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Champagne subregions of the disk whose bubbles carry harmonic measure

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Abstract. We show that for any $\varepsilon > 0$ and any region G whose outer boundary equals $\{z : |z| = 1\}$, there is a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of pairwise disjoint closed disks in G such that $\{z : |z| = 1\}$ is the set of accumulation points of $\{\Delta_n\}_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} radius(\Delta_n) < \varepsilon$ and ω_Ω (harmonic measure on the boundary of $\Omega := \{z : |z| < 1\} \setminus (\cup_{n=1}^{\infty}\Delta_n)$ for evaluation at some z_o in Ω) is supported on $\cup_{n=1}^{\infty}(\partial\Delta_n)$.

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1 Introduction

Let $\Lambda = \{z_n\}_{n=1}^{\infty}$ be a sequence of distinct points in $\mathbf{D} := \{z : |z| < 1\}$, and let Λ' be its set of accumulation points in $\overline{\mathbf{D}}$. We assume here that $\Lambda \cap \Lambda' = \emptyset$ and that $\mathbf{D} \cap \Lambda'$ is finite. Choose a sequence $\{r_n\}_{n=1}^{\infty}$ of positive real numbers so that the closed disks $\Delta_n := \{z : |z - z_n| \le r_n\}$ $(n = 1, 2, 3, ...)$ are pairwise disjoint and are contained in $\mathbf{D}$. Then $\Omega := \mathbf{D} \setminus (\cup_{n=1}^{\infty}\Delta_n)$ is called a *champagne* (*bubbles*) subregion of $\mathbf{D}$. Notice that Ω is a Dirichlet region and that $\partial\mathbf{D} \subseteq \partial\Omega$; we let ω_Ω denote harmonic measure on $\partial\Omega$ for evaluation at some point in Ω. If $\{r_n\}_{n=1}^{\infty}$ tends to zero sufficiently fast, then, by various methods, including an argument involving the Maximum Principle and estimates concerning $log|B(z)|$, where $B(z)$ is a finite Blaschke product, one finds that $\omega_\Omega(\partial\mathbf{D}) > 0$. In the other direction, if the points of Λ are sufficiently dispersed and if they do not tend to $\partial\mathbf{D}$ too quickly, then one may choose the radii r_n $(n = 1, 2, 3, ...)$ so that $\omega_\Omega(\partial\mathbf{D}) = 0$. What we mean by "dispersed" and "not tending to $\partial\mathbf{D}$ too quickly" relate closely to conditions that characterize so-called sampling sequences for Bergman spaces. Recall that a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbf{D}$ that has no accumulation points in $\mathbf{D}$ is said to be a *sampling sequence* for the Bergman space $L_a^2(\mathbf{D}) := \{f : f$ is analytic in $\mathbf{D}$ and $\int |f|^2 dA < \infty\}$ – A denotes area measure on $\mathbf{D}$ – if there is a constant $M > 1$ and there is a summable sequence $\{c_n\}_{n=1}^{\infty}$ of positive constants such that

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$$\frac{1}{M}||p||_{L^2(A)} \leq \{\sum_{n=1}^{\infty} c_n|p(z_n)|^2\}^{\frac{1}{2}} \leq M||p||_{L^2(A)}$$

for all polynomials p. This notion of sampling sequences extends to a wide variety of Banach spaces of analytic functions in **D**. Sequences in **D** that are interpolating for $H^\infty(\mathbf{D})$ have already been interpreted in terms of champagne subregions of **D** (see [GGJ]). Most of our methods make a case for a strong link between sampling sequences and champagne subregions Ω of **D** for which $\omega_\Omega(\partial\mathbf{D}) = 0$. Yet, rather than focusing on this link, our primary objective in this paper is to establish the existence of champage subregions Ω of **D** such that $\omega_\Omega(\partial\mathbf{D}) = 0$, even under the strictest conditions; that is, when $\partial\Omega$ is rectifiable and the bubbles of Ω are forced to reside in some prescribed subregion G of **D** whose outer boundary equals $\partial\mathbf{D}$ (Theorem 3.4). Recall that the outer boundary of G (denoted $d_\infty G$) is the boundary of the unbounded component of $C \setminus \overline{G}$. This result has consequences in the contexts of potential theory, sampling sequences and cyclic vectors for the shift on Hardy spaces; we explore some of these in Proposition 4.2 and Corollaries 4.3 and 4.4.

2 Preliminaries

Let G be a bounded Dirichlet region and let z_o be a point in G. If $h \in C_{\mathbf{R}}(\partial G)$, then there is a continuous extension $\hat{h}$ of h to $\overline{G}$ such that $\hat{h}$ is harmonic in G. By the Maximum Principle, $\hat{h}(z_o) \leq \|h\|_\infty$. Evidently, $z_o \longrightarrow \hat{h}(z_o)$ defines a bounded linear functional (of norm one) on $C_{\mathbf{R}}(\partial G)$. So, by the Riesz Representation Theorem, there is a unique probability measure $\omega(\cdot, G, z_o)$ with support in ∂G such that $\hat{h}(z_o) = \int_{\partial G} h(z)d\omega(z, G, z_o)$ (for all h in $C_{\mathbf{R}}(\partial G)$); $\omega(\cdot, G, z_o)$ is called *harmonic measure* on ∂G for evaluation at z_o. If z_1 is any other point in G, then, by Harnack's Inequality, $\omega(\cdot, G, z_1)$ and $\omega(\cdot, G, z_o)$ are boundedly equivalent. For this reason, we often suppress the point z_o in our notation and abbreviate $\omega(\cdot, G, z_o)$ by $\omega_G(\cdot)$. Notice that if E is a Borel subset of ∂G, then $z \longrightarrow \omega(E, G, z)$ is harmonic in G and has "boundary values" χ_E a.e. ω_G. We end this section with two results concerning harmonic measure that are useful to us later in the paper. For any compact subset K of the complex plane **C**, we follow convention and let $\hat{K}$ denote the *polynomially convex hull* of K — that is, $\{z \in \mathbf{C} : |p(z)| \leq \sup_{w\in K} |p(w)|$ for all polynomials p$\}$. If $K = \hat{K}$, then K is said to be *polynomially convex*.

Lemma 2.1. *Let G be a bounded Dirichlet region and let K be a compact subset of G such that $E := G\setminus K$ is a Dirichlet region. Then $\omega_{E|_{\partial G}}$ and ω_G are boundedly equivalent as measures on ∂G.*

Proof. Choose z_o in E. Since $E \subseteq G$, it follows from the Maximum Principle that $\omega(\cdot, E, z_o) \leq \omega(\cdot, G, z_o)$ on ∂G. For a reverse inequality, we first observe that $(\partial E) \setminus (\partial G) = \partial K$. By our hypothesis, there is a Jordan curve γ in E such that $K \subseteq \hat{\gamma} \subseteq G$; γ can be constructed from a grid of sufficiently small diameter — see [C1], Chapter VIII, Proposition 1.1 for a similar construction. Choose h in $C_{\mathbf{R}}(\partial G)$ such that $0 \leq h \leq 1$; we let $\hat{h}$ denote the continuous extension of h to $\overline{G}$ that is harmonic on G. By the Maximum Principle, there exists w_o in γ such that $\hat{h}(w_o) \geq \hat{h}(w)$ for all w in $\hat{\gamma}$. Therefore,

$$\begin{aligned}
\int_{\partial G} h(z) d\omega(z, G, w_o) = \hat{h}(w_o) & \\
&= \int_{\partial E} \hat{h}(z) d\omega(z, E, w_o) \\
&= \int_{\partial G} h(z) d\omega(z, E, w_o) + \int_{\partial K} \hat{h}(z) d\omega(z, E, w_o) \\
&\leq \int_{\partial G} h(z) d\omega(z, E, w_o) + \hat{h}(w_o)\omega(\partial K, E, w_o) \\
&\leq \int_{\partial G} h(z) d\omega(z, E, w_o) + c \cdot \int_{\partial G} h(z) d\omega(z, G, w_o),
\end{aligned}$$

where $c := \max\{\omega(\partial K, E, w) : w \in \gamma\} < 1$. So,

$$\int_{\partial G} h(z) d\omega(z, G, w_o) \leq \frac{1}{1-c} \cdot \int_{\partial G} h(z) d\omega(z, E, w_o).$$

By Harnack's Inequality, we can now find a positive constant M (independent of h; $0 \leq h \leq 1$) such that

$$\int_{\partial G} h(z) d\omega(z, G, z_o) \leq M \cdot \int_{\partial G} h(z) d\omega(z, E, z_o).$$

Consequently, $\omega(\cdot, G, z_o) \leq M\omega(\cdot, E, z_o)$ on ∂G. □

Lemma 2.2. *Choose z in $\mathbf{D}$ ($z \neq 0$), let $w = \frac{z}{|z|}$ and let $[z, w] = \{(1-t)z + tw : 0 \leq t \leq 1\}$. Then, $\omega([z, w], \mathbf{D} \setminus [z, w], 0) \leq 1 - |z|$.*

Proof. Under a rotation of $\mathbf{D}$, we may assume that $z = r$ $(0 < r < 1)$. Let η denote the sweep of $\omega_r := \omega(\cdot, \mathbf{D} \setminus [r, 1], 0)$ to $\partial\mathbf{D}$. Then, by definition,

$$d\eta = d\omega_r|_{\partial \mathbf{D}} + h dm,$$

where m denotes normalized Lebesgue measure on $\partial\mathbf{D}$ and

$$h(\zeta) = \int_r^1 \frac{1-t^2}{|\zeta - t|^2} d\omega_r(t).$$

Since ω_r is harmonic measure for evaluation at 0, it follows that $\eta = m$. Notice that h is continuous on $\partial\mathbf{D}$ and since $\eta = m$, we must have: $0 \leq h \leq 1$. Therefore,

$$1 \geq h(1) = \int_r^1 \frac{1+t}{1-t} d\omega_r(t) \geq \frac{1+r}{1-r} \cdot \omega_r([r, 1]),$$

and so $\omega_r([r, 1]) \leq 1 - r$. □

3 Champagne subregions of the disk

The first result of this section is an important ingredient in the proof of our main result and is derived from the properties of a function that was developed by K. Seip (see [S]) to establish sets of sampling and interpolation in Bergman spaces.

Theorem 3.1. *For any* $\varepsilon > 0$*, there is a sequence* $\{\Delta_n\}_{n=1}^{\infty}$ *of pairwise disjoint closed disks in* $\mathbf{D}$ *such that* $\partial\mathbf{D}$ *is the set of accumulation points of* $\{\Delta_n\}_{n=1}^{\infty}$*,* $\sum_{n=1}^{\infty} radius(\Delta_n) < \varepsilon$ *and* $\omega_{\Omega}(\partial\mathbf{D}) = 0$*, where* $\Omega := \mathbf{D} \setminus (\cup_{n=1}^{\infty}\Delta_n)$*.*

Proof. Following [S], we let $\Gamma(2,1) = \{2^m(n+i) :$ m and n are integers$\}$ and define h on $H^+ := \{z : Im(z) > 0\}$ by:

$$h(z) = \left(\prod_{k=0}^{\infty} \frac{sin(\pi(i - 2^{-k}z))}{sin(\pi(i + 2^{-k}z))}\right) \cdot \left(\prod_{m=1}^{\infty} e^{2\pi} \frac{sin(\pi(2^m z - i))}{sin(\pi(2^m z + i))}\right).$$

As was observed in [S], h is analytic in H^+ and its zero-set there is $\Gamma(2,1)$. For nonnegative integers m, let $\alpha_{m,0} = \{z : |z - i2^m| = \frac{1}{4}\}$ and for negative integers m, let $\alpha_{m,0} = \{z : |z - i2^m| = 4^{m-1}\}$. Replicate these circles horizontally for all integers n by letting $\alpha_{m,n} = \alpha_{m,0} + n2^m$; notice that $\alpha_{m,n}$ has center $2^m(n+i)$ and $\{\hat{\alpha}_{m,n}$: m and n are integers$\}$ is a pairwise disjoint collection of closed disks in H^+ whose set of accumulation points in $\mathbf{C}$ is $\mathbf{R}$. We let $W = H^+ \setminus \{\bigcup_{m,n} \hat{\alpha}_{m,n}\}$.

Let φ be the Möbius transformation from H^+ onto $\mathbf{D}$ given by $\varphi(z) = \frac{z-i}{z+i}$ and let $\beta_{m,n} = \varphi(\alpha_{m,n})$. A brief analysis of φ' (or, instead, an application of [Go], Theorem 4, page 52) gives a positive constant M (independent of m and n) such that $radius(\beta_{m,n}) \leq M|\varphi'(2^m(n+i))|radius(\alpha_{m,n})$. So, for $m < 0$, there are positive constants C_1, C_2, C_3, and C_4 (independent of m) such that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} radius(\beta_{m,n}) &< 2 \cdot \sum_{n=0}^{\infty} radius(\beta_{m,n}) \\
&\leq C_1 \cdot \sum_{n=0}^{\infty} \frac{4^m}{|2^m(n+i)+i|^2} \\
&\leq C_2 \cdot \sum_{n=0}^{\infty} \frac{1}{|n + i2^{-m}|^2} \\
&\leq C_3 \cdot \sum_{n=0}^{\infty} \frac{1}{(n + 2^{-m})^2} \\
&\leq C_4 2^m.
\end{aligned}$$

And, for $m \geq 0$, there are positive constants C_5 and C_6 such that

$$\sum_{n=-\infty}^{\infty} radius(\beta_{m,n}) \leq C_5 \cdot \sum_{n=o}^{\infty} \frac{1}{|2^m(n+i)+i|^2}$$
$$\leq C_6 \frac{1}{4^m}.$$

Evidently, therefore, $\sum_m \sum_n radius(\beta_{m,n}) < \infty$. So, $V := \varphi(W)$ is a champagne subregion of $\mathbf{D}$, and V has rectifiable boundary.

Claim. $\omega_\Omega(\partial \mathbf{D}) = 0$.

For $n = 1, 2, 3, \ldots$, let $W_n = \{z = x + iy : -n < x < n \text{ and } 0 < \mathrm{y} < \mathrm{n}\} \cap \mathrm{W}$. By the conformal invariance of harmonic measure and the *exhaustion* $\{W_n\}_{n=1}^{\infty}$ of W, the proof of our claim reduces to showing that $\omega_{W_n}([-n, n]) = 0$. To this end, we define $W_{n,m}$ (for negative integers m) to be $\{z \in W_n : Im(z) > 3 \cdot 2^{m-2}\}$ and (for the same m) we let $I_{n,m} = \{z = x + iy : -n < x < n \text{ and } \mathrm{y} = 3 \cdot 2^{\mathrm{m}-2}\}$; notice that $I_{n,m} \subseteq W_n$. By [S], estimate (3) on page 214, there is a positive constant δ such that

$$|h(z)| \geq \delta \varrho(z, \Gamma(2,1)) y^{\frac{-2\pi}{ln(2)}},$$

where $0 < y = Im(z)$ and $\varrho(\cdot, \cdot)$ denotes the pseudohyperbolic metric on H^+. Thus, there is a positive constant λ that depends only on n, such that $|h(z)| \geq \lambda$ whenever $z \in \overline{W}_{n,m}$. And, by the same estimate, there is another positive constant C (independent of m and n) such that $|h(z)| \geq C 4^{\pi(2-m)}$ for all z in $I_{n,m}$. Select z_o in $W_{n,-1}$ and let $\omega_{n,m}(\cdot) = \omega(\cdot, W_{n,m}, z_o)$. Since $log|h|$ is continuous and harmonic on $\overline{W}_{n,m}$,

$$\begin{aligned} log|h(z_o)| &= \int log|h| d\omega_{n,m} \\ &\geq log(\lambda) \cdot \omega_{n,m}((\partial W_{n,m}) \setminus I_{n,m}) + [log(C) + \pi(2-m)]\omega_{n,m}(I_{n,m}) \\ &\geq log(\lambda) + [log(C) + \pi(2-m)]\omega_{n,m}(I_{n,m}). \end{aligned}$$

Evidently, therefore, $\omega_{n,m}(I_{n,m}) \longrightarrow 0$, as $m \to -\infty$. Since, by the Maximum Principle, $\omega([-n, n], W_n, z_o) \leq \omega_{n,m}(I_{n,m})$ (independent of m and n), we conclude that $\omega_{W_n}([-n, n]) = 0$. And so our claim holds. To this point, we have produced a champagne region $V = \mathbf{D} \setminus (\bigcup_{n=1}^{\infty} \Delta_n)$, where $\{\Delta_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence of closed disks in $\mathbf{D}$ such that $\partial \mathbf{D}$ is the set of accumulation points of $\{\Delta_n\}_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} radius(\Delta_n) < \infty$ and $\omega_V(\partial \mathbf{D}) = 0$. So, for any $\varepsilon > 0$, there is a positive integer N such that $\sum_{n=N}^{\infty} radius(\Delta_n) < \varepsilon$. By Lemma 2.1, $\Omega := \mathbf{D} \setminus (\cup_{n=N}^{\infty} \Delta_n)$ satisfies the conclusion of our theorem. □

Corollary 3.2. *Suppose* $0 < r < \frac{1}{4}$ *and let* $G = \mathbf{D} \setminus \Delta$, *where* $\Delta := \{z : |z - \frac{3}{4}| \leq r\}$. *Then, for any* $\varepsilon > 0$ *and any* $\rho, r < \rho \leq \frac{1}{4}$, *there is a finite, pairwise disjiont collection* $\{\Delta_n\}_{n=1}^N$ *of closed disks in* $A_\rho := \{z : r < |z - \frac{3}{4}| < \rho\}$ *such that:*

(i) $\sum_{n=1}^N radius(\Delta_n) < \varepsilon$ *and*
(ii) $\omega(\partial\Delta, W, 0) < \varepsilon$, *where* $W := G \setminus (\cup_{n=1}^N \Delta_n)$.

Proof. Choose $\varepsilon > 0$ and $\rho, r < \rho < \frac{1}{4}$. By Theorem 3.1 and the application of a Möbius transformation, there is a sequence $\{\Delta_n\}_{n=1}^\infty$ of pairwise disjoint closed disks in A_ρ such that $\partial\Delta$ is the set of accumulation points of $\{\Delta_n\}_{n=1}^\infty$, $\sum_{n=1}^\infty radius(\Delta_n) < \varepsilon$ and $\omega_V(\partial\Delta) = 0$, where $V := G \setminus (\cup_{n=1}^\infty \Delta_n)$. By the Maximum Principle, if we choose a positive integer N so that $\sum_{n=N+1}^\infty \omega(\partial\Delta_n, V, 0) < \varepsilon$, then $W := G \setminus (\cup_{n=1}^N \Delta_n)$ satisfies the conclusion of our corollary. □

Lemma 3.3. *Let* $G = \mathbf{D} \setminus K$, *where* K *is a compact, connected, polynomially convex subset of* $\mathbf{D}$ *that contains more than one point. Suppose* $z_o \in G$ *and* $0 < \delta < dist(z_o, K)$, *and let* $G_\delta = \{z \in G : dist(z, K) < \delta\}$. *Then, for any* $\varepsilon > 0$, *there are finitely many pairwise disjoint closed disks* $\{\Delta_n\}_{n=1}^N$ *in* G_δ *such that* $\sum_{n=1}^N radius(\Delta_n) < \varepsilon$ *and* $\omega(\partial K, W, z_o) < \varepsilon$, *where* $W := G \setminus (\cup_{n=1}^N \Delta_n)$.

Proof. Placing a square grid of sufficiently small diameter on the complex plane $\mathbf{C}$, we can find finitely many closed squares $\{S_k\}_{k=1}^m$ (of equal sidelength) in G_δ so that S_k shares a side with S_{k-1} and a side with S_{k+1} (for $2 \leq k \leq m-1$), and S_1 and S_m also share a side. Moreover, these are the only sides that are common to two squares in the collection $\{S_k\}_{k=1}^m$, and $\mathbf{C} \setminus (\cup_{k=1}^m S_k)$ has two components – the bounded component contains K. So, $\{S_k\}_{k=1}^m$ forms a collar in G_δ about K. Let D_k be the closed disk of largest area in S_k $(1 \leq k \leq m)$; notice that $\cup_{k=1}^m D_k$ disconnects $\mathbf{C}$ and K is in the bounded component of $\mathbf{C} \setminus (\cup_{k=1}^m D_k)$. Slightly reduce the radius of each disk D_k $(1 \leq k \leq m)$ to get a pairwise disjoint collection of closed disks $\{D_k^*\}_{k=1}^m$ – each of the same radius – in G_δ. There are small "gaps" between successive disks in the collection $\{D_k^*\}_{k=1}^m$, and also there is a small gap between D_1^* and D_m^*, all produced by this reduction of the radii. By the Maximum Principle and majorants for harmonic measure derived from the notion of extremal length (see [B], pages 361-385), if the reduction of the radii is sufficiently slight, then we have $\omega(\partial K, E, z_o) < \frac{\varepsilon}{2}$, where $E := G \setminus (\cup_{k=1}^m D_k^*)$. By the proof of Corollary 3.2 applied to each disk D_k^* $(1 \leq k \leq m)$ successively, we can produce a finite collection $\{\Delta_n\}_{n=1}^N$ of pairwise disjoint closed disks in $E \cap G_\delta$ such that $\sum_{n=1}^N radius(\Delta_n) < \varepsilon$ and $\omega(\cup_{k=1}^m \partial D_k^*, F, z_o) < \dfrac{\varepsilon}{2}$, where $F := E \setminus (\cup_{n=1}^N \Delta_n)$. From the Maximum Principle, it now follows that $\omega(\partial K, W, z_o) < \varepsilon$, where $W := G \setminus (\cup_{n=1}^N \Delta_n)$. □

We are now in a position to establish the main result of this paper.

Theorem 3.4. *Let G be a region whose outer boundary is $\partial\mathbf{D}$ and choose $\varepsilon > 0$. Then there is a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of pairwise disjoint closed disks in G such that $\partial\mathbf{D}$ is the set of accumulation points of $\{\Delta_n\}_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} radius(\Delta_n) < \varepsilon$ and $\omega_\Omega(\partial\mathbf{D}) = 0$, where $\Omega := \mathbf{D} \setminus (\cup_{n=1}^{\infty}\Delta_n)$.*

Proof. We first produce a sequence $\{\gamma_{n,k}\}$ of pairwise disjoint Jordan arcs in G such that any compact subset of $\mathbf{D}$ has nonempty intersection with at most finitely many of the arcs and such that $\omega_V(\partial\mathbf{D}) = 0$, where $V := \mathbf{D} \setminus (\bigcup_{n,k} \gamma_{n,k})$; so, ω_V is supported on $\bigcup_{n,k} \gamma_{n,k}$.

The construction of $\{\gamma_{n,k}\}$

By a Möbius transformation from $\mathbf{D}$ onto $\mathbf{D}$, we may assume that $0 \in G$. For $n = 5, 6, 7, ...$ and $k = 1, 2, 3, ..., 2^{n+1}$, let $R_{n,k} = \{re^{i\theta} : 1 - 2^{-n} < r < 1 \text{ and } \frac{(\mathrm{k}-1)\pi}{2^{\mathrm{n}}} < \theta < \frac{\mathrm{k}\pi}{2^{\mathrm{n}}}\}$. Notice that $\{R_{n,k}\}_{k=1}^{2^{n+1}}$ is a pairwise disjoint collection of open Carleson rectangles and $\bigcup_{k=1}^{2^{n+1}} \overline{R}_{n,k} = \{z : 1 - 2^{-n} \le |z| \le 1\}$; we call $\{R_{n,k}\}_{k=1}^{2^{n+1}}$ the $\mathrm{n}^{\underline{\mathrm{th}}}$ generation of rectangles. For $n = 5, 6, 7, \ldots,$ let $\Gamma_n = \{z : |z| = 1 - \frac{1}{n}\}$. We construct the sequence of arcs $\{\gamma_{n,k}\}$ inductively, starting with $n = 5$ and $k = 1$. Now since $\partial_\infty G = \partial\mathbf{D}$, $0 \in G$ and G is connected, it follows that $G \cap R_{5,1} \neq \emptyset$ and that $G \cap \Gamma_5 \neq \emptyset$. So, there is a rectifiable Jordan arc $\gamma_{5,1}$ contained in G that has one endpoint in Γ_5 and the other endpoint in $R_{5,1}$; we may assume that $|z| \ge \frac{4}{5}$ for all z in $\gamma_{5,1}$. Let $G_{5,1} = G \setminus \gamma_{5,1}$; notice that $G_{5,1}$ is a subregion of $\mathbf{D}$, $\partial_\infty G_{5,1} = \partial\mathbf{D}$ and $0 \in G_{5,1}$. Consequently, $G_{5,1} \cap R_{5,2} \neq \emptyset$ and $G_{5,1} \cap \Gamma_5 \neq \emptyset$. So, there is a rectifiable Jordan arc $\gamma_{5,2}$ contained in $G_{5,1}$ that has one endpoint in Γ_5 and the other endpoint in $R_{5,2}$; again, we may assume that $|z| \ge \frac{4}{5}$ for all z in $\gamma_{5,2}$. Let $G_{5,2} = G_{5,1} \setminus \gamma_{5,2}$ and proceed as before. Once the arcs $\gamma_{5,1}, \gamma_{5,2}, ..., \gamma_{5,64}$ have been chosen, let $G_{6,0} = G_{5,63} \setminus \gamma_{5,64}$. Since $G_{6,0}$ is a subregion of $\mathbf{D}$, $\partial_\infty G_{6,0} = \partial\mathbf{D}$ and $0 \in G_{6,0}$, it follows that $G_{6,0} \cap R_{6,1} \neq \emptyset$ and that $G_{6,0} \cap \Gamma_6 \neq \emptyset$. So, there is a rectifiable Jordan arc $\gamma_{6,1}$ contained in $G_{6,0}$ that has one endpoint in Γ_6 and its other endpoint in $R_{6,1}$; we may assume that $|z| \ge \frac{5}{6}$ for all z in $\gamma_{6,1}$. Let $G_{6,1} = G_{6,0} \setminus \gamma_{6,1}$ and proceed as before to define $\gamma_{6,2}, ..., \gamma_{6,128}$. Continuing in this way, we get a pairwise disjoint sequence $\{\gamma_{n,k} : n = 5, 6, 7, \ldots \text{ and } \mathrm{k} = 1, 2, 3, ..., 2^{\mathrm{n}+1}\}$ of rectifiable Jordan arcs in G such that $\gamma_{n,k}$ has one endpoint in Γ_n and its other endpoint in $R_{n,k}$, and $|z| \ge 1 - \frac{1}{n}$ for all z in $\gamma_{n,k}$. Observe that $V := \mathbf{D} \setminus (\bigcup_{n,k} \gamma_{n,k})$ is a Dirichlet region, $0 \in V$ and $\partial V = (\partial\mathbf{D}) \cup (\bigcup_{n,k} \gamma_{n,k})$.

Claim. $\omega_V(\partial\mathbf{D}) = 0$.

By Harnack's Inequality, we may assume that $\omega_V(\cdot) = \omega(\cdot, V, 0)$. Now, since $V \subseteq \mathbf{D}$, the Maximum Principle gives us: $\omega_V|_{\partial\mathbf{D}} \leq \omega(\cdot, \mathbf{D}, 0) = m$ (normalized Lebesgue measure on $\partial\mathbf{D}$). So, to establish our claim, it is sufficient (by [R], Definitions 8.2 and 8.3, and Theorem 8.6) to show that, for any $e^{i\theta}$ in $\partial\mathbf{D}$, there is a sequence $\{E_k\}_{k=1}^{\infty}$ of Borel subsets of $\partial\mathbf{D}$ that "shrink to $e^{i\theta}$ nicely" such that $\frac{\omega_V(E_k)}{m(E_k)} \longrightarrow 0$ as $k \to \infty$. We show this in the case that $e^{i\theta} = 1$; the argument for any other $e^{i\theta}$ proceeds in a similar way. Now choose a sequence of (Carleson) rectangles – one from each generation – such that the closure of each contains 1; we may choose the sequence $\{R_{n,1}\}_{n=5}^{\infty}$. Corresponding to this sequence of rectangles is the sequence of Jordan arcs $\{\gamma_{n,1}\}$; recall that $\gamma_{n,1}$ has one endpoint in Γ_n and has its other endpoint in $R_{n,1}$, such that $|z| \geq 1 - \frac{1}{n}$ for all z in $\gamma_{n,1}$. Since $\gamma_{n,1} \cap \overline{R}_{n,1}$ is a compact subset of $\mathbf{D}$, there is a point ζ_n in $\gamma_{n,1} \cap \overline{R}_{n,1}$ such that $1 - |\zeta_n| = dist((\gamma_{n,1} \cap \overline{R}_{n,1}), \partial\mathbf{D})$; let $a_n = \frac{\zeta_n}{|\zeta_n|}$. Traverse $\gamma_{n,1}$ from its endpoint in Γ_n until one first encounters ζ_n and let $\gamma_{n,1}^*$ denote that Jordan subarc of $\gamma_{n,1}$. Let τ_n be the Jordan arc consisting of $\gamma_{n,1}^*$ along with the radial segment $[\zeta_n, a_n] := \{(1-t)\zeta_n + ta_n : 0 \leq t \leq 1\}$; we can by no means assert that $\{\tau_n\}_{n=1}^{\infty}$ is a pairwise disjoint collection or that $\tau_n \setminus \{a_n\} \subseteq G$. Now $a_n \in \overline{R}_{n,1} \subseteq \{z : |z-1| < \frac{1}{n}\}$ (since $n \geq 5$), and furthermore, τ_n has an endpoint in Γ_n. Therefore, τ_n intersects the circle $\{z : |z-1| = \frac{1}{n}\}$. Traverse τ_n from a_n until one first intersects $\{z : |z-1| = \frac{1}{n}\}$ and let σ_n denote that subarc of τ_n. Let $A_n = \mathbf{D} \setminus \gamma_{n,1}$, let $B_n = \mathbf{D} \setminus \gamma_{n,1}^*$, let $O_n = \mathbf{D} \setminus \tau_n$ and let $P_n = \mathbf{D} \setminus \sigma_n$. Now, $V \subseteq A_n \subseteq B_n$, $O_n \subseteq P_n$ and $1 - |\zeta_n| < 2^{-n}$. Therefore, by the Maximum Principle and Lemma 2.2, if E is any Borel subset of $\partial\mathbf{D}$, then

$$
\begin{aligned}
\omega_V(E) &\leq \omega(E, A_n, 0) \leq \omega(E, B_n, 0) \qquad (1)\\
&= \omega(E, O_n, 0) + \int_{[\zeta_n, a_n]} \omega(E, B_n, z) d\omega(z, O_n, 0)\\
&\leq \omega(E, O_n, 0) + \omega([\zeta_n, a_n], O_n, 0)\\
&\leq \omega(E, O_n, 0) + 2^{-n}\\
&\leq \omega(E, P_n, 0) + 2^{-n}.
\end{aligned}
$$

To finish the proof of our claim, it is helpful to divide the problem into two cases, based on whether or not certain portions of the arcs σ_n $(n = 5, 6, 7, \ldots)$ are contained in one of a particular class of lens-shaped regions. For $0 < \alpha < \frac{\pi}{2}$, let $W(\alpha) = \{re^{i\theta} : r > 0 \text{ and } -\alpha < \theta < \alpha\}$. Let φ be the Möbius transformation from $\{z : Re(z) > 0\}$ onto $\mathbf{D}$ given by $\varphi(z) = \frac{z-1}{z+1}$ and let $S(\alpha) = \varphi(W(\alpha))$. Notice that $S(\alpha)$ is a lens-shaped region in $\mathbf{D}$, $S(\alpha)$ is symmetric with respect to $\mathbf{R}$ and $\partial(S(\alpha))$ consists of two arcs of circles that form an angle of 2α at both -1 and 1. Now, either:

Case 1. there exists α such that $\{z \in \sigma_n : \frac{n}{2^n} \leq |1-z| \leq \frac{1}{n}\} \subseteq S(\alpha)$ for $n = 5, 6, 7, \ldots$, or

Case 2. there is a subsequence $\{\sigma_{n_k}\}_{k=1}^{\infty}$ of $\{\sigma_n\}_{n=5}^{\infty}$ and a sequence of points $\{z_k\}_{k=1}^{\infty}$ ($z_k \in \sigma_{n_k}$ and $|1-\mathrm{z_k}| \geq \frac{\mathrm{n_k}}{2^{\mathrm{n_k}}}$) such that $\frac{1-|z_k|}{|1-z_k|} \longrightarrow 0$ as $k \to \infty$; let $w_k = \frac{z_k}{|z_k|}$.

We first address Case 2. Let E_k be the shortest arc of $\partial\mathbf{D}$ that has endpoints a_{n_k} and w_k; notice that E_k shrinks to 1 nicely as $k \to \infty$. By the Maximum Principle and Lemma 2.2, $\omega(E_k, P_{n_k}, 0) \leq 1 - |z_k|$. Therefore, by (1), if k is sufficiently large, then

$$\begin{aligned}\frac{\omega_V(E_k)}{m(E_k)} &\leq \frac{2[(1-|z_k|) + 2^{-n_k}]}{|1-z_k|} \\ &\leq \frac{2(1-|z_k|)}{|1-z_k|} + \frac{2}{n_k} \longrightarrow 0, \text{ as k} \to \infty.\end{aligned}$$

We now turn to Case 1. For $n = 5, 6, 7, \ldots$, let E_n be the subarc of $\partial\mathbf{D}$ that extends from a_n counterclockwise until one first encounters $\{z : |1-z| = \frac{n}{2^n}\}$; notice that E_n shrinks to 1 nicely as $n \to \infty$. For the same n and $\frac{n}{2^n} \leq r \leq \frac{1}{n}$, let $\beta_n(r) = \{1 + re^{i\theta} : \theta_1 \leq \theta \leq \theta_2\}$, where θ_1 and θ_2 are the smallest numbers in $[0, 2\pi)$ such that $1 + re^{i\theta_1} \in \partial\mathbf{D}$ and $1 + re^{i\theta_2} \in \sigma_n$. By our hypothesis for Case 1, there is a constant $c > 1$ (independent of n and r) such that $length(\beta_n(r)) \leq \frac{\pi}{c} r$. So, we can apply [B], pages 361-385, and find a positive constant M (independent of n) such that

$$\begin{aligned}\omega(E_n, P_n, 0) &\leq M \cdot exp\left(-\pi \cdot \int_{\frac{n}{2^n}}^{\frac{1}{n}} \frac{1}{length(\beta_n(r))} dr\right) \\ &\leq M \cdot exp\left(-c \cdot \int_{\frac{n}{2^n}}^{\frac{1}{n}} \frac{1}{r} dr\right) \\ &= M \left(\frac{n^2}{2^n}\right)^c.\end{aligned}$$

It now follows from (1) that

$$\frac{\omega_V(E_n)}{m(E_n)} \leq \frac{M(\frac{n^2}{2^n})^c + 2^{-n}}{n2^{-n-1}} \longrightarrow 0, \text{ as n} \to \infty.$$

By Cases 1 and 2, our claim is established. Now, since $\{\gamma_{n,k}\}$ is countable, we can rename these Jordan arcs and enumerate them $\{\gamma_m\}_{m=1}^{\infty}$. Choose $\varepsilon > 0$. By applying Lemma 3.3 to $V_m := \mathbf{D} \setminus \gamma_m$ successively (for $m = 1, 2, 3, \ldots$), we can produce a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of pairwise disjoint closed disks in $\{z \in G :$

$|z| > \frac{1}{2}\} \setminus (\cup_{m=1}^{\infty}\gamma_m)$ such that $\partial\mathbf{D}$ contains the accumulation points of $\{\Delta_n\}_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} radius(\Delta_n) < \varepsilon$ and

$$\omega(\gamma_m, U_m, 0) \leq \frac{1}{2^m}, \qquad (2)$$

where $U_m := V_m \setminus (\cup_{n=1}^{\infty}\Delta_n)$; we let $U = \cap_{m=1}^{\infty}U_m$ $(= V \setminus (\cup_{n=1}^{\infty}\Delta_n))$. By our claim and the Maximum Principle, $\omega(\partial\mathbf{D}, U, 0) = 0$. So, by Lemma 2.1, $\omega(\partial\mathbf{D}, U_n^*, 0) = 0$ (for $n = 1, 2, 3, ...$), where $U_n^* := \cap_{m=n}^{\infty}U_m$. Moreover, by (2) and the Maximum Principle, $\omega(\cup_{m=n}^{\infty}\gamma_m, U_n^*, 0) < 2^{1-n}$. Once again applying the Maximum Principle, we can now conclude that $\omega(\partial\mathbf{D}, \Omega, 0) < 2^{1-n}$ (for $n = 1, 2, 3, ...$), where $\Omega := \mathbf{D} \setminus (\cup_{n=1}^{\infty}\Delta_n)$. Consequently, $\omega(\partial\mathbf{D}, \Omega, 0) = 0$. □

In the proof of Theorem 3.1 we used estimates of K. Seip and a special sampling sequence to construct a champagne subregion Ω of $\mathbf{D}$ with rectifiable boundary such that $\omega_{\Omega}(\partial\mathbf{D}) = 0$. The next result shows that this construction is somewhat dependent on the nature of sampling sequences; Blaschke sequences are zero sets for $H^{\infty}(\mathbf{D})$ and therefore cannot be sampling. We conclude this paper with applications of Theorem 3.4 in the contexts of sampling sequences and cyclic vectors for the shift on Hardy spaces.

Proposition 3.5. *Let $\Omega := \mathbf{D} \setminus (\cup_{n=1}^{\infty}\Delta_n)$ be a champagne subregion of $\mathbf{D}$. If there is a Blaschke sequence $\Lambda := \{z_k\}_{k=1}^{\infty}$ in $\mathbf{D}$ and there exists c, $0 < c < 1$, such that $\rho(z, \Lambda) < c$ for all z in $\cup_{n=1}^{\infty}\Delta_n$, then $\omega_{\Omega}(\partial\mathbf{D}) > 0$.*

Proof. By adding some disks to the collection $\{\Delta_n\}_{n=1}^{\infty}$, or by slightly expanding some of the existing disks, we may assume that $\Lambda \cap \overline{\Omega} = \emptyset$. Let B be the Blaschke product corresponding to the sequence Λ. Now there exists z_o in Ω such that $|B(z_o)| > c$; let $f_m(z) = \left[\frac{B(z)}{B(z_o)}\right]^m$ (for $m = 1, 2, 3, ...$). Since $z \longrightarrow log|f_m(z)|$ is harmonic in Ω and has boundary values in $L^1(d\omega_{\Omega})$, it follows that $0 = log|f_m(z_o)| = \int_{\partial\Omega} log|f_m(z)| d\omega(z, \Omega, z_o)$. Therefore, since $f_m \longrightarrow 0$ uniformly on $\cup_{n=1}^{\infty}\Delta_n$, as $n \to \infty$, we conclude that $\omega_{\Omega}(\partial\mathbf{D}) > 0$. □

4 Applications

We now turn to an application that involves some rather standard sampling techniques. To develop the subject properly, we first need to discuss point evaluations. We assume here that μ is a finite, positive Borel measure with compact support in $\mathbf{C}$ and for $1 \leq t < \infty$, we let $P^t(d\mu)$ denote the closure of the polynomials in $L^t(d\mu)$ and let $Ball(P^t(d\mu)) = \{f \in P^t(d\mu) : \|f\|_{L^t(d\mu)} \leq 1\}$. A point z in $\mathbf{C}$ is called a *bounded point evaluation* for $P^t(d\mu)$ if there is a positive constant C such that $|p(z)| \leq C\|p\|_{L^t(d\mu)}$ for all polynomials p; the collection of all such points is denoted $bpe(P^t(d\mu))$. A point z in $\mathbf{C}$ is called an *analytic bounded* *point evaluation* for $P^t(d\mu)$ if there are positive constants δ and M such that $|p(w)| \leq M\|p\|_{L^t(d\mu)}$ whenever p is a polynomial and $|z-w| < \delta$; the collection of all such points z is denoted $abpe(P^t(d\mu))$. Now $abpe(P^t(d\mu))$ is the largest open set in $\mathbf{C}$ to which every function f in $P^t(d\mu)$ has an analytic continuation and, moreover, each component of $abpe(P^t(d\mu))$ is simply connected.

Lemma 4.1. *Let μ be a finite, positive Borel measure with support in $\overline{\mathbf{D}}$ such that $\mathbf{D} = abpe(P^t(d\mu))$. Then, for $0 < r < 1$, there is a positive constant M_r such that $\left|\frac{f(w)-f(z)}{w-z}\right| \leq M_r$ whenever $f \in Ball(P^t(d\mu))$ and z and w are distinct points of moduli no greater than r.*

Proof. Let $\Gamma_r = \{\zeta : |\zeta| = \frac{1+r}{2}\}$. Now since $abpe(P^t(d\mu)) = \mathbf{D}$, there is a positive constant C_r such that $|f(\zeta)| \leq C_r$ for all f in $Ball(P^t(d\mu))$ and all ζ in Γ_r. So, if z and w are distinct points of moduli no greater than r, then, by Cauchy's Integral Formula,

$$\begin{aligned}\left|\frac{f(w)-f(z)}{w-z}\right| &\leq \frac{1}{2\pi} \cdot \int_{\Gamma_r} \frac{|f(\zeta)|}{|\zeta - z||\zeta - w|}|d\zeta| \\ &\leq \frac{4C_r}{(1-r)^2}.\end{aligned}$$

□

The proof of the next result follows a well-known technique that was pointed out to the author by D. Luecking. The result itself answers, in the affirmative, Question 3.4 of [AS]. However, the sampling methods presented here are by no means "sharp" enough to give the results found in [AS]. In what follows, we will be considering a finite, positive Borel measure μ with support in $\overline{\mathbf{D}}$. For the sake of convenience, we let $\mu_o = \mu|_{\mathbf{D}}$ and let $\mu_\infty = \mu|_{\partial\mathbf{D}}$.

Proposition 4.2. *Let μ be a finite, positive Borel measure with support in $\overline{\mathbf{D}}$ such that $\mathbf{D} = abpe(P^t(d\mu))$. Then there exist an at most countable collection of points $\{z_n\}$ in $\mathbf{D} \cap support(\mu)$, where $\{z_n\}$ has no accumulation point in $\mathbf{D}$, a summable collection $\{c_n\}$ of positive constants and a constant $M > 1$ such that $\nu := \mu_\infty + \sum_n c_n \delta_{z_n}$ satisfies:*

$$\frac{1}{M}\|p\|_{L^t(d\mu)} \leq \|p\|_{L^t(d\nu)} \leq M\|p\|_{L^t(d\mu)}$$

for all polynomials p.

Proof. We assume that μ is a probability measure; the general result follows immediately from the proof of this case. By Lemma 4.1, we can partition $\mathbf{D}$ into sets R_k ($k = 1, 2, 3, ...$) of the form $\{re^{i\theta} : a \leq r < b \text{ and } \alpha \leq \theta < \beta\}$ and with the properties:

(i) $0 < d_k := diameter(R_k) < dist(R_k, \partial\mathbf{D}) \longrightarrow 0$ as $k \to \infty$, and
(ii) for each k there exists r_k, $0 < r_k < 1$, such that $|z| \le r_k$ for all z in R_k and $d_k M_{r_k} \le \frac{1}{2}$, where M_{r_k} is the constant provided by Lemma 4.1.

Let f be a ploynomial of norm 1 in $P^t(d\mu)$ and choose w_k in R_k ($k = 1, 2, 3, ...$); we choose w_k in $support(\mu) \cap R_k$ if $\mu(R_k) \neq 0$. Then, by Lemma 4.1,

$$|f(z) - f(w_k)| \le M_{r_k}|z - w_k| \le \frac{1}{2}$$

for any z in R_k. Therefore, $|f(z)| \le \frac{1}{2} + |f(w_k)|$ for all z in R_k. Consequently, $|f(z)|^t \le 2^{t-1}(2^{-t} + |f(w_k)|^t)$ and hence $|f(z)|^t - 2^{t-1}|f(w_k)|^t \le \frac{1}{2}$ for all z in R_k. Since we are assuming that μ is a probability measure and that $\|f\|_{L^t(d\mu)} = 1$, it follows that

$$\int |f|^t d\mu_o - 2^{t-1} \cdot \int |f|^t d\sigma \le \frac{1}{2}\mu_o(\mathbf{D}) \le \frac{1}{2}\int |f|^t d\mu,$$

where $\sigma := \sum_{k=1}^{\infty} \mu(R_k)\delta_{w_k}$. Therefore,

$$\int |f|^t d\mu - 2^{t-1} \cdot \int |f|^t d\nu \le \frac{1}{2}\int |f|^t d\mu,$$

where $\nu := \mu_\infty + \sigma$. And thus,

$$\|f\|_{L^t(d\mu)} \le 2\|f\|_{L^t(d\nu)}.$$

In a similar fashion we obtain the inequality: $\|f\|_{L^t(d\nu)} \le 2\|f\|_{L^t(d\mu)}$. The proof is now complete. □

Corollary 4.3. *Let G be region whose outer boundary is $\partial\mathbf{D}$. Then there exist a sequence $\{z_n\}_{n=1}^{\infty}$ in G and a summable sequence $\{c_n\}_{n=1}^{\infty}$ of positive constants such that $\{z_n\}_{n=1}^{\infty}$ has no accumulation point in $\mathbf{D}$ and $\sigma := \sum_{n=1}^{\infty} c_n\delta_{z_n}$ satisfies:*

$$abpe(P^t(d\sigma)) = \mathbf{D} \; for \; 1 \le t < \infty.$$

Proof. By Theorem 3.4, there is a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of pairwise disjoint closed disks in G such that $\{\Delta_n\}_{n=1}^{\infty}$ has no accumulation point in $\mathbf{D}$ and $\omega_\Omega(\partial\mathbf{D}) = 0$, where $\Omega := \mathbf{D} \setminus (\cup_{n=1}^{\infty}\Delta_n)$; so ω_Ω is supported on $\cup_{n=1}^{\infty}(\partial\Delta_n)$. Since $|p|$ is subharmonic in $\mathbf{C}$ for any polynomial p, it follows from Harnack's Inequality that $\Omega \subseteq abpe(P^1(d\omega_\Omega))$ ($\subseteq \mathbf{D}$). Since the components of $abpe(P^1(d\omega_\Omega))$ are simply connected, we in fact have that $abpe(P^1(d\omega_\Omega)) = \mathbf{D}$. An application of Proposition 4.2 and Jensen's Inequality completes the proof. □

Taking the sweep of σ (σ as defined in Corollary 4.3) to ∂G and using the properties of this sweep (see [C2], Chapter V, Sect. 9), we have:

Corollary 4.4. *Let G be a simply connected region whose outer boundary is* $\partial\mathbf{D}$*. Then there exists* f *in* $H^1(G)$ *such that* $f\circ\varphi$ *is an outer function (*φ *is a conformal mapping from* $\mathbf{D}$ *onto G) and* $abpe(P^t(|\tilde{f}|d\omega_G)) = \mathbf{D}$ *for* $1 \le t < \infty$*;* $|\tilde{f}|d\omega_G$ *is the measure with support in* ∂G *that is "carried" by* φ *from* $\partial\mathbf{D}$ *as* $|\widetilde{f\circ\varphi}|dm$*, where* $\widetilde{f\circ\varphi}$ *denotes the boundary values of* $f\circ\varphi$ *on* $\partial\mathbf{D}$*.*

Question 4.5. Let $\{z_n\}_{n=1}^{\infty}$ be a sampling sequence for the Bergman space $L_a^2(\mathbf{D})$. Then does there exist a summable sequence $\{r_n\}_{n=1}^{\infty}$ of positive constants such that $\Delta_n := \{z : |z - z_n| \le r_n\}$ $(n = 1, 2, 3, ...)$ are pairwise disjoint in $\mathbf{D}$ and $\Omega := \mathbf{D} \setminus (\cup_{n=1}^{\infty}\Delta_n)$ satisfies: $\omega_\Omega(\partial\mathbf{D}) = 0$?

In a recent communication with the author, P. Poggi-Corra has shown evidence of making some progress with this question.

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