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Existence and uniqueness of a solution for a parabolic quasilinear problem for linear growth functionals with L^1 data

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Dedicated to R. Nagel on the occasion of his 60th birthday

Abstract. We introduce a new concept of solution for the Dirichlet problem for quasilinear parabolic equations in divergent form for which the energy functional has linear growth. Using Kruzhkov's method of doubling variables both in space and time we prove uniqueness and a comparison principle in L^1 for these solutions. To prove the existence we use the nonlinear semigroup theory.

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1. Introduction

Let Ω be an open bounded set in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^1. We are interested in the Dirichlet problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = div\, \mathbf{a}(x, Du) & \text{in} \quad Q = (0,\infty) \times \Omega \\ u(t,x) = \varphi(x) & \text{on} \quad S = (0,\infty) \times \partial\Omega \\ u(0,x) = u_0(x) & \text{in} \quad x \in \Omega \end{cases} \tag{1}$$

where $u_0 \in L^1(\Omega)$ and $\mathbf{a}(x,\xi) = \nabla_\xi f(x,\xi)$, f being a function with linear growth as $\|\xi\| \to \infty$. In [4] existence and uniqueness of solutions of problem (1) are proved for initial data in $L^2(\Omega)$. Our aim here is to solve this problem for initial and boundary data in $L^1(\Omega)$ using the technique introduced in [3] to solve the Dirichlet problem for the minimizing total variation flow.

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One of the more important examples of a function $f(x,\xi)$ satisfying the conditions we shall assume on f is the nonparametric area integrand $f(x,\xi)=\sqrt{1+\|\xi\|^2}$. Problem (1) for this particular f, that is, the time-dependent minimal surface equation, has been studied in [20] and [30]. Another example of a problem of type (1) is the evolution problem for plastic antiplanar shear, studied in [34], which corresponds to the plasticity functional f given by

$$f(\xi)=\begin{cases}\frac{1}{2}\|\xi\|^2 & \text{if} \quad \|\xi\|\leq 1\\ \\ \|\xi\|-\frac{1}{2} & \text{if} \quad \|\xi\|\geq 1\end{cases}$$

On the other hand, problem (1) is studied in [26] for some Lagrangians f, which do not include the nonparametric area integrand, but include instead the plasticity functional and the total variation, that is, the case $f(\xi)=\|\xi\|$. An application of this type of equations to faceted crystal growth is studied in [27]. All these results are given for initial data in $L^2(\Omega)$ and have been obtained in the framework of nonlinear semigroup theory. We point out there is a viscosity approach to (1), given in [24] and [25], when the space dimension is one. In recent years, with the introduction of the concept of entropy or renormalized solutions, the L^1-theory has been developed for problems in divergent form when the associated variational energy has growth at infinity of order p with $p>1$ (see for instance [10], [13], [5], [6]). Now, as far as we know, for energy functionals with linear growth at infinity, the only L^1 results are the one obtained in [2] and [3] for the total variation flow.

In general, problem (1) does not have a classical solution. The aim of this paper is to introduce a new concept of solution for the Dirichlet problem (1), called entropy solution, for which existence and uniqueness for initial and boundary data in $L^1(\Omega)$ is proved. To prove existence we use our previous existence results for initial data in $L^2(\Omega)$ given in [4]. To get uniqueness we use Kruzhkov's method of doubling variables (both in space and time).

The paper is organized as follows. In Section 2 we summarize several results we need about functions of a measure and functions of bounded variation. In Section 3 we give the definition of entropy solution for the problem (1) and we state the main result. In section 4 we study the problem from the point of view of nonlinear semigroup theory. The final Section is devoted to prove existence and uniqueness of entropy solutions.

2. Definitions and preliminar facts

To make precise our notion of solution, let us recall several facts concerning functions of a measure and functions of bounded variation.

In order to consider the relaxed energy we recall the definition of function of a measure (see for instance, [8] or [20]). Let $g:\Omega\times\mathbb{R}^N\to\mathbb{R}$ a Carathéodory function such that

$$|g(x,\xi)| \le M(1+\|\xi\|) \qquad \forall\,(x,\xi)\in \Omega\times\mathbb{R}^N, \tag{2}$$

for some constant $M \ge 0$. We assume furthermore that g posseses an asymptote function, i.e., for all $x \in \Omega$ there exists the finite limit

$$\lim_{t\to 0^+} g\left(x, \frac{\xi}{t}\right)t = g^0(x,\xi). \tag{3}$$

It is clear that the function $g^0(x,\xi)$ is positively homogeneous in ξ, i.e.,

$$g^0(x, s\xi) = sg^0(x,\xi) \quad \text{ for all } x,\xi \text{ and } s>0.$$

We denote by $\mathcal{M}(\Omega,\mathbb{R}^N)$ the set of all $\mathbb{R}^N$-valuated bounded Radon measures on Ω. Given $\mu \in \mathcal{M}(\Omega,\mathbb{R}^N)$, we consider its Lebesgue decomposition

$$\mu = \mu^a + \mu^s,$$

where μ^a is its absolutely continuous part with respect to the Lebesgue measure λ_N of $\mathbb{R}^N$, and μ^s is singular with respect to λ_N. We denote by $\mu^a(x)$ the density of the measure μ^a with respect to λ_N and by $(d\mu^s/d|\mu|^s)(x)$ the density of μ^s with respect to $|\mu|^s$.

Given $\mu \in \mathcal{M}(\Omega,\mathbb{R}^N)$, we define $\tilde{\mu} \in \mathcal{M}(\Omega,\mathbb{R}^{N+1})$ by

$$\tilde{\mu}(B) := \big(\mu(B), \lambda_N(B)\big),$$

for every Borel set $B \subset \mathbb{R}^N$. Then, we have

$$\tilde{\mu} = \tilde{\mu}^a + \tilde{\mu}^s = \tilde{\mu}^a(x)\lambda_N + \tilde{\mu}^s = (\mu^a(x), \mathbb{1}_\Omega)\lambda_N + (\mu^s, 0).$$

Hence, we have

$$|\tilde{\mu}^s| = |\mu^s|, \qquad \frac{d\tilde{\mu}^s}{d|\tilde{\mu}^s|} = \left(\frac{d\mu^s}{d|\mu^s|}, 0\right) \quad |\mu^s| - \text{a.e.}$$

For $\mu \in \mathcal{M}(\Omega,\mathbb{R}^N)$ and g satisfying the above conditions, we define the measure $g(x,\mu)$ on Ω as

$$\int_B g(x,\mu) := \int_B g(x,\mu^a(x))\,dx + \int_B g^0\left(x, \frac{d\mu^s}{d|\mu|^s}(x)\right)d|\mu|^s \tag{4}$$

for all Borel set $B \subset \Omega$. In formula (4) we could write $(d\mu/d|\mu|)(x)$ instead of $(d\mu^s/d|\mu|^s)(x)$, because the two functions are equal $|\mu|^s$-a.e.

Another way of writing the measure $g(x,\mu)$ is the following. Let us consider the function $\tilde{g} : \Omega \times \mathbb{R}^N \times [0,+\infty[\to \mathbb{R}$ defined as

$$\tilde{g}(x,\xi,t) := \begin{cases} g\big(x, \frac{\xi}{t}\big)t & \text{if} \quad t>0 \\ \\ g^0(x,\xi) & \text{if} \quad t=0 \end{cases} \tag{5}$$

In [8] it is proved that if g is a Carathéodory function satisfying (2), then one has

$$\int_B g(x,\mu) = \int_B \tilde{g}\left(x, \frac{d\mu}{d\alpha}(x), \frac{d\lambda_N}{d\alpha}(x)\right) d\alpha, \tag{6}$$

where α is any positive Borel measure such that $|\mu| + \lambda_N \ll \alpha$.

Due to the linear growth condition on the Lagrangian, the natural energy space for problem (1) is the space of functions of bounded variation. Let us recall several facts concerning functions of bounded variation (for further information concerning functions of bounded variation we refer to [1], [22], or [35]).

A function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. Thus $u \in BV(\Omega)$ if and only if there are Radon measures $\mu_1, \ldots, \mu_N$ defined in Ω with finite total mass in Ω and

$$\int_\Omega u D_i \varphi dx = -\int_\Omega \varphi d\mu_i \tag{7}$$

for all $\varphi \in C_0^\infty(\Omega)$, $i = 1, \ldots, N$. Thus the gradient of u is a vector valued measure with finite total variation

$$\| Du \| = \sup\{\int_\Omega u \, div \, \varphi \, dx : \varphi \in C_0^\infty(\Omega, \mathbb{R}^N), |\varphi(x)| \le 1 \text{ for } x \in \Omega\}. \tag{8}$$

The space $BV(\Omega)$ is endowed with the norm

$$\| u \|_{BV} = \| u \|_{L^1(\Omega)} + \| Du \| . \tag{9}$$

For $u \in BV(\Omega)$, the gradient Du is a Radon measure that decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u \, \lambda_N$ where ∇u is the Radon-Nikodym derivative of the measure Du with respect to the Lebesgue measure λ_N. Also there is the polar decomposition $D^s u = \overrightarrow{D^s u}|D^s u|$ where $|D^s u|$ is the total variation measure of $D^s u$.

We shall need several results from [7] (see also [28]). Following [7], let

$$X(\Omega) = \{z \in L^\infty(\Omega, \mathbb{R}^N) : div(z) \in L^1(\Omega)\}. \tag{10}$$

If $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ we define the functional $(z, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$< (z, Dw), \varphi > = -\int_\Omega w \, \varphi \, div(z) \, dx - \int_\Omega w \, z \cdot \nabla\varphi \, dx. \tag{11}$$

Then (z, Dw) is a Radon measure in Ω,

$$\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w \, dx \tag{12}$$

for all $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and

$$\left| \int_B (z, Dw) \right| \leq \int_B |(z, Dw)| \leq \|z\|_\infty \int_B \|Dw\| \tag{13}$$

for any Borel set $B \subseteq \Omega$. Moreover, (z, Dw) is absolutely continuous with respect to $\|Dw\|$ with Radon-Nikodym derivative $\theta(z, Dw, x)$ which is a $\|Dw\|$ measurable function from Ω to $\mathbb{R}$ such that

$$\int_B (z, Dw) = \int_B \theta(z, Dw, x)\|Dw\| \tag{14}$$

for any Borel set $B \subseteq \Omega$. We also have that

$$\|\theta(z, Dw, .)\|_{L^\infty(\Omega, \|Dw\|)} \leq \|z\|_{L^\infty(\Omega, \mathbb{R}^N)}. \tag{15}$$

By writing

$$z \cdot D^s u := (z, Du) - (z \cdot \nabla u)\, d\lambda_N,$$

we see that $z \cdot D^s u$ is a bounded measure, furthermore in [28], it is proved that $z \cdot D^s u$ is absolutely continuous with respect to $|D^s u|$ (and thus is singular) and

$$|z \cdot D^s u| \leq \|z\|_\infty |D^s u|. \tag{16}$$

As a consequence of Theorem 2.4 of [7], we have:

$$\text{If} \quad z \in X(\Omega) \cap C(\Omega, \mathbb{R}^N), \quad \text{then} \quad z \cdot D^s u = (z \cdot \overrightarrow{D^s u})\, |D^s u| \tag{17}$$

In [7], a weak trace on $\partial\Omega$ of the normal component of $z \in X(\Omega)$ is defined. Concretely, it is proved that there exists a linear operator $\gamma : X(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(z)\|_\infty \leq \|z\|_\infty,$$

$$\gamma(z)(x) = z(x) \cdot \nu(x) \quad \text{for all } x \in \partial\Omega \ \text{ if } \ z \in C^1(\overline{\Omega}, \mathbb{R}^N).$$

We shall denote $\gamma(z)(x)$ by $[z, \nu](x)$. Moreover, the following *Green's formula*, relating the function $[z, \nu]$ and the measure (z, Dw), for $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$, is established:

$$\int_\Omega w\, div(z)\, dx + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z, \nu] w\, dH^{N-1}. \tag{18}$$

Let g be a function satisfying (2). Then for every $u \in BV(\Omega)$ we have the measure $g(x, Du)$ defined by

$$\int_B g(x, Du) = \int_B g(x, \nabla u(x))\, dx + \int_B g^0(x, \overrightarrow{D^s u}(x))\, d|D^s u|$$

for all Borel sets $B \subset \Omega$. If we assume Ω to have Lipschitz boundary and that $g(x, \xi)$ is defined also for $x \in \partial\Omega$, we can consider the functional G in $BV(\Omega)$ defined by

$$G(u) := \int_{\Omega} g(x, Du) + \int_{\partial\Omega} g^0\big(x, \nu(x)[\varphi(x) - u(x)]\big)\, dH^{N-1}, \tag{19}$$

where $\varphi \in L^1(\partial\Omega)$ is a given function and ν is the outward unit normal to $\partial\Omega$. In [8] is proved that if $\tilde{g}(x, \xi, t)$ is continuous on $\overline{\Omega} \times \mathbb{R}^N \times [0, +\infty[$ and convex in (ξ, t) for each fixed $x \in \overline{\Omega}$, then G is the greatest functional on $BV(\Omega)$ which is lower-semicontinuous with respect to the $L^1(\Omega)$-convergence and satisfies $G(u) \leq \int_{\Omega} g(x, \nabla u(x))\, dx$ for all functions $u \in C^1(\Omega) \cap W^{1,1}(\Omega)$ with $u = \varphi$ on $\partial\Omega$.

As in [3], we need to consider a weak trace on $\partial\Omega$ of the normal component of certain vector fields in Ω. Let us recall it. We define the space

$$Z(\Omega) := \{(z, \xi) \in L^\infty(\Omega, \mathbb{R}^N) \times BV(\Omega)^* \ :\ div(z) = \xi \ \ in\ \ \mathcal{D}'(\Omega)\}.$$

We denote $R(\Omega) := W^{1,1}(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$. For $(z, \xi) \in Z(\Omega)$ and $w \in R(\Omega)$ we define

$$\langle (z, \xi), w \rangle_{\partial\Omega} := \langle \xi, w \rangle_{BV(\Omega)^*, BV(\Omega)} + \int_{\Omega} z \cdot \nabla w\, dx.$$

Then, working as in the proof of Theorem 1.1. of [7], we obtain that if $w, v \in R(\Omega)$ and $w = v$ on $\partial\Omega$ one has

$$\langle (z, \xi), w \rangle_{\partial\Omega} = \langle (z, \xi), v \rangle_{\partial\Omega} \quad \forall\, (z, \xi) \in Z(\Omega). \tag{20}$$

As a consequence of (20), we can give the following definition: Given $u \in BV(\Omega) \cap L^\infty(\Omega)$ and $(z, \xi) \in Z(\Omega)$, we define $\langle (z, \xi), u \rangle_{\partial\Omega}$ by setting

$$\langle (z, \xi), u \rangle_{\partial\Omega} := \langle (z, \xi), w \rangle_{\partial\Omega}$$

where w is any function in $R(\Omega)$ such that $w = u$ on $\partial\Omega$. Again, working as in the proof of Theorem 1.1. of [7], we can prove that for every $(z, \xi) \in Z(\Omega)$ there exists $M_{z,\xi} > 0$ such that

$$|\langle (z, \xi), u \rangle_{\partial\Omega}| \leq M_{z,\xi} \|u\|_{L^1(\partial\Omega)} \qquad \forall\, u \in BV(\Omega) \cap L^\infty(\Omega). \tag{21}$$

Now, taking a fixed $(z, \xi) \in Z(\Omega)$, we consider the linear functional $F : L^\infty(\partial\Omega) \to \mathbb{R}$ defined by

$$F(v) := \langle (z, \xi), w \rangle_{\partial\Omega}$$

where $v \in L^\infty(\partial\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ is such that $w_{|\partial\Omega} = v$. By estimate (21), there exists $\gamma_{z,\xi} \in L^\infty(\partial\Omega)$ such that

$$F(v) = \int_{\partial\Omega} \gamma_{z,\xi}(x) v(x)\, dH^{N-1}.$$

Consequently there exists a linear operator $\gamma : Z(\Omega) \to L^\infty(\partial\Omega)$, with $\gamma(z, \xi) := \gamma_{z,\xi}$, satisfying

$$\langle (z, \xi), w \rangle_{\partial\Omega} = \int_{\partial\Omega} \gamma_{z,\xi}(x) w(x)\, dH^{N-1} \qquad \forall\, w \in BV(\Omega) \cap L^\infty(\Omega).$$

In case $z \in C^1(\overline{\Omega}, \mathbb{R}^N)$, we have $\gamma_z(x) = z(x) \cdot \nu(x)$ for all $x \in \partial\Omega$. Hence, the function $\gamma_{z,\xi}(x)$ is the weak trace of the normal component of (z, ξ). For simplicity of the notation, we shall denote $\gamma_{z,\xi}(x)$ by $[z, \nu](x)$.

We need to consider the space $BV(\Omega)_2$, defined as $BV(\Omega) \cap L^2(\Omega)$ endowed with the norm

$$\|w\|_{BV(\Omega)_2} := \|w\|_{L^2(\Omega)} + \|Du\|.$$

It is easy to see that $L^2(\Omega) \subset BV(\Omega)_2^*$ and

$$\|w\|_{BV(\Omega)_2^*} \le \|w\|_{L^2(\Omega)} \qquad \forall\, w \in L^2(\Omega). \tag{22}$$

It is well known (see [33]) that the dual space $\left(L^1(0,T; BV(\Omega)_2)\right)^*$ is isometric to the space $L^\infty(0,T; BV(\Omega)_2^*, BV(\Omega)_2)$ of all weakly* measurable functions $f : [0,T] \to BV(\Omega)_2^*$, such that $v(f) \in L^\infty([0,T])$, where $v(f)$ denotes the supremum of the set $\{|\langle w, f\rangle| \ : \ \|w\|_{BV(\Omega)_2} \le 1\}$ in the vector lattice of measurable real functions. Moreover, the dual paring of the isometric is defined by

$$\langle w, f\rangle = \int_0^T \langle w(t), f(t)\rangle\, dt,$$

for $w \in L^1(0,T; BV(\Omega)_2)$ and $f \in L^\infty(0,T; BV(\Omega)_2^*, BV(\Omega)_2)$.

By $L^1_w(0,T, BV(\Omega))$ we denote the space of weakly measurable functions $w : [0,T] \to BV(\Omega)$ (i.e., $t \in [0,T] \to < w(t), \phi >$ is measurable for every $\phi \in BV(\Omega)^*$) such that $\int_0^T \|w(t)\| < \infty$. Observe that, since $BV(\Omega)$ has a separable predual (see [1]), it follows easily that the map $t \in [0,T] \to \|w(t)\|$ is measurable.

To make precise our notion of solution we need the following definitions.

Definition 1. Let $\Psi \in L^1(0,T, BV(\Omega))$. We say Ψ admits a *weak derivative* in the space $L^1_w(0,T, BV(\Omega)) \cap L^\infty(Q_T)$ if there is a function $\Theta \in L^1_w(0,T, BV(\Omega)) \cap L^\infty(Q_T)$ such that $\Psi(t) = \displaystyle\int_0^t \Theta(s)ds$, the integral being taken as a Pettis integral.

Definition 2. Let $\xi \in \left(L^1(0,T, BV(\Omega)_2)\right)^*$. We say that ξ is *the time derivative* in the space $\left(L^1(0,T, BV(\Omega)_2\right)^*$ of a function $u \in L^1((0,T) \times \Omega)$ if

$$\int_0^T < \xi(t), \Psi(t) > dt = -\int_0^T \int_\Omega u(t,x)\Theta(t,x)dxdt$$

for all test functions $\Psi \in L^1(0,T, BV(\Omega))$ with compact support in time, which admit a weak derivative $\Theta \in L^1_w(0,T, BV(\Omega)) \cap L^\infty(Q_T)$.

Note that if $w \in L^1(0,T, BV(\Omega)) \cap L^\infty(Q_T)$ and $z \in L^\infty(Q_T, \mathbb{R}^N)$ such that there exists $\xi \in \left(L^1(0,T, BV(\Omega)\right)^*$ with $div(z) = \xi$ in $\mathcal{D}'(Q_T)$, we can define, associated to the pair (z, ξ), the distribution (z, Dw) in Q_T by

$$\langle (z, Dw), \phi \rangle :=$$
$$- \int_0^T \langle \xi(t), w(t)\phi(t) \rangle \, dt - \int_0^T \int_\Omega z(t,x) w(t,x) \nabla_x \phi(t,x) \, dx dt. \tag{23}$$

for all $\phi \in \mathcal{D}(Q_T)$.

Definition 3. Let $\xi \in \left(L^1(0, T, BV(\Omega)_2)\right)^*$, $z \in L^\infty(Q_T, \mathbb{R}^N)$. We say that $\xi = div(z)$ in $\left(L^1(0, T, BV(\Omega)_2)\right)^*$ if (z, Dw) is a Radon measure in Q_T with normal boundary values $[z, \nu] \in L^\infty((0, T) \times \partial\Omega)$, such that

$$\int_{Q_T} (z, Dw) + \int_0^T < \xi(t), w(t) > dt$$
$$= \int_0^T \int_{\partial\Omega} [z(t,x), \nu] w(t,x) dH^{N-1} dt,$$

for all $w \in L^1(0, T, BV(\Omega)) \cap L^\infty(Q_T)$.

We shall denote by

$$sign_0(r) := \begin{cases} 1 & \text{if r > 0} \\ 0 & \text{if r = 0} \\ -1 & \text{if r < 0} \end{cases}$$

and by

$$sign(r) := \begin{cases} 1 & \text{if r > 0} \\ [-1, 1] & \text{if r = 0} \\ -1 & \text{if r < 0.} \end{cases}$$

Let $T_k(r) = [k - (k - |r|)^+] sign_0(r)$, $k \geq 0$, $r \in \mathbb{R}$. We consider the set $\mathcal{T} = \{T_k, T_k^+, T_k^- \ : \ k > 0\}$. We need to consider a more general set of truncature functions, concretely, the set $\mathcal{P}$ of all nondecreasing Lipschitz-continuous functions $p : \mathbb{R} \to \mathbb{R}$, such that $p'(s) \in \{0, 1\}$ and $\{r \in \mathbb{R} \ : \ p'(r) = 1\} = \cup_{j=1}^m]a_j, b_j[$. Observe that the functions of the form $p(r) = T(r - l)$, for some $T \in \mathcal{T}$ and $l \in \mathbb{R}$ are in $\mathcal{P}$, and obviously, $\mathcal{T} \subset \mathcal{P}$.

3. The main result

In this section we give the concept of solution for the Dirichlet problem (1) and we state the existence and uniqueness result for this type of solutions.

Here we assume that Ω is an open bounded set in $\mathbb{R}^N$, $N \geq 2$, with boundary $\partial\Omega$ of class C^1, and the Lagrangian $f : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumptions, which we shall refer collectively as (H):

(H_1) f is continuous on $\overline{\Omega} \times \mathbb{R}^N$, convex and differentiable with continuous gradient in ξ for each fixed $x \in \Omega$ and satisfies the linear growth condition

$$C_0\|\xi\| - C_1 \le f(x,\xi) \le M(\|\xi\| + C_2). \tag{24}$$

for some positive constants C_0, C_1, C_2 and M. Moreover, f^0 exists and $f^0(x,-\xi) = f^0(x,\xi)$ for all $\xi \in \mathbb{R}^N$ and all $x \in \overline{\Omega}$.

(H$_2$) $\tilde{f}(x,\xi,t)$ is continuous on $\overline{\Omega} \times \mathbb{R}^N \times [0,+\infty[$ and convex in (ξ,t) for each fixed $x \in \overline{\Omega}$.

We consider the function $\mathbf{a}(x,\xi) = \nabla_\xi f(x,\xi)$ associated to the Lagrangian f. By the convexity of f

$$\mathbf{a}(x,\xi)\cdot(\eta-\xi) \le f(\eta) - f(\xi), \tag{25}$$

and the following monotonicity condition is satisfied

$$(\mathbf{a}(x,\eta) - \mathbf{a}(x,\xi))\cdot(\eta-\xi) \ge 0. \tag{26}$$

Moreover, it is easy to see that

$$|\mathbf{a}(x,\xi)| \le M \quad \forall\,(x,\xi) \in \Omega \times \mathbb{R}^N. \tag{27}$$

We consider the function $h : \Omega \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$h(x,\xi) := \mathbf{a}(x,\xi)\cdot\xi.$$

From (25) and (24), it follows that

$$C_0\|\xi\| - D_1 \le h(x,\xi) \le M\|\xi\| \tag{28}$$

for some positive constant D_1. We assume that $h(x,-\xi) = h(x,\xi)$ for all $\xi \in \mathbb{R}^N$.

We assume that

(H$_3$) $h(x,\xi) \ge 0$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$, h^0 exists and the function $\tilde{h}$ is continuous on $\overline{\Omega} \times \mathbb{R}^N \times [0,+\infty[$.

We need to consider the mapping $\mathbf{a}^\infty$ defined by

$$\mathbf{a}^\infty(x,\xi) := \lim_{t\to+\infty} \mathbf{a}(x,t\xi).$$

Observe that

$$h^0(x,\xi) = \mathbf{a}^\infty(x,\xi)\cdot\xi \quad \text{and} \quad C_0\|\xi\| \le h^0(x,\xi) \le M\|\xi\|.$$

(H$_4$) $\mathbf{a}^\infty(x,\xi) = \nabla_\xi f^0(x,\xi)$ for all $\xi \neq 0$ and all $x \in \overline{\Omega}$.

In particular, as a consequence of Euler's Theorem, we have

$$f^0(x,\xi) = \mathbf{a}^\infty(x,\xi)\cdot\xi = h^0(x,\xi),$$

for all $\xi \in \mathbb{R}^N$ and all $x \in \overline{\Omega}$, and consequently,

$$C_0\|\xi\| \le f^0(x,\xi) \le M\|\xi\| \quad \forall\,\xi \in \mathbb{R}^N,\ \forall\,x \in \overline{\Omega} \tag{29}$$

(H$_5$) $\mathbf{a}(x,\xi)\cdot\eta \le h^0(x,\eta)$ for all $\xi,\eta \in \mathbb{R}^N$, and all $x \in \overline{\Omega}$.

Either from (H_4) or (H_5) it follows that $\mathbf{a}^\infty(x,\xi)\cdot\eta \le h^0(x,\eta)$ for all $\xi,\eta\in\mathbb{R}^N$, $\xi\neq 0$, and all $x\in\overline{\Omega}$. Indeed, it suffices to replace ξ by $t\xi$ in (H_5) and let $t\to+\infty$.

(H_6) We assume that

$$|a(x,\xi)-a(y,\xi)|\le\omega(\|x-y\|) \tag{30}$$

for all $x,y\in\Omega$, and all $\xi\in\mathbb{R}^N$, where $\omega(r)$ is a modulus of continuity. From the definition of $\mathbf{a}^\infty$

$$|\mathbf{a}^\infty(x,\xi)-\mathbf{a}^\infty(y,\xi)|\le\omega(\|x-y\|).$$

Hence

$$\begin{aligned}|h^0(x,\xi)-h^0(y,\xi)| &= |\mathbf{a}^\infty(x,\xi)\cdot\xi-\mathbf{a}^\infty(y,\xi)\cdot\xi| \\ &\le \omega(\|x-y\|)\|\xi\|. \end{aligned}\tag{31}$$

Remark 1. Note that assumption (H_6) is only needed to prove uniqueness. The Lipschitz continuity in x of the flux is a common assumption to prove uniqueness of Kruzhkov's solutions of scalar conservation laws ([29]).

Definition 4. A measurable function $u:(0,T)\times\Omega\to\mathbb{R}$ is an *entropy solution* of (1) in $Q_T=(0,T)\times\Omega$ if $u\in C([0,T];L^1(\Omega))$, $p(u(\cdot))\in L^1_w(0,T,BV(\Omega))$ for all $p\in\mathcal{T}$, and there exists $\xi\in\big(L^1(0,T,BV(\Omega)_2)\big)^*$ such that:

(i) $(\mathbf{a}(x,\nabla u(t)),\xi(t))\in Z(\Omega)$ a.e. $t\in[0,T]$,

(ii) ξ is the time derivative of u in $\big(L^1(0,T,BV(\Omega)_2)\big)^*$,

(iii) $\xi=div(\mathbf{a}(x,\nabla u(t)))$ in $\big(L^1(0,T,BV(\Omega)_2)\big)^*$,

(iv) $[\mathbf{a}(x,\nabla u(t)),\nu]\in\operatorname{sign}\big(p(\varphi)-p(u(t))\big)f^0(x,\nu(x))$ a.e. in $t\in[0,T]$ for all $p\in\mathcal{T}$,

(iv) The following inequality is satisfied

$$\begin{aligned}&-\int_0^T\int_\Omega j(u(t)-l)\eta_t\,dxdt+\int_0^T\int_\Omega\eta(t)h(x,Dp(u(t)-l))\,dt\\ &\quad+\int_0^T\int_\Omega\mathbf{a}(x,\nabla u(t))\cdot\nabla\eta(t)p(u(t)-l)\,dxdt\\ &\le\int_0^T\int_{\partial\Omega}[\mathbf{a}(x,\nabla u(t)),\nu]\eta(t)p(u(t)-l)\,dH^{N-1}dt,\end{aligned}$$

for all $l\in\mathbb{R}$, for all $\eta\in C^\infty(\overline{Q_T})$, with $\eta\ge 0$, $\eta(t,x)=\phi(t)\psi(x)$, being $\phi\in\mathcal{D}(]0,T[)$, $\psi\in C^\infty(\overline{\Omega})$, and $p\in\mathcal{T}$, where $j(r)=\int_0^r p(s)\,ds$.

Our main result is:

Theorem 1. *Assume we are under asumptions (H) and let* $u_0 \in L^1(\Omega)$ *and* $\varphi \in L^1(\partial\Omega)$. *Then there exists a unique entropy solution of (1) in* $(0, T) \times \Omega$ *for every* $T > 0$ *such that* $u(0) = u_0$. *Moreover, if* $u(t), \hat{u}(t)$ *are the entropy solutions corresponding to initial data* $u_0, \hat{u}_0$, *respectively, then*

$$\|\big(u(t) - \hat{u}(t)\big)^+\|_1 \leq \|\big(u_0 - \hat{u}_0\big)^+\|_1, \quad \|u(t) - \hat{u}(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1 \tag{32}$$

for all $t \geq 0$.

4. The semigroup solution

To prove the existence part of Theorem 1 we shall use the techniques of completely accretive operators and the Crandall-Liggett's semigroup generation Theorem ([19]). Let us recall the notion of completely accretive operator introduced in [11]. Let $\mathcal{M}(\Omega)$ be the space of measurable functions in Ω. Given $u, v \in \mathcal{M}(\Omega)$, we shall write

$$u \ll v \quad \text{if and only if} \int_\Omega j(u)dx \leq \int_\Omega j(v)dx \tag{33}$$

for all $j \in J_0$ where

$$J_0 = \{j : \mathbb{R} \to [0, \infty], \ \text{convex}, \ \text{l.s.c.}, \ j(0) = 0\} \tag{34}$$

(l.s.c. is an abbreviation for lower semicontinuous function). Let A be an operator (possibly multivalued) in $\mathcal{M}(\Omega)$, i.e., $A \subseteq \mathcal{M}(\Omega) \times \mathcal{M}(\Omega)$. We shall say that A is *completely accretive* if

$$u - \hat{u} \ll u - \hat{u} + \lambda(v - \hat{v}) \quad \text{for all } \lambda > 0 \text{ and all } (u, v), (\hat{u}, \hat{v}) \in A. \tag{35}$$

A completely accretive operator in $L^1(\Omega)$ is said to be m-completely accretive if $R(I+\lambda A) = L^1(\Omega)$ for any $\lambda > 0$. In that case, by Crandall-Ligget's Theorem, A generates a contraction semigroup in $L^1(\Omega)$ given by the exponential formula

$$e^{-tA}u_0 = \lim_{n\to\infty} (I + \frac{t}{n}A)^{-n}u_0 \quad \text{for any } u_0 \in L^1(\Omega).$$

Let us write $u(t) = e^{-tA}u_0$. Then $u \in C([0, T], L^1(\Omega))$ for any $T > 0$ and it is a *mild solution* (a solution in the sense of semigroups [12]) of

$$\frac{du}{dt} + Au \ni 0, \tag{36}$$

such that $u(0) = u_0$. Moreover, if $u(t), \hat{u}(t)$ are the mild solutions corresponding to initial data $u_0, \hat{u}_0$, respectively, then

$$\|\big(u(t) - \hat{u}(t)\big)^+\|_1 \leq \|\big(u_0 - \hat{u}_0\big)^+\|_1, \quad \|u(t) - \hat{u}(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1 \tag{37}$$

for all $t \geq 0$.

In [4], using nonlinear semigroups, the Dirichlet problem (1) is studied for initial data in L^2. For that we consider the energy functional $\Phi_\varphi : L^2(\Omega) \to [0, +\infty]$ defined by

$$\Phi_\varphi(u) := \int_\Omega f(x, Du) + \int_{\partial\Omega} f^0(x, \nu(x)[\varphi - u])\, dH^{N-1},$$

if $u \in BV(\Omega) \cap L^2(\Omega)$ and

$$\Phi_\varphi(u) := +\infty \quad \text{ if } \quad u \in L^2(\Omega) \setminus BV(\Omega).$$

Note that, on the boundary, the integrand can be written in the form $f^0(x, \nu(x)[\varphi - u]) = |\varphi - u| f^0(x, \nu(x))$. Functional Φ_φ is clearly convex and has the form given in (19). Then, as a consequence of Anzellotti's result ([8]), we have that Φ_φ is lower-semicontinuous, therefore the subdifferential $\partial\Phi_\varphi$ of Φ_φ, i.e. the operator in $L^2(\Omega)$ defined by

$$v \in \partial\Phi_\varphi(u) \iff \Phi_\varphi(w) - \Phi_\varphi(u) \geq \int_\Omega v(w - u)\, dx, \quad \forall\, w \in L^2(\Omega)$$

is a maximal monotone operator in $L^2(\Omega)$. Consequently, existence and uniqueness of a solution of the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Phi_\varphi(u(t)) \ni 0 & t \in]0, \infty[\\ u(0) = u_0 & u_0 \in L^2(\Omega), \end{cases} \tag{38}$$

follows immediately from the nonlinear semigroup theory (see [15]). Now, to get the full strength of the abstract result derived from semigroup theory $\partial\Phi_\varphi(u)$ is characterized. To get this characterization, the following operator $\mathcal{B}_\varphi$ in $L^2(\Omega)$ was introduced in [4].

$$(u, v) \in \mathcal{B}_\varphi \iff u \in BV(\Omega) \cap L^2(\Omega), v \in L^2(\Omega) \text{ and}$$

$$\mathbf{a}(x, \nabla u) \in X(\Omega) \text{ satisfies :}$$

$$-v = div\, \mathbf{a}(x, \nabla u) \quad \text{ in } \mathcal{D}'(\Omega), \tag{39}$$

$$\mathbf{a}(x, \nabla u) \cdot D^s u = f^0(x, D^s u) = f^0(x, \overrightarrow{D^s u})|D^s u|, \tag{40}$$

$$[\mathbf{a}(x, \nabla u), \nu] \in \text{sign}\,(p(\varphi) - p(u)) f^0(x, \nu(x)), \;\; \forall\, p \in \mathcal{P} \quad H^{N-1} - \text{a.e.} \tag{41}$$

In [4], the following result is proved.

Theorem 2. *Let $\varphi \in L^1(\partial\Omega)$. Assume we are under assumptions (H_1)-(H_5). Then the operator $\partial\Phi_\varphi$ has dense domain in $L^2(\Omega)$ and*

$$\partial\Phi_\varphi = \mathcal{B}_\varphi.$$

As a consequence of the above result the following characterization of the semigroup solution is obtained.

Theorem 3. *Let $\varphi \in L^1(\partial\Omega)$. Assume we are under assumptions (H_1)-(H_5). Let $(S(t))_{t\geq 0}$ be the semigroup in $L^2(\Omega)$ generated by $\mathcal{B}_\varphi$. Then, given $u_0 \in L^2(\Omega)$, the strong solution $u(t) = S(t)u_0$ of (38) is characterized as the only function $u \in C([0,T], L^2(\Omega))$ that verifies: $u(0) = u_0$, $u'(t) \in L^2(\Omega)$, $u(t) \in BV(\Omega) \cap L^2(\Omega)$, $\mathbf{a}(x, \nabla u(t)) \in X(\Omega)$ a.e. $t \in [0,T]$, and for almost all $t \in [0,T]$ $u(t)$ satisfies:*

$$u'(t) = div(\mathbf{a}(x, \nabla u(t)) \quad \text{in } \mathcal{D}'(\Omega), \tag{42}$$

$$\mathbf{a}(x, \nabla u(t)) \cdot D^s u(t) = f^0(x, D^s u(t)), \tag{43}$$

$$[\mathbf{a}(x, \nabla u(t)), \nu] \in \text{sign}\,(p(\varphi) - p(u(t))) f^0(x, \nu(x)), \quad \forall\, p \in \mathcal{P}. \tag{44}$$

Proposition 1. *Let $\varphi \in L^1(\partial\Omega)$. Assume we are under assumptions (H_1)-(H_5). Then the operator $\mathcal{B}_\varphi = \partial\Phi_\varphi$ is completely accretive.*

Proof. We first consider the functional $\Gamma_\varphi : L^2(\Omega) \to]-\infty, +\infty]$ given by

$$\Gamma_\varphi(u) := \int_\Omega f(x, \nabla u)\, dx + \int_{\partial\Omega} f^0(x, \nu(x)[\varphi - u])\, dH^{N-1},$$

if $u \in W^{1,1}(\Omega) \cap L^2(\Omega)$ and

$$\Gamma_\varphi(u) := +\infty \quad \text{if} \quad u \in L^2(\Omega) \setminus W^{1,1}(\Omega)$$

Let us see that Γ_φ satisfies:

$$\Gamma_\varphi(u + p(\hat{u} - u)) + \Gamma_\varphi(\hat{u} - p(\hat{u} - u)) \leq \Gamma_\varphi(u) + \Gamma_\varphi(\hat{u}) \tag{45}$$

for all $u, \hat{u} \in L^1(\Omega)$, $p \in P_0$. In fact, we may assume $u, \hat{u} \in W^{1,1}(\Omega)$. If $v := u + p(\hat{u} - u)$ and $\hat{v} := \hat{u} - p(\hat{u} - u)$, then

$$\Gamma_\varphi(v) = \int_\Omega f(x, \lambda\nabla\hat{u} + (1-\lambda)\nabla u)\, dx$$

$$+ \int_{\partial\Omega} f^0(x, \nu(x))|\alpha(\varphi - \hat{u}) + (1-\alpha)(\varphi - u)|\, dH^{N-1}$$

and

$$\Gamma_\varphi(\hat{v}) = \int_\Omega f(x, \lambda\nabla u + (1-\lambda)\nabla\hat{u})\, dx$$

$$+ \int_{\partial\Omega} f^0(x, \nu(x))|\alpha(\varphi - u) + (1-\alpha)(\varphi - \hat{u})|\, dH^{N-1},$$

where

$$\lambda = p'(\hat{u} - u) \quad \text{and} \quad \alpha = \mathbb{1}_{\{u \neq \hat{u}\}} \frac{p(\hat{u} - u)}{\hat{u} - u}.$$

By the convexity of $f(x,\xi)$ with respect to ξ, we have

$$f(x, \lambda\nabla\hat{u} + (1-\lambda)\nabla u) + f(x, \lambda\nabla u + (1-\lambda)\nabla\hat{u})$$
$$\leq \lambda f(x,\nabla\hat{u}) + (1-\lambda) f(x,\nabla u) + \lambda f(x,\nabla u) + (1-\lambda) f(x,\nabla\hat{u})$$
$$= f(x,\nabla u) + f(x,\nabla\hat{u}).$$

In the same way

$$|\alpha(\varphi-\hat{u}) + (1-\alpha)(\varphi-u)| + |\alpha(\varphi-u) + (1-\alpha)(\varphi-\hat{u})| \leq |\varphi-u| + |\varphi-\hat{u}|.$$

Then, integrating, we obtain $\Gamma_\varphi(v) + \Gamma_\varphi(\hat{v}) \leq \Gamma_\varphi(u) + \Gamma_\varphi(\hat{u})$.

Since Γ_φ satisfies (45), by Lemma 7.1 in [11], we have that the operator $\partial\Gamma_\varphi$ is completely accretive. Now, by the results in [8], the lower semicontinuous envelope of the functional Γ_φ is the functional Φ_φ. Hence, by Lemma 7.5 and Lemma 7.1 in [11], we get that $\partial\Phi_\varphi$ is completely accretive. □

To associate an m-completely accretive operator in $L^1(\Omega)$ with our problem we need to consider the function space

$$TBV(\Omega) := \{u \in L^1(\Omega) \ : \ T_k(u) \in BV(\Omega), \ \ \forall\, k > 0\},$$

and to give a sense to the Radon-Nikodym derivative ∇u of a function $u \in TBV(\Omega)$. A similar problem was treated in [10] where the authors had to give a sense to the derivative of functions whose truncatures are in a Sobolev space (in their notation, for functions in $\mathcal{T}^{1,1}_{loc}(\Omega)$, $p \geq 1$). Notice that the function space $TBV(\Omega)$ is closely related to the space $GBV(\Omega)$ of generalized functions of bounded variation introduced by E. Di Giorgi and L. Ambrosio ([17], see also [1]). Using the chain rule for BV-functions (see for instance [1]), with a similar proof to the one given in Lemma 2.1 of [10], we obtain the following result.

Lemma 1. *For every $u \in TBV(\Omega)$ there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v \mathbb{1}_{\{|u|<k\}} \qquad \lambda_N - a.e. \tag{46}$$

Thanks to this result we define ∇u for a function $u \in TBV(\Omega)$ as the unique function v which satisfies (46). This notation will be used throughout in the sequel.

Lemma 2. *If $u \in TBV(\Omega)$, then $p(u) \in BV(\Omega)$ for every Lipschitz continuous function $p : \mathbb{R} \to \mathbb{R}$ satisfying $p'(s) = 0$ for $|s|$ large enough. Moreover, $\nabla p(u) = p'(u)\nabla u$ λ_N-a.e.*

Proof. The proof of this lemma is straightforward since $p(u) = p(T_k(u))$ for k large enough. Hence, $p(u) \in BV(\Omega)$ and by the chain rule,

$$\nabla p(u) = \nabla p(T_k(u)) = p'(T_k(u))\nabla T_k(u) = p'(u)\nabla u \ \mathbb{1}_{\{|u|<k\}} = p'(u)\nabla u.$$

□

We define the operator $\mathcal{A}_\varphi$ in $L^1(\Omega)$ by:

$$(u, v) \in \mathcal{A}_\varphi \iff u, v \in L^1(\Omega),\ p(u) \in BV(\Omega) \text{ for all } p \in \mathcal{P}$$
$$\text{and } \mathbf{a}(x, \nabla u) \in X(\Omega) \text{ satisfies :}$$

$$-v = div\, \mathbf{a}(x, \nabla u) \quad \text{in } \mathcal{D}'(\Omega), \tag{47}$$

$$\mathbf{a}(x, \nabla u) \cdot D^s p(u) = f^0(x, D^s p(u)) \quad \forall\, p \in \mathcal{P}, \tag{48}$$

$$[\mathbf{a}(x, \nabla u), \nu] \in \text{sign } (p(\varphi) - p(u)) f^0(x, \nu(x)), \ \ \forall\, p \in \mathcal{P} \quad H^{N-1} - \text{a.e.} \tag{49}$$

Taking into account Lemma 2, we have that if $(u, v) \in \mathcal{A}_\varphi$, then

$$\int_\Omega (w - p(u)) v\, dx = \int_\Omega (\mathbf{a}(x, \nabla u), Dw) - \int_\Omega h(x, Dp(u))$$
$$- \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu](w - p(\varphi))\, dH^{N-1} \tag{50}$$
$$- \int_{\partial\Omega} |p(u) - p(\varphi)| f^0(x, \nu(x))\, dH^{N-1},$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$ and for all $p \in \mathcal{P}$.

Lemma 3. *If $(u, v) \in \mathcal{A}_\varphi$, for $a, b > 0$, we have*

$$\frac{1}{b} \int_{\{a<|u|<a+b\}} \mathbf{a}(x, \nabla u) \cdot \nabla u\, dx$$
$$\leq \int_{\{|u|>a\}} |v|\, dx + \int_{\{|\varphi|>a\}} |[\mathbf{a}(x, \nabla u), \nu]|\, dH^{N-1}. \tag{51}$$

In particular,

$$\lim_{a\to\infty} \int_{\{a<|u|<a+b\}} \mathbf{a}(x, \nabla u) \cdot \nabla u\, dx$$
$$= \lim_{a\to\infty} \int_{\{a<|u|<a+b\}} |\nabla u|\, dx = 0. \tag{52}$$

Proof. We consider the truncature function $T_{a,b}(s) := T_b(s - T_a(s))$. Since $T_{a,b}$ is an element of $\mathcal{P}$, applying Green's formula, we have

$$- \int_\Omega v T_{a,b}(u)\, dx = \int_\Omega div(\mathbf{a}(x, \nabla u)) T_{a,b}(u)\, dx$$
$$= - \int_\Omega (\mathbf{a}(x, \nabla u), DT_{a,b}(u)) + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu] T_{a,b}(u)\, dH^{N-1}$$

$$= -\int_{\Omega} \mathbf{a}(x, \nabla u) \cdot \nabla T_{a,b}(u)\, dx - \int_{\Omega} \mathbf{a}(x, \nabla u) \cdot D^s T_{a,b}(u)$$

$$+ \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu] T_{a,b}(u)\, dH^{N-1} = -\int_{\{a<|u|<a+b\}} \mathbf{a}(x, \nabla u) \cdot \nabla u\, dx$$

$$- \int_{\Omega} f^0(x, D^s T_{a,b}(u)) + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu] T_{a,b}(u)\, dH^{N-1}.$$

Since

$$\int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu] T_{a,b}(u)\, dH^{N-1}$$

$$= \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu](T_{a,b}(u) - T_{a,b}(\varphi))\, dH^{N-1}$$

$$+ \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu] T_{a,b}(\varphi)\, dH^{N-1}$$

$$= -\int_{\partial\Omega} |T_{a,b}(u) - T_{a,b}(\varphi)| f^0(x, \nu(x))\, dH^{N-1}$$

$$+ \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu] T_{a,b}(\varphi)\, dH^{N-1} \leq \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu] T_{a,b}(\varphi)\, dH^{N-1},$$

it follows that

$$\int_{\{a<|u|<a+b\}} \mathbf{a}(x, \nabla u) \cdot \nabla u\, dx$$

$$\leq \int_{\Omega} v T_{a,b}(u)\, dx + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u), \nu] T_{a,b}(\varphi)\, dH^{N-1}$$

$$\leq b \left(\int_{\{|u|>a\}} |v|\, dx + \int_{\{|\varphi|>a\}} |[\mathbf{a}(x, \nabla u), \nu]|\, dH^{N-1} \right),$$

and the result follows. □

The main result of this section is the following.

Theorem 4. *Let $\varphi \in L^1(\partial\Omega)$. Assume we are under assumptions (H_1)-(H_5). Then, $\mathcal{B}_\varphi \subset \mathcal{A}_\varphi$ and the operator $\mathcal{A}_\varphi$ is m-completely accretive in $L^1(\Omega)$ with dense domain. Moreover, if $(T(t))_{t\geq 0}$ is the semigroup of order preserving contractions in $L^1(\Omega)$ generated by the operator $\mathcal{A}_\varphi$, then its restriction to $L^2(\Omega)$ coincides with the semigroup generated by the operator $\mathcal{B}_\varphi$.*

Proof. Let us see first that $\mathcal{B}_\varphi \subset \mathcal{A}_\varphi$. If $(u, v) \in \mathcal{B}_\varphi$, then $u \in BV(\Omega) \cap L^2(\Omega)$, $v \in L^2(\Omega)$ and $\mathbf{a}(x, \nabla u) \in X(\Omega)$ satisfies:

$$-v = div\, \mathbf{a}(x, \nabla u) \quad \text{ in } \mathcal{D}'(\Omega), \tag{53}$$

$$\mathbf{a}(x, \nabla u) \cdot D^s u = f^0(x, D^s u) = f^0(x, \overrightarrow{D^s u})|D^s u|, \tag{54}$$

$$[\mathbf{a}(x, \nabla u), \nu] \in \text{sign } (\varphi - u) f^0(x, \nu(x)), \ \ \forall\, p \in \mathcal{P} \quad H^{N-1} - \text{a.e.} \tag{55}$$

Since the functions of $\mathcal{P}$ are nondecreasing, from (55) we get (49). On the other hand, by (54), we have

$$\theta(\mathbf{a}(x, \nabla u), Du, x) = f^0(x, \overrightarrow{D^s u}) \quad |D^s u| - a.e \tag{56}$$

Then, given $p \in \mathcal{P}$, by Proposition 2.8 of [7], for every Borel set $B \subset \Omega$, we have

$$\int_B \mathbf{a}(x, \nabla u) \cdot \nabla p(u)\, dx + \int_B \mathbf{a}(x, \nabla u) \cdot D^s p(u) = \int_B (\mathbf{a}(x, \nabla u), Dp(u))$$

$$= \int_B \theta(\mathbf{a}(x, \nabla u), Dp(u), x)|Dp(u)| = \int_B \theta(\mathbf{a}(x, \nabla u), Du, x)|Dp(u)|$$

$$= \int_B \theta(\mathbf{a}(x, \nabla u), Du, x)|\nabla p(u)| + \int_B \theta(\mathbf{a}(x, \nabla u), Du, x)|D^s p(u)|$$

$$= \int_B \theta(\mathbf{a}(x, \nabla u), Du, x) \cdot p'(u)|\nabla u|\, dx + \int_B f^0(x, \overrightarrow{D^s u})|D^s p(u)|$$

$$= \int_B \mathbf{a}(x, \nabla u) \cdot \nabla p(u)\, dx + \int_B f^0(x, \overrightarrow{D^s u})|D^s p(u)|.$$

Hence, $\mathbf{a}(x, \nabla u) \cdot D^s p(u) = f^0(x, \overrightarrow{D^s u})|D^s p(u)| = f^0(x, D^s p(u))$. Thus, (48) holds and, therefore, we have proved that $\mathcal{B}_\varphi \subset \mathcal{A}_\varphi$. Moreover, since the domain of $\mathcal{B}_\varphi$ is dense in $L^2(\Omega)$ (Theorem 2), we also obtain that the domain of $\mathcal{A}_\varphi$ is dense in $L^1(\Omega)$.

Next we are going to prove that the operator $\mathcal{A}_\varphi$ is accretive in $L^1(\Omega)$. We must show that

$$\int_\Omega |u - \hat{u}| \leq \int_\Omega |f - \hat{f}| \tag{57}$$

whenever $f \in u + \mathcal{A}_\varphi u$, $\hat{f} \in \hat{u} + \mathcal{A}_\varphi \hat{u}$. Indeed, for $r, k > 0$, we have

$$\int_\Omega \left[(f - u) - (\hat{f} - \hat{u})\right] T_r(T_k(u) - T_k(\hat{u}))\, dx$$

$$= \int_\Omega (\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u}), DT_r(T_k(u) - T_k(\hat{u})))$$

$$- \int_{\partial\Omega} [\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u}), \nu] T_r(T_k(u) - T_k(\hat{u}))\, dH^{N-1}$$

$$= \int_\Omega (\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u})) \cdot \nabla T_r(T_k(u) - T_k(\hat{u}))\, dx$$

$$+ \int_\Omega (\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u})) \cdot D^s T_r(T_k(u) - T_k(\hat{u}))$$

$$- \int_{\partial\Omega} [\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u}), \nu] T_r(T_k(u) - T_k(\hat{u}))\, dH^{N-1}.$$

If we write

$$I_{r,k} := \int_\Omega (\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u})) \cdot \nabla T_r(T_k(u) - T_k(\hat{u}))\, dx,$$

$$J_{r,k} := \int_\Omega (\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u})) \cdot D^s T_r(T_k(u) - T_k(\hat{u})),$$

since

$$\int_{\partial\Omega} [\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u}), \nu] T_r(T_k(u) - T_k(\hat{u}))\, dH^{N-1} \le 0,$$

we have

$$\int_\Omega (u - \hat{u}) T_r(T_k(u) - T_k(\hat{u}))\, dx$$
$$\le -I_{r,k} - J_{r,k} + \int_\Omega (f - \hat{f}) T_r(T_k(u) - T_k(\hat{u}))\, dx. \tag{58}$$

Now, if $\Omega_{k,r} := \{|T_k(u) - T_k(\hat{u})| < r\}$, then

$$I_{r,k} = \int_{\Omega_{k,r}} (\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u})) \cdot \nabla (T_k(u) - T_k(\hat{u}))\, dx$$

$$= \int_{\Omega_{k,r}\cap\{|u|<k, |\hat{u}|<k\}} (\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u})) \cdot (\nabla u - \nabla \hat{u})\, dx$$

$$+ \int_{\Omega_{k,r}\cap\{|u|<k, |\hat{u}|\ge k\}} (\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u})) \cdot \nabla u\, dx$$

$$- \int_{\Omega_{k,r}\cap\{|u|\ge k, |\hat{u}|<k\}} (\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u})) \cdot \nabla \hat{u}\, dx$$

$$\ge \int_{\Omega_{k,r}\cap\{|u|<k, |\hat{u}|\ge k\}} (\mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla \hat{u})) \cdot \nabla u\, dx$$

$$- \int_{\Omega_{k,r} \cap \{|u| \ge k, |\hat{u}| < k\}} (\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla \hat{u})) \cdot \nabla \hat{u} \, dx.$$

On the other hand, by Lemma 1. 2 of [26], there exists a nonnegative function ξ_r such that

$$\begin{aligned} J_{r,k} &= \int_\Omega (\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla \hat{u})) \cdot \xi_r (D^s T_k(u) - D^s T_k(\hat{u})) \\ &= \int_\Omega \xi_r \big(f^0(x, D^s T_k(u)) - \mathbf{a}(x, \nabla \hat{u}) \cdot D^s T_k(u) \big) \\ &+ \int_\Omega \xi_r \big(f^0(x, D^s T_k(\hat{u})) - \mathbf{a}(x, \nabla u) \cdot D^s T_k(\hat{u}) \big). \end{aligned}$$

Then, by (H_5), we have $J_{r,k} \ge 0$. Hence, letting $k \to \infty$ in (58), it follows that

$$\begin{aligned} &\int_\Omega (u - \hat{u}) T_r(u - \hat{u}) \, dx \le \int_\Omega (f - \hat{f}) T_r(u - \hat{u}) \, dx \\ &- \lim_{k \to \infty} \int_{\Omega_{k,r} \cap \{|u| < k, |\hat{u}| \ge k\}} (\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla \hat{u})) \cdot \nabla u \, dx \\ &+ \lim_{k \to \infty} \int_{\Omega_{k,r} \cap \{|u| \ge k, |\hat{u}| < k\}} (\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla \hat{u})) \cdot \nabla \hat{u} \, dx. \end{aligned}$$

Now, by Lemma 3,

$$\begin{aligned} &\lim_{k \to \infty} \left| \int_{\Omega_{k,r} \cap \{|u| < k, |\hat{u}| \ge k\}} (\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla \hat{u})) \cdot \nabla u \, dx \right| \\ &\le \lim_{k \to \infty} \int_{\{k - r < |u| < k\}} |(\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla \hat{u})) \cdot \nabla u| \, dx = 0, \end{aligned}$$

and

$$\begin{aligned} &\lim_{k \to \infty} \left| \int_{\Omega_{k,r} \cap \{|u| \ge k, |\hat{u}| < k\}} (\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla \hat{u})) \cdot \nabla \hat{u} \, dx \right| \\ &\le \lim_{k \to \infty} \int_{\{k - r < |\hat{u}| < k\}} |(\mathbf{a}(x, \nabla u) - \mathbf{a}(x, \nabla \hat{u})) \cdot \nabla \hat{u}| \, dx = 0. \end{aligned}$$

Consequently, we get

$$\frac{1}{r} \int_\Omega (u - \hat{u}) T_r(u - \hat{u}) \, dx \le \frac{1}{r} \int_\Omega (f - \hat{f}) T_r(u - \hat{u}) \, dx \le \int_\Omega |f - \hat{f}| \, dx.$$

Passing to the limit as $r \to 0^+$, (57) follows.

Having in mind Theorem 2 and Proposition 1, to finish the proof, we only need to prove that $\overline{\mathcal{B}_\varphi}^{L^1(\Omega)} \subset \mathcal{A}_\varphi$. Let $(u_n, v_n) \in \mathcal{B}_\varphi$ be such that $(u_n, v_n) \to (u, v)$ in

$L^1(\Omega) \times L^1(\Omega)$. Let us prove that $(u, v) \in \mathcal{A}_\varphi$. Since $(u_n, v_n) \in \mathcal{B}_\varphi \subset \mathcal{A}_\varphi$, we have $\mathbf{a}(x, \nabla u_n) \in X(\Omega)$ satisfying

$$-v_n = div\, \mathbf{a}(x, \nabla u_n) \quad \text{in } \mathcal{D}'(\Omega), \tag{59}$$

$$\mathbf{a}(x, \nabla u_n) \cdot D^s p(u_n) = f^0(x, D^s p(u_n)) \quad \forall\, p \in \mathcal{P}, \tag{60}$$

$$[\mathbf{a}(x, \nabla u_n), \nu] \in \text{sign}\,(p(\varphi) - p(u_n)) f^0(x, \nu(x)), \;\; \forall\, p \in \mathcal{P} \quad H^{N-1} - \text{a.e.} \tag{61}$$

Then, given $p \in \mathcal{P}$, we have

$$\int_\Omega v_n p(u_n)\, dx = \int_\Omega (\mathbf{a}(x, \nabla u_n), Dp(u_n))$$
$$- \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu] p(u_n)\, dH^{N-1} = \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla p(u_n)\, dx$$
$$+ \int_\Omega f^0(x, D^s p(u_n)) - \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu] p(u_n)\, dH^{N-1}$$
$$= \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla p(u_n)\, dx + \int_\Omega f^0(x, D^s p(u_n))$$
$$+ \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x))\, dH^{N-1} - \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu] p(\varphi)\, dH^{N-1}.$$

Hence, by (28) and (29), it follows that

$$\|Dp(u_n)\| = \int_\Omega |\nabla p(u_n)|\, dx + \int_\Omega |D^s p(u_n)|$$
$$\leq \frac{1}{C_0}\left(\int_\Omega \mathbf{a}(x, \nabla p(u_n)) \cdot \nabla p(u_n)\, dx + \int_\Omega f^0(x, D^s p(u_n)) + D_1 \lambda_N(\Omega) \right)$$
$$\leq \frac{1}{C_0}\left(\int_\Omega v_n p(u_n)\, dx + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu] p(\varphi)\, dH^{N-1} + D_1 \lambda_N(\Omega) \right).$$

Thus,

$$\|Dp(u_n)\| \leq M_1 \|p\|_\infty \qquad \forall\, n \in \mathbb{N}. \tag{62}$$

Therefore, $p(u) \in BV(\Omega)$ for any $p \in \mathcal{P}$. On the other hand, since

$$\|\mathbf{a}(x, \nabla u_n)\|_\infty \leq M,$$

we may assume that $\mathbf{a}(x, \nabla u_n) \rightharpoonup z$ weak* in $L^\infty(\Omega, \mathbb{R}^N)$ with $\|z\|_\infty \leq M$. Moreover, since $v_n \to v$ in $L^1(\Omega)$, we have that $v = -div(z)$ in $\mathcal{D}'(\Omega)$. By the definition of the weak trace on $\partial\Omega$ of the normal component of z, it is easy to see that

$$[\mathbf{a}(x, \nabla u_n), \nu] \rightharpoonup [z, \nu] \quad \text{weakly}^* \text{ in } L^\infty(\partial\Omega). \tag{63}$$

On the other hand,

$$\begin{aligned}
&\lim_{n\to\infty} \left(\int_\Omega h(x, Dp(u_n)) + \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x))\, dH^{N-1} \right) \\
&= \lim_{n\to\infty} \left(\int_\Omega v_n p(u_n)\, dx + \int_{\partial\Omega} [\mathbf{a}(x, \nabla u_n), \nu] p(\varphi)\, dH^{N-1} \right) \\
&= \int_\Omega v p(u)\, dx + \int_{\partial\Omega} [z, \nu] p(\varphi)\, dH^{N-1} \\
&= -\int_\Omega div(z) p(u) + \int_{\partial\Omega} [z, \nu] p(\varphi)\, dH^{N-1}.
\end{aligned}$$

Now, applying Green's formula we get

$$\begin{aligned}
&\lim_{n\to\infty} \left(\int_\Omega h(x, Dp(u_n)) + \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x))\, dH^{N-1} \right) \\
&= \int_\Omega (z, Dp(u)) + \int_{\partial\Omega} [z, \nu](p(\varphi) - p(u))\, dH^{N-1}.
\end{aligned} \tag{64}$$

It is not difficult to see that

$$\lim_{n\to\infty} \int_\Omega (\mathbf{a}(x, \nabla u_n) - \mathbf{a}(x, \nabla p(u_n))) \cdot \nabla p(u)\, dx = 0$$

for all $p \in \mathcal{P}$. Consequently,

$$\lim_{n\to\infty} \int_\Omega \mathbf{a}(x, \nabla p(u_n)) \cdot \nabla p(u)\, dx = \int_\Omega z \cdot \nabla p(u)\, dx \quad \forall\, p \in \mathcal{P}. \tag{65}$$

Let us now prove the convergence of the energies. We consider the energy functional $\Psi_\varphi : L^1(\Omega) \to [0, +\infty]$ defined by

$$\Psi_\varphi(v) := \int_\Omega f(x, Dv) + \int_{\partial\Omega} |\varphi - v| f^0(x, \nu(x))\, dH^{N-1}$$

if $v \in BV(\Omega)$ and

$$\Psi_\varphi(v) := +\infty \qquad \text{if} \quad v \in L^1(\Omega) \setminus BV(\Omega).$$

As a consequence of the Anzellotti's result ([7]), the functional Ψ_φ is convex and lower-semicontinuous. By the convexity of f, we have

$$\Psi_{p(\varphi)}(p(u_n)) = \int_\Omega f(x, \nabla p(u_n))\, dx + \int_\Omega f^0(x, D^s p(u_n))$$

$$+\int_{\partial\Omega}|p(u_n)-p(\varphi)|f^0(x,\nu(x))\,dH^{N-1}\le\int_\Omega f(x,\nabla p(u))\,dx$$

$$+\int_\Omega \mathbf{a}(x,\nabla p(u_n))\cdot\nabla p(u_n)\,dx-\int_\Omega \mathbf{a}(x,\nabla p(u_n))\cdot\nabla p(u)\,dx$$

$$+\int_\Omega \mathbf{a}(x,\nabla u_n)\cdot D^s p(u_n)+\int_{\partial\Omega}[\mathbf{a}(x,\nabla u_n),\nu](p(\varphi)-p(u_n))\,dH^{N-1}$$

$$=\int_\Omega f(x,\nabla p(u))\,dx+\int_\Omega(\mathbf{a}(x,\nabla u_n),Dp(u_n))$$

$$-\int_\Omega \mathbf{a}(x,\nabla p(u_n))\cdot\nabla p(u)\,dx+\int_{\partial\Omega}[\mathbf{a}(x,\nabla u_n),\nu](p(\varphi)-p(u_n))\,dH^{N-1}$$

$$=\int_\Omega f(x,\nabla p(u))\,dx-\int_\Omega div(\mathbf{a}(x,\nabla u_n))p(u_n)\,dx$$

$$-\int_\Omega \mathbf{a}(x,\nabla p(u_n))\cdot\nabla p(u)\,dx+\int_{\partial\Omega}[\mathbf{a}(x,\nabla u_n),\nu]p(\varphi)\,dH^{N-1}.$$

Then, by (65), letting $n\to\infty$, we obtain

$$\limsup_{n\to\infty}\Psi_{p(\varphi)}(p(u_n))\le\int_\Omega f(x,\nabla p(u))\,dx-\int_\Omega div(z)p(u)\,dx$$

$$-\int_\Omega z\cdot\nabla p(u)\,dx+\int_{\partial\Omega}[z,\nu]p(\varphi)\,dH^{N-1}=\int_\Omega f(x,\nabla p(u))\,dx$$

$$+\int_\Omega(z,Dp(u))-\int_\Omega z\cdot\nabla p(u)\,dx+\int_{\partial\Omega}[z,\nu](p(\varphi)-p(u))\,dH^{N-1}$$

$$=\int_\Omega f(x,\nabla p(u))\,dx+\int_\Omega z\cdot D^s p(u)+\int_{\partial\Omega}[z,\nu](p(\varphi)-p(u))\,dH^{N-1}.$$

Now, by (H_5)

$$\int_\Omega z\cdot D^s p(u)=\lim_{n\to\infty}\int_\Omega \mathbf{a}(x,\nabla u_n)\cdot D^s p(u)\le\int_\Omega f^0(x,D^s p(u)).$$

Moreover, by (61)

$$\int_{\partial\Omega}[z,\nu](p(\varphi)-p(u))\,dH^{N-1}$$

$$=\lim_{n\to\infty}\int_{\partial\Omega}[\mathbf{a}(x,\nabla u_n),\nu](p(\varphi)-p(u))\,dH^{N-1}$$

$$\le\int_{\partial\Omega}f^0(x,\nu(x))|p(\varphi)-p(u)|\,dH^{N-1}.$$

Hence, we have

$$\limsup_n \Psi_{p(\varphi)}(p(u_n)) \leq \int_\Omega f(x, \nabla p(u))\, dx + \int_\Omega f^0(x, D^s p(u))\, dx$$

$$+ \int_{\partial\Omega} f^0(x, \nu(x))|p(\varphi) - p(u)|\, dH^{N-1} = \Psi_{p(\varphi)}(p(u)),$$

and, having in mind the lower-semicontinuity of $\Psi_{p(\varphi)}$, this yields

$$\lim_{n\to\infty} \Psi_{p(\varphi)}(p(u_n)) = \Psi_{p(\varphi)}(p(u)). \tag{66}$$

If we consider the $\mathbb{R}^N$-valued measures μ_n, μ on $\overline{\Omega}$ which are defined by

$$\mu_n(B) := \int_{B\cap\Omega} Dp(u_n) + \int_{B\cap\partial\Omega} (p(\varphi) - p(u_n))\nu\, dH^{N-1}, \tag{67}$$

$$\mu(B) := \int_{B\cap\Omega} Dp(u) + \int_{B\cap\partial\Omega} (p(\varphi) - p(u))\nu\, dH^{N-1} \tag{68}$$

for all Borel sets $B \subset \overline{\Omega}$, we have

$$\mu_n \to \mu \qquad \text{weakly as measures in } \overline{\Omega}.$$

Moreover,

$$\Psi_{p(\varphi)}(p(u)) = \int_{\overline{\Omega}} \tilde{f}(x, \tilde{\mu}) \qquad \text{and} \qquad \Psi_{p(\varphi)}(p(u_n)) = \int_{\overline{\Omega}} \tilde{f}(x, \tilde{\mu}_n).$$

Hence, (66) yields

$$\lim_{n\to\infty} \int_{\overline{\Omega}} \tilde{f}(x, \tilde{\mu}_n) = \int_{\overline{\Omega}} \tilde{f}(x, \tilde{\mu}).$$

Then, applying Theorem 3 of [32], it follows that

$$\int_{\overline{\Omega}} \tilde{h}(x, \tilde{\mu}) = \lim_{n\to\infty} \int_{\overline{\Omega}} \tilde{h}(x, \tilde{\mu}_n)$$

$$= \lim_{n\to\infty} \int_\Omega h(x, Dp(u_n)) + \int_{\partial\Omega} |p(u_n) - p(\varphi)| f^0(x, \nu(x))\, dH^{N-1}.$$

Now, it is easy to see that

$$\int_{\overline{\Omega}} \tilde{h}(x, \tilde{\mu}) = \int_\Omega h(x, Dp(u)) + \int_{\partial\Omega} |p(\varphi) - p(u)| f^0(x, \nu(x))\, dH^{N-1},$$

consequently, we obtain

$$\lim_{n\to\infty} \int_\Omega h(x, Dp(u_n)) + \int_{\partial\Omega} |p(u_n) - p(\varphi)| f^0(x, \nu(x))\, dH^{N-1}$$
$$= \int_\Omega h(x, Dp(u)) + \int_{\partial\Omega} |p(\varphi) - p(u)| f^0(x, \nu(x))\, dH^{N-1}. \tag{69}$$

By (64) and (69), we get

$$\int_\Omega h(x, Dp(u)) + \int_{\partial\Omega} |p(\varphi) - p(u)| f^0(x, \nu(x))\, dH^{N-1}$$
$$= \int_\Omega (z, Dp(u)) + \int_{\partial\Omega} [z, \nu](p(\varphi) - p(u))\, dH^{N-1}. \tag{70}$$

Now, (70) can be written as

$$\begin{aligned} &\int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla p(u)\, dx + \int_\Omega f^0(x, D^s p(u)) \\ &+ \int_{\partial\Omega} |p(\varphi) - p(u)| f^0(x, \nu(x))\, dH^{N-1} \\ &= \int_\Omega z \cdot \nabla p(u)\, dx + \int_\Omega z \cdot D^s p(u) \\ &+ \int_{\partial\Omega} [z, \nu](p(\varphi) - p(u))\, dH^{N-1}. \end{aligned} \tag{71}$$

On the other hand, by Lemma 4 (i) in [4], we get

$$|[z, \nu]| \le f^0(x, \nu(x)) \quad H^{N-1} - a.e. \text{ on } \partial\Omega. \tag{72}$$

Let $v_j \in C^1(\overline{\Omega})$ be a sequence such that $v_j \to p(u)$ in $L^2(\Omega)$ and $f^0(x, \nabla v_j) \to f^0(x, Dp(u))$ weakly as measures. According to (H_5), we have

$$|a(x, \nabla u_n) \cdot \nabla v_j| \le f^0(x, \nabla v_j).$$

Then, if $\psi, \phi \in C_c^1(\Omega)$, with $0 \le \psi \le \phi$, we have

$$\left| \int_\Omega \mathbf{a}(x, \nabla u_n) \cdot \nabla v_j\ \psi\, dx \right| \le \int_\Omega f^0(x, \nabla v_j) \psi\, dx,$$

and, letting $n \to \infty$, we get

$$\left| \int_\Omega z \cdot \nabla v_j\ \psi\, dx \right| \le \int_\Omega f^0(x, \nabla v_j) \psi\, dx.$$

Now, since

$$\left| \int_\Omega z \cdot \nabla v_j \; \psi \; dx \right| = \left| - \int_\Omega div(z) v_j \psi \; dx - \int_\Omega v_j z \cdot \nabla \psi \; dx \right|,$$

letting $j \to \infty$ we obtain that

$$|\langle (z, Dp(u)), \psi \rangle| = \left| - \int_\Omega div(z) p(u) \psi \; dx - \int_\Omega p(u) z \cdot \nabla \psi \; dx \right|$$
$$\leq \int_\Omega \psi f^0(x, Dp(u)) \leq \int_\Omega \phi f^0(x, Dp(u)).$$

Hence

$$\langle |(z, Dp(u))|, \phi \rangle \leq \int_\Omega \phi f^0(x, Dp(u)).$$

Thus, we have

$$|(z, Dp(u))| \leq f^0(x, Dp(u)) \quad \text{ as measures in } \Omega.$$

Then, the singular parts also satisfy a similar inequality,

$$|z \cdot D^s p(u)| \leq f^0(x, D^s p(u)) \quad \text{ as measures in } \Omega. \tag{73}$$

By the convexity of f, we have

$$\int_\Omega \mathbf{a}(x, \nabla p(u_n)) \cdot \nabla p(u) \; dx \leq \int_\Omega \mathbf{a}(x, \nabla p(u_n)) \cdot \nabla p(u_n) \; dx$$
$$+ \int_\Omega f(x, \nabla p(u)) \; dx - \int_\Omega f(x, \nabla p(u_n)) \; dx$$
$$\leq \int_\Omega \mathbf{a}(x, \nabla p(u_n)) \cdot \nabla p(u_n) \, dx + \int_\Omega h^0(x, D^s p(u_n))$$
$$+ \int_{\partial\Omega} |p(u_n) - p(\varphi)| h^0(x, \nu(x)) \; dH^{N-1} + \int_\Omega f(x, \nabla p(u)) \; dx$$
$$- \Big(\int_\Omega f(x, \nabla p(u_n)) \; dx + \int_\Omega f^0(x, D^s p(u_n))$$
$$+ \int_{\partial\Omega} |p(u_n) - p(\varphi)| f^0(x, \nu(x)) \; dH^{N-1} \Big) = \int_\Omega h(x, Dp(u_n))$$
$$+ \int_{\partial\Omega} |p(u_n) - p(\varphi)| f^0(x, \nu(x)) \; dH^{N-1}$$
$$+ \int_\Omega f(x, \nabla p(u)) \; dx - \Psi_{p(\varphi)}(p(u_n)).$$

Letting $n \to \infty$, and using (65), (66) and (69), we obtain

$$\int_\Omega z \cdot \nabla p(u)\, dx \le \int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla p(u)\, dx. \tag{74}$$

Now, from (71), (72), (73) and (74), we finally obtain

$$\int_\Omega z \cdot \nabla p(u)\, dx = \int_\Omega h(x, \nabla p(u))\, dx = \int_\Omega \mathbf{a}(x, \nabla u) \cdot \nabla p(u)\, dx, \tag{75}$$

$$z \cdot D^s p(u) = f^0(x, D^s p(u)), \tag{76}$$

$$[z, \nu] \in \text{sign}\,(p(\varphi) - p(u)) f^0(x, \nu(x)) \qquad H^{N-1} - \text{a.e. on } \partial\Omega\ . \tag{77}$$

Let us see now that

$$z(x) = \mathbf{a}(x, \nabla u(x)) \qquad \text{a.e. } x \in \Omega. \tag{78}$$

Let $0 \le \phi \in C_0^1(\Omega)$ and $g \in C^1(\overline{\Omega})$. We observe that

$$\int_\Omega \phi[(\mathbf{a}(x, \nabla u_n), Dp(u_n - g)) - \mathbf{a}(x, \nabla g) Dp(u_n - g)]$$

$$= \int_\Omega \phi[\mathbf{a}(x, \nabla u_n) - \mathbf{a}(x, \nabla g)) \cdot \nabla p(u_n - g)]\, dx$$

$$+ \int_\Omega \phi[\mathbf{a}(x, \nabla u_n) - \mathbf{a}(x, \nabla g)] \cdot D^s p(u_n - g)).$$

Since both terms at the right hand side of the above expression are positive, we have

$$\int_\Omega \phi[(\mathbf{a}(x, \nabla u_n), Dp(u_n - g)) - \mathbf{a}(x, \nabla g) Dp(u_n - g)] \ge 0.$$

Since

$$\int_\Omega \phi(\mathbf{a}(x, \nabla u_n), Dp(u_n - g)) = -\int_\Omega div(\mathbf{a}(x, \nabla u_n))\phi p(u_n - g)\, dx$$

$$- \int_\Omega p(u_n - g)\mathbf{a}(x, \nabla u_n) \cdot \nabla\phi\, dx,$$

we get

$$\lim_{n\to\infty} \int_\Omega \phi(\mathbf{a}(x, \nabla u_n), Dp(u_n - g)) = -\int_\Omega div(z)\phi p(u - g)\, dx$$

$$- \int_\Omega p(u - g) z \cdot \nabla\phi\, dx = \int_\Omega \phi(z, Dp(u - g)).$$

On the other hand,

$$\lim_{n\to\infty} \int_\Omega \phi\, \mathbf{a}(x, \nabla g) Dp(u_n - g) = \int_\Omega \phi\, \mathbf{a}(x, \nabla g) Dp(u - g).$$

Consequently, we obtain

$$\int_{\Omega} \phi[(z, Dp(u-g)) - \mathbf{a}(x, \nabla g) Dp(u-g)] \geq 0, \quad \forall\, 0 \leq \phi \in C_0^1(\Omega).$$

Thus the measure $(z, Dp(u-g)) - \mathbf{a}(x, \nabla g) Dp(u-g) \geq 0$. Then its absolutely continuous part

$$(z - \mathbf{a}(x, \nabla g)) \cdot \nabla p(u-g)) \geq 0 \quad \text{a.e. in } \Omega.$$

Hence,

$$(z - \mathbf{a}(x, \nabla g)) \cdot \nabla (u-g)) \geq 0 \quad \text{a.e. in } \Omega.$$

Since we may take a countable set dense in $C^1(\overline{\Omega})$, we have that the above inequality holds for all $x \in \tilde{\Omega}$, where $\tilde{\Omega} \subset \Omega$ is such that $\lambda_N(\Omega \setminus \tilde{\Omega}) = 0$, and all $g \in C^1(\overline{\Omega})$. Now, fixed $x \in \tilde{\Omega}$ and given $\xi \in \mathbb{R}^N$, there is $g \in C^1(\overline{\Omega})$ such that $\nabla g(x) = \xi$. Then

$$(z(x) - \mathbf{a}(x, \xi)) \cdot (\nabla u(x) - \xi) \geq 0, \quad \forall\, \xi \in \mathbb{R}^N.$$

These inequalities imply (78) by an application of the Minty-Browder's method in $\mathbb{R}^N$. Now, since $v = -div(z)$ in $\mathcal{D}'(\Omega)$, by (78) we get

$$v = -div\, \mathbf{a}(x, \nabla u) \quad \text{in } \mathcal{D}'(\Omega).$$

Finally, by (73), (78), (76) and (77), we get

$$\mathbf{a}(x, \nabla u) \cdot D^s p(u) = f^0(x, D^s p(u)) \quad \forall\, p \in \mathcal{P}$$

and

$$[\mathbf{a}(x, \nabla u), \nu] \in \text{sign}\,(p(\varphi) - p(u)) f^0(x, \nu(x)) \;\; H^{N-1}-\text{a.e. on } \partial\Omega, \;\; \forall\, p \in \mathcal{P}.$$

Therefore, $(u, v) \in \mathcal{A}_\varphi$ and the proof concludes. □

5. Existence and uniqueness for data in $L^1(\Omega)$

In this section we are going to prove Theorem 1.

5.1. Proof of Theorem 1. Existence

Let $u_0 \in L^1(\Omega)$ and $(T(t))_{t\geq 0}$ the contraction semigroup in $L^1(\Omega)$ generated by $\mathcal{A}_\varphi$. We shall prove that $u(t) := T(t)u_0$ is an entropy solution of problem (1). We divide the proof in several steps.

Step 1. Since $\mathcal{D}(\mathcal{A}_\varphi) \cap L^\infty(\Omega)$ is dense in $L^1(\Omega)$, given $u_0 \in L^1(\Omega)$ there exists a sequence $u_{0,n} \in \mathcal{D}(\mathcal{A}_\varphi) \cap L^\infty(\Omega)$ such that $u_{0,n} \to u_0$ in $L^1(\Omega)$. Then, if $u_n(t) := T(t)u_{0,n}$, we have that $u_n \to u$ in $C([0,T]; L^1(\Omega))$ for every $T > 0$. As a consequence of Theorem 3, $u_n(t), u_n'(t) \in L^2(\Omega)$, $u_n(t) \in BV(\Omega)$, $z_n(t) := \mathbf{a}(x, \nabla u_n(t)) \in X(\Omega)$ a.e. $t \in [0,T]$, and for almost all $t \in [0,T]$ $u_n(t)$ satisfies:

$$u_n'(t) = div(z_n(t)) \quad \text{in } \mathcal{D}'(\Omega), \tag{79}$$

$$\begin{cases} z_n(t) \cdot D^s u_n(t) = f^0(x, D^s u_n(t)), \\ z_n(t) \cdot D^s p(u_n(t)) = f^0(x, D^s p(u_n(t))) \quad \forall\, p \in \mathcal{P}, \end{cases} \tag{80}$$

$$[z_n(t), \nu] \in \text{sign}\,(p(\varphi) - p(u_n(t))) f^0(x, \nu(x)) \;\; \forall\, p \in \mathcal{P} \quad H^{N-1} - \text{a.e.} \tag{81}$$

From (79), (80) and (81), it follows that

$$\begin{aligned} &-\int_\Omega (w - u_n(t))u_n'(t)\,dx = \int_\Omega (z_n(t), Dw) - \int_\Omega h(x, Du_n(t)) \\ &-\int_{\partial\Omega} [z_n(t), \nu](w - \varphi)\,dH^{N-1} - \int_{\partial\Omega} |u_n(t) - \varphi| f^0(x, \nu(x))\,dH^{N-1} \end{aligned} \tag{82}$$

for every $w \in BV(\Omega) \cap L^2(\Omega)$.

On the other hand, since $\|[z_n(t), \nu]\|_\infty \leq \|z_n(t)\|_\infty \leq M$, we may assume (up to extraction of a subsequence, if necessary) that

$$[z_n(\cdot), \nu] \to \rho \quad \sigma(L^\infty(S_T), L^1(S_T)).$$

Step 2. Convergence of the derivatives and identification of the limit. Since the map $t \mapsto u_n'(t)$ is strongly measurable from $[0,T]$ into $L^2(\Omega)$, and, by (22),

$$\|u_n'(t)\|_{BV(\Omega)_2^*} \leq \|u_n'(t)\|_{L^2(\Omega)},$$

it follows that this map is strongly measurable from $[0,T]$ into $BV(\Omega)_2^*$. Moreover, for every $w \in BV(\Omega)_2$, if we take $u_n(t) - w$ as test function in (82), since

$$\int_\Omega h(x, Du_n(t)) = \int_\Omega (z_n(t), Du_n(t)),$$

we get

$$\int_\Omega u_n'(t)w\,dx = -\int_\Omega (z_n(t), Dw) + \int_{\partial\Omega} [z_n(t), \nu]w\,dH^{N-1}.$$

Hence

$$\left|\int_\Omega u_n'(t)w\,dx\right| \leq M\|Dw\| + M\int_{\partial\Omega} |w|\,dH^{N-1} \leq M_1\|w\|_{BV(\Omega)_2}$$

for all $n \in \mathbb{N}$. Thus,

$$\|u_n'(t)\|_{BV(\Omega)_2^*} \leq M_1 \quad \forall\, n \in \mathbb{N} \text{ and } t \in [0, T].$$

Consequently, $\{u_n'\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(0, T; BV(\Omega)_2^*)$. Now, since the space $L^\infty(0, T; BV(\Omega)_2^*)$ is a vector subspace of the dual space $\left(L^1(0, T; BV(\Omega)_2)\right)^*$, we can find a subnet $\{u_\alpha'\}$ such that

$$u_\alpha' \to \xi \in \left(L^1(0, T; BV(\Omega)_2)\right)^* \quad \text{weakly}^*. \tag{83}$$

Since $\|z_n(t)\|_\infty \leq M$ for all $n \in \mathbb{N}$ and a.e. $t \in [0, T]$, we may also assume that

$$z_n \to z \in L^\infty(Q_T, \mathbb{R}^N) \qquad \text{weakly}^*. \tag{84}$$

Given $\eta \in \mathcal{D}(Q_T)$, since $\eta \in L^1(0, T; BV(\Omega)_2)$, we have

$$\langle \xi, \eta\rangle = \lim_\alpha \langle u_\alpha', \eta\rangle = \lim_\alpha \int_0^T \langle u_\alpha'(t), \eta(t)\rangle\,dt$$

$$= \lim_\alpha \int_0^T \int_\Omega u_\alpha'(t)\eta(t)\,dxdt = \lim_\alpha \int_0^T \int_\Omega div(z_\alpha(t))\eta(t)\,dxdt$$

$$= -\lim_\alpha \int_0^T \int_\Omega z_\alpha(t)\cdot\nabla\eta(t)\,dxdt = -\int_{Q_T} z\cdot\nabla\eta\,dxdt = \langle div_x(z), \eta\rangle.$$

Hence,

$$\xi = div_x(z) \qquad \text{in } \mathcal{D}'(Q_T). \tag{85}$$

On the other hand, if we take $\eta(t, x) = \phi(t)\psi(x)$ with $\phi \in \mathcal{D}(]0, T[)$ and $\psi \in \mathcal{D}(\Omega)$, the same calculation as above shows that

$$\xi(t) = div_x(z(t)) \qquad \text{in } \mathcal{D}'(\Omega) \text{ a.e. } t \in [0, T]. \tag{86}$$

Consequently, $(z(t), \xi(t)) \in Z(\Omega)$ for almost all $t \in [0, T]$, therefore we can consider $[z(t), \nu]$ defined as in Section 2.

Lemma 4. *ξ is the time derivative of u in the sense of the Definition 2.*

Proof. Let $\Psi \in L^1(0,T,BV(\Omega))$ be the weak derivative of the function $\Theta \in L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$, i.e., $\Psi(t) = \int_0^t \Theta(s)ds$, the integral being taken as a Pettis integral. By (83) we have that

$$\int_0^T < \xi(t), \Psi(t) > dt = \lim_\alpha \int_0^T < u'_\alpha(t), \Psi(t) > .$$

Now,

$$\begin{aligned}\int_0^T < u'_\alpha(t), \Psi(t) >&= \lim_h \int_0^T \int_\Omega \Psi(t) \frac{u_\alpha(t+h) - u(t)}{h} \, dxdt \\ &= \lim_h \int_0^T \int_\Omega \frac{\Psi(t-h) - \Psi(t)}{h} u_\alpha(t) \, dxdt \\ &= -\lim_h \int_0^T \int_\Omega \frac{1}{h} \int_{t-h}^t \Theta(s)ds \, u_\alpha(t) \, dxdt \\ &= -\int_0^T \int_\Omega \Theta(t,x) u_\alpha(t,x) \, dxdt.\end{aligned}$$

Passing to the limit in α in the above expression, we obtain

$$\int_0^T < \xi(t), \Psi(t) > dt = -\int_0^T \int_\Omega \Theta(t,x) u(t,x) \, dxds. \tag{87}$$

□

Step 3. The boundary condition. Let us now prove that

$$\rho(t) = [z(t), \nu] \qquad H^{N-1} - \text{a.e. on } \partial\Omega, \quad \text{a.e. } t \in [0,T]. \tag{88}$$

Ideed, if $w \in BV(\Omega) \cap L^\infty(\Omega)$, and $v \in R(\Omega)$ are uch that $v_{|\partial\Omega} = w_{|\partial\Omega}$, we have that

$$\int_0^t \langle z_\alpha(s), w \rangle_{\partial\Omega} \, ds = \int_0^t \langle div(z_\alpha(s)), v \rangle \, ds + \int_0^t \int_\Omega z_\alpha(s) \cdot \nabla v \, dxds.$$

Hence

$$\lim_\alpha \int_0^t \langle z_\alpha(s), w \rangle_{\partial\Omega} \, ds$$

$$= \int_0^t \langle \xi(s), v \rangle \, ds + \int_0^t \int_\Omega z(s) \cdot \nabla v \, dxds \tag{89}$$

$$= \int_0^t \langle z(s), w \rangle_{\partial\Omega} \, ds = \int_0^t \int_{\partial\Omega} [z(s), \nu] w \, dH^{N-1} ds.$$

On the other hand, since $z_\alpha(s) \in X(\Omega)$, if we apply Green's formula we have that

$$\int_0^t \langle div(z_\alpha(s)), v\rangle ds$$

$$= -\int_0^t \int_\Omega z_\alpha(s) \cdot \nabla v \, dxds + \int_0^t \int_{\partial\Omega} [z_\alpha(s), \nu] w \, dH^{N-1} ds.$$

Consequently,

$$\int_0^t \langle z_\alpha(s), w\rangle_{\partial\Omega} \, ds = \int_0^t \int_{\partial\Omega} [z_\alpha(s), \nu] w \, dH^{N-1} ds.$$

Taking limits in α, we get

$$\int_0^t \int_{\partial\Omega} \rho(s) w \, dH^{N-1} ds = \int_0^t \int_{\partial\Omega} [z(s), \nu] w \, dH^{N-1} ds \tag{90}$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$ and $t \in [0, T]$. Now, if $w \in L^1(\partial\Omega)$, we take $w_k \in BV(\Omega) \cap L^\infty(\Omega)$ such that $w_{k|\partial\Omega} = T_k(w)$. By (90), we have

$$\int_0^t \int_{\partial\Omega} \rho(s) w_k \, dH^{N-1} ds = \int_{\partial\Omega} [z(s), \nu] w_k \, dH^{N-1} ds.$$

Letting $k \to \infty$, it follows that

$$\int_0^t \int_{\partial\Omega} \rho(s) w \, dH^{N-1} ds = \int_0^t \int_{\partial\Omega} [z(s), \nu] w \, dH^{N-1} ds$$

for all $w \in L^1(\partial\Omega)$ and $t \in [0, T]$. Therefore, (88) holds.

Step 4. Next, we prove that $\xi = div(z)$ in $\left(L^1(0, T, BV(\Omega)_2)\right)^*$ in the sense of the Definition 3. To do that, let us first observe that (z, Dw), defined by (23), is a Radon measure in Q_T for all $w \in L^1_w(0, T, BV(\Omega)) \cap L^\infty(Q_T)$. Let $\phi \in \mathcal{D}(Q_T)$, then

$$\langle (z, Dw), \phi\rangle = -\int_0^T \langle \xi(t) - u'_\alpha(t), w(t)\phi(t)\rangle \, dt$$

$$-\int_{Q_T} w(z - z_\alpha) \cdot \nabla_x \phi \, dxdt + \int_0^T \langle (z_\alpha(t), Dw(t)), \phi(t)\rangle \, dt.$$

Then, taking limits in α, and using (83), we get

$$\langle (z, Dw), \phi\rangle = \lim_\alpha \int_0^T \langle (z_\alpha(t), Dw(t)), \phi(t)\rangle \, dt. \tag{91}$$

Therefore

$$|\langle (z, Dw), \phi\rangle| \le M\|\phi\|_\infty \int_0^T \|Dw(t)\| \, dt.$$

Hence, (z, Dw) is a Radon measure in Q_T. Moreover, from (91), applying Green's formula we obtain that

$$\int_{Q_T} (z, Dw) = \lim_{\alpha} \int_0^T (z_\alpha(t), Dw(t))\, dt$$

$$= \lim_{\alpha} \left(- \int_0^T \int_\Omega div(z_\alpha(t)) w(t)\, dxdt + \int_0^T \int_{\partial\Omega} [z_\alpha(t), \nu] w(t)\, dH^{N-1} dt \right)$$

$$= - \int_0^T \langle \xi(t), w(t) \rangle\, dt + \int_0^T \int_{\partial\Omega} [z(t), \nu] w(t)\, dH^{N-1} dt,$$

that is,

$$\begin{aligned} &\int_{Q_T} (z, Dw) + \int_0^T \langle \xi(t), w(t) \rangle\, dt \\ &= \int_0^T \int_{\partial\Omega} [z(t), \nu] w(t)\, dH^{N-1} dt. \end{aligned} \tag{92}$$

As a consequence of the boundedness of $\{u_n'\}$, (83) and the above statement, we have

$$u_n' \to \xi \in \left(L^1(0, T; BV(\Omega)_2) \right)^* \quad \text{weakly}^*. \tag{93}$$

Step 5. Convergence of the energy. Let us first prove the following result.

Lemma 5. *Let $w \in L^1(Q_T)$ be such that $w(t) \in BV(\Omega)$ for almost all $t \in [0, T]$. Then the map $t \mapsto \|w(t)\|_{BV(\Omega)}$ from $[0, T]$ into R is measurable, and the map $t \mapsto w(t)$ from $[0, T]$ into $BV(\Omega)$ is weakly measurable.*

Proof. Let $E := C_c(\Omega)^{N+1}$ and $S : BV(\Omega) \to E^*$ be the map defined by

$$S(w) := \left(w\, dx, \frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_N} \right).$$

Then, $\|w\|_{BV(\Omega)} \le \|S(w)\|_{E^*} \le N \|w\|_{BV(\Omega)}$. If we denote by F the closure in E of the set

$$\{(\phi_0, \phi_1, \cdots, \phi_N) \ : \ \phi_i \in \mathcal{D}(\Omega), \text{ and } \phi_0 = div\, (\phi_1, \cdots, \phi_N)\},$$

then, it is proved in [1] that $S(BV(\Omega))$ is isomorphic to $(\frac{E}{F})^*$, that is, $G := \frac{E}{F}$ is the predual of the space $BV(\Omega)$. Now, if $\phi = (\phi_0, \phi_1, \cdots, \phi_N)$ with $\phi_i \in \mathcal{D}(\Omega)$,

$$\langle S(w(t)), \phi \rangle = \int_\Omega w(t) \phi_0\, dx - \sum_{i=1}^N \int_\Omega w(t) \frac{\partial \phi}{\partial x_i}\, dx.$$

Hence, the map $t \mapsto \langle w(t), \phi\rangle$ is measurable. Now, aproximating the functions of $C_c(\Omega)^{N+1}$ by functions in $\mathcal{D}(\Omega)^{N+1}$, we get that for every $\phi \in G$, the function $t \mapsto \langle w(t), \phi\rangle$ is measurable. Thus, since G is separable, it follows that the map

$$t \mapsto \|w(t)\|_{BV(\Omega)} = \sup_{\phi\in G,\ \|\phi\|\le 1} \langle S(w(t)), \phi\rangle$$

is measurable.

Given $v \in BV(\Omega)^*$, let $g(t) := \langle w(t), v\rangle$. To see that g is measurable, consider $v_\alpha \in G$, such that $v_\alpha \to v$ with respect to $\sigma(G^{**}, G^*) = \sigma(BV(\Omega)^*, BV(\Omega))$. From the above, we know that if $g_\alpha(t) := \langle S(w(t)), v_\alpha\rangle$, g_α is measurable, and $g_\alpha(t) \to g(t)$. Now, since

$$|g_\alpha(t)| \le \|w(t)\|_{BV(\Omega)}\|v_\alpha\|_{BV(\Omega)^*} \le R\|w(t)\|_{BV(\Omega)} = F(t) \in L^1(0, T),$$

and the order interval $[-F, F]$ in $L^1(0, T)$ is $\sigma(L^1(0, T), L^\infty(0, T))$-relatively compact, there exists a sequence g_{α_n}, such that

$$g_{\alpha_n} \to g \qquad \text{in } \sigma(L^1(0, T), L^\infty(0, T)).$$

Hence, g is measurable. □

Multiplying (79) by $w - p(u_n(t))$ and integrating in Ω, using (80) and (81), we have that

$$\begin{aligned} -\int_\Omega (w - p(u_n(t)))u_n'(t)\, dx &= \int_\Omega (z_n(t), Dw) \\ &- \int_\Omega h(x, Dp(u_n(t))) - \int_{\partial\Omega} [z_n(t), \nu](w - p(\varphi))\, dH^{N-1} \\ &- \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| f^0(x, \nu(x))\, dH^{N-1} \end{aligned} \tag{94}$$

for every $w \in BV(\Omega) \cap L^\infty(\Omega)$ and all $p \in \mathcal{P}$.

First, we observe that setting $w = 0$ in (94), and integrating in $(0, T)$, we obtain

$$\begin{aligned} &\int_\Omega J_p(u_n(T))dx + \int_0^T \int_\Omega h(x, Dp(u_n))\, dxdt \\ &+ \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x))\, dH^{N-1}dt \\ &= \int_0^T \int_{\partial\Omega} [z_n, \nu]p(\varphi)\, dH^{N-1}dt + \int_\Omega J_p(u_{0,n})\, dx, \end{aligned}$$

where $J_p'(r) = p(r)$. In particular, we have

$$\int_0^T \int_\Omega J_p(u_n)\, dxdt \le C, \tag{95}$$

$$\begin{aligned} &\int_0^T \int_\Omega h(x, Dp(u_n))\, dxdt \\ &+ \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x))\, dH^{N-1} dt \le C \end{aligned} \tag{96}$$

where C is a constant depending on $\|u_0\|_1$, $\|\varphi\|_1$ and $\|p\|_\infty$. Hence

$$\int_0^T \|Dp(u_n)\|\, dt + \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x))\, dH^{N-1} dt \le C \tag{97}$$

where C is a constant depending on $\|u_0\|_1$, $\|\varphi\|_1$, $\|p\|_\infty$ and the constants in (28). Since the functional $\Phi_p : L^1(\Omega) \to]-\infty, +\infty]$, defined by

$$\Phi_p(w) = \begin{cases} \|Dw\| + \displaystyle\int_{\partial\Omega} |w - p(\varphi)| f^0(x, \nu(x)) & \text{if } w \in BV(\Omega) \\ +\infty & \text{if } w \in L^1(\Omega) \setminus BV(\Omega), \end{cases}$$

is lower semicontinuous in $L^1(\Omega)$, we have that

$$\Phi_p(p(u(t))) \le \liminf_{n\to\infty} \Phi_p(p(u_n(t))) =$$

$$= \liminf_{n\to\infty} \left(\|Dp(u_n(t))\| + \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| f^0(x, \nu(x))\, dH^{N-1} \right).$$

On the other hand, by Lemma 5, the map $t \mapsto \|p(u_n(t))\|_{BV(\Omega)}$ is measurable, then by Fatou's Lemma and (97), it follows that

$$\begin{aligned} &\int_0^T \liminf_{n\to\infty} \left(\|Dp(u_n(t))\| + \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| f^0(x, \nu(x)) \right) dt \le \\ &\liminf_{n\to\infty} \int_0^T \left(\|Dp(u_n(t))\| + \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| f^0(x, \nu(x)) \right) dt \le C. \end{aligned} \tag{98}$$

As a consequence of (98), we obtain that $p(u(t)) \in BV(\Omega)$ for almost all $t \in [0, T]$.

From Lemma 5, if $0 \le \eta \in \mathcal{D}(]0, T[)$, the map $t \mapsto p(u(t))\eta(t)$, from $[0, T]$ into $BV(\Omega)$ is weakly measurable.

Lemma 6. *For any $\tau > 0$, we define the function ψ^τ, as the Dunford integral (see [21])*

$$\psi^\tau(t) := \frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)p(u(s))\, ds \in BV(\Omega)^{**},$$

that is,

$$\langle \psi^\tau(t), w\rangle = \frac{1}{\tau}\int_{t-\tau}^{t} \langle \eta(s)p(u(s)), w\rangle\, ds,$$

for any $w \in BV(\Omega)^*$. Then $\psi^\tau \in C([0,T]; BV(\Omega))$. Moreover, $\psi^\tau(t) \in L^2(\Omega)$, and, thus, $\psi^\tau(t) \in BV(\Omega)_2$, and ψ^τ admits a weak derivative in $L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$.

Proof. Given $\phi \in \mathcal{D}(\Omega)$,

$$|\langle \psi^\tau(t), \phi\rangle| \leq \frac{1}{\tau}\int_{t-\tau}^{t} |\eta(s)||\langle p(u(s)), \phi\rangle|\, ds$$

$$= \frac{1}{\tau}\int_{t-\tau}^{t} |\eta(s)|\left(\int_\Omega |p(u(s))||\phi|\, dx\right) ds \leq C\|\phi\|_\infty.$$

Consequently, $\psi^\tau(t)$ is a finite Radon measure in Ω. Moreover, a similar calculation shows that for every $i = 1, 2, \ldots, N$, $\frac{\partial \psi^\tau(t)}{\partial x_i}$ is also a finite Radon measure in Ω. Hence, we have $\psi^\tau(t) \in BV(\Omega)$ (see, Exercise 3.2. in [1]), and the Dunford integral of the definition of $\psi^\tau(t)$ is a Pettis integral. Moreover, if $a_n \to 0$ (for simplicity suppose that $a_n > 0$), given $w \in BV(\Omega)^*$ with $\|w\| \leq 1$, we have

$$|\langle \psi^\tau(t+a_n) - \psi^\tau(t), w\rangle|$$

$$= \left|\frac{1}{\tau}\int_{t+a_n-\tau}^{t+a_n} \eta(s)\langle p(u(s)), w\rangle\, ds - \frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)\langle p(u(s)), w\rangle\, ds\right|$$

$$\leq \left|\frac{1}{\tau}\int_{t}^{t+a_n} \eta(s)\langle p(u(s)), w\rangle\, ds - \frac{1}{\tau}\int_{t-\tau}^{t-\tau+a_n} \eta(s)\langle p(u(s)), w\rangle\, ds\right|$$

$$\leq \frac{1}{\tau}\int_{t}^{t+a_n} |\eta(s)|\|p(u(s))\|_{BV(\Omega)}\, ds + \frac{1}{\tau}\int_{t-\tau}^{t-\tau+a_n} |\eta(s)|\|p(u(s))\|_{BV(\Omega)}\, ds.$$

Since the function $s \mapsto |\eta(s)|\|p(u(s))\|_{BV(\Omega)}$ is in $L^1([0,T])$,

$$\lim_{n\to\infty} \|\psi^\tau(t+a_n) - \psi^\tau(t)\|_{BV(\Omega)} = 0.$$

Thus, $\psi^\tau \in C([0,T]; BV(\Omega))$.

Moreover, $\psi^\tau(t) \in L^2(\Omega)$. In fact, given $g \in L^\infty(\Omega)$, with $\|g\|_2 \leq 1$, since $g \in BV(\Omega)^*$, we have

$$|\langle \psi^\tau(t), g\rangle| = \left|\frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)\langle p(u(s)), g\rangle\, ds\right|$$

$$= \left| \frac{1}{\tau} \int_{t-\tau}^{t} \eta(s) \left(\int_\Omega p(u(s)) g \, dx \right) ds \right|$$
$$\leq \frac{1}{\tau} \int_{t-\tau}^{t} |\eta(s)| \|p(u(s))\|_2 \|g\|_2 \leq M.$$

From the density of $L^\infty(\Omega)$ in $L^2(\Omega)$, we obtain that $\psi^\tau(t) \in L^2(\Omega)$. □

Lemma 7. *For $\tau > 0$ small enough, we have*

$$\int_0^T \langle \psi^\tau(t), \xi(t) \rangle \, dt \leq - \int_0^T \int_\Omega \frac{\eta(t-\tau) - \eta(t)}{-\tau} J_p(u(t)) \, dxdt. \tag{99}$$

Proof. Since $\psi^\tau \in C([0,T], BV(\Omega))$ admits a weak derivative in the space $L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$, using (87) we have for $\tau > 0$ small enough that

$$\int_0^T \langle \psi^\tau(t), \xi(t) \rangle \, dt = \int_0^T \int_\Omega \frac{u(t+\tau) - u(t)}{\tau} \eta(t) p(u(t)) \, dxdt.$$

Now, since p is nondecreasing, we have

$$J_p(u(t)) - J_p(u(t+\tau)) \leq \big(u(t) - u(t+\tau)\big) p(u(t))$$

and consequently, for $\tau > 0$ small enough, we obtain

$$\int_0^T \int_\Omega \frac{u(t+\tau) - u(t)}{\tau} \eta(t) p(u(t)) \, dxdt$$
$$\leq \int_0^T \int_\Omega \frac{J_p(u(t+\tau)) - J_p(u(t))}{\tau} \eta(t) \, dxdt$$
$$= \int_0^T \int_\Omega \frac{\eta(t-\tau) - \eta(t)}{\tau} J_p(u(t)) \, dxdt,$$

and we finish the proof of (99). □

Lemma 8. *Let*

$$A_n := \int_0^T \int_\Omega h(x, Dp(u_n)) \, dt$$
$$+ \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x)) \, dH^{N-1} dt,$$

$\eta \in \mathcal{D}(0,T)$, and

$$(\eta p(u))^\tau(t) = \frac{1}{\tau} \int_{t-\tau}^{t} \eta(s) p(u(s)) ds.$$

Then

$$\limsup_{n\to\infty} A_n \le \int_0^T \int_\Omega z(t)\cdot \nabla p(u(t))\,dxdt$$
$$+\liminf_{\eta\uparrow 1}\liminf_{\tau\to 0}\int_0^T (z(t), D^s(\eta p(u))^\tau(t))\,dt$$
$$+\int_0^T\int_{\partial\Omega}[z(t),\nu](p(\varphi)-p(u(t)))\,dH^{N-1}dt.$$

Proof. Let $w\in W^{1,1}((0,T)\times\Omega)$ be such that $w(t)|_{\partial\Omega}=\varphi$. We use as test function in (94) $(\eta p(w(t)))^\tau$ and integrate in $(0,T)$ to obtain

$$-\int_0^T\int_\Omega (\eta p(w(t)))^\tau u_n'(t)\,dxdt+\int_0^T\int_\Omega p(u_n(t))u_n'(t)\,dxdt$$
$$+\int_0^T\int_\Omega h(x,Dp(u_n(t)))\,dt$$
$$+\int_0^T\int_{\partial\Omega}|p(\varphi)-p(u_n(t))|f^0(x,\nu(x))\,dH^{N-1}dt$$
$$=\int_0^T\int_\Omega (z_n(t), D(\eta p(w(t)))^\tau)\,dt$$
$$-\int_0^T\int_{\partial\Omega}[z_n(t),\nu](\eta^\tau(t)-1)p(\varphi)\,dH^{N-1}dt,$$

where $\eta^\tau(t)=\frac{1}{\tau}\int_{t-\tau}^t\eta(s)ds$. Our purpose is to take limits in the above expression as $n\to\infty$, $w\to u$ in $L^1((0,T)\times\Omega)$, $\tau\to 0$ and $\eta\uparrow 1$. We take $\tau>0$ small enough. Let us analyze the first term

$$-\int_0^T\int_\Omega(\eta p(w(t)))^\tau u_n'(t)\,dxdt=\int_0^T\int_\Omega(\eta p(w(t)))_t^\tau u_n(t)\,dxdt$$
$$\to\int_0^T\int_\Omega(\eta p(w(t)))_t^\tau u(t)\,dxdt\quad\text{as } n\to\infty.$$

Now, using (87) and Lemma 7,

$$\int_0^T\int_\Omega(\eta p(w(t)))_t^\tau u$$
$$=\int_0^T\int_\Omega\frac{\eta(t)p(w(t))-\eta(t-\tau)p(w(t-\tau))}{\tau}u(t)\,dxdt$$
$$\to\int_0^T\int_\Omega\frac{\eta(t)p(u(t))-\eta(t-\tau)p(u(t-\tau))}{\tau}u(t)\,dxdt,\quad\text{as } w\to u \text{ in } L^1,$$

$$= -\int_0^T \langle \xi(t), (\eta p(u(t)))^\tau \rangle \, dt \geq \int_0^T \int_\Omega \frac{\eta(t-\tau) - \eta(t)}{-\tau} J_p(u(t)) \, dx dt$$

$$\to \int_0^T \int_\Omega \eta_t J_p(u(t)) \, dx dt, \quad \text{as } \tau \to 0$$

$$\to \int_\Omega (J_p(u(0)) - J_p(u(T))) \, dx \quad \text{as } \eta \uparrow 1.$$

The analysis of the second term is easy. Letting $n \to \infty$ we have

$$\int_0^T \int_\Omega p(u_n(t)) u_n'(t) \, dx dt = \int_0^T \frac{d}{dt} \int_\Omega J_p(u_n(t)) \, dx$$

$$= \int_\Omega (J_p(u_n)(T) - J_p(u_n(0))) \, dx \to \int_\Omega (J_p(u(T)) - J_p(u(0))) \, dx.$$

Let us analyze the fifth term. Having in mind Steps 3 and 4, and (93), taking limits as $n \to \infty$, $w \to u$ in L^1 and $\tau \to 0$, we get:

$$\int_0^T \int_\Omega (z_n(t), D(\eta p(w))^\tau) \, dt \to$$

$$-\int_0^T \langle \xi(t), (\eta p(w))^\tau \rangle \, dt + \int_0^T \int_{\partial\Omega} [z(t), \nu] \eta^\tau p(\varphi) \, dH^{N-1} dt$$

$$= \int_0^T \int_\Omega u(t) (\eta p(w))_t^\tau \, dx dt + \int_0^T \int_{\partial\Omega} [z(t), \nu] \eta^\tau p(\varphi) \, dH^{N-1} dt$$

$$= \int_0^T \int_\Omega u(t) \frac{\eta(t) p(w)(t) - \eta(t-\tau) p(w)(t-\tau)}{\tau} \, dx dt$$

$$+ \int_0^T \int_{\partial\Omega} [z(t), \nu] \eta^\tau p(\varphi) \, dH^{N-1} dt$$

$$\to \int_0^T \int_\Omega u(t) \frac{\eta(t) p(u)(t) - \eta(t-\tau) p(u)(t-\tau)}{\tau} \, dx dt$$

$$+ \int_0^T \int_{\partial\Omega} [z(t), \nu] \eta^\tau p(\varphi) \, dH^{N-1} dt$$

$$= \int_0^T \int_\Omega u(t) (\eta p(u))_t^\tau \, dx dt + \int_0^T \int_{\partial\Omega} [z(t), \nu] \eta^\tau p(\varphi) \, dH^{N-1} dt$$

$$= -\int_0^T \langle \xi(t), (\eta p(u))^\tau \rangle \, dt + \int_0^T \int_{\partial\Omega} [z(t), \nu] \eta^\tau p(\varphi) \, dH^{N-1} dt$$

$$= \int_0^T \int_\Omega z(t) \cdot \nabla (\eta p(u))^\tau \, dx dt + \int_0^T \int_\Omega z(t) \cdot D^s (\eta p(u))^\tau \, dt$$

$$+\int_0^T\int_{\partial\Omega}[z(t),\nu](\eta^\tau p(\varphi)-(\eta p(u))^\tau)\,dH^{N-1}dt$$

$$\to\int_0^T\int_\Omega z(t)\cdot\nabla(\eta p(u))\,dxdt+\liminf_{\tau\to 0}\int_0^T\int_\Omega z(t)\cdot D^s(\eta p(u))^\tau\,dt$$

$$+\int_0^T\int_{\partial\Omega}[z(t),\nu]\eta(p(\varphi)-p(u(t)))\,dH^{N-1}dt$$

$$\to\int_0^T\int_\Omega z(t)\cdot\nabla p(u(t))\,dxdt+\liminf_{\eta\uparrow 1}\liminf_{\tau\to 0}\int_0^T\int_\Omega z(t)\cdot D^s(\eta p(u))^\tau\,dt$$

$$+\int_0^T\int_{\partial\Omega}[z(t),\nu](p(\varphi)-p(u(t)))\,dH^{N-1}dt,\quad\text{as}\quad\eta\uparrow 1.$$

Finally let us analyze the last term. Using Step 3, we have

$$\int_0^T\int_{\partial\Omega}[z_n(t),\nu](\eta^\tau(t)-1)p(\varphi)\,dH^{N-1}dt$$

$$\to\int_0^T\int_{\partial\Omega}[z(t),\nu](\eta^\tau(t)-1)p(\varphi)\,dH^{N-1}dt,\quad\text{as } n\to\infty$$

$$\to\int_0^T\int_{\partial\Omega}[z(t),\nu](\eta(t)-1)p(\varphi)\,dH^{N-1}dt,\quad\text{as }\tau\to 0,$$

$$\to 0\qquad\text{as }\eta\uparrow 1.$$

The lemma follows by collecting all these facts. □

Lemma 9. *Let*

$$\Psi_{p(\varphi)}(p(u(t)))=\int_\Omega f(x,Dp(u(t)))+\int_{\partial\Omega}|p(\varphi)-p(u(t))|f^0(x,\nu(x))\,dH^{N-1}.$$

Then

$$\int_0^T\Psi_{p(\varphi)}(p(u(t)))\,dt=\lim_{n\to\infty}\int_0^T\Psi_{p(\varphi)}(p(u_n(t)))\,dt.\tag{100}$$

As a consequence, we also have that

$$\Psi_{p(\varphi)}(p(u(t)))=\lim_{n\to\infty}\Psi_{p(\varphi)}(p(u_n(t)))\quad\text{a.e. in } t.\tag{101}$$

Proof. From the convexity of f, we have

$$\int_0^T \Psi_{p(\varphi)}(p(u_n(t)))\,dt = \int_0^T \int_\Omega f(x, \nabla p(u_n))\,dxdt$$

$$+ \int_0^T \int_\Omega f^0(x, D^s p(u_n))\,dt + \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x))\,dH^{N-1}dt$$

$$\leq \int_0^T \int_\Omega f(x, \nabla p(u))\,dxdt + \int_0^T \int_\Omega a(x, \nabla u_n) \cdot \nabla p(u_n)\,dxdt$$

$$- \int_0^T \int_\Omega a(x, \nabla p(u_n)) \nabla p(u)\,dxdt + \int_0^T \int_\Omega a(x, \nabla u_n) D^s p(u_n)\,dt$$

$$+ \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x))\,dH^{N-1}dt = \int_0^T \int_\Omega f(x, \nabla p(u))\,dxdt$$

$$+ \int_0^T \int_\Omega (a(x, \nabla u_n), Dp(u_n))\,dt - \int_0^T \int_\Omega a(x, \nabla p(u_n)) \cdot \nabla p(u)\,dxdt$$

$$+ \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x))\,dH^{N-1}dt.$$

Taking limits as $n \to \infty$ and using Lemma 8, we obtain

$$\limsup_{n\to\infty} \int_0^T \Psi_{p(\varphi)}(p(u_n(t)))\,dt \leq \int_0^T \int_\Omega f(x, \nabla p(u))\,dxdt + \limsup_{n\to\infty} A_n$$

$$- \int_0^T \int_\Omega z(t) \cdot \nabla p(u(t))\,dxdt \leq \int_0^T \int_\Omega f(x, \nabla p(u))\,dxdt$$

$$+ \liminf_{\eta\uparrow 1} \liminf_{\tau\to 0} \int_0^T z(t) \cdot D^s(\eta p(u))^\tau\,dt$$

$$+ \int_0^T \int_{\partial\Omega} [z(t), \nu](p(\varphi) - p(u))\,dH^{N-1}dt.$$

Now, having in mind that $f^0(x, \xi)$ is positively homogeneous in ξ, and applying Jensen's inequality,

$$\int_\Omega f^0(x, D^s(\eta p(u))^\tau) \leq \frac{1}{\tau} \int_{t-\tau}^t \eta(r) \int_\Omega f^0(x, D^s p(u(r)))\,dr. \tag{102}$$

Since $f^0(x, D^s(\eta p(u))^\tau)$ is a measure, by (102), using (H_5), we have that

$$\begin{aligned}
&\liminf_{\eta\uparrow 1}\liminf_{\tau\to 0}\int_0^T\int_\Omega z(t)\cdot D^s(\eta p(u))^\tau(t)\,dt\\
&\le \liminf_{\eta\uparrow 1}\liminf_{\tau\to 0}\int_0^T\int_\Omega f^0(x, D^s(\eta p(u))^\tau)\,dt\\
&\le \liminf_{\eta\uparrow 1}\liminf_{\tau\to 0}\int_0^T \frac{1}{\tau}\int_{t-\tau}^t \eta(r)\int_\Omega f^0(x, D^s p(u(r)))\,drdt\\
&= \int_0^T\int_\Omega f^0(x, D^s p(u(t)))\,dt.
\end{aligned} \tag{103}$$

Hence

$$\begin{aligned}
&\limsup_{n\to\infty}\int_0^T \Psi_{p(\varphi)}(p(u_n(t))\,dt\\
&\le \int_0^T\int_\Omega f(x, \nabla p(u))\,dxdt + \int_0^T\int_\Omega f^0(x, D^s p(u(t)))\,dt\\
&\quad + \int_0^T\int_{\partial\Omega}[z(t), \nu](p(\varphi) - p(u))\,dH^{N-1}dt.
\end{aligned}$$

Since $[z, \nu] = \lim_n [z_n, \nu]$, we have that $|[z, \nu]| \le f^0(x, \nu(x))$, from the above inequalities, we conclude that

$$\begin{aligned}
&\limsup_{n\to\infty}\int_0^T \Psi_{p(\varphi)}(p(u_n(t))\,dt\\
&\le \int_0^T\int_\Omega f(x, Dp(u(t)))\,dt + \int_0^T\int_{\partial\Omega}|p(\varphi) - p(u(t))| f^0(x, \nu(x))\,dH^{N-1}dt\\
&= \int_0^T \Psi_{p(\varphi)}(u)\,dt.
\end{aligned}$$

Now, from the lower semicontinuity of $\Psi_{p(\varphi)}$, we obtain

$$\begin{aligned}
&\int_0^T \Psi_{p(\varphi)}(p(u(t)))\,dt \le \liminf_n \int_0^T \Psi_{p(\varphi)}(p(u_n(t)))\,dt\\
&\le \limsup_n \int_0^T \Psi_{p(\varphi)}(p(u_n(t)))\,dt \le \int_0^T \Psi_{p(\varphi)}(p(u(t)))\,dt.
\end{aligned}$$

The proof of (100) is concluded. To prove (101) we observe that, since $u_n(t) \to u(t)$ in $L^1(\Omega)$ for all $t \in [0, T]$, we have that

$$\Psi_{p(\varphi)}(p(u(t))) \le \liminf_n \Psi_{p(\varphi)}(p(u_n(t)) \quad \text{for all } t \in [0, T].$$

Using Fatou's Lemma and (100), we have

$$\int_0^T \Psi_{p(\varphi)}(p(u(t)))\,dt \le \int_0^T \liminf_n \Psi_{p(\varphi)}(p(u_n(t))\,dt$$

$$\le \liminf_n \int_0^T \Psi_{p(\varphi)}(p(u_n(t))\,dt = \int_0^T \Psi_{p(\varphi)}(p(u(t)))\,dt,$$

and, therefore, (101) holds. □

Remark 2. Let $\eta(t,x) = \phi(t)\psi(x)$ with $\eta \ge 0$, $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C^\infty(\overline{\Omega})$. Multiplying (79) by $\hat{w} - p(u_n(t))\eta$ and integrating in Ω, using (80) and (81), we have that

$$\begin{aligned} -\int_\Omega (\hat{w} - p(u_n(t))\eta)u_n'(t)\,dx &= \int_\Omega (z_n(t), D\hat{w}) \\ -\int_\Omega h(x, Dp(u_n(t)))\eta &- \int_\Omega p(u_n(t))z_n(t)\cdot\nabla_x\eta\,dx \\ &- \int_{\partial\Omega}[z_n(t),\nu](\hat{w} - p(\varphi)\eta)\,dH^{N-1} \\ &- \int_{\partial\Omega}|p(u_n(t)) - p(\varphi)|f^0(x,\nu(x))\eta\,dH^{N-1} \end{aligned} \tag{104}$$

for every $\hat{w} \in BV(\Omega)\cap L^\infty(\Omega)$ and all $p \in \mathcal{P}$. Now, let $w \in W^{1,1}((0,T)\times\Omega)$ be such that $w(t)|_{\partial\Omega} = \varphi$. Using $(\eta p(w(t)))^\tau$ as test function in (104), integrating in $(0,T)$, and proceeding as in the proof of Lemma 8 (this time we do not let $\eta \uparrow 1$), we obtain that

$$\begin{aligned} \limsup_{n\to\infty} A_n(\eta) &\le \int_0^T \int_\Omega z(t)\cdot\nabla p(u(t))\eta\,dxdt \\ &+ \liminf_{\tau\to 0}\int_0^T\int_\Omega z(t)\cdot D^s(\eta p(u))^\tau(t)\,dt \\ &+ \int_0^T\int_{\partial\Omega}[z(t),\nu](p(\varphi) - p(u(t)))\eta\,dH^{N-1}dt. \end{aligned}$$

where

$$\begin{aligned} A_n(\eta) &:= \int_0^T\int_\Omega h(x, Dp(u_n))\eta\,dt \\ &+ \int_0^T\int_{\partial\Omega}|p(\varphi) - p(u_n)|f^0(x,\nu(x))\eta\,dH^{N-1}dt. \end{aligned}$$

Now, proceeding as in Lemma 9, we prove that

$$\int_0^T \int_\Omega f(x, Dp(u(t)))\eta \, dxdt$$

$$+ \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u(t))| f^0(x, \nu(x))\eta \, dH^{N-1} dt$$

$$= \lim_n \int_0^T \int_\Omega f(x, Dp(u_n(t)))\eta \, dxdt$$

$$+ \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u_n(t))| f^0(x, \nu(x))\eta \, dH^{N-1} dt,$$

and

$$\int_\Omega f(x, Dp(u(t)))\eta \, dx + \int_{\partial\Omega} |p(\varphi) - p(u(t))| f^0(x, \nu(x))\eta \, dH^{N-1}$$

$$= \lim_n \int_\Omega f(x, Dp(u_n(t)))\eta \, dx + \int_{\partial\Omega} |p(\varphi) - p(u_n(t))| f^0(x, \nu(x))\eta \, dH^{N-1},$$

for almost all $t \in (0, T)$.

From Lemma 9, and arguing as in the proof of Theorem 4, it follows that

$$\int_\Omega h(x, Dp(u(t))) + \int_{\partial\Omega} |p(\varphi) - p(u(t))| f^0(x, \nu(x)) \, dH^{N-1}$$

$$= \lim_{n\to\infty} \int_\Omega h(x, Dp(u_n(t))) \tag{105}$$

$$+ \lim_{n\to\infty} \int_{\partial\Omega} |p(\varphi) - p(u_n(t))| f^0(x, \nu(x)) \, dH^{N-1},$$

a.e. in $t \in (0, T)$. Let us now prove that

$$\int_0^T \int_\Omega z(t) \cdot \nabla p(u(t)) \, dxdt$$

$$\leq \int_0^T \int_\Omega a(x, \nabla u(t)) \cdot \nabla p(u(t)) \, dxdt. \tag{106}$$

In fact, from the convexity of f in ξ, we have

$$\int_0^T \int_\Omega a(x, \nabla p(u_n)) \cdot \nabla p(u) \, dxdt \leq \int_0^T \int_\Omega a(x, \nabla p(u_n)) \cdot \nabla p(u_n) \, dxdt$$

$$+\int_0^T\int_\Omega f(x,\nabla p(u))\,dxdt-\int_0^T\int_\Omega f(x,\nabla p(u_n))\,dxdt$$

$$=\int_0^T\int_\Omega h(x,\nabla p(u_n))\,dxdt+\int_0^T\int_\Omega f^0(x,D^s p(u_n))\,dt$$

$$+\int_0^T\int_{\partial\Omega}|p(\varphi)-p(u_n)|f^0(x,\nu(x))\,dH^{N-1}dt-\int_0^T\int_\Omega f^0(x,D^s p(u_n))\,dt$$

$$-\int_0^T\int_{\partial\Omega}|p(\varphi)-p(u_n)|f^0(x,\nu(x))\,dH^{N-1}dt+\int_0^T\int_\Omega f(x,\nabla p(u))\,dxdt$$

$$-\int_0^T\int_\Omega f(x,\nabla p(u_n))\,dxdt=\int_0^T\int_\Omega h(x,Dp(u_n))\,dt$$

$$+\int_0^T\int_{\partial\Omega}|p(\varphi)-p(u_n)|f^0(x,\nu(x))\,dH^{N-1}dt-\int_0^T\int_\Omega f(x,Dp(u_n))\,dt$$

$$-\int_0^T\int_{\partial\Omega}|p(\varphi)-p(u_n)|f^0(x,\nu(x))\,dH^{N-1}dt+\int_0^T\int_\Omega f(x,\nabla p(u))\,dxdt.$$

Now, since

$$\lim_{n\to\infty}\int_0^T\int_\Omega \mathbf{a}(x,\nabla u_n(t))-\mathbf{a}(x,\nabla p(u_n(t)))\cdot\nabla p(u(t))\,dxdt=0,$$

we have

$$\lim_{n\to\infty}\int_0^T\int_\Omega \mathbf{a}(x,\nabla p(u_n(t)))\cdot\nabla p(u(t))\,dxdt=\int_0^T\int_\Omega z(t)\cdot\nabla p(u(t))\,dxdt.$$

Then, letting $n\to\infty$, and using Lemma 9 and (105), we deduce that

$$\int_0^T\int_\Omega z(t)\cdot\nabla p(u(t))\,dxdt\le\int_0^T\int_\Omega h(x,Dp(u))\,dt$$

$$+\int_0^T\int_{\partial\Omega}|p(\varphi)-p(u)|f^0(x,\nu(x))\,dH^{N-1}dt-\int_0^T\int_\Omega f(x,Dp(u))\,dt$$

$$-\int_0^T\int_{\partial\Omega}|p(\varphi)-p(u)|f^0(x,\nu(x))\,dH^{N-1}dt+\int_0^T\int_\Omega f(x,\nabla p(u))\,dt$$

$$=\int_0^T\int_\Omega a(x,\nabla u(t))\cdot\nabla p(u(t))\,dxdt,$$

and (106) holds.

Lemma 10. *We have*

$$\int_0^T \int_{\partial\Omega} [z(t), \nu(x)](p(\varphi) - p(u(t)))\, dt$$
$$= \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u(t))| f^0(x, \nu(x))\, dt. \tag{107}$$

In particular, since $|[z, \nu]| \le f^0(x, \nu(x))$*, (107) implies that*

$$[z, \nu] \in sign(p(\varphi) - p(u)) f^0(x, \nu(x)). \tag{108}$$

Proof. Using (105), Fatou's Lemma, Lemma 8, (103) and (106), we have

$$\int_0^T \int_\Omega h(x, Dp(u(t)))\, dt + \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u(t))| f^0(x, \nu(x))\, dH^{N-1} dt$$
$$\le \liminf_{n\to\infty} \int_0^T \int_\Omega h(x, Dp(u_n(t)))\, dt$$
$$+ \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u_n(t))| f^0(x, \nu(x))\, dH^{N-1} dt$$
$$\le \int_0^T \int_\Omega z(t) \cdot \nabla p(u(t))\, dxdt$$
$$+ \liminf_{\eta\uparrow 1} \liminf_{\tau\to 0} \int_0^T \int_\Omega z(t) \cdot D^s(\eta p(u(t)))^\tau\, dt$$
$$+ \int_0^T \int_{\partial\Omega} [z(t), \nu(x)](p(\varphi) - p(u(t)))\, dH^{N-1} dt$$
$$\le \int_0^T \int_\Omega z(t) \cdot \nabla p(u(t))\, dxdt + \int_0^T \int_\Omega f^0(x, D^s p(u(t)))\, dt$$
$$+ \int_0^T \int_{\partial\Omega} [z(t), \nu(x)](p(\varphi) - p(u(t)))\, dH^{N-1}\, dt$$
$$\le \int_0^T \int_\Omega a(x, \nabla u(t)) \cdot \nabla p(u(t))\, dxdt + \int_0^T \int_\Omega f^0(x, D^s p(u(t)))\, dt$$
$$+ \int_0^T \int_{\partial\Omega} [z(t), \nu(x)](p(\varphi) - p(u(t)))\, dH^{N-1} dt$$
$$= \int_0^T \int_\Omega h(x, Dp(u(t)))\, dt$$
$$+ \int_0^T \int_{\partial\Omega} [z(t), \nu(x)](p(\varphi) - p(u(t)))\, dH^{N-1} dt$$

$$\leq \int_0^T \int_\Omega h(x, Dp(u(t)))\, dt$$
$$+ \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u(t))| f^0(x, \nu(x))\, dH^{N-1} dt.$$

From this series of inequalities, the identity (107) follows. □

Remark 3. From last series of inequalities, we also have the following identities

$$\begin{aligned} &\int_0^T \int_\Omega h(x, Dp(u(t)))\, dt \\ &+ \int_0^T \int_{\partial\Omega} |p(\varphi) - p(u(t))| f^0(x, \nu(x))\, dH^{N-1} dt \\ &= \lim_{n\to\infty} \int_0^T \int_\Omega h(x, Dp(u_n(t)))\, dt \\ &\int_0^T \int_{\partial\Omega} |p(\varphi) - p(u_n(t))| f^0(x, \nu(x))\, dH^{N-1} dt, \end{aligned} \tag{109}$$

$$\begin{aligned} &\int_0^T \int_\Omega z(t) \cdot \nabla p(u(t))\, dxdt = \\ &\int_0^T \int_\Omega a(x, \nabla u(t)) \cdot \nabla p(u(t))\, dxdt, \end{aligned} \tag{110}$$

$$\begin{aligned} &\liminf_{\eta\uparrow 1} \liminf_{\tau\to 0} \int_0^T \int_\Omega z(t) \cdot D^s(\eta p(u))^\tau\, dt \\ &= \int_0^T \int_\Omega f^0(x, D^s p(u(t)))\, dt. \end{aligned} \tag{111}$$

Step 6. Identification of the limit. Let us now prove that

$$z(t,x) = \mathbf{a}(x, \nabla u(t,x)) \qquad \text{a.e. } (t,x) \in (0,T) \times \Omega. \tag{112}$$

Let $0 \leq \phi \in C_0^1((0,T) \times \Omega)$ and $g \in C^1([0,T] \times \overline{\Omega})$. We observe that

$$\int_0^T \int_\Omega \phi[(\mathbf{a}(x, \nabla u_n), Dp(u_n - g)) - \mathbf{a}(x, \nabla g) Dp(u_n - g)]$$
$$= \int_0^T \int_\Omega \phi[\mathbf{a}(x, \nabla u_n) - \mathbf{a}(x, \nabla g)] \cdot \nabla p(u_n - g)\, dxdt$$

$$+ \int_\Omega \phi[\mathbf{a}(x, \nabla u_n) - \mathbf{a}(x, \nabla g)] \cdot D^s p(u_n - g).$$

Since both terms at the right hand side of the above expression are positive, we have

$$\int_0^T \int_\Omega \phi[(\mathbf{a}(x, \nabla u_n), Dp(u_n - g)) - \mathbf{a}(x, \nabla g) Dp(u_n - g)] \geq 0. \tag{113}$$

Our purpose is to take limits as $n \to \infty$ in the above inequality. We assume that $\phi(t, x) = \eta(t)\psi(x)$, where $\eta \in \mathcal{D}(0, T)$, $\psi \in \mathcal{D}(\Omega)$, $\eta \geq 0$, $\psi \geq 0$. First, integrating by parts in the first term, we have

$$\int_0^T \int_\Omega \phi(\mathbf{a}(x, \nabla u_n), Dp(u_n - g))\, dt$$

$$= - \int_0^T \int_\Omega p(u_n - g) \nabla_x \phi \cdot \mathbf{a}(x, \nabla u_n)\, dxdt$$

$$- \int_0^T \int_\Omega \phi div(\mathbf{a}(x, \nabla u_n)) p(u_n - g)\, dxdt$$

$$= - \int_0^T \int_\Omega p(u_n - g) \nabla_x \phi \cdot \mathbf{a}(x, \nabla u_n)\, dxdt - \int_0^T \int_\Omega \phi u_n'(t) p(u_n - g)\, dxdt$$

$$= - \int_0^T \int_\Omega p(u_n - g) \nabla_x \phi \cdot \mathbf{a}(x, \nabla u_n)\, dxdt$$

$$- \int_0^T \int_\Omega \phi \frac{d}{dt} J_p(u_n - g)\, dxdt - \int_0^T \int_\Omega \phi g_t p(u_n - g)\, dxdt$$

$$= - \int_0^T \int_\Omega p(u_n - g) \nabla_x \phi \cdot \mathbf{a}(x, \nabla u_n)\, dxdt + \int_0^T \int_\Omega \phi_t J_p(u_n - g)\, dxdt$$

$$- \int_0^T \int_\Omega \phi g_t p(u_n - g)\, dxdt.$$

Letting $n \to \infty$ in (113), taking into account the above equalities, we obtain

$$- \int_0^T \int_\Omega p(u - g) \nabla_x \phi \cdot z\, dxdt + \int_0^T \int_\Omega \phi_t J_p(u - g)\, dxdt$$

$$- \int_0^T \int_\Omega \phi g_t p(u - g)\, dxdt - \int_0^T \int_\Omega \phi(a(x, \nabla g), Dp(u - g))\, dt \geq 0. \tag{114}$$

Now,

$$\int_0^T \int_\Omega \phi_t J_p(u-g)\, dxdt$$

$$= \lim_{\tau \to 0} \int_0^T \int_\Omega \frac{\phi(t-\tau) - \phi(t)}{-\tau} J_p(u-g)\, dxdt \tag{115}$$

$$= \lim_{\tau \to 0} \int_0^T \int_\Omega \psi(x) \frac{\eta(t-\tau) - \eta(t)}{-\tau} J_p(u-g)\, dxdt.$$

For simplicity, let us write $v = u - g$. Since $J_p(v(t)) - J_p(v(t+\tau)) \leq (v(t) - v(t+\tau))p(v(t))$, for τ small enough, we have

$$\int_0^T \int_\Omega \frac{v(t+\tau) - v(t)}{\tau} \eta(t)\psi(x) p(v(t))\, dxdt$$

$$\leq \int_0^T \int_\Omega \frac{J_p(v(t+\tau)) - J_p(v(t))}{\tau} \eta(t)\psi(x)\, dxdt \tag{116}$$

$$= \int_0^T \int_\Omega \frac{\eta(t-\tau) - \eta(t)}{\tau} \psi(x) J_p(v)\, dxdt.$$

By Lemma 4, we have

$$\int_0^T \int_\Omega \frac{v(t+\tau) - v(t)}{\tau} \eta(t)\psi(x) p(v(t))\, dxdt$$

$$= -\int_0^T \int_\Omega v(t) \frac{d}{dt} (\eta p(v))^\tau (t) \psi(x)\, dxdt \tag{117}$$

$$= \int_0^T \int_\Omega (\xi - g_t)(t) (\eta p(v))^\tau (t) \psi(x)\, dxdt.$$

Collecting these inequalities, we obtain

$$\int_0^T \int_\Omega \frac{\eta(t-\tau) - \eta(t)}{-\tau} \psi J_p(v)\, dxdt$$

$$\leq -\int_0^T \int_\Omega (\xi - g_t)(t) (\eta p(v))^\tau (t) \psi(x)\, dxdt$$

$$= -\lim_n \int_0^T < u_n'(t) - g_t, (\eta p(v))^\tau (t) \psi > dt$$

$$= -\lim_n \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) < div(z_n(t)) - g_t(t), p(v(s))\psi > \, dsdt$$

$$= \lim_n \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s)[\int_\Omega (z_n(t), D(p(v(s))\psi)) + < g_t, p(v(s))\psi >] \, dsdt$$

$$= \lim_n \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_\Omega (z_n(t), Dp(v(s)))\psi \, dsdt$$

$$+ \lim_n \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) p(v(s)) \int_\Omega z_n(t) \cdot \nabla\psi \, dsdt$$

$$+ \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) < g_t, p(v(s))\psi > \, dsdt.$$

Since

$$Dp(v(s)) = \nabla p(u(s) - g(s)) + D^s p(u(s) - g(s))$$

and

$$z_n(t) \cdot D^s p(u(s) - g(s)) = a(x, \nabla u_n(t, x)) \cdot D^s p(u(s) - g(s))$$

$$\leq h^0(x, D^s p(u(s) - g(s))),$$

from the above inequality, it follows that

$$\int_0^T \int_\Omega \frac{\eta(t-\tau) - \eta(t)}{-\tau} \psi J_p(u - g) \, dxdt$$

$$\leq \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_\Omega z(t) \cdot \nabla p(u(s) - g(s))\psi \, dxdsdt$$

$$+ \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_\Omega \psi h^0(x, D^s p(u(s) - g(s))) \, dsdt \qquad (118)$$

$$+ \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_\Omega p(u(s) - g(s)) z(t) \cdot \nabla\psi \, dxdsdt$$

$$+ \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_\Omega g_t(t) p(u(s) - g(s))\psi(x) \, dxdsdt.$$

Hence, letting $\tau \to 0$ in (118), we obtain

$$
\begin{aligned}
\int_0^T \int_\Omega \phi_t J_p(u-g) \leq \int_0^T \int_\Omega \eta(t) z(t) \cdot \nabla p(u(t)-g(t)) \psi \, dxdt \\
+ \int_0^T \eta(t) \int_\Omega \psi h^0(x, D^s p(u(t)-g(t))) \, dt \\
+ \int_0^T \int_\Omega \eta(t) p(u(t)-g(t)) z(t) \cdot \nabla \psi \, dxdt \\
+ \int_0^T \int_\Omega \eta(t) g_t(t) p(u(t)-g(t)) \psi(x) \, dxds \, dt.
\end{aligned}
\tag{119}
$$

Taking into account (114) and (119), we obtain

$$
\int_0^T \int_\Omega \phi \Big([z - \mathbf{a}(x, \nabla g)] \cdot \nabla p(u-g) \\
+ h^0(x, D^s p(u-g)) - \mathbf{a}(x, \nabla g) \cdot D^s p(u-g) \Big) \geq 0
$$

for all $\phi(t,x) = \eta(t)\psi(x)$, $\eta \in \mathcal{D}(0,T)$, $\psi \in \mathcal{D}(\Omega)$, $\eta, \psi \geq 0$. Thus, the measure

$$
\Big([z - \mathbf{a}(x, \nabla g)] \cdot \nabla p(u-g) + h^0(x, D^s p(u-g)) \\
- \mathbf{a}(x, \nabla g) \cdot D^s p(u-g) \Big) \geq 0.
$$

Then its absolutely continuous part

$$
[z - \mathbf{a}(x, \nabla g)] \cdot \nabla p(u-g) \geq 0 \quad \text{a.e. in } \Omega.
$$

Hence

$$
[z - \mathbf{a}(x, \nabla g)] \cdot \nabla (u-g) \geq 0 \quad \text{a.e. in } \Omega.
$$

Since we may take a countable set dense in $C^1([0,T] \times \overline{\Omega})$, we have that the above inequality holds for all $(t,x) \in S$ where $S \subseteq (0,T) \times \Omega$ is such that $\lambda_N((0,T) \times \Omega \setminus S) = 0$, and all $g \in C^1([0,T] \times \overline{\Omega})$. Now, fixe $(t,x) \in S$, and, given $y \in \mathbb{R}^N$, there is $g \in C^1([0,T] \times \overline{\Omega})$ such that $\nabla g(t,x) = y$. Then

$$
\big(z(t,x) - \mathbf{a}(x,y) \big) \cdot \big(\nabla u(t,x) - y \big) \geq 0 \quad \forall y \in \mathbb{R}^N,
$$

and we get that

$$
z(t,x) = \mathbf{a}(x, \nabla u(t,x)) \qquad \text{a.e.} \quad (t,x) \in Q_T. \tag{120}
$$

Then, we have

$$
div(z(t)) = div(\mathbf{a}(x, \nabla u(t)) \quad \text{in } \mathcal{D}'(\Omega), \quad a.e.\ t \in [0,T],
$$

and, since

$$|[z, \nu]| \leq f^0(x, \nu(x)) \;\; H^{N-1} - \text{a.e. on } \; \partial\Omega,$$

we also get

$$|[a(x, \nabla u(t)), \nu]| \leq f^0(x, \nu(x)) \quad H^{N-1}\text{-a.e. on } \partial\Omega, \text{ a.e. in } t \in (0, T).$$

Finally, from (120) and (108), we obtain that

$$[a(x, \nabla u(t)), \nu] \in sign(p(\varphi) - p(u(t))) f^0(x, \nu(x))$$

H^{N-1}-a.e. on $\partial\Omega$, a.e. in $t \in (0, T)$, and all $p \in \mathcal{P}$.

Step 7. Conclusion. Finally, we are going to prove that u verifies:

$$\begin{aligned} -\int_0^T \int_\Omega j(u(t)-l)\eta_t \, dxdt + \int_0^T \int_\Omega \eta(t) h(x, Dp(u(t)-l)) \, dt \\ + \int_0^T \int_\Omega z(t) \cdot \nabla\eta(t) p(u(t)-l) \, dxdt \\ \leq \int_0^T \int_{\partial\Omega} [z(t), \nu]\eta(t) p(u(t)-l) \, dH^{N-1} dt, \end{aligned} \tag{121}$$

for all $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geq 0$, $\eta(t,x) = \phi(t)\psi(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C^\infty(\overline{\Omega})$, and $p \in \mathcal{T}$, where $j(r) = \displaystyle\int_0^r p(s)\, ds$.

Let $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geq 0$, $\eta(t,x) = \phi(t)\psi(x)$, $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C^\infty(\overline{\Omega})$, $p \in \mathcal{P}$ and $a \in \mathbb{R}$. Let $G_p(r) = \int_a^r p(s)\, ds$. Since $u_n'(t) = div(z_n(t))$, multiplying by $p(u_n(t))\eta(t)$ and integrating, we obtain that

$$\begin{aligned} \int_0^T \int_\Omega \frac{d}{dt} G_p(u_n(t))\eta(t) \, dxdt &= \int_0^T \int_\Omega div(z_n(t)) p(u_n(t))\eta(t) \, dxdt \\ &= -\int_0^T \int_\Omega \big(z_n(t), D(p(u_n(t))\eta(t))\big) \, dt \\ &\quad + \int_0^T \int_{\partial\Omega} [z_n(t), \nu] p(u_n(t))\eta(t) \, dH^{N-1} dt \\ &= -\int_0^T \int_\Omega \eta(t) h(x, Dp(u_n(t))) \, dt - \int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t) \; p(u_n(t)) \, dxdt \\ &\quad + \int_0^T \int_{\partial\Omega} [z_n(t), \nu] p(u_n(t))\eta(t) \, dH^{N-1} dt \\ &= -\int_0^T \int_\Omega \eta(t) h(x, Dp(u_n(t))) \, dt - \int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t) \; p(u_n(t)) \, dxdt \end{aligned}$$

$$-\int_0^T \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| f^0(x, \nu(x))\eta(t)\, dH^{N-1}dt$$
$$+\int_0^T \int_{\partial\Omega} [z_n(t), \nu] p(\varphi)\eta(t)\, dH^{N-1}dt.$$

Hence, having in mind that $\eta(0) = \eta(T) = 0$, we get

$$\int_0^T \int_{\Omega} \eta(t) h(x, Dp(u_n(t)))\, dt$$
$$+\int_0^T \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| f^0(x, \nu(x))\eta(t)\, dH^{N-1}dt$$
$$= -\int_0^T \int_{\Omega} z_n(t) \cdot \nabla\eta(t)\; p(u_n(t))\, dxdt + \int_0^T \int_{\partial\Omega} [z_n(t), \nu] p(\varphi)\eta(t)\, dH^{N-1}dt$$
$$-\int_0^T \int_{\Omega} \frac{d}{dt} G_p(u_n(t))\eta(t)\, dxdt = -\int_0^T \int_{\Omega} z_n(t) \cdot \nabla\eta(t)\; p(u_n(t))\, dxdt$$
$$+\int_0^T \int_{\partial\Omega} [z_n(t), \nu] p(\varphi)\eta(t)\, dH^{N-1}dt + \int_0^T \int_{\Omega} G_p(u_n(t))\; \eta_t\, dxdt.$$

Now, observe that, by the Remark 2, we have that

$$\int_{\Omega} \eta(t,x) f(x, Dp(u_n)) + \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x))\eta(t,x)\; dH^{N-1}$$
$$\to \int_{\Omega} \eta(t,x) f(x, Dp(u)) + \int_{\partial\Omega} |p(\varphi) - p(u)| f^0(x, \nu(x))\eta(t,x)\; dH^{N-1}$$

a.e. in $t \in (0, T)$, and, therefore,

$$\int_{\Omega} \eta(t,x) h(x, Dp(u_n)) + \int_{\partial\Omega} |p(\varphi) - p(u_n)| f^0(x, \nu(x))\eta(t,x)\; dH^{N-1}$$
$$\to \int_{\Omega} \eta(t,x) h(x, Dp(u)) + \int_{\partial\Omega} |p(\varphi) - p(u)| f^0(x, \nu(x))\eta(t,x)\; dH^{N-1}$$

a.e. in $t \in (0, T)$. Hence, integrating in $(0, T)$ and using Fatou's Lemma, it follows that

$$\int_0^T \int_{\Omega} \eta(t) h(x, Dp(u(t)))\, dt$$
$$+\int_0^T \int_{\partial\Omega} |p(u(t)) - p(\varphi)| f^0(x, \nu(x))\eta(t)\, dH^{N-1}dt$$
$$\leq \lim_{n\to\infty} \Big[\int_0^T \int_{\Omega} \eta(t) h(x, Dp(u_n(t)))\, dt$$

$$+\int_0^T \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| f^0(x, \nu(x))\eta(t)\, dH^{N-1} dt\Bigg]$$

$$= \lim_{n\to\infty}\Bigg[-\int_0^T \int_\Omega z_n(t)\cdot\nabla\eta(t)\ p(u_n(t))\, dxdt$$

$$+\int_0^T \int_{\partial\Omega} [z_n(t), \nu] p(\varphi)\eta(t)\, dH^{N-1} dt$$

$$+\int_0^T \int_\Omega G_p(u_n(t))\ \eta_t\, dxdt\Bigg] = -\int_0^T \int_\Omega z(t)\cdot\nabla\eta(t)\ p(u(t))\, dxdt$$

$$+\int_0^T \int_{\partial\Omega} [z(t), \nu] p(\varphi)\eta(t)\, dH^{N-1} dt + \int_0^T \int_\Omega G_p(u(t))\ \eta_t\, dxdt.$$

Now, using that $|p(u(t)) - p(\varphi)| f^0(x, \nu(x)) = [z(t), \nu](p(\varphi) - p(u(t)))$, we have

$$-\int_0^T \int_\Omega G_p(u(t))\eta_t\, dxdt + \int_0^T \int_\Omega \eta(t) h(x, Dp(u(t)))\, dt$$

$$+\int_0^T \int_\Omega z(t)\cdot\nabla\eta(t)\ p(u(t))\, dxdt \tag{122}$$

$$\le \int_0^T \int_{\partial\Omega} [z(t), \nu] p(u(t))\eta(t)\, dH^{N-1} dt.$$

Finally, given $l \in \mathbb{R}$ and $p \in \mathcal{T}$, since $q(r) := p(r - l)$ is an element of $\mathcal{P}$, and taking $a = l$, we obtain (121) as a consequence of (122). The proof of the existence is finished. □

5.2. Proof of Theorem 1. Uniqueness

To prove uniqueness of entropy solutions, we follow the same technique as in [3], which was inspired by the doubling of variables method introduced by Kruzhkov [29] (see also [16], [31] and [18]) to prove L^1-contraction for entropy solutions for scalar conservation laws.

Since the operator $\mathcal{A}_\varphi$ is m-completely accretive in $L^1(\Omega)$, if we prove that the entropy solution coincides with the semigroup solution, then, by (37), (32) holds. So we only need to prove that any entropy solution is a semigroup solution.

Let $u(t)$ be an entropy solution with initial datum $u_0 \in L^1(\Omega)$ and $\overline{u}(t) = T(t)\overline{u}_0$ the semigroup solution with initial datum $\overline{u}_0 \in L^\infty(\Omega)$. Then, there exist $\xi, \overline{\xi} \in \left(L^1(0,T,BV(\Omega)_2)\right)^*$ such that if $z(t) := \mathbf{a}(x, \nabla u(t))$ and $\overline{z}(t) := \mathbf{a}(x, \nabla \overline{u}(t))$, we have $(z(t), \xi(t)), (\overline{z}(t), \overline{\xi}(t)) \in Z(\Omega)$ for almost all $t \in [0, T]$,

$$[z(t), \nu] \in \operatorname{sign}\left(T_k^+(\varphi) - T_k^+(u(t))\right) f^0(x, \nu(x)), \tag{123}$$

$$[\overline{z}(t), \nu] \in \operatorname{sign}\left(T_k^+(\varphi) - T_k^+(\overline{u}(t))\right) f^0(x, \nu(x)), \tag{124}$$

and, if $r, \overline{r} \in \mathbb{R}^N$ and $l_1, l_2 \in \mathbb{R}$, then

$$\begin{aligned} -\int_0^T \int_\Omega j_k^+(u(t)-l_1)\eta_t + \int_0^T \int_\Omega \eta(t) h(x, DT_k^+(u(t)-l_1)) \\ + \int_0^T \int_\Omega (z(t) - r)\cdot \nabla\eta(t)\, T_k^+(u(t)-l_1) \\ + \int_0^T \int_\Omega r \cdot \nabla \eta(t)\, T_k^+(u(t)-l_1) \\ \leq \int_0^T \int_{\partial\Omega} [z(t), \nu]\eta(t) T_k^+(u(t)-l_1) \end{aligned} \tag{125}$$

and

$$\begin{aligned} -\int_0^T \int_\Omega j_k^-(\overline{u}(t)-l_2)\eta_t + \int_0^T \int_\Omega \eta(t) h(x, DT_k^-(\overline{u}(t)-l_2)) \\ + \int_0^T \int_\Omega (\overline{z}(t) - \overline{r})\cdot \nabla\eta(t)\, T_k^-(\overline{u}(t)-l_2) \\ + \int_0^T \int_\Omega \overline{r} \cdot \nabla \eta(t)\, T_k^-(\overline{u}(t)-l_2) \\ \leq \int_0^T \int_{\partial\Omega} [\overline{z}(t), \nu]\eta(t) T_k^-(\overline{u}(t)-l_2), \end{aligned} \tag{126}$$

for all $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geq 0$, $\eta(t,x) = \phi(t)\psi(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C^\infty(\overline{\Omega})$, and $j_k^+(r) = \int_0^r T_k^+(s)\,ds$, $j_k^-(r) = \int_0^r T_k^-(s)\,ds$.

We choose two different pairs of variables (t,x), (s,y) and consider u, z as functions in (t,x); $\overline{u}$, $\overline{z}$ in (s,y). Let $0 \leq \phi \in \mathcal{D}(]0,T[)$, $0 \leq \psi \in \mathcal{D}(\Omega)$, ρ_n a classical sequence of mollifiers in $\mathbb{R}^N$ and $\tilde{\rho}_n$ a sequence of mollifiers in $\mathbb{R}$. Define

$$\eta_n(t,x,s,y) := \rho_n(x-y)\tilde{\rho}_n(t-s)\phi\left(\frac{t+s}{2}\right)\psi\left(\frac{x+y}{2}\right).$$

Note that for n sufficiently large,

$$(t,x)\mapsto \eta_n(t,x,s,y)\in \mathcal{D}(]0,T[\times\Omega)\quad \forall\,(s,y)\in Q_T,$$
$$(s,y)\mapsto \eta_n(t,x,s,y)\in \mathcal{D}(]0,T[\times\Omega)\quad \forall\,(t,x)\in Q_T.$$

Hence, for (s,y) fixed, if we take in (125) $l_1=\overline{u}(s,y)$ and $r=\overline{z}(s,y)$, we get

$$\begin{aligned}
&-\int_0^T\int_\Omega j_k^+(u(t,x)-\overline{u}(s,y))(\eta_n)_t\,dxdt\\
&+\int_0^T\int_\Omega \eta_n h(x,D_xT_k^+(u(t,x)-\overline{u}(s,y)))\,dt\\
&+\int_0^T\int_\Omega (z(t,x)-\overline{z}(s,y))\cdot\nabla_x\eta_n\;T_k^+(u(t,x)-\overline{u}(s,y))\,dxdt\\
&+\int_0^T\int_\Omega \overline{z}(s,y)\cdot\nabla_x\eta_n\;T_k^+(u(t,x)-\overline{u}(s,y))\,dxdt\le 0.
\end{aligned}\tag{127}$$

Similarly, for (t,x) fixed, if we take in (126) $l_2=u(t,x)$ and $\overline{r}=z(t,x)$, we get

$$\begin{aligned}
&-\int_0^T\int_\Omega j_k^-(\overline{u}(s,y)-u(t,x))(\eta_n)_s\,dyds\\
&+\int_0^T\int_\Omega \eta_n h(y,D_yT_k^-(\overline{u}(s,y)-u(t,x)))\,dyds\\
&+\int_0^T\int_\Omega (\overline{z}(s,y)-z(t,x))\cdot\nabla_y\eta_n\;T_k^-(\overline{u}(s,y)-u(t,x))\,dyds\\
&+\int_0^T\int_\Omega z(t,x)\cdot\nabla_y\eta_n\;T_k^-(\overline{u}(s,y)-u(t,x))\,dyds\le 0.
\end{aligned}$$

Now, since $T_k^-(r)=-T_k^+(-r)$, $j_k^-(r)=j_k^+(-r)$ and $h(x,-\xi)=h(x,\xi)$, we can rewrite the last inequlity as

$$\begin{aligned}
&-\int_0^T\int_\Omega j_k^+(u(t,x)-\overline{u}(s,y))(\eta_n)_s\,dyds\\
&+\int_0^T\int_\Omega \eta_n h(y,D_yT_k^+(u(t,x)-\overline{u}(s,y)))\,ds\\
&+\int_0^T\int_\Omega (z(t,x)-\overline{z}(s,y))\cdot\nabla_y\eta_n\;T_k^+(u(t,x)-\overline{u}(s,y))\,dyds\\
&-\int_0^T\int_\Omega z(t,x)\cdot\nabla_y\eta_nT_k^+(u(t,x)-\overline{u}(s,y))\,dyds\le 0.
\end{aligned}\tag{128}$$

Integrating (127) in (s, y), (128) in (t, x) and taking their sum yields

$$
\begin{aligned}
&-\int_{Q_T\times Q_T} j_k^+(u(t,x)-\overline{u}(s,y))\big((\eta_n)_t+(\eta_n)_s\big)\\
&+\int_{Q_T\times Q_T} \eta_n h(x, D_x T_k^+(u(t,x)-\overline{u}(s,y)))\\
&+\int_{Q_T\times Q_T} \eta_n h(y, D_y T_k^+(u(t,x)-\overline{u}(s,y)))\\
&+\int_{Q_T\times Q_T} \big(z(t,x)-\overline{z}(s,y)\big)\cdot\big(\nabla_x\eta_n+\nabla_y\eta_n\big)T_k^+(u(t,x)-\overline{u}(s,y))\\
&+\int_{Q_T\times Q_T} \overline{z}(s,y)\cdot\nabla_x\eta_n T_k^+(u(t,x)-\overline{u}(s,y))\\
&-\int_{Q_T\times Q_T} z(t,x)\cdot\nabla_y\eta_n T_k^+(u(t,x)-\overline{u}(s,y))\le 0.
\end{aligned}
\tag{129}
$$

Now, by Green's formula and the identities $z(t,x)=a(x,\nabla u(t,x))$, $\overline{z}(s,y)=a(y,\nabla\overline{u}(s,y))$, we have

$$
\begin{aligned}
J_n := &\int_{Q_T\times Q_T} \overline{z}(s,y)\cdot\nabla_x\eta_n\; T_k^+(u(t,x)-\overline{u}(s,y))\\
&+\int_{Q_T\times Q_T} \eta_n h(x, D_x T_k^+(u(t,x)-\overline{u}(s,y)))\\
&-\int_{Q_T\times Q_T} z(t,x)\cdot\nabla_y\eta_n\; T_k^+(u(t,x)-\overline{u}(s,y))\\
&+\int_{Q_T\times Q_T} \eta_n h(y, D_y T_k^+(u(t,x)-\overline{u}(s,y)))\\
=&-\int_{Q_T\times Q_T} \eta_n\big(\overline{z}(s,y), D_x T_k^+(u(t,x)-\overline{u}(s,y))\big)\\
&+\int_{Q_T\times Q_T} \eta_n h(x, D_x T_k^+(u(t,x)-\overline{u}(s,y)))\\
&+\int_{Q_T\times Q_T} \eta_n\big(z(t,x), D_y T_k^+(u(t,x)-\overline{u}(s,y))\big)\\
&+\int_{Q_T\times Q_T} \eta_n h(y, D_y T_k^+(u(t,x)-\overline{u}(s,y)))
\end{aligned}
$$

$$= \int_{Q_T\times Q_T} \eta_n (T_k^+)'(u(t,x)-\overline{u}(s,y))[z(t,x)-\overline{z}(s,y)]\cdot(\nabla_x u(t,x)-\nabla_y \overline{u}(s,y))$$
$$- \int_{Q_T\times Q_T} \eta_n \overline{z}(s,y)\cdot D_x^s T_k^+(u(t,x)-\overline{u}(s,y))$$
$$+ \int_{Q_T\times Q_T} \eta_n h^0(x, D_x^s T_k^+(u(t,x)-\overline{u}(s,y)))$$
$$+ \int_{Q_T\times Q_T} \eta_n z(t,x)\cdot D_y^s T_k^+(u(t,x)-\overline{u}(s,y))$$
$$+ \int_{Q_T\times Q_T} \eta_n h^0(y, D_y^s T_k^+(u(t,x)-\overline{u}(s,y))).$$

We claim that

$$J_n \geq o(1), \tag{130}$$

where $o(1)$ is an expression that tends to 0 as $n \to \infty$. In fact: Let us analyze the term

$$- \int_{Q_T\times Q_T} \eta_n \overline{z}(s,y)\cdot D_x^s T_k^+(u(t,x)-\overline{u}(s,y))$$
$$+ \int_{Q_T\times Q_T} \eta_n h^0(x, D_x^s T_k^+(u(t,x)-\overline{u}(s,y))).$$

By assumption (H_5) we have

$$- \int_{Q_T\times Q_T} \eta_n \overline{z}(s,y)\cdot D_x^s T_k^+(u(t,x)-\overline{u}(s,y))$$
$$+ \int_{Q_T\times Q_T} \eta_n h^0(x, D_x^s T_k^+(u(t,x)-\overline{u}(s,y)))$$
$$\geq \int_{Q_T\times Q_T} \eta_n h^0(x, D_x^s T_k^+(u(t,x)-\overline{u}(s,y)))$$
$$- \int_{Q_T\times Q_T} \eta_n h^0(y, D_x^s T_k^+(u(t,x)-\overline{u}(s,y))).$$

Let us prove that the term in the right hand side tends to 0 as $n \to \infty$. Let $l = k + \|\overline{u}\|_\infty$. Indeed, using (31), and having in mind that

$$|D_x^s T_k^+(u(t,x)-\overline{u}(s,y))| = |D_x^s T_k^+(T_l(u(t,x))-\overline{u}(s,y))|$$
$$\leq |D_x^s(T_l(u(t,x))-\overline{u}(s,y))| = |D_x^s T_l(u(t,x))|,$$

we have

$$\left| \int_{Q_T\times Q_T} \eta_n h^0(x, D_x^s T_k^+(u(t,x)-\overline{u}(s,y)))\right.$$

$$-\int_{Q_T\times Q_T}\eta_n h^0(y, D_x^s T_k^+(u(t,x)-\overline{u}(s,y)))\Bigg|$$
$$\le \int_{Q_T\times Q_T}\eta_n\omega(\|x-y\|)|D_x^s T_k^+(u(t,x)-\overline{u}(s,y))|$$
$$\le \int_{Q_T} dy\int_{Q_T} dx\eta_n\omega(\|x-y\|)|D_x^s T_l(u(t,x))|$$
$$= \int_{Q_T} dx|D_x^s T_l(u(t,x))|\int_{Q_T} dy\eta_n\omega(\|x-y\|).$$

Now, we observe that

$$\int_{Q_T} dy\eta_n\omega(\|x-y\|)$$
$$=\int_0^T \tilde{\rho}_n(t-s)\phi\left(\frac{t+s}{2}\right)ds\int_\Omega dy\rho_n(x-y)\psi\left(\frac{x+y}{2}\right)\omega(\|x-y\|)$$

and

$$\left|\int_\Omega \rho_n(x-y)\psi\left(\frac{x+y}{2}\right)\omega(\|x-y\|)dy\right|$$
$$\le \|\psi\|_\infty\int_{\mathbb{R}^N}\rho_n(x-y)\omega(\|x-y\|)dy$$
$$=\|\psi\|_\infty\int_{\mathbb{R}^N}\rho_n(z)\omega(\|z\|)dz\to 0,\quad \text{as } n\to\infty.$$

Hence

$$\int_{Q_T\times Q_T}\eta_n h^0(x, D_x^s T_k^+(u(t,x)-\overline{u}(s,y)))$$
$$-\int_{Q_T\times Q_T}\eta_n h^0(y, D_x^s T_k^+(u(t,x)-\overline{u}(s,y)))\to 0,\quad \text{as } n\to\infty.$$

In a similar way, we prove that

$$\int_{Q_T\times Q_T}\eta_n z(t,x)\cdot D_y^s T_k^+(u(t,x)-\overline{u}(s,y))$$
$$+\int_{Q_T\times Q_T}\eta_n h^0(y, D_y^s T_k^+(u(t,x)-\overline{u}(s,y)))\ge o(1).$$

Let us prove that

$$\int_{Q_T\times Q_T}\eta_n (T_k^+)'(u(t,x)-\overline{u}(s,y))[z(t,x)-\overline{z}(s,y)]$$
$$\cdot(\nabla_x u(t,x)-\nabla_y\overline{u}(s,y))\ge o(1).$$

Indeed, let $\overline{z}(s,x,y) = a(x, \nabla_y \overline{u}(s,y))$, then

$$\int_{Q_T\times Q_T} \eta_n (T_k^+)'(u(t,x)-\overline{u}(s,y))[z(t,x)-\overline{z}(s,y)] \cdot(\nabla_x u(t,x) - \nabla_y \overline{u}(s,y))$$
$$= \int_{Q_T\times Q_T} \eta_n (T_k^+)'(u(t,x)-\overline{u}(s,y))[z(t,x)-\overline{z}(s,x,y)] \cdot(\nabla_x u(t,x) - \nabla_y \overline{u}(s,y))$$
$$+ \int_{Q_T\times Q_T} \eta_n (T_k^+)'(u(t,x)-\overline{u}(s,y))[\overline{z}(s,x,y)-\overline{z}(s,y)] \cdot(\nabla_x u(t,x) - \nabla_y \overline{u}(s,y))$$
$$\geq \int_{Q_T\times Q_T} \eta_n (T_k^+)'(u(t,x)-\overline{u}(s,y))[\overline{z}(s,x,y)-\overline{z}(s,y)] \cdot(\nabla_x u(t,x) - \nabla_y \overline{u}(s,y))$$
$$= \int_{Q_T\times Q_T} \eta_n [\overline{z}(s,x,y)-\overline{z}(s,y)]\cdot \nabla_x T_k^+(u(t,x)-\overline{u}(s,y))$$
$$+ \int_{Q_T\times Q_T} \eta_n [\overline{z}(s,x,y)-\overline{z}(s,y)]\cdot \nabla_y T_k^+(u(t,x)-\overline{u}(s,y)).$$

Now, we observe that, using the same argument as above, both terms in the right hand side in the last inequality tend to zero as $n\to\infty$. With this we finish the proof of (130).

From (130) and (129), it follows that

$$\begin{aligned} &-\int_{Q_T\times Q_T} j_k^+(u(t,x)-\overline{u}(s,y))\big((\eta_n)_t+(\eta_n)_s\big) \\ &+\int_{Q_T\times Q_T} \big(z(t,x)-\overline{z}(s,y)\big)\cdot\big(\nabla_x\eta_n+\nabla_y\eta_n\big) T_k^+(u(t,x)-\overline{u}(s,y)) \\ &\leq o(1). \end{aligned} \tag{131}$$

Since,

$$(\eta_n)_t + (\eta_n)_s = \rho_n(x-y)\tilde{\rho}_n(t-s)\phi'\left(\frac{t+s}{2}\right)\psi\left(\frac{x+y}{2}\right)$$

and

$$\nabla_x\eta_n + \nabla_y\eta_n = \rho_n(x-y)\tilde{\rho}_n(t-s)\phi\left(\frac{t+s}{2}\right)\nabla\psi\left(\frac{x+y}{2}\right),$$

passing to the limit in (131), it yields

$$
\begin{aligned}
&-\int_{Q_T} j_k^+(u(t,x)-\overline{u}(t,x))\phi'(t)\psi(x) \\
&+\int_{Q_T} \big(z(t,x)-\overline{z}(t,x)\big)\cdot\nabla\psi(x)\,\phi(t)T_k^+(u(t,x)-\overline{u}(t,x)) \leq 0.
\end{aligned}
\tag{132}
$$

We have to prove that

$$
\lim_n \int_{Q_T} \big(z(t,x)-\overline{z}(t,x)\big)\cdot\nabla\psi_n(x)\,\phi(t)T_k^+(u(t,x)-\overline{u}(t,x)) \geq 0
$$

for any sequence $\psi_n \uparrow 1\!\!1_\Omega$. Since

$$
\xi = div(z),\;\; \overline{\xi} = div(\overline{z}) \;\text{ in }\; \big(L^1(0,T,BV(\Omega)_2)\big)^*,
$$

the following integration by parts formula holds

$$
\begin{aligned}
&\int_{Q_T}(z-\overline{z},Dw) + \int_0^T <\xi(t)-\overline{\xi}(t), w(t)> dt \\
&= \int_0^T \int_{\partial\Omega}[z(t,x)-\overline{z}(t,x),\nu]w(t,x)dH^{N-1}dt
\end{aligned}
$$

for all $w \in L^1(0,T,BV(\Omega))\cap L^\infty(Q_T)$. Set

$$
w(t) = ((\psi-1)\phi T_k^+(u-\overline{u}))^\tau(t,x) = (\psi(x)-1)(\phi T_k^+(u-\overline{u}))^\tau(t,x),
$$

where $\psi \in \mathcal{D}(\Omega)$, $\quad 0\leq\psi\leq 1$ and

$$
(\phi T_k^+(u-\overline{u}))^\tau(t,x) = \frac{1}{\tau}\int_t^{t+\tau} \phi(s)T_k^+(u(s,x)-\overline{u}(s,x))ds,
$$

in the above formula to obtain

$$
\begin{aligned}
&\int_{Q_T}\big(z(t,x)-\overline{z}(t,x)\big)\cdot\nabla(\psi(x)-1)\;(\phi T_k^+(u-\overline{u}))^\tau(t,x)dxdt \\
&= -\int_0^T\int_\Omega(\psi-1)\big(z(t)-\overline{z}(t), D(\phi T_k^+(u-\overline{u}))^\tau(t)\big)dt \\
&-\int_{Q_T}(\xi(t)-\overline{\xi}(t))(\psi(x)-1)(\phi T_k^+(u-\overline{u}))^\tau(t,x) \\
&+\int_0^T\int_{\partial\Omega}[z(t,x)-\overline{z}(t,x),\nu](\psi(x)-1)(\phi T_k^+(u-\overline{u}))^\tau(t,x)dH^{N-1}dt.
\end{aligned}
$$

Since

$$
\int_{Q_T}\big(z-\overline{z}\big)\cdot\nabla(\psi-1)\phi T_k^+(u-\overline{u})dxdt
$$

$$= \lim_{\tau \to 0+} \int_{Q_T} \left(z - \overline{z}\right) \cdot \nabla(\psi - 1)\ (\phi T_k^+(u - \overline{u}))^\tau dxdt,$$

and, using that $\psi|_{\partial\Omega} = 0$, also

$$-\int_0^T \int_{\partial\Omega} [z - \overline{z}, \nu] \phi T_k^+(u - \overline{u}) dH^{N-1} dt$$

$$= \lim_{\tau \to 0+} \int_0^T \int_{\partial\Omega} [z - \overline{z}, \nu](\psi - 1)(\phi T_k^+(u - \overline{u}))^\tau dH^{N-1} dt,$$

we may write

$$\int_{Q_T} (z - \overline{z}) \nabla\psi \phi T_k^+(u - \overline{u}) = \int_{Q_T} (z - \overline{z}) \nabla(\psi - 1) \phi T_k^+(u - \overline{u})$$

$$= \lim_{\tau \to 0+} \int_0^T \int_{\Omega} (\psi - 1)\big(z(t) - \overline{z}(t), D(\phi T_k^+(u - \overline{u}))^\tau(t)\big) dt$$

$$+ \int_{Q_T} (\xi - \overline{\xi})(1 - \psi)(\phi T_k^+(u - \overline{u}))^\tau) - \int_0^T \int_{\partial\Omega} [z - \overline{z}, \nu] \phi T_k^+(u - \overline{u}) dH^{N-1} dt.$$

Since $\xi, \overline{\xi}$ are the time derivatives of u, resp. $\overline{u}$, in $\left(L^1(0, T, BV(\Omega)_2)\right)^*$, we have that

$$\int_0^T \int_{\Omega} (\xi - \overline{\xi})(1 - \psi)(\phi T_k^+(u - \overline{u}))^\tau = \int_0^T \int_{\Omega} (\xi - \overline{\xi})((1 - \psi)\phi T_k^+(u - \overline{u}))^\tau$$

$$= \int_0^T \int_{\Omega} (1 - \psi)\phi T_k^+(u - \overline{u}) \frac{1}{\tau} \Delta_\tau^-(u - \overline{u}),$$

where $\Delta_\tau^-(u - \overline{u}) = (u - \overline{u})(t) - (u - \overline{u})(t - \tau)$. Let $v = u - \overline{u}$. Since

$$T_k^+(v(t))(v(t) - v(t - \tau)) \geq J_{T_k^+}(v(t)) - J_{T_k^+}(v(t - \tau))$$

($J_{T_k^+}$ being the primitive of T_k^+), and ϕ, $(1 - \psi) \geq 0$, we have for τ small enough that

$$\int_0^T \int_{\Omega} (\xi - \overline{\xi})(1 - \psi)(\phi T_k^+(u - \overline{u}))^\tau$$

$$\geq \int_0^T \int_{\Omega} (1 - \psi)\phi \frac{J_{T_k^+}(v(t)) - J_{T_k^+}(v(t - \tau))}{\tau}$$

$$= -\int_0^T \int_{\Omega} \frac{\phi(t + \tau) - \phi(t)}{\tau} (1 - \psi) J_{T_k^+}(u - \overline{u}).$$

Thus, we have

$$\int_{Q_T} (z - \overline{z}) \nabla\psi \phi T_k^+(u - \overline{u})$$

$$\geq \lim_{\tau\to 0+}\left(\int_0^T\int_\Omega(\psi-1)\big(z(t)-\overline{z}(t), D(\phi T_k^+(u-\overline{u}))^\tau(t)\big)dt\right.$$
$$\left.-\int_0^T\int_\Omega\frac{\phi(t+\tau)-\phi(t)}{\tau}(1-\psi)J_{T_k^+}(u-\overline{u})\right)$$
$$-\int_0^T\int_{\partial\Omega}[z-\overline{z},\nu]\phi T_k^+(u-\overline{u})dH^{N-1}dt.$$

Finally, we observe that

$$\lim_{\tau\to 0+}\left|\int_0^T\int_\Omega(\psi-1)\big(z(t)-\overline{z}(t), D(\phi T_k^+(u-\overline{u}))^\tau(t)\big)dt\right|$$
$$\leq 2M\int_{Q_T}(1-\psi)\phi\|DT_k^+(u-\overline{u})\|dxdt,$$

which enables us to write that

$$\int_{Q_T}(z-\overline{z})\nabla\psi\phi T_k^+(u-\overline{u})\geq -2M\int_{Q_T}(1-\psi)\phi\|DT_k^+(u-\overline{u})\|dxdt$$
$$-\int_0^T\int_\Omega\phi'(t)(1-\psi)J_{T_k^+}(u-\overline{u})-\int_0^T\int_{\partial\Omega}[z-\overline{z},\nu]\phi T_k^+(u-\overline{u})dH^{N-1}dt.$$

Let $\psi=\psi_n$ where $\psi_n\uparrow 1\!\!1_\Omega$ in the last expression. Using that

$$\|DT_k^+(u(t)-\overline{u}(t))\|$$

is a Radon measure a.e. in t with $\|DT_k^+(u(t)-\overline{u}(t))\|\in L^1(0,T)$, which follows from Lemma 5, letting $n\to\infty$, we obtain

$$\lim_n\int_{Q_T}(z-\overline{z})\nabla\psi_n\phi T_k^+(u-\overline{u})$$
$$\geq -\int_0^T\int_{\partial\Omega}[z-\overline{z},\nu]\phi T_k^+(u-\overline{u})dH^{N-1}dt.$$

Thus, using (132), we get

$$\int_{Q_T} j_k^+(u(t,x)-\overline{u}(t,x))\phi'(t)$$
$$\geq -\int_0^T\int_{\partial\Omega}[z-\overline{z},\nu]\phi T_k^+(u-\overline{u})dH^{N-1}dt\geq 0.$$

Since this is true for all $0\leq\phi\in\mathcal{D}(]0,T[)$, we have

$$\frac{d}{dt}\int_\Omega j_k^+(u(t,x)-\overline{u}(t,x))\leq 0.$$

Hence
$$\int_\Omega j_k^+(u(t,x)-\overline{u}(t,x)) \le \int_\Omega j_k^+(u_0-\overline{u}_0).$$
Then, letting $k\to 0$, we obtain
$$\int_\Omega (u(t,x)-\overline{u}(t,x))^+ \le \int_\Omega (u_0-\overline{u}_0)^+.$$
From this, we deduce that
$$\|u(t)-\overline{u}(t)\|_1 \le \|u_0-\overline{u}_0\|_1, \quad \forall\, t\ge 0.$$
Hence, taking $u_n(t)=T(t)u_{0,n}$, $u_{0,n}\in L^\infty(\Omega)$ and $u_{0,n}\to u_0$ in $L^1(\Omega)$, we have
$$\|u(t)-u_n(t)\|_1 \le \|u_0-u_{0,n}\|_1, \quad \forall\, t\ge 0.$$
Consequently, letting $n\to\infty$, we obtain that $u(t)=T(t)u_0$. Then we have that entropy solutions coincide with semigroup solutions. This proves the uniqueness of entropy solutions and concludes the proof. □

6. Appendix

It is well known that if $u_n, u\in BV(\Omega)$, satisfy: $u_n\to u$ in $L^1(\Omega)$, $\|Du_n\|\to\|Du\|$ and $\nabla u_n\to F$ a.e in Ω, then F does not coincide with ∇u, in general, as the following example shows. Consider $\Omega=]0,1[$ and $u_n\in BV(\Omega)$, defined by
$$u_n := \sum_{i=1}^n \frac{i}{n}\,\mathbb{1}_{[\frac{i-1}{n},\frac{i}{n}[}.$$
Then, $u_n(x)\to u(x)=x$ for almost all $x\in\Omega$, but $\nabla u_n=0$ for all $n\in\mathbb{N}$, and $\nabla u=1$.

Now, in the proof of Theorem 4, we have seen that if $(u_n,v_n)\in\mathcal{B}_\varphi$ is such that $(u_n,v_n)\to(u,v)$ in $L^1(\Omega)\times L^1(\Omega)$, then $(u,v)\in\mathcal{A}_\varphi$. Thus, a natural question is when $\nabla u_n\to\nabla u$ a.e. in Ω. We are going to prove that, if we assume that the Lagrangian f is strictly convex, then the answer to this question is positive.

Firstly, observe that from the strict convexity of f, we deduce the following strict monotonicity condition on **a**:
$$(\mathbf{a}(x,\eta)-\mathbf{a}(x,\xi))\cdot(\eta-\xi)>0 \quad \text{if } \xi\ne\eta. \tag{133}$$

Let us prove that $\{\nabla u_n\}$ is a Cauchy sequence in measure. To do that, we follow the same technique as in [14]. Let $t,\epsilon>0$. For $a>1$, we set
$$C(x,a,t):=$$

$$\inf\{(\mathbf{a}(x,\xi)-\mathbf{a}(x,\eta))\cdot(\xi-\eta)\ :\ \|\xi\|\le a,\ \|\eta\|\le a,\ \|\xi-\eta\|\ge t\}.$$

Having in mind that the function $\xi \mapsto \mathbf{a}(x,\xi)$ is continuos for almost all $x \in \Omega$, and the set $\{(\xi,\eta)\ :\ \|\xi\|\le a,\ \|\eta\|\le a,\ \|\xi-\eta\|\ge t\}$ is compact, the infimum in the definition of $C(x,a,t)$ is a minimum. Hence by (133), it follows that

$$C(x,a,t) > 0 \quad \text{for almost all } \ x \in \Omega. \tag{134}$$

For $n, m \in \mathbb{N}$, and any $k > 0$, we have

$$\begin{aligned}
&\{\|\nabla u_n - \nabla u_m\| > t\} \subset \{\|\nabla T_a u_n\| \ge a^2\} \cup \{\|\nabla T_a u_m\| \ge a^2\} \\
&\cup\{|u_n| \ge a\} \cup \{|u_m| \ge a\} \cup \{|u_n - u_m| \ge k^2\} \cup \{C(x,a^2,t) \le k\} \\
&\cup\{|u_n - u_m| < k^2, |u_n| < a, |u_m| < a, C(x,a^2,t) \ge k, \\
&\|\nabla T_a u_n\| \le a^2, \|\nabla T_a u_m\| \le a^2, \|\nabla u_n - \nabla u_m\| > t\}.
\end{aligned} \tag{135}$$

Since $\{u_n\}$ is bounded in $L^1(\Omega)$ we can choose a large enough in order to have

$$\lambda_N\big(\{|u_n| \ge a\} \cup \{|u_m| \ge a\}\big) \le \frac{\epsilon}{5} \quad \forall\, n, m \in \mathbb{N}. \tag{136}$$

Similarly, by (62), we can choose a large enough in order to have

$$\lambda_N\big(\{\|\nabla T_a u_n\| \ge a^2\} \cup \{\|\nabla T_a u_m\| \ge a^2\}\big) \le \frac{\epsilon}{5} \ \ \forall\, n, m \in \mathbb{N}. \tag{137}$$

Fixing a satisfying (136) and (137), by (134), taking k small enough, we have

$$\lambda_N\big(\{C(x,a^2,t) \le k\}\big) \le \frac{\epsilon}{5}. \tag{138}$$

On the other hand, since $v_n = -div\,\mathbf{a}(x,\nabla u_n)$, using Green's formula, we have

$$\begin{aligned}
&\int_\Omega (\mathbf{a}(x,\nabla u_n) - \mathbf{a}(x,\nabla u_m), DT_r(u_n - u_m)) \\
&= \int_\Omega (v_n - v_m) T_r(u_n - u_m)\, dx \\
&+ \int_{\partial\Omega} [\mathbf{a}(x,\nabla u_n) - \mathbf{a}(x,\nabla u_m), \nu] T_r(u_n - u_m)\, dH^{N-1} \le 2Qr,
\end{aligned}$$

for all $n, m \in \mathbb{N}$. Now,

$$\begin{aligned}
&\int_\Omega (\mathbf{a}(x,\nabla u_n) - \mathbf{a}(x,\nabla u_m), DT_r(u_n - u_m)) \\
&= \int_\Omega (\mathbf{a}(x,\nabla u_n) - \mathbf{a}(x,\nabla u_m)) \cdot \nabla T_r(u_n - u_m)\, dx
\end{aligned}$$

$$+\int_\Omega (\mathbf{a}(x,\nabla u_n)-\mathbf{a}(x,\nabla u_m))\cdot D^s T_r(u_n-u_m).$$

Moreover, by chain's rule, there exists a positive function η such that

$$\begin{aligned}&\int_\Omega (\mathbf{a}(x,\nabla u_n)-\mathbf{a}(x,\nabla u_m))\cdot D^s T_r(u_n-u_m)\\ &=\int_\Omega \eta[(\mathbf{a}(x,\nabla u_n)-\mathbf{a}(x,\nabla u_m))\cdot D^s(u_n-u_m)]\\ &=\int_\Omega \eta[f^0(x,D^s u_n)-\mathbf{a}(x,\nabla u_m)\cdot D^s u_n\\ &\quad +f^0(x,D^s u_m)-\mathbf{a}(x,\nabla u_m)\cdot D^s u_m]\geq 0,\end{aligned}$$

by (H_5). Therefore, we obtain

$$\int_\Omega (\mathbf{a}(x,\nabla u_n)-\mathbf{a}(x,\nabla u_m))\cdot \nabla T_r(u_n-u_m)\,dx\leq 2Qr. \tag{139}$$

If

$$\begin{aligned}S:=\{&|u_n-u_m|<k^2, |u_n|<a, |u_m|<a, C(x,a^2,t)\geq k,\\ &\|\nabla T_a u_n\|\leq a^2, \|\nabla T_a u_m\|\leq a^2, \|\nabla u_n-\nabla u_m\|>t\},\end{aligned}$$

since $\nabla T_a u_n=\nabla u_n$ a.e in S, by (139), we get

$$\begin{aligned}&\lambda_N(S)\\ &\leq \lambda_n\{|u_n-u_m|<k^2, (\mathbf{a}(x,\nabla u_n)-\mathbf{a}(x,\nabla u_m))\cdot(\nabla u_n-\nabla u_m)\geq k\}\\ &\leq \frac{1}{k}\int_{|u_n-u_m|<k^2}(\mathbf{a}(x,\nabla u_n)-\mathbf{a}(x,\nabla u_m))\cdot(\nabla u_n-\nabla u_m)\,dx\leq 2Qk.\end{aligned}$$

Hence, for k small enough, we have

$$\lambda_N(S)\leq \frac{\epsilon}{5}. \tag{140}$$

Since a and k have already been choosen, if n_0 is large enough, we have for $n,m\geq n_0$ the estimate $\lambda_n(\{|u_n-u_m|\geq k^2\})\leq \frac{\epsilon}{5}$. Now, using (135), (136), (137), (138) and (140), it follows that

$$\lambda_N(\{\|\nabla u_n-\nabla u_m\|>t\})\leq \epsilon \quad \text{for } n,m\geq n_0.$$

Consequently, $\{\nabla u_n\}$ is a Cauchy sequence in measure. Then, up to extraction of a subsequence, we have convergence a.e., and we can say that there exists a measurable function F, such that

$$\nabla u_n\to F \quad \text{a.e. in } \Omega. \tag{141}$$

Now, $\mathbf{a}(x,\nabla u_n)\rightharpoonup \mathbf{a}(x,\nabla u)$ in the weak* topology of $L^\infty(\Omega,\mathbb{R}^N)$, and by (141), $\mathbf{a}(x,\nabla u_n)\to \mathbf{a}(x,F)$ a.e. in Ω. Hence, $\mathbf{a}(x,F)=\mathbf{a}(x,\nabla u)$ a.e. in Ω. Therefore, by (133), we deduce that

$$\nabla u_n\to \nabla u \quad \text{a.e. in } \Omega.$$

Remark 4. Coming back to the observation made at the beginning of this section, the only thing we can expect is that there exists $\lambda : \Omega \to \mathbb{R}$, $0 \le \lambda(x) \le 1$ a.e., such that $F = \lambda(x)\nabla u(x)$ a.e.. Indeed, we have: if μ_k, μ are vector measures in Ω (with values in $\mathbb{R}^N$) such that $\mu_k \to \mu$, $|\mu_k| \to |\mu|$ weakly* as measures in Ω, and $\mu_k^{ac} \to F$ in measure in Ω, then there is $\lambda : \Omega \to \mathbb{R}$, $0 \le \lambda(x) \le 1$ a.e., such that $F = \lambda(x)\mu^{ac}(x)$ a.e.. This can be proved using Reshetnyak's Theorem ([23], p. 90, Thm. 19). Indeed, Reshetnyak's Theorem implies that

$$\int_\Omega N_u(x, \mu_k)\phi \to \int_\Omega N_u(x, \mu)\phi$$

for any $\phi \in C_0(\Omega)$, where $C_0(\Omega)$ denotes the space of continuous functions with compact support in Ω, and $N_u^+(x, v) = (< u, v >)^+$, $u \in \mathbb{R}^N$. Now, for any $k > 0$ we have

$$\begin{aligned}
\int_\Omega (N_u(x, \mu) \wedge k)\phi &= \int_\Omega N_u(x, \mu)\phi - \int_\Omega (N_u(x, \mu) - k)^+\phi \\
&\ge \lim_n \int_\Omega N_u(x, \mu_n)\phi - \liminf_n \int_\Omega (N_u(x, \mu_n) - k)^+\phi \\
&\ge \liminf_n \int_\Omega (N_u(x, \mu_n) \wedge k)\phi = \liminf_n \int_\Omega (N_u(x, \mu_n^{ac}) \wedge k)\phi \\
&= \int_\Omega N_u(x, F)\phi.
\end{aligned}$$

Since

$$\int_\Omega (N_u(x, \mu) \wedge k)\phi = \int_\Omega (N_u(x, \mu^{ac}) \wedge k)\phi$$

and the previous inequality holds for any $k \in \mathbb{R}$, any $u \in \mathbb{R}^N$, and any $\phi \in C_0(\Omega)$, and all these spaces are separable we obtain that

$$N_u(x, F(x)) \le N_u(x, \mu^{ac}(x))$$

for all $x \in Q$, where $\lambda_N(\Omega \setminus Q) = 0$, and all $u \in \mathbb{R}^N$. Now, we observe that if $v, w \in \mathbb{R}^N$ are such that

$$(< u, v >)^+ \le (< u, w >)^+ \tag{142}$$

for all $u \in \mathbb{R}^N$, then there is $\lambda \in [0, 1]$ such that $v = \lambda w$. If we fix $x \in Q$, applying the last observation, we conclude that there is $\lambda(x) \in [0, 1]$ such that $F(x) = \lambda(x)\mu^{ac}(x)$. These observations can be used to prove that there exists $\lambda : \Omega \to \mathbb{R}$, $0 \le \lambda(x) \le 1$ a.e., such that $F = \lambda(x)\nabla u(x)$ a.e., once we know that $\nabla u_n(x) \to F(x)$ in measure. Then, we need structural assumptions on $\mathbf{a}(x, \xi)$ to obtain either that $\mathbf{a}(x, F) = \mathbf{a}(x, \nabla u)$, or $F(x) = \nabla u(x)$. Since, to prove that $\nabla u_n(x) \to F(x)$ in measure, we need to use the strict convexity of f, and this also gives that $F = \nabla u$, we do not need the more involved approach of this remark.

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