are concerned with the question of existence of a pair of smooth functions u and v satisfying (I). Our interest partly lies in the fact that semi-linear systems such as (I) need not have a variational structure (cf. the case of single equations) nor be cooperative. Specifically, we shall establish a priori estimates and existence of positive solutions to general semilinear elliptic system (I). In particular, we do not require (I) be variational or cooperative.

It is well known that the (algebraic) growth of the nonlinearity f plays a critical role in treating semilinear equations

$$\Delta u + f(u) = 0.$$

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For instance, when $f = u^p$ (i.e., the celebrated Lane-Emden equation), the Sobolev exponent $(n+2)/(n-2)$ is the dividing number for existence of positive solutions with homogeneous Dirichelt condition $u = 0$ (at least for star-shaped domain in some sense, see for instance [9, 10, 17, 19]), in terms of the exponent p. Corresponding a priori estimates can also be established [11].

There are two natural variational structures associated with (I). That is, the so called potential structure with

$$I(u, v) = \int \left\{(|\nabla u|^2 + |\nabla v|^2) - 2H(x, u, v)\right\} \tag{1.1}$$

and the Hamiltonian structure with

$$J(u, v) = \int \{\nabla u \cdot \nabla v - H(x, u, v)\}\,. \tag{1.2}$$

A prototype model of (I) to exhibit the distinct nature resulting from the structures (1.1) and (1.2), respectively, occurs when

$$H(x, u, v) = \frac{u^{p+1}}{p+1} + \frac{v^{q+1}}{q+1}.$$

For (1.1), (I) simply decouples to two single equations

$$\Delta u + u^p = 0, \qquad \Delta v + v^q = 0$$

then there is no (significant) impact of the exponents p and q on each other and techniques used for single equations remain applicable. For (1.2), on the other hand, (I) becomes

$$\Delta v + u^p = 0, \qquad \Delta u + v^q = 0, \tag{1.3}$$

which is a natural extension of the well-known Lane-Emden equation and thus is referred to as the Lane-Emden system. This is the case where the exponents p and q interplay, compensating each other. It has been known for certain cases that the hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}, \tag{1.4}$$

plays the role of dividing curve in terms of exponents p and q for existence for the Lane-Emden system, directly extending the well-known results for the Lane-Emden equation, see Sects 3 and 5 for details.

Let $a,\ b,\ c,\ d$ and $p,\ q,\ r,\ s$ be non-negative numbers satisfying

$$p,\ q > 0, \qquad pq > 1, \qquad r,\ s > 1.$$

Put

$$\alpha = \frac{2(p+1)}{pq-1} > 0, \qquad \beta = \frac{2(q+1)}{pq-1} > 0.$$

We first have the following a priori estimate for the dimension $n = 3$.

Theorem 1.1 *Let $n = 3$ and (u, v) be a non-negative C^2-solution of* (I) *with*

$$f(x, u, v) = au^r + bv^q, \qquad g(x, u, v) = cu^p + dv^s. \tag{1.5}$$

Then there exists a positive constant $M = M(n, a, b, c, d, p, q, r, s)$ such that

$$||u||_{L^\infty(\Omega)} \leq M, \qquad ||v||_{L^\infty(\Omega)} \leq M,$$

provided that

(i) $\alpha + \beta > 1$,
(ii) $\max\{r, s\} < 5$, *and*
(iii) $\beta \neq 2/(r - 1)$ *and* $\alpha \neq 2/(s - 1)$.

Theorem 1.1 is optimal in the sense that if either condition (i) or (ii) is violated, the conclusion need no longer hold, in view of non-existence of positive solutions for non-linearities with *super-critical* growth on bounded star-shaped domains. Moreover, the existence for *super-critical* non-linearities on the entire space $\mathbb{R}^n$ shows that the proof of Theorem 1.1 breaks-down, see section 5 for details.

In Sect. 3, we extend Theorem 1.1 (Theorem 3.1) to systems with a general non-linearity which satisfies a growth condition (G). The extension is clearly as sharp in the same spirit. That is, Theorem 3.1 need not hold, if either condition (i) or (ii) does not meet. Moreover, Theorem 1.1 extends to higher dimension $n > 3$ for general non-linearities, see Theorem 3.2. However, the extension for $n > 3$ is not sharp in the above sense, due to a lack of non-existence for the Lane-Emden system in the *full sub-critical* range, see Sect. 5 again.

The proof of Theorem 1.1 is based on a blow-up argument, see for example [1, 11, 24] and the references therein. The argument, in turn, relies on non-existence theorems for positive solutions of the limiting system after blow-up (either a system of equations or a single equation on the entire space $\mathbb{R}^n$ or on the half space $\mathbb{R}^n_+$).

We say a solution (u, v) of (I) is non-trivial if one of the components u and v is non-trivial. With the help of Theorem 1.1, we are able to prove the following existence result.

Theorem 1.2 *Let $n = 3$ and let f and g be given by* (1.5). *Suppose the conditions in Theorem* 1.1 *are satisfied. We further assume that*

$$p,\ q > 1, \qquad a + b > 0, \qquad c + d > 0.$$

Then the system (I) *has a classical non-negative nontrivial solution.*

The proof of Theorem 1.2, which does not require a variational structure, is based on a fixed point theorem on positive cones as well as the a priori estimate Theorem 1.1. In particular, Theorem 1.2 is sharp in the same sense of Theorem 1.1, see Sect. 5. One important feature of the proof of Theorem 1.1 is that it applies to general non-linearities with certain growth restriction in u and v at infinity. Therefore, both Theorems 1.1 and 1.2 apply to a wide class of general non-linearities f and g with suitable growth, see Sectis. 3 and 4. It is also worth to point out that the non-linearities may depend on the independent variable x and the gradients ∇u and ∇v, as well as change sign. Needless to say, suitable conditions must be imposed and it is left to the reader .

When (I) is irreducible in the sense that

$$f(x, 0, v) \neq 0 \text{ for } v > 0; \qquad g(x, u, 0) \neq 0 \text{ for } u > 0,$$

then Theorem 1.2 can be strengthened.

Corollary 1.1 *Let $n = 3$ and let f and g be given by* (1.5). *Suppose the conditions in Theorem* 1.1 *are satisfied. We further assume that*

$$p,\ q > 1, \qquad b > 0, \qquad c > 0.$$

Then the system (I) *has a classical positive* (*component-wise*) *solution.*

This is an immediate consequence of Theorem 1.2 and the strong maximum principles.

The following example shows that Theorems 1.1–1.2 and Corollary 1.1 can still hold for nonlinearities f and g changing sign for $u, v > 0$.

Theorem 1.3 *Let $n = 3$ and let*

$$f(x, u, v) = u^r - u^a v^b + v^q, \qquad g(x, u, v) = u^p - u^c v^d + v^s,$$

where $a, b, c, d \geq 1$ are positive numbers. Suppose that (i)–(iii) *of Theorem* 1.1 *hold. Then the conclusion of Theorems* 1.1–1.2 *and Corollary* 1.1 *continues to hold, provided that*

(iv) *$a < r, b < q, c < p, d < s$ and*

$$\max\left\{\left(\frac{a}{r} + \frac{b}{q}\right), \left(\frac{c}{p} + \frac{d}{s}\right)\right\} < 1.$$

Clearly for $u > 0$ small, one has

$$f(u, u^{r/q}) < 0; \qquad g(u, u^{p/s}) < 0.$$

That is, both f and g are negative for some values of small positive u and v. Nevertheless Theorem 1.3 assures that (I) has a positive solution (component-wise). Note particularly that (I) is neither variational nor cooperative.

The organization of the paper is as follows. In Sect. 2 we prove Theorem 1.1. We extend Theorem 1.1 to general functions f and g and to the dimension $n > 3$ in Sect. 3. The existence will be established in Sect. 4. We finally discuss several examples in Sect. 5.

2 A priori estimates I: $n = 3$

In this section, we shall establish a priori estimates for positive solutions of (I) when the dimension $n = 3$.

Recall that $p,\ q > 0$ with $pq > 1$, $r,\ s > 1$, and

$$\alpha = \frac{2(p+1)}{pq-1} > 0, \qquad \beta = \frac{2(q+1)}{pq-1} > 0.$$

We first prove a special case of Theorem 1.1.

Theorem 2.1 *Let* $n = 3$ *and* (u, v) *be a non-negative* C^2*-solution of* (I) *with*

$$f(u, v) = u^r + v^q, \qquad g(u, v) = u^p + v^s.$$

Then there exists a constant $M = M(n, p, q, r, s) > 0$ *such that*

$$||u||_{L^\infty(\Omega)} \le M, \qquad ||v||_{L^\infty(\Omega)} \le M, \tag{2.1}$$

provided that

(i) $\alpha + \beta > 1$,
(ii) $\max\{r, s\} < 5$, *and*
(iii) $\beta \ne 2/(r-1)$ *and* $\alpha \ne 2/(s-1)$.

As mentioned in the introduction, by a blow-up argument, the proof of Theorem 2.1 reduces to one of Lemmas 2.1–2.3 below. The first two lemmas are non-existence results for the Lane-Emden system.

Lemma 2.1 (Serrin and Zou [21]) *Let* $n = 3$ *and suppose that* p, q *are positive numbers such that either* $pq \le 1$ *or* $\alpha + \beta > 1$*. Then the Lane-Emden system*

$$\Delta u + v^q = 0, \quad \Delta v + u^p = 0, \qquad x \in \mathbb{R}^n \tag{2.2}$$

does not admit non-trivial non-negative solutions (u, v) *with algebraic growth at infinity.*

When $\Omega = \mathbb{R}^n_+ = \{x_n > 0\}$, Lemma 2.1 was extended by Birindelli and Mitidieri to arbitrary dimension $n \ge 3$ for bounded solutions.

Lemma 2.2 (Birindelli and Mitidieri [1]) *Let* $n \ge 3$ *and* $p, q > 1$*. Then the Lane-Emden system*

$$\Delta u + v^q = 0, \quad \Delta v + u^p = 0, \qquad x \in \mathbb{R}^n_+ \tag{2.3}$$

together with zero boundary condition $u = v = 0$ *on* $\partial\mathbb{R}^n_+ = \{x_n = 0\}$ *does not admit bounded non-trivial non-negative solutions* (u, v)*, provided*

$$\max(\alpha, \beta) \ge (n-3).$$

Remarks. 1. Note that Lemma 2.2 holds for all $p, q > 1$ when $n = 3$.

2. When $n = 4$, with the aid of Lemma 2.1, one slightly improves Lemma 2.2 under the condition

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{1}{3}.$$

The following non-existence result for the Lane-Emden equation is due to Gidas and Spruck.

Lemma 2.3 (Gidas and Spruck [10, 11]) *Let $p \in (1, (n+2)/(n-2))$ and suppose that u is a non-negative solution of*

$$\Delta u + u^p = 0, \qquad x \in \Omega. \tag{2.4}$$

Then $u \equiv 0$ if either $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$ with $u = 0$ on $\partial\mathbb{R}^n_+$.

Remark. When $\Omega = \mathbb{R}^n_+$, Dancer [5] extended the above range $p \in (1, (n+2)/(n-2))$ to $p \in (1, (n+1)/(n-3))$ for bounded solutions.

The following formula is by direct calculations.

Lemma 2.4 *Let (u, v) be a positive solution of* (I) *with $f(u, v) = u^r + v^q$ and $g(u, v) = u^p + v^s$. For $\xi \in \Omega$ and $S,\ l_1,\ l_2 > 0$, put*

$$\bar{u}(y) = S^{-1}u(x), \quad \bar{v}(y) = S^{-l_1}v(x), \qquad y = (x-\xi)S^{l_2}. \tag{2.5}$$

Then

$$\Delta\bar{u} + S^{r-1-2l_2}\bar{u}^r + S^{ql_1-1-2l_2}\bar{v}^q = 0,$$
$$\Delta\bar{v} + S^{p-l_1-2l_2}\bar{u}^p + S^{sl_1-l_1-2l_2}\bar{v}^s = 0.$$

After these preparations, we can prove Theorem 2.1.

Proof of Theorem 2.1. The proof is based on contradiction. Suppose that Theorem 2.1 is false. Then there exists a sequence of solutions $\{u_k(x), v_k(x)\}_{k=1}^{\infty}$ of (I) such that

$$\lim_{k\to\infty}(||u_k||_{L^\infty(\Omega)} + ||v_k||_{L^\infty(\Omega)}) = \infty. \tag{2.6}$$

For $k = 1, 2, \cdots$, put

$$M_k = \sup_{x\in\Omega_k} u_k(x) = u_k(\tau_k), \quad N_k = \sup_{x\in\Omega_k} v_k(x) = v_k(\zeta_k),$$

where $\tau_k, \zeta_k \in \Omega$.

In Lemma 2.4, we take

$$l_1 = \frac{\alpha}{\beta} > 0, \qquad l_2 = \frac{1}{\beta} > 0,$$

and

$$u = u_k, \qquad v = v_k, \qquad S = S_k = M_k + N_k^{1/l_1} \to \infty.$$

The choice of ξ will be determined later. Clearly

$$ql_1 - 1 - 2l_2 = p - l_1 - 2l_2 = 0.$$

Moreover, the pair $(\bar{u}_k, \bar{v}_k)$ given by (2.5) satisfy

$$\bar{u}_k \le 1, \qquad \bar{v}_k \le 1, \tag{2.7}$$

and

$$\Delta \bar{u}_k + S_k^l \bar{u}_k^r + \bar{v}_k^q = 0, \qquad \Delta \bar{v}_k + \bar{u}_k^p + S_k^m \bar{v}_k^s = 0, \tag{2.8}$$

where

$$l = r-1-2l_2 = r-1-2/\beta \neq 0, \qquad m = sl_1 - l_1 - 2l_2 = (\alpha(s-1)-2)/\beta \neq 0$$

by assumption (iii).

We shall consider several cases in terms of the parameter l and m values. Clearly (2.8) is symmetric in l and m (i.e., in u and v) and we shall only treat different l values in the following two cases.

Case 1. $l > 0$. We further divide the proof into two subcases.

(i). $m > 0$. We first show

$$\lim_{k\to\infty} \frac{M_k}{N_k^{1/l_1}} = 0. \tag{2.9}$$

Taking $\xi = \tau_k$, then obviously it is equivalent to show

$$\lim_{k\to\infty} \bar{u}_k(0) = 0.$$

Suppose for contradiction this is not true. Then there exist $\epsilon_0 > 0$ and a subsequence (still using same subscripts) such that

$$\bar{u}_k(0) \ge \epsilon_0, \qquad k = 1, 2, \cdots. \tag{2.10}$$

Put

$$\tilde{u}_k(z) = \bar{u}_k(y), \qquad \tilde{v}_k(z) = \bar{v}_k(y), \qquad z = y S_k^{l/2}.$$

Therefore, by $(2.8)_1$, $\tilde{u}_k(z)$ and $\tilde{v}_k(z)$ are bounded and satisfy

$$\tilde{u}_k(0) \in [\epsilon_0, 1), \qquad \Delta \tilde{u}_k + \tilde{u}_k^r + S_k^{-l/2} \tilde{v}_k^q = 0.$$

For each k, denote

$$d_k = \operatorname{dist}(\tau_k, \partial\Omega), \qquad n_k = S_k^{-l_2 - l/2} \to 0.$$

There are two possibilities. First, assume that the sequence $\{d_k/n_k\}$ is unbounded. Then, by standard elliptic theory, the sequence $\{\tilde{u}_k\}$ (extracting a subsequence if necessary) converges uniformly to a non-negative function $\tilde{u} \in C^2(\mathbb{R}^n)$ on any compact subset $\Sigma \subset \mathbb{R}^n$. Moreover, $\tilde{u}$ satisfies (2.4) with $\Omega = \mathbb{R}^n$ since obviously by (2.7) and the assumption $l > 0$

$$\lim_{k\to\infty} S_k^{-l/2}\tilde{v}_k(z) = 0$$

uniformly on $\mathbb{R}^n$. Thus $\tilde{u} \equiv 0$ by Lemma 2.3 since $r \in (1, 5)$ by assumption (ii), an immediate contradiction in view of (2.10).

Next suppose that $\{d_k/n_k\}$ is bounded. Thanks to the smooth (C^1) boundary condition, the sequence $\{d_k/n_k\}$ is bounded away from zero (standard by elliptic estimates, see [11] or [12]). In this case, there exist $s > 0$ and a non-negative function $\tilde{u} \in C^2(\mathbb{R}^n_s)$, satisfying (2.4) with

$$\Omega = \mathbb{R}^n_s = \mathbb{R}^n \cap \{x^n > -s\}, \qquad \tilde{u} = 0 \quad \text{on} \quad \partial\mathbb{R}^n_s.$$

Thus $\tilde{u} \equiv 0$ by Lemma 2.3, which yields a contradiction again. And (2.9) is proved.

Next, since $m > 0$, we utilize $(2.8)_2$ and take $\xi = \zeta_k$ to derive

$$\lim_{k\to\infty} \frac{N_k^{1/l_1}}{M_k} = 0.$$

This is impossible, in view of (2.9).

(ii). $m < 0$. Since $l > 0$, thus (2.9) holds. Now taking $\xi = \zeta_k$, then clearly one has

$$\bar{v}_k(0) \to 1, \qquad \bar{u}_k(y) \le \bar{u}_k((\tau_k - \zeta_k)S_k^{l_2}) = \max \bar{u}_k \to 0 \tag{2.11}$$

as $k \to \infty$. Proceeding as in (i), with the aid of the fact $m < 0$ and (2.11), we pass to a limit in $(2.8)_2$ to infer that there exists $\bar{v} \in C^2(\mathbb{R}^n)$ such that

$$\Delta\bar{v} = 0, \qquad x \in \mathbb{R}^n, \qquad \bar{v}(0) = 1,$$

provided the sequence $\{d_k/n_k\}$ is unbounded, and for some $s > 0$

$$\Delta\bar{v} = 0, \qquad x \in \mathbb{R}^n_s, \qquad \bar{v}(0) = 1, \qquad \bar{v}\Big|_{\partial\mathbb{R}^n_s} = 0,$$

provided $\{d_k/n_k\}$ is bounded, where

$$d_k = \text{dist}(\zeta_k, \partial\Omega), \qquad n_k = S_k^{-l_2} \to 0.$$

The second case cannot happen, since the Phragmèn-Lindelöf principle [18] implies $\bar{v}$ vanishes identically, contradicting the fact $\bar{v}(0) = 1$. If the first possibility occurs, then

$$\bar{v} \equiv \bar{v}(0) = 1,$$

since all bounded harmonic functions on $\mathbb{R}^n$ must be constant. In turn,

$$\lim_{k\to\infty} \bar{v}_k(y) = 1$$

uniformly for $y \in B = B_1(0)$. Moreover, by (2.9)

$$\lim_{k\to\infty} \bar{u}_k(y) = 0$$

uniformly on B. On the other hand, applying Green's formula to $(2.8)_1$ on B yields

$$\begin{aligned} 0 \leftarrow \bar{u}_k(0) &= \int_{\partial B} \bar{u}_k(x)\frac{\partial G}{\partial \nu}(x,0)d\sigma + \int_B [S_k^l \bar{u}_k^r + \bar{v}_k^q]G(x,0)dx \\ &\geq \int_{\partial B} \bar{u}_k(x)\frac{\partial G}{\partial \nu}(x,0)d\sigma + \int_B \bar{v}_k^q G(x,0)dx \to \int_B G(x,0)dx = c_n \end{aligned}$$

as $k \to \infty$, where $G(x, y)$ is the Green function on B, an absurdity.

Case 2. $l < 0$. Again we consider two cases.

(i). $m > 0$. The proof is essentially the same as that of (ii) of Case 1 (being a mirror image) and the detail is left to the reader.

(ii). $m < 0$. Plainly,

$$M_k S_k^{-1} + N_k S_k^{-l_1} \geq c > 0.$$

In turn, without loss of generality (by taking $\xi = \tau_k$ or $\xi = \zeta_k$ accordingly), we may assume

$$\bar{u}_k(0) + \bar{v}_k(0) \geq c > 0.$$

Letting $k \to \infty$ in (2.8), similarly as in Case 1, one deduces that there exist $\bar{u} \geq 0$ and $\bar{v} \geq 0$ satisfying either (2.2) or (2.3). Moreover

$$\bar{u}(0) + \bar{v}(0) \geq c > 0, \qquad \bar{u} + \bar{v} \leq 1.$$

This is impossible, in view of either Lemma 2.1 or Lemma 2.2.

It follows that (2.6) cannot hold and the proof is complete.

We conclude the section with the following proof of Theorem 1.1.

Proof of Theorem 1.1. The proof essentially reduces to that of Theorem 2.1. We shall consider different possibilities for the values of a, b, c and d.

Case 1. Either $b = 0$ or $c = 0$. It simply reduces to the case of single equations and the conclusion is well known.

Case 2. $a = 0$. Thus $b,\ c,\ d > 0$. Since $a = 0$, the term involving l does not appear and therefore one simply treats it as $l < 0$. If $m > 0$, (i) of Case 2 of Theorem 2.1 applies. If $m < 0$, one then proceeds exactly as (ii) of Case 2 of Theorem 2.1.

Case 3. $d = 0$. Thus $a,\ b,\ c > 0$. Since $d = 0$, the term involving m does not appear and thus one simply treats it as $m < 0$. It follows that arguments in Case 2 above ($a = 0$) apply.

Case 4. $a = d = 0$. Thus $b,\ c > 0$. Since $a = d = 0$, the terms involving l or m do not appear and thus one simply treats it as $l < 0$ and $m < 0$. It follows that arguments in (ii) of Case 2 of Theorem 2.1. apply.

This completes the proof of Theorem 1.1.

3 A priori estimates II: general cases

In this section, we consider system (I) with general functions f and g for $n \geq 3$, which may also depend on the independent variable x.

Suppose that

$$f:\ \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}; \qquad g:\ \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

are continuous functions and that

$$a:\ \overline{\Omega} \to \mathbb{R}; \quad b:\ \overline{\Omega} \to \mathbb{R}; \quad c:\ \overline{\Omega} \to \mathbb{R}; \quad d:\ \overline{\Omega} \to \mathbb{R}$$

are non-negative continuous functions. We further assume that $a(x)$, $b(x)$, $c(x)$ or $d(x)$ is either strictly positive or identically zero on $\overline{\Omega}$. We shall be needing the following growth condition of f and g at infinity.

(G) There exist positive numbers p, q with $pq > 1$ and $r, s > 1$ such that for $u,\ v \geq 0$ and fixed $x \in \Omega$

$$\lim_{u+v\to\infty} \frac{f(x,u,v)}{a(x)u^r + b(x)v^q} = 1, \qquad \lim_{u+v\to\infty} \frac{g(x,u,v)}{c(x)u^p + d(x)v^s} = 1.$$

We first generalize Theorem 1.1 to general functions f and g for $n = 3$.

Theorem 3.1 *Let $n = 3$ and (u, v) be a non-negative C^2-solution of* (I). *Suppose that the assumption* (G) *holds. Then there exists a positive constant $M = M(n, a, b, c, d, p, q, r, s)$ such that*

$$||u||_{L^\infty(\Omega)} \leq M, \qquad ||v||_{L^\infty(\Omega)} \leq M, \tag{3.1}$$

provided that

(i) $\alpha + \beta > 1$,
(ii) $\max\{r, s\} < 5$, *and*
(iii) $\beta \neq 2/(r-1)$ *and* $\alpha \neq 2/(s-1)$.

Remark. As pointed in the introduction, Theorem 3.1 is optimal, see Sect. 5.

Proof. The proof is essentially the same as before, here we only sketch it for the case $l > 0$ and $m > 0$. Suppose for contradiction that (3.1) is false. Then there exists a sequence of solutions $\{u_k(x), v_k(x)\}$ of (I) such that

$$\lim_{k\to\infty} (||u_k||_{L^\infty(\Omega)} + ||v_k||_{L^\infty(\Omega)}) = \infty. \tag{3.2}$$

We want to derive a contradiction to (3.2). In the light of Theorem 1.1, we shall assume that all functions $a(x), b(x), c(x), d(x)$ are strictly positive. As in Sect. 2, we use the same transform given in Lemma 2.4 with the same notation and want to show

$$\lim_{k\to\infty} \frac{M_k}{N_k^{1/l_1}} = 0. \tag{3.3}$$

Suppose the contrary. That is (up-to a subsequence, again),

$$\xi_k = \tau_k, \qquad \bar{u}_k(0) \geq \epsilon_0. \tag{3.4}$$

Put

$$\tilde{u}_k(z) = \bar{u}_k(y), \qquad \tilde{v}_k(z) = \bar{v}_k(y), \qquad z = yS_k^{l/2},$$

and

$$a_k(z) = a(\xi_k + S_k^{-l_2-l/2}z), \qquad b_k(z) = b(\xi_k + S_k^{-l_2-l/2}z)$$

and

$$c_k(z) = c(\xi_k + S_k^{-l_2-l/2}z), \qquad d_k(z) = b(\xi_k + S_k^{-l_2-l/2}z).$$

Without loss of generality, we may assume $\xi_k \to \xi_0$. In turn

$$a_0 = \lim_{k\to\infty} a_k(z) = a(\xi_0) > 0, \qquad b_0 = \lim_{k\to\infty} b_k(z) = b(\xi_0) > 0$$

and

$$c_0 = \lim_{k\to\infty} c_k(z) = c(\xi_0) > 0, \qquad d_0 = \lim_{k\to\infty} d_k(z) = d(\xi_0) > 0$$

uniformly on any compact subset of Γ (Γ is either $\mathbb{R}^3$ or $\mathbb{R}^3_s$, see below also) since $S_k \to \infty$ and $l_2 + l/2 > 0$.

By direct calculations, $\tilde{u}_k(z)$ and $\tilde{v}_k(z)$ are bounded globally and satisfy

$$\Delta\tilde{u}_k + S_k^{-1-2l_2-l} f(\xi_k + S_k^{-l_2-l/2}z, S_k\tilde{u}_k(z), S_k^{l_1}\tilde{v}_k(z)) = 0. \tag{3.5}$$

Using (G), there exists $M > 0$ such that for $u, v \geq 0$

$$|f(x, u, v)| \leq 2[a(x)u^r + b(x)v^q] + M.$$

It follows that

$$\begin{aligned} \left|S_k^{-1-2l_2-l} f(\xi_k + S_k^{-l_2-l/2}z, S_k\tilde{u}_k(z), S_k^{l_1}\tilde{v}_k(z))\right| &\leq (a_k\tilde{u}_k^r + b_k S_k^{-l/2}\tilde{v}_k^q) + o(1) \\ &\leq \bar{M}. \end{aligned} \tag{3.6}$$

Therefore, via standard elliptic theory and with the aid of (3.5) and (3.6), we deduce that there exists $\tilde{u} \in C^{1,\alpha}(\Gamma)$ (up-to a subsequence) such that

$$\lim_{k\to\infty} \tilde{u}_k(z) = \tilde{u}(z)$$

uniformly on any compact subset of Γ in $C^{1,\alpha}$-topology for any $\alpha \in (0,1)$. Moreover,

$$\tilde{u}(0) \geq \epsilon_0 \tag{3.7}$$

by (3.4). Hence clearly the condition (G) implies that (note $1 + 2l_2 + l = r$)

$$\begin{aligned}
&\lim_{k\to\infty} S_k^{-1-2l_2-l} f(\xi_k + S_k^{-l_2-l/2} z, S_k\tilde{u}_k(z), S_k^{l_1}\tilde{v}_k(z)) \\
&= \lim_{k\to\infty} \frac{f(\xi_k + S_k^{-l_2-l/2} z, S_k\tilde{u}_k(z), S_k^{l_1}\tilde{v}_k(z))}{a_k S_k^r \tilde{u}_k^r(z) + b_k S_k^{pl_1}\tilde{v}_k^q(z)} \cdot [a_k\tilde{u}_k^r(z) + b_k S_k^{-l/2}\tilde{v}_k^q(z)] \\
&= \lim_{k\to\infty} [a_k\tilde{u}_k^r(z) + b_k S_k^{-l/2}\tilde{v}_k^q(z)] = a_0\tilde{u}^r(z)
\end{aligned}$$

uniformly on any compact subset of Γ (up to a subsequence). It follows that $\tilde{u}(z)$ satisfies (in weak sense)

$$\Delta\tilde{u} + a_0\tilde{u}^r = 0 \quad \text{in } \Gamma$$

with appropriate boundary condition. It follows that $\tilde{u}(z) \equiv 0$ as before and yields a contradiction to (3.7). Thus (3.3) holds.

Proceeding similarly and using the fact $m > 0$ (taking $\xi_k = \zeta_k$) to derive

$$\lim_{k\to\infty} \frac{N_k^{1/l_1}}{M_k} = 0.$$

Apparently this is impossible in view of (3.3) and the proof is complete.

When $n > 3$, the Liouville type non-existence results for positive solutions of the Lane-Emden system are not available for all subcritical $(p,q)'s$, see Lemmas 3.1–3.3 below. Consequently, we have similar but weaker results than the case $n = 3$.

Theorem 3.2 *Let $n > 3$ and let (u, v) be a non-negative C^2-solution of* (I). *Suppose that* (G) *holds. Then there exists a positive constant $M = M(n, a, b, c, d, p, q, r, s)$ such that*

$$||u||_{L^\infty(\Omega)} \leq M, \qquad ||v||_{L^\infty(\Omega)} \leq M, \tag{3.8}$$

provided that all the following three conditions hold.

(i) $\max(\alpha, \beta) \geq n - 2$, *or,* $\max(p,q) \leq (n+2)/(n-2)$ *and* $\min(p,q) < (n+2)/(n-2)$,

(ii) $\max\{r, s\} < (n+2)/(n-2)$, *and*
(iii) $\beta \neq 2/(r-1)$ *and* $\alpha \neq 2/(s-1)$.

The proof is essentially the same as that of Theorem 3.1. However, we shall also be using the following non-existence results for $n > 3$.

Lemma 3.1 *Suppose that* $n \geq 3$ *and* p, q *are positive numbers satisfying either* $pq \leq 1$ *or*

$$\max(\alpha, \beta) \geq n-2.$$

Then the Lane-Emden system (2.2) *does not admit any non-negative and non-trivial solutions.*

The lemma was first proved by Mitidieri [15] for $p, q > 1$, and later extended to general cases in [21].

Lemma 3.2 (de Figueiredo and Felmer[7]) *Let* $n \geq 3$ *and Suppose that* p, q *are positive numbers satisfying*

$$\max(p, q) \leq \frac{n+2}{n-2} \text{ and } \min(p, q) < \frac{n+2}{n-2}.$$

Then the Lane-Emden system (2.2) *does not admit any non-negative and non-trivial solutions.*

Finally, one has a half-space version of Lemma 3.2 for bounded solutions.

Lemma 3.3 *Let* $n > 3$ *and suppose*

$$\max(p, q) \leq \frac{n+1}{n-3} \text{ and } \min(p, q) < \frac{n+1}{n-3}.$$

Then the Lane-Emden system (2.3) *does not admit any non-negative and non-trivial bounded solutions.*

The proof is the same as that of Lemma 2.2, using an argument of Dancer [5] and the nonexistence Lemma 3.2.

Remark. The approach applies to non-linearities depending on the gradients ∇u and ∇v. Indeed, the gradient part with suitable growth restrictions can be blown-out, i.e., disappearing from the limiting equation(s), and the proof carries over with little change.

4 Existence

In this section, we prove the existence result Theorem 1.2. First, we establish existence for system (I) under suitable assumptions on general functions f and g. Throughout this section, we use $\mathbf{w}$ to denote a pair of functions u and v, that is,

$$\mathbf{w} = (u, v), \qquad \text{with } ||\mathbf{w}|| = ||u|| + ||v||,$$

and assume that f and g satisfy the following hypotheses.

(H1) $f, g \in C^1(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R})$ with $f, g \not\equiv 0$.
(H2) There exist constants $\lambda, \mu \geq 0$ such that for $u,\ v \geq 0$

$$f(x, u, v) + \lambda u \geq 0, \qquad g(x, u, v) + \mu v \geq 0.$$

(H3) For $u,\ v \geq 0$

$$f(x, u, v) = o(u + v), \quad g(x, u, v) = o(u + v),$$

as $u + v \to 0$ uniformly on Ω.
(H4) There exists $M > 0$ such that for $u,\ v \geq 0$ and $x \in \Omega$

$$f(x, u, v) + g(x, u, v) \geq \lambda_1(u + v) - M,$$

where λ_1 is the first eigenvalue of $(-\Delta, H_0)$.

We say that system (I) has property (AP), provided that the following holds.

(AP) For $(\phi(x), \psi(x)) \in C(\overline{\Omega})$, let $\mathbf{w} = (u, v)$ be a non-negative solution of

$$\Delta u + f(x, u, v) = \phi,\ \ \Delta v + g(x, u, v) = \psi,\ \text{ in } \Omega; \quad u = v = 0,\ \text{ on } \partial\Omega.$$

Then there exists a positive constant $C = C(||(\phi, \psi)||_{L^\infty(\Omega)}) > 0$ (independent of $\mathbf{w}$) such that

$$||\mathbf{w}||_{L^\infty(\Omega)} \leq C.$$

Remark. Property (AP), via standard elliptic theory, implies

$$||\mathbf{w}||_{C^{2,\alpha}(\overline{\Omega})} \leq C_1$$

for all $\alpha \in (0, 1)$.

We have the following existence result.

Theorem 4.1 *The system* (I) *admits a classical non-negative non-trivial solution* $\mathbf{w}$, *provided that* (I) *has the property* (AP).

We shall apply the fixed point theory on a (positive) cone to prove Theorem 3.1. For $\alpha \in (0, 1)$, put

$$E = C_0^{2,\alpha}(\overline{\Omega}) \times C_0^{2,\alpha}(\overline{\Omega}), \qquad H = C^\alpha(\overline{\Omega}) \times C^\alpha(\overline{\Omega}), \tag{4.1}$$

where

$$C_0^{2,\alpha}(\overline{\Omega}) = C^{2,\alpha}(\overline{\Omega}) \cap C_0(\overline{\Omega})$$

is the usual Banach space equipped with a standard norm. Consider the operator

$$T = \begin{pmatrix} -\Delta + \lambda & 0 \\ 0 & -\Delta + \mu \end{pmatrix} :\ E \to H,$$

with λ, μ given before. It is obvious that T has a bounded inverse $T^{-1} : H \to E$. Set

$$T_0(\mathbf{w}) = \begin{pmatrix} f(x,u,v) + \lambda u \\ g(x,u,v) + \mu v \end{pmatrix} : H \to H, \qquad \mathbf{w} = (u,v) \in H.$$

Now consider

$$F = T^{-1} \circ T_0 : H \to H.$$

It is by standard elliptic theory that the operator F is compact.

The following fixed point theorem on a (positive) cone is due to de Figueiredo, Lions and Naussbaum [8], which is a modified version of a theorem of Krasnosel'skii [13].

Proposition 4.1 *Let C be a cone in a Banach space X and $T : C \to C$ a compact mapping such that $T(0) = 0$. Suppose that there exist numbers $t_0 > 0$ and $0 < r < R$ and a vector $v \in C - \{0\}$ such that*

1. *$x \neq tT(x)$ for $0 \leq t \leq 1$ and $\|x\| = r$,*
2. *$x \neq T(x) + tv$ for $t \geq 0$ and $\|x\| = R$.*
3. *$x \neq T(x) + tv$ for $t \geq t_0$ and $\|x\| \leq R$.*

Then if $U = \{x \in C : r < \|x\| < R\}$ and $B_\rho = \{x \in C : \|x\| < \rho\}$, one has

$$i_C(T, B_R) = 0, \quad i_C(T, B_r) = 1, \quad i_C(T, U) = -1.$$

In particular, T has a fixed point in U.

Proof. We refer the readers to [8] for a proof.

Proof of Theorem 4.1. We want to apply Proposition 4.1 to the operator F. Define

$$C = \{(u,v) \in H \mid u, v \geq 0\}.$$

Clearly C is a (positive) cone. We divide the proof into several steps.

Step 1. $F : C \to C$ is compact with $F(0) = 0$. Clearly $T_0(0) = 0$ since

$$f(x,0,0) = g(x,0,0) = 0$$

by (H3). Thus $F(0) = T^{-1} \circ T_0(0) = T^{-1}(0) = 0$. By standard elliptic theory, F is compact. Finally, we want to use a maximum principle to show $F : C \to C$. For $\mathbf{w_0} \in C$, consider

$$\mathbf{w} = F\mathbf{w_0}, \quad \text{i.e.,} \quad T\mathbf{w} = T_0\mathbf{w_0},$$

that is,

$$\begin{aligned} \Delta u - \lambda u + [f(x,u_0,v_0) + \lambda u_0] = 0, \\ \Delta v - \mu v + [g(x,u_0,v_0) + \mu v_0] = 0, \end{aligned} \qquad x \in \Omega.$$

By (H2), we have

$$f(x, u_0, v_0) + \lambda u_0 \geq 0, \qquad g(x, u_0, v_0) + \mu v_0 \geq 0$$

since $\mathbf{w}_0 \in C$. It is immediate that $u \geq 0$ and $v \geq 0$ by the maximum principle, that is, $\mathbf{w} \in C$.

Step 2. For $t \in [0, 1]$, there exists a positive number r such that

$$\mathbf{w} \neq tF(\mathbf{w}) \tag{4.2}$$

for $||x|| = r$. Consider

$$\mathbf{w} = tF(\mathbf{w})$$

for $t \in [0, 1]$, that is,

$$\begin{aligned} -\Delta u + \lambda u &= t[f(x, u, v) + \lambda u] \\ -\Delta v + \mu v &= t[g(x, u, v) + \mu v]. \end{aligned} \tag{4.3}$$

Multiply $(4.3)_1$ by u and $(4.3)_2$ by v respectively, and integrate over Ω to obtain

$$\int_\Omega (|\nabla u|^2 + |\nabla v|^2) + (1-t) \int_\Omega (\lambda u^2 + \mu v^2) = t \int_\Omega (f(x, u, v)u + g(x, u, v)v).$$

In turn,

$$\lambda_1 \int_\Omega (u^2 + v^2) \leq \int_\Omega |f(x, u, v)u + g(x, u, v)v|,$$

since $t \in [0, 1]$ and $\lambda, \mu \geq 0$. Therefore by (H3), there exists $\sigma > 0$ such that $||(u, v)||_H > \sigma$ or $(u, v) = 0$, and consequently (4.2) holds with the choice $r = \sigma$.

Step 3. There exist positive numbers t_0 and R and a vector $\mathbf{w}_0 = (u_0, v_0) \in C - \{0\}$ such that

$$\mathbf{w} \neq F(\mathbf{w}) + t\mathbf{w}_0 \tag{4.4}$$

for $t \geq t_0$ and $||x|| \leq R$.

Let ϕ_1 be a (normalized) first eigenfunction of $(-\Delta, H_0)$ and take

$$\mathbf{w}_0 = (\phi_1, \phi_1) \in C - \{0\}.$$

We shall show that (4.4) holds for this $\mathbf{w}_0$. Consider

$$\mathbf{w} = F(\mathbf{w}) + t\mathbf{w}_0,$$

that is,

$$\begin{aligned} \Delta u + t(\lambda_1 + \lambda)\phi_1 + f(x, u, v) &= 0, \\ \Delta v + t(\lambda_1 + \mu)\phi_1 + g(x, u, v) &= 0, \end{aligned} \qquad x \in \Omega. \tag{4.5}$$

Multiply both of (4.5) by ϕ_1 and integrate over Ω to obtain

$$t(2\lambda_1 + \lambda + \mu)\int_\Omega \phi_1^2 + \int_\Omega \phi_1[f(x,u,v) + g(x,u,v)] = \lambda_1 \int_\Omega \phi_1(u+v).$$

By (H4), there exists $M > 0$ such that

$$f(x,u,v) + g(x,u,v) \geq \lambda_1(u+v) - M$$

for $(u, v) \in C$. Therefore,

$$t \leq \frac{M\int_\Omega \phi_1}{2\lambda_1 + \lambda + \mu}.$$

It follows immediately that (4.4) holds by taking any $R > 0$ and

$$t_0 = \frac{2M\int_\Omega \phi_1}{2\lambda_1 + \lambda + \mu}.$$

Note particularly the choice of $R > 0$ can be arbitrary.

Step 4. There exists a positive number R such that

$$\mathbf{w} \neq F(\mathbf{w}) + t\mathbf{w}_0 \tag{4.6}$$

for $t \geq 0$ and $||x|| = R$, where the vector $\mathbf{w}_0 = (u_0, v_0) \in C - \{0\}$ is given in Step 3. Consider

$$\mathbf{w} = F(\mathbf{w}) + t\mathbf{w}_0. \tag{4.7}$$

Then $t \leq t_0/2$ for all $\mathbf{w} \in C$ by Step 3. By the assumption (AP), there exists $K > 0$ (depending on t_0) such that

$$||(u,v)||_H \leq K$$

since (u, v) satisfies (4.7). It follows that (4.6) holds for all $t \geq 0$ with the choice of $R = 2K$.

Step 5. Now we can finish the proof applying Proposition 4.1 with $X = H$. Plainly, all conditions of Proposition 4.1 are satisfied. Therefore the mapping F has a fixed point $\mathbf{w} \in C$ with $||\mathbf{w}|| \in [r, R]$, which is a non-negative non-trivial solution of (I). And the proof is complete.

Remarks. 1. As we remarked at the end of Sect. 3, our approach applies to nonlinearities depending on the gradients ∇u and ∇v with suitable restriction.

2. It is easy to see that the components u and v are either strictly positive or identically zero, via strong maximum principle.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. This is a special case of Theorem 4.1 since one readily verifies that f and g satisfy (H1)–(H4) and (I) has property (AP) (Theorem 3.1).

Corresponding to Theorems 3.1 and 3.2, we have the following existence theorems.

Corollary 4.1 *Let* $n = 3$ *and suppose* f *and* g *satisfy the condition* (G) *with*

(i) $\alpha + \beta > 1$,
(ii) $\max\{r, s\} < 5$, *and*
(iii) $\alpha \neq 2/(s-1)$ *and* $\beta \neq 2/(r-1)$.

Then the system (I) *admits a classical non-negative non-trivial solution* $\mathbf{w}$.

Corollary 4.2 *Let* $n > 3$ *and suppose* f *and* g *satisfy the condition* (G) *with*

(i) $\max(\alpha, \beta) \geq n-2$, *or,* $\max(p, q) \leq (n+2)/(n-2)$ *and* $\min(p, q) < (n+2)/(n-2)$,
(ii) $\max\{r, s\} < (n+2)/(n-2)$, *and*
(iii) $\alpha \neq 2/(s-1)$ *and* $\beta \neq 2/(r-1)$.

Then the system (I) *admits a classical non-negative non-trivial solution* $\mathbf{w}$.

The proofs are essentially the same as before and are left to the reader.
When (I) is irreducible, namely,

$$f(x, 0, v) \neq 0 \text{ for } v > 0; \qquad g(x, u, 0) \neq 0 \text{ for } u > 0,$$

Then the solutions of (I) obtained in Theorem 4.1 are necessarily positive.

Theorem 4.2 *Suppose the conditions given in Theorem* 4.1 *hold. Then the solutions of* (I) *obtained in Theorem* 4.1 *are necessarily positive, provided that* (I) *is irreducible.*

Proof. We need to show that $\mathbf{w}$ is strictly positive. Following the argument in step 1 of the proof of Theorem 4.1, a strong maximum principle argument shows that the components u and v are either strictly positive or identically zero. We claim neither component u or v can vanish identically. For otherwise, suppose $u \equiv 0$. Then we have

$$f(x, 0, v) = 0, \quad \text{i.e.,} \quad v \equiv 0$$

since (I) is irreducible. Therefore $u = v \equiv 0$. This impossible since (u, v) is a non-trivial solution.

Proof of Corollary 1.1. By Theorem 1.2, (I) has a non-trivial and non-negative solution $\mathbf{w}$. By Theorem 4.2, $\mathbf{w}$ is strictly positive since one readily sees that (I) is irreducible.

Proof of Theorem 1.3. (i). A priori estimates. By Theorem 3.1, we only need to verify that f and g satisfy the property (G). By (iv), we can choose positive numbers $l \in (1, r/a)$ and $m \in (1, a/b)$ such that

$$\frac{1}{l} + \frac{1}{m} = 1.$$

By the Young inequality, we have for $u, v \geq 0$,

$$u^a v^b \leq u^{la} + v^{mb}.$$

It follows that

$$\lim_{u+v\to\infty} \frac{f(x,u,v)}{u^r + v^q} = 1,$$

since $la < r$ and $mb < q$. Similarly, one readily checks that

$$\lim_{u+v\to\infty} \frac{g(x,u,v)}{u^p + v^s} = 1.$$

That is, f and g satisfy the property (G) and Theorem 3.1 applies. The first conclusion is proved.

(ii). Existence of a positive solution. We shall apply Theorem 4.1 and need to verify (H1)–(H4) and (AP).

Verification of (H1): Obvious.

Verification of (H2): For $u, v \geq 0$, take $l = q/(q-b)$ and $m = q/b$ and apply the Young inequality,

$$u^a v^b \leq (q-b)u^{aq/(q-b)} + bv^q/q.$$

It follows that

$$f(x,u,v) \geq u^r - (q-b)u^{aq/(q-b)} + (1-b/q)v^q \geq u^r - (q-b)u^{aq/(q-b)},$$

since $b < q$. Similarly,

$$g(x,u,v) \geq (1-c/p)u^p - (p-c)v^{dp/(p-c)} + v^s \geq -(p-c)v^{dp/(p-c)} + v^s,$$

since $c < p$. Therefore one can choose $\lambda, \mu \geq 0$ such that

$$f(x,u,v) + \lambda u \geq u^r - (q-b)u^{aq/(q-b)} + \lambda u \geq 0$$

since $aq/(q-b) \in [1, r)$, and

$$g(x,u,v) + \mu v \geq \mu v - (p-c)v^{dp/(p-c)} + v^s \geq 0,$$

since $dp/(p-c) \in [1, s)$ and (H2) is verified immediately.

Verification of (H3): Obvious.

Verification of (H4): Obvious.

Verification of (AP): Obvious by (i).

Therefore Theorem 4.1 applies. That is (I) has a non-trivial and non-negative solution (u, v). Plainly (I) is irreducible under our assumption and whence (u, v) must be positive by Theorem 4.2. The proof is complete.

5 Concluding remarks

Concerning existence as well as a priori estimates for positive solutions, it is well known that the Sobolev exponent

$$2^* = \frac{n+2}{n-2}$$

is the dividing number for the celebrated Lane-Emden equation. For the system (I), the a priori estimates and existence results for $n = 3$ (Theorems 1.1–1.3) established in this paper are also optimal in a similar spirit. We shall include several examples here to demonstrate the point.

The first example is the classical existence result of Fowler for the Lane-Emden equation with a super-critical growth.

Theorem 5.1 (Fowler [9]) *Let $n \geq 3$ and suppose $p \geq 2^*$. Then the Lane-Emden equation*

$$\Delta u + u^p = 0$$

has infinitely many positive solutions on the entire space $\mathbb{R}^n$.

In [22], the authors extended the above result to the Lane-Emden system.

Theorem 5.2 (Serrin and Zou [22]) *Let $n \geq 3$ and suppose*

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n}. \tag{5.1}$$

Then the Lane-Emden system

$$\Delta u + v^q = 0, \qquad \Delta v + u^p = 0$$

has infinitely many positive solutions on the entire space $\mathbb{R}^n$.

On bounded domains, non-existence is known when the domain is star-shaped for super-critical non-linearities.

Theorem 5.3 (Pohozaev [17]) *Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be a bounded smooth star-shaped domain. Then the Lane-Emden equation has no non-trivial non-negative solution provided $p \geq 2^*$.*

Theorem 5.4 (Mitidieri [14] and van der Vorst [23]) *Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be a bounded smooth star-shaped domain. Then the Lane-Emden system has no non-trivial non-negative solution provided that* (5.1) *holds.*

Theorems 5.1-5.4 clearly show that all a priori estimate and existence results for $n = 3$ are best possible when $n = 3$ in the sense that either condition (i) or (ii) is violated, then the conclusion can fail to hold.

Indeed, suppose that condition (i) is violated. That is, $\alpha + \beta \leq 1$, which is equivalent to (5.1) for $n = 3$ and thus Theorem 5.4 applies. Take $a = d = 0, b = c = 1$ and $p, q > 1$. Then (I) reduces to the Lane-Emden system and Theorem 5.4 says that (I) cannot have non-trivial non-negative solutions on bounded smooth star-shaped domains. Hence Theorem 1.2 and Corollaries 1.1 and 4.1 (note (I) is irreducible) do not hold. Plainly Theorems 1.1 and 3.1 must not hold either. For otherwise, system (I) would possess the (AP) property and Theorems 4.1 and 4.2 would apply, indicating (I) with a positive solution. This is of course impossible. A closer examination also reveals that the blow-up procedure breaks down. In fact, the blown-up equation in this case is precisely the Lane-Emden system, which does have positive solutions on the entire space $\mathbb{R}^n$ by Theorem 5.2 since (5.1) is satisfied.

Next, assume that condition (ii) is violated, say, $r \geq 5$. Take $b = c = 0$, $a = d = 1$ and $s > 1$. Apparently (I) reduces to the case of single equations and Theorems 5.1 and 5.3 apply. Therefore Theorems 1.1–1.2 and 3.1, and Corollaries 1.1 and 4.1 must fail, argued as above.

When $n > 3$, Theorem 3.2 and Corollary 4.2 are not optimal in the sense above, as shown by the following example.

Theorem 5.5 (Clement, de Figueiredo and Mitidieri [3]) *Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Then the Lane-Emden system has a positive solution provided that $p, q \geq 1$ satisfies*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}.$$

For the Lane-Emden system, the curve

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n} \tag{5.2}$$

is precisely the dividing curve for existence (at least for star-shaped domains) by Theorems 5.4–5.5. However, both Theorem 3.2 and Corollary 4.2 only cover the region below the curve given by

$$\max(\alpha, \beta) = n - 2, \quad \text{or,} \quad \max(p, q) \leq \frac{n+2}{n-2} \text{ and } \min(p, q) < \frac{n+2}{n-2}. \tag{5.3}$$

There is a gap between the lower curve (5.3) and the so called critical (higher) curve (5.2).

When system (I) is *fully irreducible* (in the sense below, see Theorem 5.6), it is not known if Theorems 1.1–1.3 are still optimal even for $n = 3$. Nevertheless, the following non-existence result for a prototype *fully irreducible* f and g should shed some light on this issue.

Theorem 5.6 (Reichel and Zou [20]) *Let* $n \geq 3$ *and let* $\Omega \subset \mathbb{R}^n$ *be a bounded smooth star-shaped domain. Suppose*

$$f(x,u,v) = u^r + v^q, \qquad g(x,u,v) = u^p + v^s$$

with $\min(p,q,r,s) \geq 2^*$. *Then system* (I) *has no non-trivial non-negative solution.*

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