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On singular fibres of Lagrangian fibrations over holomorphic symplectic manifolds

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Abstract. We classify singular fibres over general points of the discriminant locus of projective Lagrangian fibrations over 4-dimensional holomorphic symplectic manifolds. The singular fibre F is the following either one: F is isomorphic to the product of an elliptic curve and a Kodaira singular fibre up to finite unramified covering or F is a normal crossing variety consisting of several copies of a minimal elliptic ruled surface of which the dual graph is Dynkin diagram of type A_n, $\tilde{A_n}$ or $\tilde{D_n}$. Moreover, we show all types of the above singular fibres actually occur.

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1. Introduction

We begin with the definition of *Lagrangian fibrations*.

Definition 1.1. *Let (X, ω) be a Kähler manifold with a d-closed holomorphic symplectic two form ω and S a normal variety. A proper surjective morphism $f : X \to S$ is said to be a Lagrangian fibration if a general fibre F of f is a Lagrangian submanifold with respect to ω, that is, the restriction of 2-form $\omega|_F$ is identically zero and* $\dim F = (1/2) \dim X$.

Remark. A general fibre F of a Lagrangian fibration is a complex torus by Liouville's theorem.

The plainest example of a Lagrangian fibration is an elliptic fibration of $K3$ surface over $\mathbb{P}^1$. In higher dimension, every fibre space of a projective irreducible symplectic manifold is a Lagrangian fibration ([7, Theorem 2] and [8, Theorem 1]). When the dimension of fibre is one, a Lagrangian fibration is a minimal elliptic fibration and whose singular fibre is completely classified by Kodaira [6, Theorem 6.2]. In this note, we investigate singular fibres of a projective Lagrangian fibration whose fibre is 2-dimensional.

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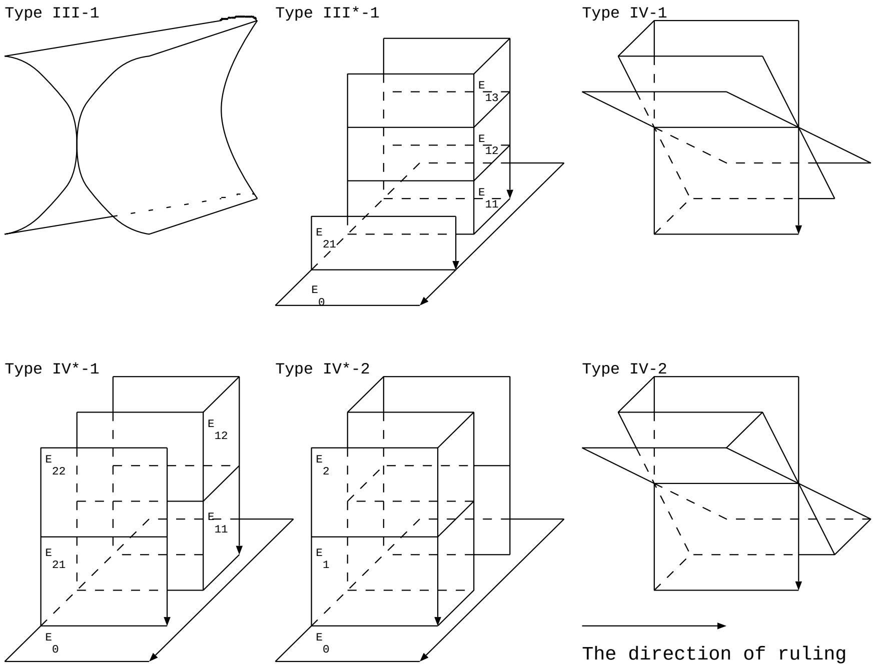

Fig. 1. Figures of singular fibres of type III or type IV

Theorem 1.2. *Let $f : X \to S$ be a Lagrangian fibration and D the discriminant locus of f. Assume that f is projective and* $\dim X = 4$. *Then there exists finite set D_0 of D and $f^{-1}(x)$ is the following one for $x \in D \setminus D_0$:*

(1) *There is a morphism from $f^{-1}(x)$ to an elliptic curve C and an étale morphism $\tilde{C} \to C$ such that $f^{-1}(x) \times_C \tilde{C}$ is isomorphic to the product of an elliptic curve $\tilde{C}$ and a Kodaira singular fibre of type I_0, I_0^*, II, II^*, III, III^*, IV or IV^*. Such a $f^{-1}(s)$ is classified as 18 types (see Tables 4, 5).*

(2) *$f^{-1}(x)$ is isomorphic to a normal crossing variety consisting of several copies of a minimal elliptic ruled surface. The dual graph of $f^{-1}(x)$ is the Dynkin diagram of type A_n, $\tilde{A}_n$ or $\tilde{D}_n$. If the dual graph is of type $\tilde{A}_n$ or $\tilde{D}_n$, each double curve is a section of the ruling. In the other cases, the double curve on each edge components is a bisection and other double curve is a section (see Figs. 2 and 3).*

Moreover, all types of the above singular fibres actually occur.

Combining Theorem 1.2 with [7, Theorem 2] and [8, Theorem 1], we obtain the following corollary.

Corollary 1.3. *Let $f : X \to B$ be a fibre space of a projective irreducible symplectic manifold. Assume that* $\dim X = 4$. *Then, for a general point x of the*

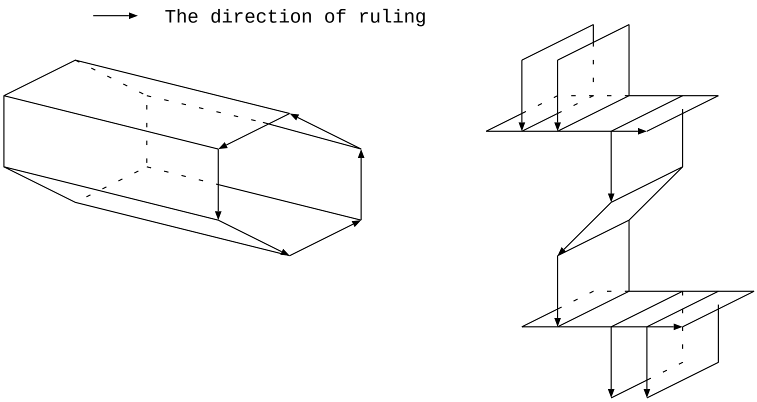

Fig. 2. Figures of $\tilde{A_n}$ and $\tilde{D_n}$ case

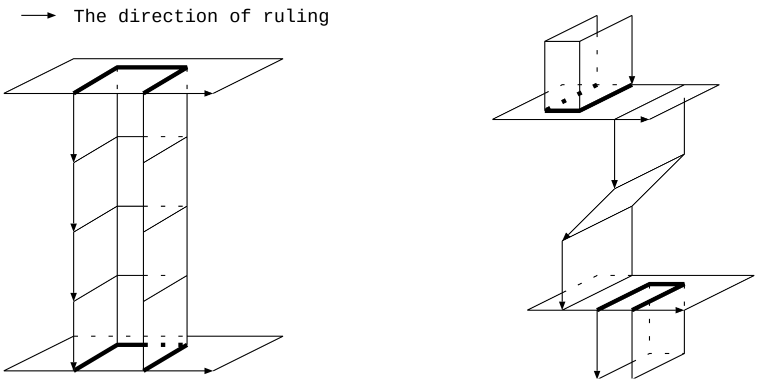

Fig. 3. Figures of A_n case. Bold line represents bisection and line represents section

discriminant locus of f, $f^{-1}(x)$ satisfies the properties of Theorem 1.2 (1) or (2).

Remark. Let S be a $K3$ surface and $\pi : S \to \mathbb{P}^1$ an elliptic fibration. The induced morphism $f : \mathrm{Hilb}^2 S \to \mathbb{P}^2$ gives examples of singular fibres above except whose dual graphs are A_n. The author does not know whether a normal crossing variety whose dual graph is A_n occur as a singular fibre of a fibre space of an irreducible symplectic manifold.

This paper is organized as follows. In section 2, we set up the proof of Theorem 1.2. The key proposition is stated and proved in section 3. Section 4 and 5 are devoted to the proof of the classification of singular fibres. Examples of all types of singular fibres are constructed in Section 6.

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2. Preliminary

(2.1) In this section, we collect definitions and some fundamental material which are necessary for the proof of Theorem 1.2.

Definition 2.1. *Let $f : X \to \Delta^1$ be a proper surjective morphism from algebraic variety to an unit disk. f is said to be semistable degeneration if f satisfies the following two properties:*

(1) *f is smooth over $\Delta^1 \setminus 0$.*
(2) *$f^*\{0\}$ is a reduced normal crossing divisor.*

Definition 2.2. *Let $f : X \to \Delta^1$ and $f' : X' \to \Delta^1$ be proper surjective morphisms from algebraic varieties to unit disks. We call f is isomorphic to f' (resp. birational) if there exists an isomorphism (resp. a birational map) $g : X \to X'$ such that $f' \circ g = f$.*

(2.2) We review the fundamental properties of an Abelian fibration.

Lemma 2.3. *Let $f : (X, \omega) \to S$ be a Lagrangian fibration, F be an irreducible component of a fibre of f and $j : \tilde{F} \to F$ a resolution of F. Assume that f is flat. Then $j^*(\omega|_F)$ is identically zero.*

Proof. Let A be a Kähler form on X. We consider the following function:

$$\lambda(s) := \int_{X_s} \omega \wedge \bar{\omega} A^{\dim S - 2},$$

where $X_s := f^{-1}(s)$ $(s \in S)$. Since f is flat, $\lambda(s)$ is a continuous function on S by [3, Corollary 3.2]. Thus $\lambda(s) \equiv 0$ on S and

$$\int_F \omega \wedge \bar{\omega} A^{\dim S - 2} = 0.$$

Since F and $\tilde{F}$ is birational, $j^*\omega$ is identically zero on $\tilde{F}$. $\square$

(2.3) We review basic properties of the mixed hodge structure on a simple normal crossing variety.

Lemma 2.4. *Let $X := \sum X_i$ be a simple normal crossing variety. Then*

$$F^1 H^1(X, \mathbb{C}) = \{(\alpha_i) \in \oplus H^0(X_i, \Omega^1_{X_i}) | \alpha_i|_{X_i \cap X_j} = \alpha_j|_{X_i \cap X_j}\}.$$

Proof. Let

$$X^{[k]} := \cup_{i_0<\cdots<i_k} X_{i_0} \cap \cdots \cap X_{i_k} \quad \text{(disjoint union)}.$$

For an index set $I = \{i_0, \cdots i_k\}$, we define an inclusion δ_j^I

$$\delta_j^I : X_{i_0} \cap \cdots \cap X_{i_k} \to X_{i_0} \cap \cdots \cap X_{i_{j-1}} \cap X_{i_{j+1}} \cap \cdots X_{i_k} \quad (0 \le j \le k).$$

We consider the following spectral sequence [4, Chapter 4]:

$$E_1^{p,q} = H^q(X^{[p]}, \mathbb{C}) \Longrightarrow E^{p+q} = H^{p+q}(X, \mathbb{C}),$$

where $D : E_1^{p,q} \to E_1^{p+1,q}$ is defined by the

$$\bigoplus_{|I|=p} \sum_{j=0}^{p} (-1)^j (\delta_j^I)^*.$$

Since this spectral sequence degenerates at E_2 level ([4, Chapter 4.8]), we deduce

$$\mathrm{Gr}_1^W(H^1(X, \mathbb{C})) = \mathrm{Ker}(\oplus_i H^1(X_i, \mathbb{C}) \xrightarrow{D} \oplus_{i<j} H^1(X_i \cap X_j, \mathbb{C})).$$

Moreover $F^1 \cap W_0 = 0$, $F^1 H^1(X, \mathbb{C}) = F^1 \mathrm{Gr}_1^W(H^1(X, \mathbb{C}))$. Thus we obtain the assertion of Lemma 2.4 from the definition of D. $\square$

Lemma 2.5. *Let $f : X' \to X$ be a birational morphism between smooth algebraic varieties. Assume that there exists a simple normal crossing divisor Y on X such that f is isomorphic on $X \backslash Y$ and the pull back of of Y of f is a simple normal crossing divisor on X'. Let $Y' := (f^*Y)_{\mathrm{red}}$. Then $F^1H^1(Y', \mathbb{C}) \cong F^1H^1(Y, \mathbb{C})$.*

Proof. We consider the following exact sequence of morphisms of Mixed Hodge structures.

$$\begin{aligned} H^0(Y', \mathbb{C}) &\xrightarrow{\alpha} H^1(X, \mathbb{C}) \to H^1(X', \mathbb{C}) \oplus H^1(Y, \mathbb{C}) \to H^1(Y', \mathbb{C}) \\ &\xrightarrow{\beta} H^2(X, \mathbb{C}) \to \end{aligned}$$

Note that each morphism has weight $(0, 0)$. Since $H^i(X, \mathbb{C})$ carries the pure Hodge structure of weight i, α and β are 0-map. Moreover $F^1H^1(X, \mathbb{C}) \cong F^1H^1(X', \mathbb{C})$. Thus we deduce that $F^1H^1(Y', \mathbb{C}) \cong F^1H^1(Y, \mathbb{C})$. $\square$

3. Kulikov model

(3.1) In this section, we prove the key proposition of the proof of Theorem 1.2. First we refer the the folloing theorem due to Kulikov, Morrison [9, Classification Theorem I] and Persson [11, Proposition 3.3.1].

Theorem 3.1. *Let $g' : T' \to \Delta$ be a semistable degeneration whose general fibre is an abelian surface. Then there exists a semistable degeneration $k : \mathcal{K} \to \Delta$ such that k and g' is birational and $K_{\mathcal{K}} \sim_k 0$. Moreover, exactly one of the following cases occurs:*

(1) *$\mathcal{K}_0$ is an abelian surface.*
(2) *$\mathcal{K}_0$ consists of a cycle of minimal elliptic ruled surfaces, meeting along disjoint sections. The selfintersection number of each double curve is 0.*
(3) *$\mathcal{K}_0$ consists of a collection of rational surfraces, such that the double curves on each component form a cycle of rational curves; the dual graph Γ of $Y_0^{'}$ is a triangulation of $S^1 \times S^1$.*

We call $\mathcal{K}$ a Kulikov model of type I, II or III according to the case occurs (1), (2) or (3).
(3.2) We state the key propositon.

Proposition 3.2. *Let $f : (X, \omega) \to S$ be a projective Lagrangian fibration on 4-dimensional symplectic manifold X and D the discriminant locus of f. Then there exists finite sets D_0 of D which has the following three properties.*

(1) *For a point $x \in D \setminus D_0$, there exists an unit disk Δ^1 on S such that Δ^1 and D intersects transversally at x and $T := X \times_S \Delta^1$ is smooth.*
(2) *$t : T \to \Delta^1$ is birational to the quotient of Kulikov model $\mathcal{K}$ of Type I or Type II by a cyclic group G.*
(3) *There exists a nonzero G-equivariant element of $F^1H^1(\mathcal{K}, \mathbb{C})$.*

(3.3) For the proof of Proposition 3.2, we need the following Lemmas.

Lemma 3.3. *Let $\nu : Y \to X$ be a birational morphism such that $(f \circ \nu)^* D$ is a simple normal crossing divisor. Then there exists finite sets D_0 of D and*

$$F^1H^1(Y_x, \mathbb{C}) \neq 0$$

for all $x \in D \setminus D_0$, where $Y_x := f^{-1}(x)$.

Proof. Let $E := ((f \circ \nu)^* D)_{\mathrm{red}}$ and $E = \sum E_i$. We take an open set U of S which satisfies the following three conditions:

(1) U is smooth.
(2) $D|_U$ is a smooth curve.
(3) $f \circ \nu|_U : (E|_{f^{-1}(U)})^{[k]} \to D|_U$ is a smooth morphism for every k.

Note that $\dim S \setminus U = 0$ since S is normal and $\dim S = 2$. We consider the following exact sequences:

$$0 \to \mathcal{F} \to \Omega^2_{E_i} \to \Omega^2_{E_i/D} \to 0$$

$$0 \to (f \circ \nu)^* \Omega^2_D \to \mathcal{F} \xrightarrow{\alpha} (f \circ \nu)^* \Omega^1_D \otimes \Omega^1_{E_i/D} \to 0$$

Since ω is nondegenerate, $\nu^*\omega \neq 0$ on a non ν-exceptional divisor E_i. By condition (3), $f \circ \nu|_U$ is flat. Therefore the restriction of ω on every irreducible component of a fibre of f is zero by Lemma 2.3 and $\nu^*\omega = 0$ in $\Omega^2_{E_i/D}$. On the contrary, $(f \circ \nu)^*\Omega^2_D = 0$, we deduce $\alpha(\nu^*\omega) \neq 0$ for non ν-exceptional divisor E_i. Thus, for an element of $\partial/\partial t \in H^0((f \circ \nu)^* T_D)$,

$$\nu^*\omega\left(\frac{\partial}{\partial t}, *\right) \neq 0$$

in $H^0(\Omega^1_{E_i/D})$. Hence, for a general point x of $D \cap U$, $H^0(E_{i,x}, \Omega^1_{E_{i,x}}) \neq 0$ where $E_{i,x}$ is the fibre of $E_i \to D$ over x. We denote by α_i the restriction $\nu^*\omega(\partial/\partial t, *)$ to $E_{i,x}$. By the construction, if $E_{i,x} \cap E_{j,x} \neq \emptyset$, $\alpha_i = \alpha_j$ on $E_{i,x} \cap E_{j,x}$. Thus there exists finite sets D_0 of D such that $F^1H^1(Y_x, \mathbb{C}) \neq 0$ for $x \in D \setminus D_0$ from Lemma 2.4. □

Lemma 3.4. *Let $k : \mathcal{K} \to \Delta^1$ be a Kulikov model of type I or type II. Assume that k is birational to a projective abelian fibration $t' : T' \to \Delta^1$. Then*

(1) *k is a projective morphism.*
(2) *Every birational map $\Phi : \mathcal{K} \dashrightarrow \mathcal{K}$ which commutes with k is a birational morphism.*
(3) *If $\mathcal{K}$ is Kulikov model of type II, then every component of the central fibre of $\mathcal{K}$ is isomorphic to each other.*

Proof. (1) Taking the resolution of indeterminancy, we may assume that there is a morphism $\nu : T' \to \mathcal{K}$ such that $k \circ \nu = t'$. Let H' be a t'-ample divisor on T' and $H := \nu_* H'$. Then H is k-big. We will prove that H is k-ample. Since every big divisor on abelian surface is ample, H is k-ample if $\mathcal{K}$ is of type I. In the case that $\mathcal{K}$ is of type II, we investigate the nef cone of each component of the central fibre of $\mathcal{K}$. Let V be a component of the central fibre. Then $K_V \sim -2e$, where e is a double curve. Since e is a section and $e^2 = 0$, the nef cone of V is spanned by e and a fibre l of the ruling of V. Therefore every big divisor on V is ample and H is k-ample.
(2) From (1), $k : \mathcal{K} \to \Delta^1$ is a relative minimal model over Δ^1. Since Φ is commutes with k and $\mathcal{K}$ is a relative minimal model, Φ is isomorphic in codimension one. Moreover, $\mathcal{K}$ has no flopping curve. Therefore Φ is an isomorphism.
(3) Let

$$\mathcal{K}^\circ := \mathcal{K} \setminus (\text{double curves on the central fibre}).$$

Then $\mathcal{K}^\circ$ is a Neron model of the abelian scheme $k|_{\Delta^1\setminus\{0\}} : \mathcal{K}\setminus k^{-1}(0) \to \Delta^1\setminus\{0\}$ and there exists a multiplication morphism

$$\mathcal{K}^\circ \times_{\Delta^1} \mathcal{K}^\circ \to \mathcal{K}^\circ.$$

Thus for a section of $\mathcal{K}^\circ \to \Delta^1$, we obtain the birational map $\mathcal{K} \dashrightarrow \mathcal{K}$ which commutes with k. From (2) of Lemma, the above birational map is an isomorphism. Thus there exists an action of $\mathcal{K}^\circ$ on $\mathcal{K}$. Since the action of $\mathcal{K}^\circ$ on $\mathcal{K}^\circ$ is transitive, every component of the central fibre is mapped to each other by this action. Hence every component is isomorphic each other. □

(3.4) *Proof of Proposition 3.2.* Let $\nu : Y \to X$ be a birational morphism such that $(f \circ \nu)^* D$ is a simple normal crossing divisor. By Lemma 3.3, there exists a finite sets D_0 of D such that $F^1H^1((\nu^*T)_0, \mathbb{C}) \neq 0$ for $x \in D\setminus D_0$, where $(\nu^*T)_0$ is the central fibre of $\nu^*T \to \Delta^1$. Let $(\nu^*T)_0 = \sum e_i E_i$ and $e =$ L.C.M.(e_i). We define a cyclic cover $d : \Delta^1(s) \to \Delta^1(t)$ by $t = s^e$ and we denote the Galois group of d by G. Let T'' be the normalization of $\nu^*T \times_{\Delta^1(t)} \Delta^1(s)$. By [5, Theorem 11*], if we take a suitable resolution of T'', we obtain a semistable degeneration T'.

Claim. Let $T_0^{'}$ be the central fibre of $t' : T' \to \Delta^1$. Then $\eta^* : H^1((\nu^*T)_0, \mathbb{C}) \to H^1(T_0^{'}, \mathbb{C})$ is injection, where $\eta : T' \to \nu^*T$.

Proof. Since ν^*T and T' are deformation retract to each central fibre, it is enough to show that $\eta^* : H^1(\nu^*T, \mathbb{C}) \to H^1(T', \mathbb{C})$ is injective. Since $T''/G \cong \nu^*T$, $H^1(\nu^*T, \mathbb{C}) \cong H^1(T'', \mathbb{C})^G$ and $H^1(\nu^*T, \mathbb{C}) \to H^1(T'', \mathbb{C})$ is injective. Moreover, T'' has only quotient singularities, T'' is a homology manifold by [12, Proposition 1.4]. Hence $H^1(T'', \mathbb{C}) \to H^1(T', \mathbb{C})$ is injective by [2, Théorème 8.2.4]. □

We go back to the proof of Proposition. By Theorem 3.1, there exists the Kulikov model $k : \mathcal{K} \to \Delta^1$ which is birational to t'. We denote by $\mathcal{K}_0$ the central fibre of $\mathcal{K}$. By Claim 3 and Proposition 3.2, $F^1H^1(T_0^{'}, \mathbb{C}) \neq 0$. Due to Lemma 2.5, $F^1H^1(\mathcal{K}_0, \mathbb{C}) \cong F^1H^1(T_0^{'}, \mathbb{C})$. Hence $F^1H^1(\mathcal{K}_0, \mathbb{C}) \neq 0$ and $\mathcal{K}$ is of type I or type II. Let g be a generator of G. Since T' is a resolution of T'', there is a birational action G of T' which commutes t'. Thus there exists a birational map $\Phi_g : \mathcal{K} \dashrightarrow \mathcal{K}$ correponding to g which commutes with k, because k is birational to t'. By Lemma 3.4 (2), Φ_g is an isomorphism and G acts on $\mathcal{K}$ holomorphically. Therefore T is birational to the quotient $\mathcal{K}/G$. We claim that $F^1H^1(\mathcal{K}_0, \mathbb{C})$ carries a nonzero G-equivariant element. Let Z be a G-equivariant resolution of indeterminancy of $T' \dashrightarrow \mathcal{K}$. Then $F^1H^1(T_0^{'}, \mathbb{C}) \cong F^1H^1(Z_0, \mathbb{C}) \cong F^1H^1(\mathcal{K}_0, \mathbb{C})$ by Lemma 2.5, where Z_0 is the central fibre of $Z \to \Delta^1$. Let α be a nonzero element of $F^1H^1((\nu^*T)_0, \mathbb{C})$. The pull back of α in $F^1H^1(Z_0, \mathbb{C})$ is an non zero element by Claim 3 and hence a G-equivariant element. Thus there exists a nonzero G-equivariant element in $F^1H^1(\mathcal{K}_0, \mathbb{C})$. □

4. Classification of type I degeneration

(4.1) In this section, we prove the following proposition.

Proposition 4.1. *Let $t : T \to \Delta^1$ be an abelian fibration which is birational to the quotient of a Kulikov model $\mathcal{K}$ of type I by a cyclic group G. Assume that*

(1) *T is smooth.*
(2) $K_T \sim_t 0$.
(3) *There exists a nonzero G-equivariant element of $F^1H^1(\mathcal{K}_0, \mathbb{C})$.*

Then the central fibre T_0 satisfies the properties of Theorem 1.2 (1).

(4.2) For a proof of Proposition 4.1, we will construct a suitable resolution Z of $\mathcal{K}/G$ and a relative minimal model W of Z over Δ^1. If the representation $\rho : G \to \mathrm{Aut} H^1(\mathcal{K}_0, \mathbb{C})$ is trivial, then $\mathcal{K}/G$ is smooth and it contains no rational curve. Hence $\mathcal{K}/G$ is the unique minimal model over Δ^1. Since T is a relative minimal model over Δ^1, $T \cong \mathcal{K}/G$. However, $K_{\mathcal{K}/G} \not\sim 0$, because $K_{\mathcal{K}}$ is not G-equivariant. Thus we may assume that the representation ρ is not trivial. We need the following lemma to prove Proposition 4.1.

Lemma 4.2. *There is a G-equivariant elliptic fibration $\mathcal{K}_0 \to C'$ which satisfies the following diagram:*

$$\begin{array}{ccc} \mathcal{K}_0 & \to & \mathcal{K}_0/G \\ \downarrow & & \downarrow \\ C' & \to & C, \end{array}$$

where C and C' are elliptic curves and $C'/G \cong C$.

Proof. Since $\mathcal{K}_0$ is an Abelian surface, it is enough to show that there is a G-equivariant fibration on $\mathcal{K}_0$. Let N be a representation matrix of $\rho : G \to \mathrm{Aut} H^0(\mathcal{K}_0, \Omega^1_{\mathcal{K}_0})$. By Proposition 3.2, one of eigenvalues of N is one. Since G is a finite cyclic group, there exists a basis of $H^0(\mathcal{K}_0, \Omega^1_{\mathcal{K}_0})$ under which $N = \mathrm{diag}(1, \zeta)$, where ζ is a n-th root of unity. Note that $\zeta \neq 1$ because ρ is not trivial. Around a fixed point p of $\mathcal{K}_0$, the action of G on $\mathcal{K}_0$ can be written $(x, y) \mapsto (x, \zeta y)$, where x, y are local coordinates of p. Hence the finite map $\mathcal{K}_0 \to \mathcal{K}_0/G$ is branched along smooth curves and $\mathcal{K}_0/G$ is a smooth surface. Since $\dim H^0(\mathcal{K}_0, \Omega^1_{\mathcal{K}_0})^G = 1$, the irregularity of $\mathcal{K}_0/G$ is one and there is the Albanese map $\mathcal{K}_0/G \to C$ over an elliptic curve. This morphism is not constant, namely, this morphism is surjective. We consider the composition morphism $\mathcal{K}_0 \to \mathcal{K}_0/G \to C$. If we take the stein factorization $\mathcal{K}_0 \to C'$ of the above morphism, we obtain desired morphisms. □

(4.3) *Proof of Proposition 4.2.* By Lemma 4.2, there exists a G-equivariant elliptic fibration $\mathcal{K}_0 \to C'$. Since C and C' are elliptic curves, the action of G on C' is translation. Let g be a generator of G and m the minimal integer such that the action of g^m on C' is trivial. We define the subgroup H of G by $H := \langle g^m \rangle$.

(4.3.1) First we consider the case that $H = \{1\}$. In this case, $\mathcal{K}/G$ is smooth. Moreover $\mathcal{K}/G$ is the unique relative minimal model over Δ^1 since it has no rational curves. On the contrary, T is a relative minimal model over Δ^1, $T \cong \mathcal{K}/G$. By the construction, the central fibre $\mathcal{K}_0/G$ of $\mathcal{K}/G$ is a hyperelliptic surface. Since every hyperelliptic surface is the étale quotient of the product of elliptic curves, $T_0 \cong \mathcal{K}_0/G$ is type of I_0.
(4.3.2) Next we consider the case that $H \neq \{1\}$. Since the action of H on C' is trivial, $\pi' : \mathcal{K}_0/H \to C'$ is a $\mathbb{P}^1$-bundle and singular locus of $\mathcal{K}/H$ consists of several copies of the products of a surface quotient singularity and an elliptic curve. Moreover each connected component of $\mathrm{Sing}(\mathcal{K}/H)$ forms a multisection of π' and $\mathcal{K}/H$ is equisingular along each connected component of $\mathrm{Sing}(\mathcal{K}/H)$. By definition, the action of G/H on C' is non trivial translation. Thus the quotient morphism $\mathcal{K}/H \to \mathcal{K}/G$ is an étale morphism. Therefore singularities of $\mathcal{K}/G$ also consists of several copies of the products of a surface singularity and an elliptic curve, $\pi : \mathcal{K}_0/G \to C$ is a $\mathbb{P}^1$-bundle, each connected component of $\mathrm{Sing}(\mathcal{K}/G)$ forms a multisection of π and $\mathcal{K}/G$ is equisingular along each connected component of $\mathrm{Sing}(\mathcal{K}/G)$. The list of surface quotient singularities which occur above is found in [1, Table 5]. According to this table, the possibily of singulaties of $\mathcal{K}/G$ is one of Table 1 and Table 2. Note that $\mathbb{C}^3/\mathbb{Z}_n(a, b, c)$ stands for the cyclic quotient singularity $\mathbb{C}^3/\mathbb{Z}$ whose character is (a, b, c) in the above tables. We construct the minimal resolution Z of $\mathcal{K}/G$ by the minimal resolution of surface quotient singularities. If the singularities of $\mathcal{K}/G$ consists of the product of Du Val singularities and an elliptic curve only, Z is a relative minimal model over Δ^1 and we put $W = Z$. In other cases, we obtain a relative minimal model W after birational contractions of Z. (cf. [1, pp 158]) In both cases, W has no flopping curve. Namely, W is the unique minimal model over Δ^1. Since W is birational to T and T is a relative minimal model over Δ^1, $T \cong W$. By construction, it is easy to see that the singular fibre of W is isomorphic to one of singular fibres in Table 4 and Table 5 according to the type of singularities of $\mathcal{K}/G$. The remain object what we will show is that there is an étale covering $\tilde{C} \to C$ such that $W_0 \times_C \tilde{C}$ is isomorphic to the product of a Kodaira singular fibre and $\tilde{C}$. Since $\pi : \mathcal{K}_0/G \to C$ is a $\mathbb{P}^1$-bundle and every fibre of π intersects $\mathrm{Sing}(\mathcal{K}/G)$ with at least 3 points, there is an étale morphism $\tilde{C} \to C$ such that $\mathcal{K}_0/G \times_C \tilde{C} \cong \mathbb{P}^1 \times \tilde{C}$ and the pull back of each connected component of $\mathrm{Sing}(\mathcal{K}/G)$ forms a section of second projection. Note that $\mathcal{K}_0/G \cong \mathbb{P}^1 \times C$ if singularities of $\mathrm{Sing}(\mathcal{K}/G)$ is of type $I_0^* - 0$, II, II^*, $III - 0$, $III^* - 0$, $IV - 0$ or $IV^* - 0$. Every exceptional divisor $Z \to \mathcal{K}/G$ is isomorphic to the product of an elliptic curve and $\mathbb{P}^1$, because this resolution is the product of the minimal resolution of a surface singularity and an elliptic curve. Thus $Z_0 \times_C \tilde{C}$ is isomorphic to the product of a tree of $\mathbb{P}^1$ and $\tilde{C}$, where Z_0 is the central fibre of Z. Since $Z \to W$ is a composition of contracting $\mathbb{P}^1$-bundle along its ruling, $W_0 \times_C \tilde{C}$ is isomorhpic to the product of a Kodaira singular fibre and $\tilde{C}$. □

Table 1. List of sigularities 1

Type $I_0^* - 0$	$\mathrm{Sing}(\mathcal{K}/G)$ has 4 connected components. Every irreducible component is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_2(1,1,0)$ for all points p of $\mathrm{Sing}(\mathcal{K}/G)$.
Type $I_0^* - 1$	$\mathrm{Sing}(\mathcal{K}/G)$ has 3 connected components C_i $(1 \leq i \leq 3)$. C_1 is a 2-section of π and C_i $(i = 2, 3)$ are sections of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_2(1,1,0)$ for all points p of $\mathrm{Sing}(\mathcal{K}/G)$.
Type $I_0^* - 2$	$\mathrm{Sing}(\mathcal{K}/G)$ has 2 connected components C_i $(i = 1, 2)$. C_i $(i = 1, 2)$ are 2-section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_2(1,1,0)$ for all points p of $\mathrm{Sing}(\mathcal{K}/G)$.
Type $I_0^* - 3$	$\mathrm{Sing}(\mathcal{K}/G)$ has 2 connected components C_i $(i = 1, 2)$. C_1 is 3-section of π and C_2 is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_2(1,1,0)$ for all points p of $\mathrm{Sing}(\mathcal{K}/G)$.
Type $I_0^* - 4$	$\mathrm{Sing}(\mathcal{K}/G)$ has 1 connected components. It forms a 4-section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_2(1,1,0)$ for all points p of $\mathrm{Sing}(\mathcal{K}/G)$.
Type II	$\mathrm{Sing}(\mathcal{K}/G)$ has 3 connected components C_i $(i = 1, 2, 3)$. Every irreducible component is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_6(1,1,0)$ for all points p of C_1, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_3(1,1,0)$ for all points p of C_2 and $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_2(1,1,0)$ for all points p of C_3.
Type II^*	$\mathrm{Sing}(\mathcal{K}/G)$ has 3 connected components C_i $(i = 1, 2, 3)$. Every irreducible component is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_6(5,1,0)$ for all points p of C_1, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_3(2,1,0)$ for all points p of C_2 and $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_2(1,1,0)$ for all points p of C_3.
Type $III - 0$	$\mathrm{Sing}(\mathcal{K}/G)$ has 3 connected components C_i $(i = 1, 2, 3)$. Every irreducible component is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_4(1,1,0)$ for all points p of C_1 and C_2 and $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_2(1,1,0)$ for all points p of C_3.
Type $III - 1$	$\mathrm{Sing}(\mathcal{K}/G)$ has 2 connected components C_i $i = 1, 2$. C_1 is a 2-section of π and C_2 is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_4(1,1,0)$ for all points p of C_1 and $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_2(1,1,0)$ for all points p of C_2.
Type $III^* - 0$	$\mathrm{Sing}(\mathcal{K}/G)$ has 3 connected components C_i $(i = 1, 2, 3)$. Every irreducible component is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_4(3,1,0)$ for all points p of C_1 and C_2 and $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_2(1,1,0)$ for all points of p of C_3.
Type $III^* - 1$	$\mathrm{Sing}(\mathcal{K}/G)$ has 2 connected components C_i $i = 1, 2$. C_1 is a 2-section of π and C_2 is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_4(3,1,0)$ for all points p of C_1 and $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_2(1,1,0)$ for all points p of C_2.

Table 2. List of sigularities 2

Type $IV-0$	$\mathrm{Sing}(\mathcal{K}/G)$ has 3 connected components C_i $(i=1,2,3)$. Every irreducible component is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_3(1,1,0)$ for all points p of $\mathrm{Sing}(\mathcal{K}/G)$.
Type $IV-1$	$\mathrm{Sing}(\mathcal{K}/G)$ has 2 connected components C_i $i=1,2$. C_1 is a 2-section of π and C_2 is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_3(1,1,0)$ for all points p of $\mathrm{Sing}(\mathcal{K}/G)$.
Type $IV-2$	$\mathrm{Sing}(\mathcal{K}/G)$ has one connected components C_1. It forms a 3-section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_3(1,1,0)$ for all points p of $\mathrm{Sing}(\mathcal{K}/G)$.
Type IV^*-0	$\mathrm{Sing}(\mathcal{K}/G)$ has 3 connected components C_i $(i=1,2,3)$. Every irreducible component is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_3(2,1,0)$ for all points p of $\mathrm{Sing}(\mathcal{K}/G)$.
Type IV^*-1	$\mathrm{Sing}(\mathcal{K}/G)$ has 2 connected components C_i $i=1,2$. C_1 is a 2-section of π and C_2 is a section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_3(2,1,0)$ for all points p of $\mathrm{Sing}(\mathcal{K}/G)$.
Type IV^*-2	$\mathrm{Sing}(\mathcal{K}/G)$ has one connected components C_1. It fomrs a 3-section of π. Analytic locally, $(p, \mathcal{K}/G) \cong \mathbb{C}^3/\mathbb{Z}_3(2,1,0)$ for all points p of $\mathrm{Sing}(\mathcal{K}/G)$.

Table 3. The list of actions

Type	I_0^*-1	I_0^*-2	I_0^*-3	I_0^*-4
G	$\mathbb{Z}_6$	$\mathbb{Z}_2^{\oplus 2}$	$\mathbb{Z}_4$	$\mathbb{Z}_2^{\oplus 3}$
τ	arbitrary	arbitrary	arbitrary	arbitrary
Action	$g:(\zeta_6, 0; \frac{1}{3}\tau, \zeta_6^{-1})$	$g_1:(\zeta_2, 0; 0, \zeta_2)$ $g_2:(1, \frac{1}{2}; \frac{1}{2}, 1)$	$g:(\zeta_4, 0; \frac{1}{2}\tau, \zeta_4^{-1})$	$g_1:(\zeta_2, 0; 0, \zeta_2)$ $g_2:(1, \frac{1}{2}; \frac{1}{2}, 1)$ $g_3:(1, \frac{1}{2}\tau; \frac{1}{2}, 1)$
Type	$III-1$	III^*-1	$IV-1$	IV^*-1
G	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_6$
τ	ζ_4	ζ_4	ζ_3	ζ_3
Action	$g_1:(\zeta_4, 0; 0, \zeta_4)$ $g_2:(1, \frac{1}{2}+\frac{1}{2}\zeta_4; \frac{1}{2}, 1)$	$g_1:(\zeta_4, 0; 0, \zeta_4^{-1})$ $g_2:(1, \frac{1}{2}+\frac{1}{2}\zeta_4; \frac{1}{2}, 1)$	$g:(\zeta_6^5, 0; \frac{1}{4}, \zeta_{12})$	$g:(\zeta_6, 0; \frac{1}{2}, \zeta_6^{-1})$
Type	$IV-2$	IV^*-2	I_0	
G	$\mathbb{Z}_3^{\oplus 2}$	$\mathbb{Z}_3^{\oplus 2}$	$\mathbb{Z}_m$	
τ	ζ_3	ζ_3	arbitary	
Action	$g_1:(\zeta_3, 0; 0, \zeta_3)$ $g_2:(1, \frac{1}{3}+\frac{2}{3}\zeta_3; \frac{1}{3}, 1)$	$g_1:(\zeta_3, 0; 0, \zeta_3^{-1})$ $g_2:(1, \frac{1}{3}+\frac{2}{3}\zeta_3; \frac{1}{3}, 1)$	$g:(\zeta_m, 0; \frac{1}{m}\tau, \zeta_m^{-1})$	

Table 4. Classification Table 1

$f^{-1}(s)^{\sim}$ is a product of a Kodaira singular fibre of type I_0 and $\tilde{C}$.	
Type I_0	$f^{-1}(s) = mE$, where E is a hyperelliptic surface.
$f^{-1}(s)^{\sim}$ is a product of a Kodaira singular fibre of type I_0^* and $\tilde{C}$.	
Type $I_0^* - 0$	$f^{-1}(s)$ is isomorphic to the product of the Kodaira singular fibre of type I_0^* and an elliptic curve.
Type $I_0^* - 1$	$f^{-1}(s) = m(2E_0 + E_1 + E_2 + E_3)$. Every component is a minimal elliptic ruled surface. $E_3 \cap E_0$ is 2-section of the ruling of E_0, $E_i \cap E_0$ $(i = 1, 2)$ are sections.
Type $I_0^* - 2$	$f^{-1}(s) = m(2E_0 + E_1 + E_2)$. Every component is a minimal elliptic ruled surface. $E_i \cap E_0$ $(i = 1, 2)$ are 2-sections of the ruling of E_0.
Type $I_0^* - 3$	$f^{-1}(s) = m(2E_0 + E_1 + E_2)$. Every component is a minimal elliptic ruled surface. E_1 is a 3-section of the ruling of E_0 and E_2 is a section.
Type $I_0^* - 4$	$f^{-1}(s) = m(2E_0 + E_1)$. Every component is a minimal elliptic ruled surface. $E_1 \cap E_0$ is 4-section of the ruling of E_0.
$f^{-1}(s)^{\sim}$ is a product of a Kodaira singular fibre of type II and $\tilde{C}$.	
Type II	$f^{-1}(s)$ is isomorphic to the product of the Kodaira singular fibre of type II and an elliptic curve.
$f^{-1}(s)^{\sim}$ is a product of a Kodaira singular fibre of type II^* and $\tilde{C}$.	
Type II^*	$f^{-1}(s)$ is isomorphic to the product of the Kodaira singular fibre of type II^* and an elliptic curve.
$f^{-1}(s)^{\sim}$ is a product of a Kodaira singular fibre of type III and $\tilde{C}$.	
Type $III - 0$	$f^{-1}(s)$ is isomorphic to the product of the Kodaira singular fibre of type III and an elliptic curve.
Type $III - 1$	$f^{-1}(s) = mE$, The normalization of E is a minimal elliptic ruled surface (see Fig. 1).
If $f^{-1}(s)^{\sim}$ is a product of a Kodaira singular fibre of type III^* and $\tilde{C}$	
Type $III^* - 0$	$f^{-1}(s)$ is isomorphic to the product of the Kodaira singular fibre of type III^* and an elliptic curve.
Type $III^* - 1$	$f^{-1}(s) = m(E_0 + 3E_{11} + 2E_{12} + E_{13} + E_{21})$. Every component is a minimal elliptic ruled surface. $E_{11} \cap E_0$ is 2-section of the ruling of E_0 (see Fig. 1).
$f^{-1}(s)^{\sim}$ is a product of a Kodaira singular fibre of type IV and $\tilde{C}$.	
Type $IV - 0$	$f^{-1}(s)$ is isomorphic to the product of the Kodaira singular fibre of type IV and an elliptic curve.
Type $IV - 1$	$f^{-1}(s) = m(E_1 + E_2)$. The normalization of E_i is a minimal elliptic ruled surface (see Fig. 1).
Type $IV - 2$	$f^{-1}(s) = mE$. The normalization of E is a minimal elliptic ruled surface (see Fig. 1).

Table 5. Classification Table 2

$f^{-1}(s)^{\sim}$ is a product of a Kodaira singular fibre of type IV^* and $\tilde{C}$.	
Type $IV^* - 0$	$f^{-1}(s)$ is isomorphic to the product of the Kodaira singular fibre of type IV^* and an elliptic curve.
Type $IV^* - 1$	$f^{-1}(s) = m(E_0 + \sum_{i=1}^{2}(2E_{i1} + E_{i2}))$. Every component is a minimal elliptic ruled surface. $E_{11} \cap E_0$ is 2-section of the ruling of E_0 (see Fig. 1).
Type $IV^* - 2$	$f^{-1}(s) = m(E_0 + 2E_1 + E_2)$. Every component is a minimal elliptic ruled surface. $E_1 \cap E_0$ is 3-section of the ruling of E_0 (see Fig. 1).

5. Classification of type II degeneration

(5.1) In this section, we prove the following proposition and Theorem 1.2.

Proposition 5.1. *Let $t : T \to \Delta^1$ be an abelian fibration which is birational to the quotient of a Kulikov model $\mathcal{K}$ of type II by a cyclic group G. Assume that*

(1) *T is smooth.*
(2) *$K_T \sim_t 0$.*
(3) *There exists a nonzero G-equivariant element of $F^1H^1(\mathcal{K}_0, \mathbb{C})$.*

Then the central fibre T_0 of T satisfies the properties of Theorem 1.2 (2).

(5.2) For the proof of Proposition 5.1, we investigate the action of G on the central fibre of $\mathcal{K}$.

Lemma 5.2. *Let g be a generator of G and m the smallest positive interger such that every component is stable under the action of H. We denote by H the subgroup of G generated by g^m. Then*

(1) *Every element of $F^1H^1(\mathcal{K}_0, \mathbb{C})$ is G-invariant.*
(2) *The action of H is free and the central fibre of the quotient $\mathcal{K}/H$ is a cycle of mininal elliptic ruled surfaces.*

Proof.

(1) By Proposition 2.5, there exists a G-equivariant element in $F^1H^1(\mathcal{K}_0, \mathbb{C})$. Since $\dim F^1H^1(\mathcal{K}_0, \mathbb{C}) = 1$, every element of $F^1H^1(\mathcal{K}_0, \mathbb{C})$ is G-invariant.
(2) From the assumption there exists an action of H on each component of the central fibre of $\mathcal{K}$. Let V be a component of the central fibre and $\pi : V \to C$ the ruling. Since every fibre of π is $\mathbb{P}^1$ and C is an elliptic curve, π is H-equivariant. From Lemma 5.2 (1) and Lemma 2.4, holomorphic one forms on V are invariant under the action of g^m. Thus, the action of H on C is translation. Therefore the action of H on V is free and V/H is a minimal elliptic ruled surface. From the assumption that each component is stable under the action of H, the central fibre of the quotient $\mathcal{K}/H$ is a cycle of minimal elliptic ruled surfaces. □

(5.3) *Proof of Proposition 5.1.* From Lemma 5.2, $\mathcal{K}/H$ is smooth and the central fibre of $\mathcal{K}/H$ is a cycle of minimal elliptic ruled surfaces. Let Γ be the dual graph of the central fibre of $\mathcal{K}$ and g a generator of G. Considering $\mathcal{K}/H$ instead of $\mathcal{K}$, we may assume that the action of g^m is trivial if the action of g^m on Γ is trivial.

(5.3.1) If the action of G is free, $\mathcal{K}/G$ is smooth and this is a relative minimal model over Δ^1. Since Γ is a Dynkin diagram of type $\tilde{A}_n$ and G is a cyclic group, the action of G on Γ is either rotation or reflection.

(1) If the action of G on Γ is rotation, the central fibre $\mathcal{K}_0/G$ of $\mathcal{K}/G$ is a cycle of minimal elliptic ruled surfaces. Each double curve is a section of a minimal elliptic ruled surface.
(2) If the action of G on Γ is reflection, the central fibre $\mathcal{K}_0/G$ of $\mathcal{K}/G$ is a chain of minimal elliptic ruled surfaces. We denote each component of $\mathcal{K}_0/G$ by $\bar{V}_i$. Since $\tau : \mathcal{K}/H \to \mathcal{K}/G$ is an étale morphism of degree 2, every component of $\mathcal{K}_0/G$ is a minimal elliptic ruled surface and $\mathcal{K}_0/G = 2m\sum \bar{V}_i$. We investigate double curves of $\mathcal{K}_0/G$. It is obvious that double curves forms a section on non edge components of $\mathcal{K}_0/G$. We will show that double curves forms a bisection on edge components. Let $\bar{V}_0$ be one of the edge component and $\bar{V}_1$ the next component. By adjuction formula, $K_{\bar{V}_0} \equiv -\bar{V}_1|_{\bar{V}_0}$. Since $\bar{V}_0$ is a minimal elliptic ruled surface, $K_{\bar{V}_0}$ is linearly equivalent to two sections. Combining with that the double curve $\bar{V}_0 \cap \bar{V}_1$ is connected, $\bar{V}_0 \cap \bar{V}_1$ is a bisection.

In both cases, $\mathcal{K}/G$ has no flopping curve. Thus $\mathcal{K}/G$ is the unique relative minimal model and we obtain $T \cong \mathcal{K}/G$.

(5.3.2) If the action of g is not free, we need the following lemma.

Lemma 5.3. *If the action of G has fixed points, then the action of G on Γ is reflection and it preserves two vertices. Furthermore, the fixed locus consists of four sections of the ruling or two bisections of the ruling.*

Assuming this Lemma, the central fibre of the quotient $\mathcal{K}/G$ is a chain of minimal elliptic ruled surfaces. The singularities of $\mathcal{K}/G$ consists of several copies of the product of A_1 singlarity and an elliptic curve. Thus a relative minimal model W over Δ^1 is obtained by blowing up along singlar locus. Since W has no flopping curve, W is the unique relative minimal model and we obtain $W \cong T$. From the construction of W and the above Lemma, the dual graph of the central fibre of W is A_n or $\tilde{D}_n$, the double curve on the edge component is a bisection or a section. and every other double curve is section.

(5.4) *Proof of Lemma 5.2.* If the action of G on Γ is rotation, there exists no fixed points. Thus the action of G on Γ is reflection. We derive the contradiction assuming that G fixes one of edges of Γ. Let C be the elliptic curve corresponding to the edge which is fixed by G. From Lemma 2.4 and Lemma 5.2, the action of G on C preserves holomorphic one form on C. Therefore C is fixed locus of the action of G. The singularities of the quotient $\mathcal{K}/G$ consist of several copies of the product of A_1 singularity and an elliptic curve. Let $w : W \to \mathcal{K}/G$ be the blowing up along C. The central fibre W_0 of $w : W \to \Delta^1$ is a chain of minimal ellitic ruled surfaces. We denote by V_i each components of W_0. Let V_0, V_1 and V_2 be the exceptional divisor coming from the blowing up along C, the next component of V_0 and the next component of V_1 respectively.

$$\overset{V_0}{\circ} - \overset{V_1}{\circ} - \overset{V_2}{\circ} - \cdots$$

Then $W_0 = m(V_0 + 2V_1 + 2V_2 + (\text{Other components}))$. Since W is smooth along V_1 and K_W is numerically trivial,

$$K_{V_1} \equiv K_W + V_1|_{V_1} \equiv (-\frac{1}{2}V_0 - V_2)|_{V_1}.$$

by adjunction formula. Let l be a fibre of ruling of V_1. Then

$$K_l \equiv K_{V_1} + l|_l \equiv (-\frac{1}{2}V_0 - V_2).l.$$

Since every double curve of W_0 is a section, $\deg K_l = -3/2$. However this is a contradiction because $l \cong \mathbb{P}^1$. Therefore G fixes two vertices. In the following, we investigate the fixed locus on $\mathcal{K}/G$. By Lemma 3.4 (3), every component of the central fibre of $\mathcal{K}$ is isomorphic to each other, it is enough to investigate the fixed locus on one of the components correponding to the fixed vertices. We denote V this component and $\pi : V \to C$ the ruling of V. Since C is an elliptic curve and every fibre π is $\mathbb{P}^1$, π is G-equivariant. By Lemma 5.2 (1), the action of G on V preserves a one form on V. Since the action of G is not free, G acts on C trivially. Thus there exists two fixed points on each fibre of the ruling of V. If V is not isomorphic to $\mathbb{P}^1 \times C$, then there exist only two sections of the ruling and these curves are double curves. Since no dobule curve is stable under the action of G , we obtain the fixed locus consists of a bisection. If V is isomorphic to $\mathbb{P}^1 \times C$, there exist no bisection of the ruling. Therefore the fixed locus consists of sections. Thus we obtain the rest of assertion of Lemma 5.2. □

The proof of Proposition 5.1 is completed. □

(5.5) *Proof of Theorem 1.2.* Let $f : (X, \omega) \to S$ be a projective Lagrangian fibration over 4-dimensional holomorphic symplectic manifold. By Proposition 3.2, there exists a finite sets D_0 of D which has the following properties: For $x \in D \setminus D_0$, there exists an abelian fibraton $T \to \Delta^1$ which satisfies assumptions of Proposition 4.1 or 5.1 and $\Delta^1 \cap D = \{x\}$. Then $T_0 = f^{-1}(x)$ satisfies the assertions of Theorem 1.2 by Proposition 4.1 and 5.1. All types of singular fibre actually occur by Propsition 6.1. □

6. Examples

Proposition 6.1. *All types of singular fibre actually occur.*

Proof. First we construct examples of singular fibre in Proposition 4.1. We fix some notations. Let $E := \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ be an elliptic curve and ζ_n a n-th root of unity. Since the singular fibre of type $I_0^* - 0$, II, II^*, $III - 0$, $III^* - 0$, $IV - 0$ or $IV^* - 0$ is isomorphic to the product of a Kodaira singular fibre and an elliptic curve, it is easy to construct examples. We concentrate the construction of other types of singular fibres. We consider the quotient of $E \times E \times \Delta^2$ by an Abelian group G. The list of G and its action is Table 3. In this table, action $(a, b; c, d)$ means

$$\begin{aligned} E \times E \times \Delta^2 &\overset{g}{\to} \qquad E \times E \times \Delta^2 \\ (z_1, z_2, t_1, t_2) &\mapsto (az_1 + b, z_2 + c, dt_1, t_2). \end{aligned}$$

We will show a relative minimal model W over Δ^2 of $E \times E \times \Delta^2/G$ gives a desired example. By direct calculation, singularities of $E \times E \times \Delta^2/G$ is one of types of Table 1 and Table 2. According to the proof of Proposition 4.1, each singular fibre of W is isomorphic to a singular fibre which we want to construct. Thus it is enough to show that W is a Lagrangian fibration. If the action of G is of type I_0, $I_0^* - 1, 2, 3, 4$, $III^* - 1$ and $IV^* - 1, 2$, singularities of the quotient $E \times E \times \Delta^2/G$ consists of several copies of the product of a Du Val singularity and an elliptic curve. Thus the minimal resolution of $E \times E \times \Delta^2/G$ is a relative minimal model over Δ^2. Moreover the symplectic form $dz_1 \wedge dt_1 + dz_2 \wedge dt_2$ is G-invariant and it vanishes on a general fibre of projection $E \times E \times \Delta^2 \to \Delta^2$. Thus $E \times E \times \Delta^2/G$ has a nondegenerate holomorphic 2-form ω. Since K_W is nef, the pull back of ω is nondegenerate. Hence W is a symplectic manifold and $W \to \Delta^2$ is a Lagrangian fibration. If the action of G is of type $III - 1$, $IV - 1, 2$, we consider the blowing up $\tilde{\nu} : \tilde{Z} \to E \times E \times \Delta^2$ along the fixed locus of G. Then the action of G can be lifted on $\tilde{Z}$ and the minimal resolution Z of $E \times E \times \Delta^2/G$ is isomorphic to $\tilde{Z}/G$. Let D be the discriminat locus of $w : W \to \Delta^2$, γ the quotient morphism $\tilde{Z} \to Z$, ν the birational morphism $Z \to E \times E \times \Delta^2/G$, F the proper transform on Z of $w^{-1}(D)$ and $\tilde{F} := \gamma^{-1}(F)$. Note that F is the ν-exceptional divisor coming from the minimal resolution of the quotient singularity $\mathbb{C}^4/\mathbb{Z}_m(1, -1, 0, 0)$, where $m = 4$, (resp. $m = 3$.) if the action of G is of type $III - 1$. (resp. type $IV - 1, 2$.) (cf. [1, pp 158].) We define the holomorphic 2-form ω on $E \times E \times \Delta^2$ by $t_1^{m-2} dz_1 \wedge dt_1 + dz_2 \wedge dt_2$. Then ω is G-invariant and it vanishes on a general fibre of projection $E \times E \times \Delta^2 \to \Delta^2$. Thus ω induces a holomorpic 2-form ω' on $E \times E \times \Delta^2/G$. Moreover, $\wedge^2(\tilde{\nu}^*\omega)$ has order $m - 1$ zero along each irreducible component of $\tilde{F}$. On the contrary, the order of isotorpie group of each irreducible component of $\tilde{F}$ is m, the branching order of the quotient morphism $\tilde{Z} \to Z$ along each irreducible component of F is m. Hence $\wedge^2(\nu^*\omega')$ is nonzero along F and $\nu^*\omega'$ is non degenerate along F.

Therefore $\nu^*\omega'$ defines a symplectic form on W and $W \to \Delta^2$ is a Lagrangian fibration. Next we construct an example of singular fibre which satisfies the properties of Theorem 1.2 (2). We begin with the construction of an elliptic fibration whose singular fibre is a Kodaira singular fibre of type I_{2n} according to [10, Section 2]. Let R_k be a subring of $\mathbb{C}[v^{\pm 1}, t^{\pm 1}]$ defined by

$$R_k := \mathbb{C}[vt^{-k}, v^{-1}t^{k+1}].$$

We define a smooth scheme

$$\mathcal{M} := \bigcup_{k \in \mathbb{Z}} \operatorname{Spec} R_k.$$

Note that there is a morphism $\mathcal{M} \to \mathbb{C}[t]$. Let $\mathcal{M}' := \mathcal{M} \times_{\mathbb{C}[t]} \Delta^1$. We define the action of $m \in \mathbb{Z}$ on $\mathcal{M}'$ by

$$\begin{array}{rccc} & R_k & \to & R_{k+2nm} \\ m : & (vt^{-k}, v^{-1}t^{k+1}) & \mapsto & (vt^{-k-2nm}, v^{-1}t^{k+1+2nm}) \end{array}$$

Then this action is properly discontinuous and fixed point free by [10, Theorem 2.6]. The quotient $s : S \to \Delta^1$ is an elliptic fibration whose singular fibre is a Kodaira singular fibre of type I_{2n}. We define two involutions η_i, $(i = 1, 2)$ on $\mathbb{C}[v^{\pm 1}, t^{\pm 1}]$ by

$$\begin{array}{rccc} & \mathbb{C}[v^{\pm 1}, t^{\pm 1}] & \to & \mathbb{C}[v^{\pm 1}, t^{\pm 1}] \\ \eta_1 : & (v, t) & \mapsto & (1/v, -t) \\ \eta_2 : & (v, t) & \mapsto & (-v, t). \end{array}$$

Then these actions induce involutions on R_k and $\mathcal{M}'$. Moreover, these involutions are compatible with the action of $\mathbb{Z}$ on $\mathcal{M}'$. Therefore, η_i, $(i = 1, 2)$ defines an involution of S. We denote these involutions by same character, The symplectic form on $\mathcal{M}'$ defined by $dv/v \wedge dt$ is $\mathbb{Z}$-invariant and preserved by η_i, $(i = 1, 2)$. Thus the induced symplectic form ω on S is preserved by induced involutions. Note that ω vanishes on a general fibre of $S \to \Delta^1$. Now we construct examples of a Lagrangian fibration such that its singular fibre is a normal crossing variety whose dual graph is A_n. We consider the following action of $\mathbb{Z}_2$ on $S \times E \times \Delta^1$:

$$\begin{array}{ccc} S \times E \times \Delta^1 & \to & S \times E \times \Delta^1 \\ (v_1, z_2, t_2) & \mapsto & (\eta_1(v_1), z_2 + \frac{1}{2}, t_2). \end{array}$$

Then this action is fixed point free and a symplectic form $\omega + dz_2 \wedge dt_2$ is $\mathbb{Z}_2$-invariant. Since $\omega + dz_2 \wedge dt_2$ vanishes on a general fibre of $S \times E \times \Delta^1 \to \Delta^2$, the quotient of $S \times E \times \Delta^1/\mathbb{Z}_2$ gives a desired example (see left one of Fig. 3). We construct another example. The minimal resolution $\bar{S}$ of S/τ_1 admits an elliptic fibration $\bar{s} : \bar{S} \to \Delta^1$ whose singular fibre is a Kodaira singular fibre of type I_n^*. Since η_1 and η_2 are compatible, η_2 induces an involution on S/η_1 and $\bar{S}$. We denote this induced involution by $\bar{\eta}_2$. Then the action of $\bar{\eta}_2$ on the dual graph of the singular fibre of $\bar{s}$ interchange edge vertices with each other. (Note that the dual graph of the singular fibre of $\bar{s}$ is the Dynkin diagram of type $\tilde{D}_n$.) We consider the following action of $\mathbb{Z}_2$ on $\bar{S} \times E \times \Delta^1$.

$$\begin{array}{rcl} \bar{S} \times E \times \Delta^1 & \to & \bar{S} \times E \times \Delta^1 \\ (v_1, v_2, t) & \mapsto & (\eta_2(v_1), z_2 + \frac{1}{2}, t). \end{array}$$

Then this action is fixed point free and every singular fibre of the quotient $\bar{S} \times E \times \Delta^1/\mathbb{Z}_2$ is a normal crossing variety whose dual graph is the Dynkin diagram of type A_n (see right one of Fig. 3). We will show that $\bar{S} \times E \times \Delta^1/\mathbb{Z}_2 \to \Delta^2$ is a Lagrangian fibration. There is an induced symplectic form ω' from ω on $\bar{S}$. Then the symplectic form $\omega' + dz_2 \wedge dt_2$ is $\mathbb{Z}_2$-invariant and it vanishes on a general fibre of $\bar{S} \times E \times \Delta^1 \to \Delta^2$. Therefore $\bar{S} \times E \times \Delta^1/\mathbb{Z}_2 \to \Delta^2$ is a Lagrangian fibration. □

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