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A new geometric construction of compact complex manifolds in any dimension

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Abstract. We consider holomorphic linear foliations of dimension m of $\mathbb{C}^n$ (with $n > 2m$) fulfilling a so-called weak hyperbolicity condition and equip the projectivization of the leaf space (for the foliation restricted to an adequate open dense subset) with a structure of compact, complex manifold of dimension $n - m - 1$. We show that, except for the limit-case $n = 2m + 1$ where we obtain any complex torus of any dimension, this construction gives non-symplectic manifolds, including the previous examples of Hopf, Calabi-Eckmann, Haefliger (linear case), Loeb-Nicolau (linear case) and López de Medrano-Verjovsky. We study some properties of these manifolds, that is to say meromorphic functions, holomorphic vector fields, forms and submanifolds. For each manifold, we construct an analytic space of deformations of dimension $m(n - m - 1)$ and show that, under some additional conditions, it is universal. Lastly, we give explicit examples of new compact, complex manifolds, in particular of connected sums of products of spheres and show the existence of a momentum-like map which classifies these manifolds, up to diffeomorphism.

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Introduction

The aim of this article is to construct and study a class of compact, complex manifolds, which are not algebraic and even not Kählerian nor symplectic, except for a particular case. Unlike the Riemann surfaces, which are all algebraic, compact complex manifolds of dimension bigger than one have to satisfy very particular properties in order to be algebraic (see [We]). Moreover, if we except dimension 2, for which the Kodaira classification keeps close links with the classification of algebraic surfaces (see [B-P-V]), the set of compact, complex, non algebraic manifolds is much larger than that of algebraic manifolds: for example, Taubes' theorem [Ta] on conformal anti-self-dual structures implies that every finitely presented group is the fundamental group of a compact complex 3-manifold. These manifolds are twistor spaces over real 4-manifolds, so are not Kählerian (except for the simply-connected case) by a theorem of Hitchin [Hi].

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However, there are few explicit examples of such manifolds. This is particularly due to the fact that they admit holomorphic embeddings neither in $\mathbb{C}^n$ (by compacity), nor in the complex projective space $\mathbb{C}P^{n-1}$ (otherwise, by Chow's theorem, they would be algebraic). Besides, surgery is much trickier in the complex domain (see [M-K]). Thus, the connected sum of complex manifolds does not have a natural complex structure: the almost complex structures do not generally even extend to the entire connected sum and there are many examples of connected sums of complex manifolds which do not admit any almost complex structure (see [Au]). The class of manifolds that we consider here is obtained by generalizing a construction of Santiago López de Medrano and Alberto Verjovsky [LdM-Ve], which follows other works recalled now.

The first example of compact, complex, non-Kählerian manifolds is Hopf's example [Ho] of complex structures on products of spheres $S^{2n-1} \times S^1$ for every n, which are obtained by taking the quotient of $\mathbb{C}^n - \{0\}$ by a holomorphic, totally discontinuous action of $\mathbb{Z}$. The second de Rham cohomology group of these manifolds is trivial, so they are not symplectic, therefore not Kählerian and finally not algebraic (see [We] for these implications).

The second example is the existence of complex structures on $S^{2k-1} \times S^{2l-1}$ by Calabi and Eckmann [C-E]. The process here is different from Hopf's one: it is a matter of putting a conformal structure on the torus fiber of the bundle $(S^{2k-1} \times S^{2l-1}) \to (\mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1})$ in order to get, in association with the complex structure of the projective space, a complex structure on the products of spheres.

Haefliger [Ha] has generalized Hopf's construction (see also [Bor]) by using the following trick, which is the key fact of the construction presented in this paper: **a smooth manifold embedded in $\mathbb{C}^n$ transversely to a holomorphic foliation is in fact a complex manifold**; transverse holomorphic foliated charts form, when restricted to the transverse embedding, a complex atlas for the manifold. Following this trick, Loeb and Nicolau give in [L-N1] a unified description of the construction of Hopf and that of Calabi-Eckmann and thus find a much larger class of complex structures on products of odd dimensional spheres. To achieve that, they consider a holomorphic vector field in the neighbourhood of $0 \in \mathbb{C}^N$ whose linear part is in Jordan normal form and whose diagonal linear part belongs to the Poincaré domain (i.e. 0 does not belong to the convex hull of the coefficients of the linear part), and show that, if this field satisfies a so-called weak hyperbolicity (m, n) condition, there is an embedding of $S^{2m-1} \times S^{2n-1}$ in $\mathbb{C}^N$ transverse to the flow, and inducing a complex structure on this manifold. Moreover, this description allows them to study Dolbeault's cohomology and the deformations of these manifolds.

Lastly, López de Medrano and Verjovsky [LdM-Ve] have used a linear holomorphic diagonal vector field of $\mathbb{C}^n$ in the Siegel domain this time (i.e. 0 belongs to the convex hull of the linear coefficients) and have shown that, under a weak hyperbolicity condition, the projectivization of the leaf space, when restricted to the closed leaves of the induced flow, is a compact complex manifold. They find again the linear examples of Loeb-Nicolau, but other non-symplectic examples with a more complicated topology too, which they classify. Besides, they construct a smooth space of deformations and show that, in some cases, it is universal.

In this paper, we generalize the latter construction to the case of m vector fields in $\mathbb{C}^n$, with $n > 2m$. The first goal is to obtain and study numerous new examples of compact, complex, non algebraic manifolds, in particular of connected sums of products of spheres, something which is not achieved by López de Medrano and Verjovsky (they find manifolds which are the basis of a non trivial circle-bundle whose total space is a connected sum of products of spheres). In the first part, we adapt the construction. Then we show that, in the limit-case $n = 2m+1$, we obtain complex tori and that every complex torus can be obtained in this way. The third part is devoted to demonstrate that, for $n > 2m + 1$, the constructed manifolds are not symplectic, therefore not algebraic, and do not admit any Kählerian modification. The fourth part studies meromorphic functions and holomorphic 1-forms: we compute, under a generic condition, for which we give a geometric meaning, the degree of transcendence of the field of meromorphic functions and the dimension of the space of global holomorphic 1-forms on these manifolds. The fifth part describes holomorphic vector fields and submanifolds. The sixth part contains the description of an analytic deformation space and we show that this space is universal in some cases. Lastly, the seventh part gives some elements about the classification up to diffeomorphism, showing in particular the existence of a momentum-like map. The article ends with new examples of compact, complex, non symplectic manifolds, including examples of connected sums of products of spheres. Theorems 2, 10 and 11 are generalizations, to the case $m > 1$, of similar statements in [LdM-Ve]. In the same way, Theorems 7 and 8 refer to [L-N2]. In the two cases, when the demonstration is an immediate generalization of the one of these articles, we content ourselves with referring the reader to them. On the contrary, Theorems 1, 3, 4, 5, 6, 9, 12 and 13 do not have any equivalent, either in [LdM-Ve], or in [L-N2], and thus their application to the case $m = 1$ (except for Theorem 1 which reduces in these conditions to a remark and Theorems 12 and 13) specifies the properties of the manifolds of [LdM-Ve]. Lastly, Theorems 14 and 15 describe new families of compact, complex manifolds, which are not obtained in [LdM-Ve]. Some of these results are stated, with a sketch of the proof, in [Me].

Let us indicate that a particular example of compact, complex manifold obtained by a construction very close to this one can be found in [Le2].

I would like to thank my advisor, Alberto Verjovsky, for having guided me so well all over these years, Santiago López de Medrano and Etienne Ghys for

their advice, as well as François Lescure for having made me discover another aspect of this construction.

I. Construction of the manifolds N

In this section, we generalize and adapt the construction of [LdM-Ve]. **The notations that we use here will be maintained throughout the article.**

Let m and n be two positive integers such that $n > 2m$. Let $(\Lambda_1, \dots, \Lambda_n)$ be an n-uple of vectors of $\mathbb{C}^m$ and $\Lambda_i = (\lambda_i^1, \dots, \lambda_i^m)$ for i between 1 and n. Let $\mathcal{H}(\Lambda_1, \dots, \Lambda_n)$ be the convex hull of $(\Lambda_1, \dots, \Lambda_n)$ in $\mathbb{C}^m$.

Definition. *We call admissible configuration an n-uple $(\Lambda_1, \dots, \Lambda_n)$ fulfilling*

(i) the Siegel condition: $0 \in \mathcal{H}(\Lambda_1, \dots, \Lambda_n)$;
(ii) the weak hyperbolicity condition: for every $2m$-uple of integers $(i_1, \dots, i_{2m})$ such that $1 \leq i_1 < \dots < i_{2m} \leq n$, we have $0 \notin \mathcal{H}(\Lambda_{i_1}, \dots, \Lambda_{i_{2m}})$.

This definition can be reformulated geometrically in the following way: the convex polytope $\mathcal{H}(\Lambda_1, \dots, \Lambda_n)$ contains 0, but neither external nor internal facet of this polytope (that is to say hyperplane passing through $2m$ vertices) contains 0. An admissible configuration satisfies the following regularity property (we omit the proof)

Lemma I.1. *Let $\Lambda'_i = (\Lambda_i, 1)$ in $\mathbb{C}^{m+1}$, for i between 1 and n. For all set of integers J between 1 and n such that $0 \in \mathcal{H}((\Lambda_j)_{j\in J})$, the complex rank of the matrix whose columns are the vectors $(\Lambda'_j)_{j\in J}$ is equal to $m+1$, therefore maximal.*

To an admissible configuration $(\Lambda_1, \dots, \Lambda_n)$, we associate the linear foliation of $\mathbb{C}^n$ generated by the m holomorphic commuting vector fields $(1 \leq j \leq m)$

$$\xi_j \ : \ (z_1, \dots, z_n) \in \mathbb{C}^n \mapsto \sum_{i=1}^{n} \lambda_i^j z_i \frac{\partial}{\partial z_i} \ ,$$

and corresponding to the following holomorphic action

$$(T, z) \in \mathbb{C}^m \times \mathbb{C}^n \mapsto (z_1 e^{<\Lambda_1, T>}, \dots, z_n e^{<\Lambda_n, T>}) \in \mathbb{C}^n \ ,$$

where $< \Lambda_i, T >$ means the **scalar product** and not the Hermitian one.

The so-defined foliation is degenerate, in particular 0 is a singular point. Such foliations have been studied in [C-K-P] and [Ku]. The behaviour in the neighbourhood of 0 determines two different sorts of leaves.

Definition. *Let L be a leaf of the previous foliation. If 0 belongs to the closure of L, we say that L is a Poincaré leaf. In the opposite case, we talk of a Siegel leaf.*

The Poincaré leaves do not suit us for, their closure having a common point, they cannot be separated in the quotient space (for the quotient topology). Let us consider the Siegel leaves. A direct generalization of [C-K-P] (see also [LdM-Ve], Sect. 2) shows that the function $\|z\|^2$ on $\mathbb{C}^n$ has a unique minimum when restricted to a Siegel leaf and that the set $\mathcal{T}$ of these minima can be written

$$\mathcal{T} = \left\{ z \in \mathbb{C}^n - \{0\} \ \mid \ \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0 \right\}.$$

From this, the union S of the Siegel leaves is

$$S = \{z \in \mathbb{C}^n - \{0\} \ \mid \ 0 \in \mathcal{H}(\Lambda_j)_{j \in I_z}\} \quad \text{with} \quad j \in I_z \iff z_j \neq 0\,,$$

and the Siegel condition implies that S contains $(\mathbb{C}^*)^n$, therefore is dense in $\mathbb{C}^n$. A more flexible presentation of S consists in writing $S = \mathbb{C}^n - E$ with E an analytic set, whose different components correspond to subspaces of $\mathbb{C}^n$ where some coordinates vanish.

Besides, the leaf space of the foliation restricted to S, that we call M, is identified with $\mathcal{T}$. But the weak hyperbolicity condition implies that the system of equations which defines it is non-degenerate at each point of S (see Lemma I.1) and $\mathcal{T}$ is thus a C^∞-manifold, so, in particular, M is Hausdorff.

As M is Hausdorff and as its embedding $\mathcal{T}$ is transverse to the foliation, a fact that can be proven by a direct computation, we may use transverse holomorphic foliated charts as atlas of M, making it a complex manifold (see the introduction).

Remark now that the previous construction can be projectivized. We may consider the vector fields ξ_j in $\mathbb{C}P^{n-1}$, define $V = S/\mathbb{C}^*$, as well as the transverse submanifold to the foliation

$$\mathcal{N} = \left\{ [z] \in \mathbb{C}P^{n-1} \ \mid \ \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0 \right\}.$$

This transverse submanifold is identified with N, the leaf space of the projectivized foliation restricted to V, but, by the same argument, we put a complex structure on N turning it a compact, complex manifold of dimension $n - m - 1$. The manifold N is the object we wanted to construct.

Remark. The projectivization can be seen as resulting from the action induced by the vector field

$$R(z) = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}, \quad \text{for } z \in \mathbb{C}^n - \{0\}$$

which commutes with the vector fields ξ_1, ..., ξ_m.

Remark. The manifold N appears as a differentiable submanifold $\mathcal{N}$ transverse to a holomorphic foliation of $\mathbb{C}P^{n-1}$ and equipped with the complex structure inherited from this foliation. By a conjecture of Bogomolov [Bo], every compact, complex manifold can be obtained in this way, and this sort of construction seems to be general.

Remark (I would like to thank Blaine Lawson for suggesting the following construction). Let X be a compact, complex manifold and $p : E \to X$ a holomorphic vector bundle with fiber $\mathbb{C}^n$. As previously described, $\mathbb{C}^m \times \mathbb{C}^*$ acts on the fiber. Moreover, suppose that E admits a decomposition as Whitney sum $E = E_1 \oplus \ldots \oplus E_n$ of holomorphic line bundles. Then, this action can be extended to the whole bundle (locally defined as an action on the fiber), so that we may thus construct a (locally trivial) holomorphic fiber bundle $\tilde{p} : \tilde{E} \to X$ with fiber the manifold N defined above.

We now give an alternate description of the manifold N. To achieve that, note that the algebraic torus $(\mathbb{C}^*)^n$ acts holomorphically on S with a dense orbit

$$\Phi : (u, z) \in (\mathbb{C}^*)^n \times S \mapsto (u_1.z_1, \ldots, u_n.z_n) \in S.$$

This action commutes with the previous ones, according to the diagram

$$\begin{array}{ccc}
(\mathbb{C}^*)^n \times S & \xrightarrow{\Phi} & S \\
{\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\
(\mathbb{C}^*)^{n-1}/\mathbb{C}^m \times N & \longrightarrow & N\,,
\end{array}$$

where π is the natural projection of S onto the leaf space N; it is in fact a principal bundle with fiber $\mathbb{C}^* \times \mathbb{C}^m$. From this, there is an action of the complex Abelian Lie group $G = (\mathbb{C}^*)^{n-1}/\mathbb{C}^m$ on the compact manifold N with a dense orbit, so that N is an equivariant compactification of G (see [Le1]). Notice that, in the same way, there is an action of the complex abelian Lie group $\tilde{G} = (\mathbb{C}^*)^n/\mathbb{C}^m$ on M with a dense orbit.

Let now $M_1 = \mathcal{T} \cap S^{2n-1}$. As $\mathcal{T}$ is a cone, it intersects transversely S^{2n-1} and M_1 is a compact, differentiable manifold. We have the following commutative diagram of principal bundles

$$\begin{array}{ccc}
S & \xrightarrow{\pi_0} & M \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow \\
N & \xleftarrow{\pi_1} & M_1\,,
\end{array}$$

where π_0 is the natural projection of S onto M, and π_1 the natural projection of

M_1 onto N (see the remark above about the vector field R). It is easy to verify that π_0 is a principal bundle with fiber $\mathbb{C}^m$ and that π_1 is a principal bundle with fiber S^1. Observe that the existence of the transverse submanifold $\mathcal{T}$ implies that the bundle π_0 is differentiably trivial. Nevertheless, it is not holomorphically trivial. We have in fact

Lemma I.2. *The bundle $(\mathbb{C}^*)^n \to \tilde{G}$, restriction of π_0 to $\tilde{G}$, is not holomorphically trivial.*

Proof. As $\tilde{G}$ is a complex, connected Abelian Lie group, it is isomorphic (in the sense of Lie) to (see [Mor])

$$\mathbb{C}^p \times (\mathbb{C}^*)^q \times C \text{ for some } p \text{ and } q\ ,$$

where C is a Cousin group, i.e. a connected complex Abelian Lie group which does not have any non constant global holomorphic function.

Suppose now that the bundle $(\mathbb{C}^*)^n \to \tilde{G}$ is holomorphically trivial, then we have a biholomorphism

$$(\mathbb{C}^*)^n \simeq \mathbb{C}^{m+p} \times (\mathbb{C}^*)^q \times C\ ,$$

therefore C admits a holomorphic embedding in $(\mathbb{C}^*)^n$, so, as C does not have any non constant holomorphic function, is reduced to 0, and the biholomorphism is

$$(\mathbb{C}^*)^n \simeq \mathbb{C}^{m+p} \times (\mathbb{C}^*)^{n-m-p}$$

which is absurd. □

Remark. The same lemma works for the bundle $\tilde{\pi} : V = (S/\mathbb{C}^*) \to N$ and the Lie group G.

On the contrary, the bundles π and π_1 are not always differentiably trivial when $(\Lambda_1, \dots, \Lambda_n)$ varies. It depends on the existence of an indispensable point (see the definition below).

The action of equivariant compactification of the algebraic torus on S, when restricted to the maximal compact subgroup of this torus, is transformed into an action of the real torus on M_1, namely

$$(e^{i\theta}, z) \in (S^1)^n \times M_1 \mapsto (e^{i\theta_1}.z_1, \dots, e^{i\theta_n}.z_n) \in M_1\ .$$

The quotient of M_1 by this action can be written

$$K = \left\{ r \in (\mathbb{R}^+)^n \ \mid\ \sum_{i=1}^n r_i \Lambda_i = 0,\ \ \sum_{i=1}^n r_i = 1 \right\}\ .$$

The set K is defined as the set of coefficients of the convex hull of $(\Lambda_1, \dots, \Lambda_n)$, so it is a full convex polytope of dimension $n-2m-1$. Above the interior of any face of codimension p of this polytope, the projection $M_1 \to K$ is a trivial fibration with fiber an $(n-p)$-dimensional torus.

We call this polytope the associate polytope of M_1. Remark that the knowledge of this polytope is sufficient to reconstruct S: its different faces correspond to the orbits of different dimensions of the action, and therefore to the different components of S.

We finish this part with the following definition.

Definition (see [LdM1] and [LdM2]). *Let $(\Lambda_1, \dots, \Lambda_n)$ be an admissible configuration, and let S be the corresponding union of the Siegel leaves. We say that the coordinate z_i is an indispensable point of the configuration if the open set S is contained in $\{z \in \mathbb{C}^n \mid z_i \neq 0\}$. We denote by k the number of indispensable points of a configuration.*

The associate polytope has $n-k$ facets.

II. The case $n = 2m+1$

In the limit-case $n = 2m+1$, we shall show that we obtain any complex torus of any dimension. In [LdM-Ve] and [L-N1], only elliptic curves were obtained. Let

$$A = \begin{pmatrix} \lambda_2^1 - \lambda_1^1 & \dots & \lambda_2^m - \lambda_1^m \\ \vdots & & \vdots \\ \lambda_{m+1}^1 - \lambda_1^1 & \dots & \lambda_{m+1}^m - \lambda_1^m \end{pmatrix},$$

and $(A)_{i,j}$ be the minors associated to the matrix A. Let us set

$$\alpha_i = \left(\frac{\displaystyle\sum_{j=1}^m (-1)^{i+j} \det\ (A)_{i,j} (\lambda_{m+1+p}^j - \lambda_1^j)}{\det A} \right)_{p=1}^m,$$

which are well defined by weak hyperbolicity: the rank of A is maximal by application of Lemma I.1.

Theorem 1. *Let $(\Lambda_1, \dots, \Lambda_{2m+1})$ be an admissible configuration. Then*

(i) The manifold N is a complex torus of complex dimension m;
(ii) The lattice is $(e_1, \dots, e_m, \alpha_1, \dots, \alpha_m)$ with e_i vectors of the canonical basis of $\mathbb{C}^m$ and with the α_i as previously defined;
(iii) Any complex torus of any dimension can be obtained in this way.

Proof.
(i) By weak hyperbolicity, we have $S = (\mathbb{C}^*)^{2m+1}$, and the manifold N can be identified with the Lie group G in the presentation of N as an equivariant compactification of G (see Sect. I). This means that N is a connected, compact, complex, Abelian Lie group, therefore a complex torus.

(ii) Let us compute the intersection between a leaf and a transverse plane of dimension $m+1$. The action is given by

$$(\alpha, T, w) \in \mathbb{C}^* \times \mathbb{C}^m \times S \mapsto (\alpha e^{<\Lambda_1, T>} w_1, ..., \alpha e^{<\Lambda_{2m+1}, T>} w_{2m+1}) \in S \ .$$

Let $w \in (\mathbb{C}^*)^{2m+1}$. The orbit of w under the action of the subgroup of $\mathbb{C}^* \times \mathbb{C}^m$ which fixes its m first coordinates can be identified with the lattice of the torus. But (α, T) belongs to this subgroup as soon as it verifies

$$\begin{cases} < \Lambda_2 - \Lambda_1, T > & = 2i\pi k_1 \\ & \vdots \\ < \Lambda_{m+1} - \Lambda_1, T > & = 2i\pi k_m \\ \alpha & = e^{-<\Lambda_1, T>} \end{cases}$$

with $(k_1, \dots, k_m)$ any relative integers. The last equation determines α once T is fixed. The others equations form a Cramer system whose determinant is not zero by Lemma I.1. Let A_i be the matrix obtained from A by deleting the i-th column and by putting as last column the m-uple $(2i\pi k_1, \dots, 2i\pi k_m)$. Thus the solution is, with the notations introduced above,

$$T_i = (-1)^{m-i} \frac{\det\ A_i}{\det\ A} = \frac{\displaystyle\sum_{j=1}^{m} 2i\pi k_j (-1)^{j+i} \det\ (A)_{j,i}}{\det\ A} \quad \text{for} \quad 1 \le i \le m \ .$$

By reinjecting this in the coordinates which are left, one finds that the exponential $e^{2i\pi}$ realizes a biholomorphism (in fact a Lie isomorphism) between N and the complex torus of dimension m with lattice $(e_1, \dots, e_m, \alpha_1, \dots, \alpha_m)$ where the e_i are vectors of the canonical basis and the α_i as previously defined.

(iii) We now show that we can obtain any lattice. Let L be a lattice in $\mathbb{C}^m$ and let $(e_1, \dots, e_m, \alpha_1, \dots, \alpha_m)$ be generators of this lattice. We may assume that $(e_1, \dots, e_m)$ is the canonical basis of $\mathbb{C}^m$ (see [M-K] p. 22). We have to find an admissible configuration which gives this lattice.

Let us choose $\Lambda_2 = e_1, \ \ldots, \ \Lambda_{m+1} = e_m$. Now, det A and det $(A)_{i,j}$ are functions of Λ_1 only. Let $a_{i,j}(\Lambda_1) = (-1)^{i+j}\det\,(A)_{i,j}$. Consider the system

$$\begin{cases} \det A \times \alpha_1 = \left(\displaystyle\sum_{i=1}^{m} a_{1,i}(\lambda^i_{m+1+p} - \lambda^i_1)\right)_{p=1}^{m} \\ \qquad\qquad\vdots \\ \det A \times \alpha_m = \left(\displaystyle\sum_{i=1}^{m} a_{m,i}(\lambda^i_{m+1+p} - \lambda^i_1)\right)_{p=1}^{m} \end{cases}$$

For almost all Λ_1, this linear vectorial $m \times m$ system, with $(\Lambda_{m+2}, \ldots, \Lambda_{2m+1})$ as unknowns, has a non zero determinant, and therefore a unique solution.

Choose such a Λ_1. As $(e_1, \ldots, e_m, \alpha_1, \ldots, \alpha_m)$ has real rank $2m$, we obtain a solution of rank $2m$. Then, changing Λ_1 if necessary, we may obtain an admissible configuration. □

Remark that, from this theorem, we may theoretically give a new description of the Siegel moduli space of m-dimensional complex tori: the quotient of the set of admissible configurations for $n = 2m + 1$ by a group which can be computed by using the explicit expression of the lattice can be identified with this space.

III. For $n > 2m + 1$, the manifold N is not symplectic

The aim of this section is the generalization, to the case of several vector fields, of the following proposition (see [LdM-Ve] p. 258)

Theorem 2. *For $n > 2m + 1$, the manifold N is not symplectic, therefore not algebraic.*

Proof. Let $S = \mathbb{C}^n - E$ and let d be the minimal complex codimension of E.

1st case: Suppose $d > 1$. Then, by transversality, any sphere S^2 embedded in S can be contracted to a point in S. Thus, S is a 2-connected open set, and by the exact sequence in homotopy of the fibration $S \to M_1$, the manifold M_1 is 2-connected too. We may apply the proof of [LdM-Ve].

2nd case: We have $d = 1$ and the proof of [LdM-Ve] cannot be used. Under these conditions, the bundle $\pi_1 : M_1 \to N$ is differentiably trivial. Indeed, as $d = 1$, this means that there is at least an indispensable point, for example z_1, and the action of the torus $(S^1)^n$ can be concentrated on the first coordinate. This gives, up to diffeomorphism

$$M_1 \simeq N \times S^1 \quad \text{and} \quad N \simeq \{r \in (\mathbb{R}^+_*),\ (z_2, \dots, z_n) \in \mathbb{C}^{n-1} \ | \\ r\Lambda_1 + \sum_{i=2}^{n} \Lambda_i |z_i|^2 = 0, r + \sum_{i=2}^{n} |z_i|^2 = 1\} .$$

Besides, if we assume that the k indispensable points are $z_1, \dots, z_k$, there is an identification $M_1 \simeq (S^1)^k \times M_0$, with

$$M_0 \simeq \left\{ (r_1, \dots, r_k) \in (\mathbb{R}^+_*)^k, (z_{k+1}, \dots, z_n) \in \mathbb{C}^{n-k} \ | \right. \\ \left. \sum_{i=1}^{k} r_i\Lambda_i + \sum_{i=k+1}^{n} |z_i|^2 \Lambda_i = 0,\ \sum_{i=1}^{k} r_i + \sum_{i=k+1}^{n} |z_i|^2 = 1 \right\} .$$

By Lemma I.1, M_0 is a differentiable manifold. This decomposition corresponds to the decomposition $S = (\mathbb{C}^*)^k \times S_0$ where $S_0 = \mathbb{C}^{n-k} - F$ and F is an analytic set of complex codimension at least two at each point. But then, using the same argument as in the first case, we deduce from this that, by transversality, S_0 and therefore M_0 are 2-connected.

Moreover, as the two decompositions of M_1 are compatible, we have in fact $N \simeq (S^1)^{k-1} \times M_0$. This diffeomorphism implies an isomorphism between the de Rham cohomology groups

$$H^2(N, \mathbb{R}) \simeq H^2((S^1)^{k-1}, \mathbb{R}) .$$

Let ω be a closed non exact real 2-form on N. If ω^{n-m-1} is a volume form on N, then ω and therefore N are symplectic. But, according to the last equality, this is possible only if

$$2n - 2m - 2 \leq k - 1 .$$

Now, using the fact that the associate polytope is a polytope of dimension $n - 2m - 1$ with $n - k$ facets (see Sect. I), and that such a polytope has at least $n - 2m$ facets (it is the simplex case), we conclude that this inequality can never be satisfied. $\square$

As a consequence of Theorem 2, the manifolds N are not algebraic. Moreover, we shall now prove that they do admit neither algebraic nor Kählerian modifications, so that they cannot be obtained from an algebraic nor Kählerian manifold by a finite sequence of blows-up along analytic sets of codimension 2. Thus, the manifolds N are far from being Kählerian.

Theorem 3. *For $n > 2m + 1$, the manifolds N do not admit any Kählerian modification.*

Proof. Let $h^{1,0} = \dim_{\mathbb{C}} \ H^0(N, \Omega^1)$ and $h^{0,1} = \dim_{\mathbb{C}} \ H^1(N, \mathcal{O})$, where Ω^1 is the sheaf of germs of holomorphic 1-forms on N, and $\mathcal{O}$ the sheaf of germs of holomorphic functions on N. By a theorem of Lescure ([Le1]), a smooth equivariant compactification of a connected complex Abelian Lie group has a Kählerian modification if and only if $h^{1,0} = h^{0,1}$.

We shall first prove that $h^{0,1}$ is greater than or equal to m. Let us consider the following short exact sequence on $V = S/\mathbb{C}^*$

$$0 \to \mathcal{O}_V^{inv} \to \mathcal{O}_V \xrightarrow{L_{\tilde{\xi}_1}\oplus\ldots\oplus L_{\tilde{\xi}_m}} \mathcal{O}_{etr} \to 0\ ,$$

where $\mathcal{O}_V$ is the sheaf of germs of holomorphic functions on V, the vector fields $\tilde{\xi}_1, \ldots, \tilde{\xi}_m$ are the projections onto V of $\xi_1, \ldots, \xi_m$ and $\mathcal{O}_V^{inv}$ is the sheaf of germs of holomorphic functions on V which are constant along the linear foliation generated by $\tilde{\xi}_1, \ldots, \tilde{\xi}_m$. Besides, $L_{\tilde{\xi}_i}$ denotes the Lie derivative with respect to $\tilde{\xi}_i$ and $\mathcal{O}_{etr}$ is the image of $\mathcal{O}_V$ in $\mathcal{O}_V^{\oplus m}$ by the linear operator $L = L_{\tilde{\xi}_1} \oplus \ldots \oplus L_{\tilde{\xi}_m}$.

Remark that, from this exact sequence, we have

$$\text{(I)} \qquad H^1(N, \mathcal{O}) \simeq H^1(V, \mathcal{O}_V^{inv})$$

A class in $H^1(V, \mathcal{O}_V^{inv})$ is represented by an open cover $(V_\alpha)_{\alpha\in A}$ (where A is a finite set of integers; we will omit to write the subscript $\alpha \in A$ from now on) together with holomorphic functions $g_{\alpha\beta}$ defined on the intersection $V_\alpha \cap V_\beta$ and satisfying

$$\text{(C)} \quad \left\{ \begin{array}{ll} \text{the cocycle conditions:} & \left\{ \begin{array}{l} g_{\alpha\beta} + g_{\beta\alpha} = 0 \text{ on } V_\alpha \cap V_\beta \\ g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = 0 \text{ on } V_\alpha \cap V_\beta \cap V_\gamma \end{array} \right. \\ \text{the invariance conditions:} & L_{\tilde{\xi}_1}.g_{\alpha\beta} = \ldots = L_{\tilde{\xi}_m}.g_{\alpha\beta} = 0 \end{array} \right.$$

In particular, the invariance conditions imply that each $g_{\alpha\beta}$ is the pullback by $\tilde{\pi} : V \to N$ of a function $\tilde{g}_{\alpha\beta}$ defined on a open set of N, so that the set $(\tilde{\pi}(V_\alpha), \tilde{g}_{\alpha\beta})$ defines a class in $H^1(N, \mathcal{O})$. This explains relation (I).

Reciprocally, a $\mathbb{C}$-principal bundle over N is defined by an open cover (U_α) of N together with a cocycle $\tilde{g}_{\alpha\beta}$ whose pull back by $\tilde{\pi}$, that is to say $(\tilde{\pi}^{-1}(U_\alpha), \tilde{\pi}^*\tilde{g}_{\alpha\beta})$, is an open cover of V together with a cocycle verifying (C). Observe that this cocycle is trivial if and only if the corresponding principal bundle is trivial.

We shall construct m non trivial cocycles of this type and show that there is no linear relation between them. This shall prove that they define a free system of classes in $H^1(N, \mathcal{O})$, whose dimension is then greater than or equal to m.

Let $\tilde{\chi}$ be a linear combination $a_1\tilde{\xi}_1 + \ldots + a_m\tilde{\xi}_m$. By weak hyperbolicity, there is a vector field

$$\chi = \sum_{i=1}^n \mu_i z_i \frac{\partial}{\partial z_i}$$

on $\mathbb{C}^n$ such that

(i) $(\mu_1, \ldots, \mu_n)$ is an admissible configuration,
(ii) The projection of χ onto V is $\tilde{\chi}$,
(iii) There exists $\alpha \in \mathbb{C}$ such that

$$(1 \leq i \leq m) \qquad \mu_i = a_1\lambda_i^1 + \ldots + a_m\lambda_i^m + \alpha \ .$$

To this admissible configuration is associated an open set of Siegel leaves S'. Remark that, from the characterization of this set given in part I, if we choose wisely α in the previous definition of χ, we have an inclusion $S \subset S'$ and therefore $V \subset V'$, where $V' = S'/\mathbb{C}^*$. Besides, we have an action of $\tilde{\chi}$ on V' whose quotient space is a compact complex manifold N'. And there is a $\mathbb{C}$-principal bundle $p : V' \to N'$. This bundle is defined by an open cover of N' and a cocycle. If we pull back the cover and the cocycle by p and restrict them to V, we obtain an open cover (V_α) of V and a cocycle $h_{\alpha\beta}$. As $\tilde{\chi}$ commutes (as a diagonal linear vector field) with $(\tilde{\xi}_1, \ldots, \tilde{\xi}_m)$, this cocycle verifies (C), so represents a class in $H^1(V, \mathcal{O}_V^{inv})$, as explained above. Now this class is non trivial, because, by Lemma I.2, the corresponding bundle (the restriction of p) is non trivial.

In particular, using the $\tilde{\xi}_j$, we get in this way m non trivial elements of $H^1(N, \mathcal{O})$ represented by cocycles $g^j_{\alpha\beta}$ satisfying (C) and an open cover (V_α) of V (which we assume to be the same for all j).

Let $h_{\alpha\beta}$ be a linear combination $a_1 g^1_{\alpha\beta} + \ldots + a_m g^m_{\alpha\beta}$ defined on the cover (V_α). By a direct computation, it may be verified that this cocycle, via the construction explained above, comes from the vector field

$$\tilde{\chi} = b_1\tilde{\xi}_1 + \ldots + b_m\tilde{\xi}_m \text{ where } \begin{cases} b_i = \dfrac{1}{a_i} \text{ if } a_i \neq 0. \\ b_i = 0 \text{ otherwise.} \end{cases}$$

But, as noticed above, this means that the cocycle $h_{\alpha\beta}$ is not trivial. As a consequence, the family $(g^1_{\alpha\beta}, \ldots, g^m_{\alpha\beta})$ injects as a free family in $H^1(V, \mathcal{O}_V^{inv})$ and we have $h^{0,1} \geq m$.

On the other hand, let $b_1(N)$ be the first Betti number of N. We have

$$2h^{1,0} \leq b_1(N) \leq \max(k-1, 0) \leq 2m - 1 \ , \tag{M}$$

where the first inequality comes from [Bl] and from the fact that every 1-form on N is closed (see [Le1]), the second from the exact sequence in homotopy of the fibration $\pi : S \to N$, and the third from the proof of Theorem 2.

Therefore $h^{1,0} < h^{0,1}$, which achieves the proof. □

Remark. Let $\mathfrak{G}$ be the Lie algebra of G. Then the sheaf $\mathcal{O}_{etr}$ may be identified with a subsheaf of $\mathcal{C}^1(\mathfrak{G}, \mathcal{O})$, the set of 1-cochains of $\mathfrak{G}$ with values in $\mathcal{O}$ (see [Le3]).

IV. Meromorphic functions and holomorphic 1-forms on N

Theorem 3 has as a consequence that the manifold N is not Moïshezon, and therefore that the degree of transcendence of the field of meromorphic functions on N is strictly lower than its complex dimension (see [Mo]). We may prove more. Let $(\Lambda_1, \ldots, \Lambda_n)$ be an admissible configuration and let a be the dimension over $\mathbb{Q}$ of the vector space of the rational solutions of the system

$$\text{(S)} \qquad \begin{cases} \displaystyle\sum_{i=1}^{n} s_i \Lambda_i = 0 \\ \displaystyle\sum_{i=1}^{n} s_i \quad\;\; = 0 \end{cases}$$

Recall that d is the minimal codimension of E with $S = \mathbb{C}^n - E$. We have

Theorem 4.

(i) The degree of transcendence of the field of meromorphic functions on N is greater than or equal to a.
(ii) Moreover, if $d > 1$, this degree is equal to a.

Proof. (i) Let $s^u = (s_1^u, \ldots, s_n^u)$, for u between 1 and a, be a basis of the vector space of the rational solutions of the system (S) satisfying

$$\begin{cases} s_i^u \in \mathbb{N} \text{ for all } i \text{ and all } u \\ \mathrm{GCD}(s_1^u, \ldots, s_n^u) = 1 \text{ for all } u\ . \end{cases}$$

Let us associate to this basis the meromorphic functions $M_u = z_1^{s_1^u} \ldots z_n^{s_n^u}$ on S, for $1 \leq u \leq a$. These functions verify

$$\begin{aligned} &\forall z \in S,\ \forall \alpha \in \mathbb{C}^*,\ \forall T \in \mathbb{C}^m, \\ &M_u(\alpha e^{<\Lambda_1, T>} z_1, \ldots, \alpha e^{<\Lambda_n, T>} z_n) = M_u(z_1, \ldots, z_n)\ , \end{aligned}$$

and therefore can be projected onto meromorphic functions on N. The rational independence of the exponents s^α implying the algebraic independence of the monomials M_α, the degree of transcendence of the field of meromorphic functions on N is at least equal to a.

(ii) Let f_0 be a meromorphic function on N. It can be lifted to a meromorphic function f on S, which is constant along the leaves. As $d > 1$, we have $S = \mathbb{C}^n - E$ with E of complex codimension strictly greater than one at each point. Therefore, by Levi's theorem, f can be extended to $\mathbb{C}^n$ as a meromorphic function, and by continuity, f is constant along the leaves of the singular foliation defined on the whole $\mathbb{C}^n$. The function f is in particular invariant under the action of $\mathbb{C}^*$, and projects onto a meromorphic function on $\mathbb{C}P^{n-1}$. But, according to a classical result, we then have $f = P/Q$ with P and Q homogeneous polynomials with n variables of the same degree g.

Let

$$P(z) = \sum_{|p|=g} a_p z^p \quad \text{and} \quad Q(z) = \sum_{|p|=g} b_p z^p \ ,$$

where we write z instead of $(z_1, \ldots, z_n)$, as well as p instead of $(p_1, \ldots, p_n)$ and z^p instead of $z_1^{p_1} \ldots z_n^{p_n}$. Lastly, $|p|$ means $p_1 + \ldots + p_n$. Moreover, remark that, in this notation, the a_p and b_p cover all the indexes and only a finite number of them are not zero. The polynomials P and Q must verify

$$\frac{P}{Q}(z) = \frac{P}{Q}(e^{<\Lambda,T>} z) \ ,$$

that is to say

$$\sum_{|p|=g} \Big(a_p Q(z) - P(z) b_p \Big) z^p e^{<p\Lambda,T>} \equiv 0 \ . \tag{E}$$

Let us consider the finite set $P = \{(p_1, \ldots, p_n) \ | \ |p| = g\}$ and the equivalence relation

$$p \sim q \iff \sum_{i=1}^{n} p_i \Lambda_i = \sum_{i=1}^{n} q_i \Lambda_i \ .$$

We may then decompose P into equivalence classes $P = P^1 \sqcup \ldots \sqcup P^h$, generated by $p^1, \ldots, p^h$ and rewrite (E) as

$$\sum_{i=1}^{h} \left(\sum_{p \in P^i} \bigg(\Big(a_p Q(z) - b_p P(z) \Big) z^p \bigg) \right) e^{<p^i \Lambda,T>} \equiv 0 \ .$$

By induction on the number of classes, it can be shown that (E) implies

$$\forall \ p^0 \in P, \quad \sum_{p \in P_0} \Big(a_p Q(z) - b_p P(z) \Big) z^p \equiv 0 \ ,$$

where P_0 denotes the equivalence class of $p^0 \in P$. This can be written

$$\frac{P}{Q}(z) = \frac{\displaystyle\sum_{p \in P_0} a_p z^p}{\displaystyle\sum_{p \in P_0} b_p z^p} \tag{F}$$

for a p^0 satisfying $b_{p^0} \neq 0$.

Recall that an integer basis of the space of solutions of (S) is given by $(s^1, \dots, s^a)$. In Equation (F), each equivalence class P_0 is constituted by elements

$$p^0 + \text{ integer linear combination of } (s^1, \dots, s^a).$$

This means that if we factorize z^{p^0} in the numerator and denominator of (F), we may find P_1 and Q_1 elements of $\mathbb{C}[z_1, \dots, z_n]$ such that

$$f(z) = \frac{P}{Q}(z) = \frac{\sum_{p \in P_0} a_p z^{(p-p^0)}}{\sum_{p \in P_0} b_p z^{(p-p^0)}} = \frac{P_1(M_1, \dots, M_a)}{Q_1(M_1, \dots, M_a)} \,.$$

As a consequence, f is algebraically dependent of $M_1, \dots, M_a$, which completes the proof. □

Definition. *We say that an admissible configuration* $(\Lambda_1, \dots, \Lambda_n)$ *fulfills condition (H) if* a *is equal to* 0*, i.e. if it verifies*

$$\begin{cases} \sum_{i=1}^n \Lambda_i s_i = 0 \\ \sum_{i=1}^n s_i = 0 \\ s_i \in \mathbb{Q} \text{ for all } i \end{cases} \qquad \Longrightarrow \quad s_1 = \dots = s_n = 0 \,.$$

Remark. It is a generic condition.

Under the generic condition (H), the manifolds N have very few meromorphic functions. In particular,

Corollary. *Let* $(\Lambda_1, \dots, \Lambda_n)$ *be an admissible configuration fulfilling condition (H). Suppose* $d > 1$*. Then any meromorphic function on* N *is constant.*

This gives a correction to Theorem 4 of [Me] where the generic condition is missing.

On the contrary, note that if we take a configuration with rational coordinates, the dimension a is maximal and equal to $n - 2m - 1$. As any linear Loeb-Nicolau manifold can be obtained by an admissible configuration fulfilling $m = 1$ and $d > 1$ (see [LdM-Ve] p. 261), we may specify a result of [L-N1]

Corollary. *The degree of transcendence of the field of meromorphic functions on* $S^{2k-1} \times S^{2l-1}$ *equipped with a linear Loeb-Nicolau complex structure (with*

$k > 1$ and $l > 1$) is contained between 0 and $k + l - 2$, and these two values can be obtained.

The following proposition gives the geometric meaning of condition (H). Recall that the manifold N is the compactification of a complex Lie group G and that a Cousin group is a complex connected Lie group without any non-constant global holomorphic functions (see part I).

Proposition IV.1. *Let $(\Lambda_1, \dots, \Lambda_n)$ be an admissible configuration. Then the configuration $(\Lambda_1, \dots, \Lambda_n)$ fulfills condition (H) if and only if G is a Cousin group.*

Proof. Notice that we may apply the proof of Theorem 1 to G and have thus a Lie isomorphism between $G = (\mathbb{C}^*)^{n-1}/\mathbb{C}^m$ and $\mathbb{C}^{n-m-1}$ quotiented by the lattice generated by $(e_1, \dots, e_{n-m-1}, \alpha_1, \dots, \alpha_m)$, with $(e_1, \dots, e_{n-m-1})$ canonical basis of $\mathbb{C}^{n-m-1}$ and, as in Theorem 1, one has

$$\alpha_i = \left(\frac{\displaystyle\sum_{j=1}^{m} (-1)^{i+j} \det\ (A)_{i,j} (\lambda^j_{m+1+p} - \lambda^j_1)}{\det A} \right)_{p=1}^{n-m-1} .$$

From now on, we take this model as definition of G. Remark that we assume for this identification that $(\Lambda_1, \dots, \Lambda_{2m+1})$ is an admissible subconfiguration, in order to have $\det A$ not zero. This assumption is always possible by Lemma I.1. Suppose that G is not a Cousin group. This means it is isomorphic to $G' = (\mathbb{C}^*)^p \times C$ with $p > 0$ and C a Cousin group (see [Mor]; the presence of the canonical basis in the lattice defining G proves that there is no $\mathbb{C}$-factor in its decomposition). We may identify G' with the quotient of $\mathbb{C}^{n-m-1}$ by the lattice involving the canonical basis and m vectors whose p first coordinates are zero. Taking this model as definition of G', we then have

$$\begin{array}{ccc} \mathbb{C}^{n-m-1} & \xrightarrow{L} & \mathbb{C}^{n-m-1} \\ \downarrow & & \downarrow \\ G & \longrightarrow & G' , \end{array}$$

where L is a Lie isomorphism of $\mathbb{C}^{n-m-1}$ carrying the lattice defining G onto the lattice defining G'. This implies that L is a linear transformation of $\mathbb{C}^{n-m-1}$ completely determined by a matrix Z of $\mathrm{PSL}_{n-1}(\mathbb{Z})$ which gives the coordinates of the vectors $(e_1, \dots, e_{n-m-1}, \alpha_1, \dots, \alpha_m)$ in the basis of $\mathbb{C}^{n-1}$ given by the vectors $(e_1, \dots e_{n-m-1}, \beta_1, \dots, \beta_m)$, the $n-1$ vectors generating the lattice of G'.

Let $(a_{i,j})$ be the integer coefficients of Z. Then, for $1 \leq s \leq m$, we have

$$\begin{aligned} L(\alpha_s) &= \alpha_s^1\ L(e_1) + \dots + \alpha_s^{n-m-1} L(e_{n-m-1}) \\ L(\alpha_s) &= \alpha_s^1\ (a_{1,1}e_1 + \dots + a_{1,n-1}\beta_m) + \dots + \\ &\qquad\qquad \alpha_s^{n-m-1}(a_{n-1,1}e_1 + \dots + a_{n-1,n-1}\beta_m)\ . \end{aligned}$$

Projecting onto z_1, as the first coordinate of each β_i is zero, one gets

$$1 \leq s \leq m \qquad\qquad a_{1,1}\alpha_s^1 + \dots + a_{1,n-m-1}\alpha_s^{n-m-1} = a_{1,n-m-1+s}\ ,$$

and there are others similar relations for the projections onto $z_2, \dots, z_n$.

Now, it is straightforward to verify that the solutions of these equations are given by the solutions of the system

$$\begin{aligned} &a_{1,1}(\Lambda_2 - \Lambda_1) + \dots + a_{1,n-m-1}(\Lambda_{n-m} - \Lambda_1) = \\ &\quad a_{1,n-m}(\Lambda_{n-m+1} - \Lambda_1) + \dots + a_{1,n-1}(\Lambda_n - \Lambda_1)\ , \end{aligned}$$

that is to say that there are integer solutions if and only if condition (H) is not satisfied. □

Remark. If $(\Lambda_1, \dots, \Lambda_n)$ does not satisfy condition (H), then, by Theorem 4, the manifold N has rational meromorphic functions. The restrictions of these functions to G give global non constant holomorphic functions on G, which thus cannot be a Cousin group.

As a consequence of this proposition, we state

Theorem 5. *Let $(\Lambda_1, \dots, \Lambda_n)$ be an admissible configuration fulfilling condition (H). Then we have*

$$h^{1,0} = \dim_{\mathbb{C}}\ H^0(N, \Omega^1) = \max(0, k-m-1).$$

Proof. Under condition (H), the group G is a Cousin group. The holomorphic 1-forms on G can then be written, in the model given in proposition IV.1,

$$\Omega = \sum_{i=1}^{n-m-1} \alpha_i dz_i\ ,$$

where the α_i are constants. Now, using the Lie isomorphism between G and $(\mathbb{C}^*)^{n-1}/\mathbb{C}^m$ and pulling back by π, these forms are transformed into

$$\omega = \sum_{i=1}^{n-m-1} \frac{a_i}{z_i} dz_i \text{ on } (\mathbb{C}^*)^n \subset S$$

with a_i constants verifying

$$\begin{cases} \displaystyle\sum_{i=1}^{n} a_i \Lambda_i = 0 \\ \displaystyle\sum_{i=1}^{n} a_i = 0 \,. \end{cases}$$

If z_i is not an indispensable point of the configuration, the form ω extends to S if and only if a_i is zero. Therefore, if we assume that the k indispensable points of the configuration are the k first coordinates, the 1-forms of N are projection by π of the 1-forms

$$\omega = \sum_{i=1}^{k} \frac{a_i}{z_i} dz_i \text{ on } S$$

with a_i constants verifying

$$\begin{cases} \displaystyle\sum_{i=1}^{k} a_i \Lambda_i = 0 \\ \displaystyle\sum_{i=1}^{k} a_i = 0 \,. \end{cases}$$

By weak hyperbolicity, this system has maximal rank, so the space of solutions has dimension $\max(0, k - m - 1)$. □

Corollary. *Let* $(\Lambda_1, \ldots, \Lambda_n)$ *be an admissible configuration. In the following cases, there are no global holomorphic forms of any degree on* N*:*

(i) $n > 2m + 1$ *and* $k < 3$.
(ii) $(\Lambda_1, \ldots, \Lambda_n)$ *fulfills condition (H) and* $k < m + 2$.

Proof. For 1-forms, this is an immediate consequence of majoration (M) in case (i) and of Theorem 5 in case (ii). But in the case of an equivariant compactification, this implies that there are no holomorphic forms of greater degree (see [Le1]). □

Remark. The condition $k < 3$ is always fulfilled for the manifolds of [LdM-Ve].

V. Holomorphic vector fields and complex geometry of N

Throughout this section, we denote by $\mathcal{F}$ the foliation induced by the vector fields $(\xi_1, \dots, \xi_m, R)$ on S, and use the decomposition of the tangent bundle of S

$$TS = T\mathcal{F} \oplus N\mathcal{F} \ ,$$

where the normal bundle to the foliation $N\mathcal{F}$ is defined as a **smooth** orthogonal complement to $T\mathcal{F}$ for the standard Hermitian product of $\mathbb{C}^n$. Observe that $N\mathcal{F}$ is holomorphic above $(\mathbb{C}^*)^n \subset S$, where the foliation is holomorphically trivial. Lastly, we call Θ the sheaf of germs of holomorphic vector fields on N.

Theorem 6. *Let $(\Lambda_1, \dots, \Lambda_n)$ be an admissible configuration, and let N be the corresponding manifold.*

(i) The manifold N has at least $n-m-1$ globally linearly independent holomorphic vector fields, i.e. we have $\dim_{\mathbb{C}} H^0(N, \Theta) \geq n-m-1$.
(ii) If $d > 1$ and if the Λ_i are all different, one has an equality $\dim_{\mathbb{C}} H^0(N, \Theta) = n-m-1$.

Proof. (i) By commutation with ξ_1, ..., ξ_m and R, every diagonal linear vector field

$$\chi = \sum_{i=1}^{n} \alpha_i z_i \frac{\partial}{\partial z_i}$$

on $\mathbb{C}^n$ defines a global holomorphic vector field $\tilde{\chi}$ on N.

Let $(\chi_1, \dots, \chi_{n-m-1})$ be holomorphic diagonal linear vector fields of S, forming a basis of $N\mathcal{F}$ at each point of $(\mathbb{C}^*)^n$, and let $(\tilde{\chi}_1, \dots, \tilde{\chi}_{n-m-1})$ be their projections. The vector fields $(\chi_1, \dots, \chi_{n-m-1}, \xi_1, \dots, \xi_m, R)$ are linearly independent at each point of $(\mathbb{C}^*)^n$, therefore globally linearly independent on S. From the fibration $\pi : S \to N$ and the isomorphism $N_z\mathcal{F} \simeq T_{\pi(z)}N$ for all $z \in S$, we deduce the global linear independence of $(\tilde{\chi}_1, \dots, \tilde{\chi}_{n-m-1})$ on N. Thus, we construct $n-m-1$ globally linearly independent holomorphic vector fields on N.

(ii) Let $\tilde{\chi}$ be a global holomorphic vector field on N. In the following commutative diagram

$$\begin{array}{ccc} N\mathcal{F} & \longrightarrow & TN \\ \downarrow & & \downarrow \\ S & \xrightarrow{\pi} & N \ , \end{array}$$

the bundle $N\mathcal{F} \to S$ is isomorphic to the pullback of the tangent bundle $TN \to N$ by π. This property allows us to lift $\tilde{\chi}$ to a smooth vector field χ on S, which commutes with ξ_1, ..., ξ_m and R. Moreover, this vector field is holomorphic on $(\mathbb{C}^*)^n \subset S$, so is in fact holomorphic on S.

As $d > 1$, this vector field can be extended holomorphically to $\mathbb{C}^n$, and this extension, by density of S and continuity of χ, commutes with ξ_1, ..., ξ_m and R on the whole of $\mathbb{C}^n$. In particular, the commutation with R implies that χ defines a global holomorphic vector field on $\mathbb{C}P^{n-1}$, therefore is linear (see [C-K-P]). Moreover, the commutation with ξ_1, ..., ξ_m implies then, as the Λ_i are different, that χ is diagonal linear and that it is one of the vector fields constructed in part (i). □

We now generalize the results of [L-N2].

Definition (see [L-N2]). *Let N be a complex manifold equipped with a regular holomorphic foliation $\mathcal{G}$. Let ω be a closed real 2-form on N. The foliation $\mathcal{G}$ is called transversely Kählerian with respect to ω if and only if*

(i) the form ω is J-invariant (where J is the almost complex structure of N),
(ii) for all $z \in N$, the kernel of $\omega(z)$ is the tangent space to the foliation $T_z\mathcal{G}$,
(iii) the quadratic form $h(u_1, u_2) = \omega(Ju_1, u_2) + i\omega(u_1, u_2)$ is positive definite on $N\mathcal{G}$, the normal bundle to the foliation.

Theorem 7. *Let $(\Lambda_1, \ldots, \Lambda_n)$ be an admissible configuration and let N be the corresponding compact, complex manifold. There exists on N a regular holomorphic foliation of dimension m, which is transversely Kählerian with respect to the Euler class of the bundle $\pi_1 : M_1 \to N$.*

Proof. As in [L-N2], we consider the vector fields of $\mathbb{C}^n$

$$\eta_j = \sum_{i=1}^{n} \mathrm{Re}\ (\lambda_i^j) z_i \frac{\partial}{\partial z_i} \quad \text{for} \quad 1 \leq j \leq m\ .$$

By Lemma I.1, the vector fields $(\eta_1, \ldots, \eta_m, \xi_1, \ldots, \ldots, \xi_m, R)$ are linearly independent at each point of S, so their projections $(\tilde{\eta}_1, \ldots, \tilde{\eta}_m)$ are linearly independent at each point of N and generate a regular holomorphic foliation of dimension m.

A direct generalization of the proof given in [L-N2] shows that the so-defined foliation is transversely Kählerian with respect to ω, the projection onto N of the standard Kählerian 2-form of $\mathbb{C}^n$.

Finally, the bundle $\pi_1 : M_1 \to N$ is the pullback of the bundle $S^{2n-1} \to \mathbb{C}P^{n-1}$ by the smooth embedding of N into the projective space (see part I). Therefore its Euler class is the restriction of the Kähler form of $\mathbb{C}P^{n-1}$ to this embedding, that is to say, is 2ω. □

As in [L-N2], this allows us to describe analytic sets and holomorphic submanifolds of N in a generic case. To achieve that, recall that N contains holomorphic submanifolds, that we shall call standard, which are constructed in the following way: to every subset J of $\{1, 2, \ldots, n\}$ fulfilling the Siegel condition $0 \in \mathcal{H}((\Lambda_j)_{j\in J})$, we associate the standard submanifold N_J of N obtained as leaf space for the foliation restricted to

$$S_J = \{z \in S \mid z_j = 0 \text{ for } j \notin J\} .$$

Theorem 8. *Let $(\Lambda_1, \ldots, \Lambda_n)$ be an admissible configuration fulfilling condition (H). Then any analytic set (respectively holomorphic submanifold) of N of dimension greater than or equal to $d+m-1$ is a union of standard submanifolds (respectively a standard submanifold).*

Recall that d is the minimal codimension of E in $S = \mathbb{C}^n - E$. The proof of this theorem is rigorously identical to that of [L-N2], once we have established the following modification of Proposition 1 of [L-N2].

Proposition V.1. *Let Y be an analytic set of N of dimension greater than or equal to $d + m - 1$. Then Y is tangent to $\mathcal{G}$, the foliation of Theorem 7: for all y regular point of Y, we have: $T_y\mathcal{G} \subset T_yY$.*

Proof. Let ω represent the Euler class of the bundle $\pi_1 : M_1 \to N$, therefore of the transversely Kählerian 2-form on N. From this, the exterior product ω^d is exact (see [LdM-Ve] p. 259), and, as ω is closed, ω^l is exact for every $l \geq d$. Now, it is sufficient to show the result for an analytic set Y of dimension $d + m - 1$. Let $\overline{Y}$ be the regular part of Y, and let

$$c = \min_{y\in\overline{Y}}(\dim(T_yY \cap T_y\mathcal{G})) .$$

Let K be a holomorphic distribution on $\overline{Y}$ of c-planes such that

$$\text{For all } y \in \overline{Y}, \quad K_y \subset T_yY \cap T_y\mathcal{G} .$$

We want to show that $K = T\mathcal{G}_{|\overline{Y}}$, therefore that $c = m$. Let us suppose the contrary. As K is holomorphic, there is a volume form V on it. Let

$$\omega_1 = \omega^{d+m-1-c} \wedge V .$$

This is an exact form of dimension $d + m - 1$ on Y of dimension $d + m - 1$. Let $y \in \overline{Y}$.

1st case: $\dim(T_yY \cap T_y\mathcal{G}) > c$.

Then $\omega_1(y) = 0$.

2nd case: $\dim(T_yY \cap T_y\mathcal{G}) = c$.

Then $\omega_1(y)$ is strictly positive.

By exactness of ω_1, Stokes theorem for analytic sets implies

$$\int_Y \omega_1 = 0 \text{ , hence } \forall\ y \in \overline{Y},\ \omega_1(y) = 0 \text{ .}$$

But this is absurd, because this implies that, at each point y of $\overline{Y}$, we are in the first case. This finishes the proof. □

Remark. This proposition is not specific to our construction and is still valid in the case of a compact, complex manifold N equipped with a transversely Kählerian foliation $\mathcal{G}$ with respect to a 2-form ω, such that ω^d is exact for a fixed integer d.

We close this part with a short study of the quotient space of the transversely Kählerian foliation on N. This study is not contained in [L-N2].

Definition. *We say that an admissible configuration $(\Lambda_1, \ldots, \Lambda_n)$ fulfills condition (K) if and only if, for the space of solutions of System (S),*

$$(S) \qquad \begin{cases} \displaystyle\sum_{i=1}^{n} s_i \Lambda_i = 0 \\ \displaystyle\sum_{i=1}^{n} s_i \qquad = 0 \end{cases}$$

we may choose a basis with rational coordinates.

Remark. Condition (H) can be restated as: there is no rational solution of System (S), so is totally opposite to condition (K).

We now have

Theorem 9. *Let $(\Lambda_1, \ldots, \Lambda_n)$ be an admissible configuration fulfilling condition (K), let N be the corresponding manifold and let $\mathcal{G}$ be the transversely Kählerian foliation of Theorem 7. Then*

(i) All leaves of $\mathcal{G}$ are complex tori of complex dimension m;
(ii) The quotient space $N/\mathcal{G}$ is a Kählerian orbifold.

We will not prove this theorem and state it as an announcement, as we intend to show it in another paper where we shall study more extensively the quotient space $N/\mathcal{G}$.

VI. Affine equivalence and deformation space of N

This part deals with the generalization of the results of [LdM-Ve] on the deformation space of N.

Definition. *We say that* $(\Lambda_1, ..., \Lambda_n)$ *and* $(\Lambda'_1, ..., \Lambda'_n)$ *are equivalent configurations if there is a continuous map* $H : [0, 1] \to (\mathbb{C}^m)^n$ *such that*

(i) $H(0) = (\Lambda_1, ..., \Lambda_n)$,
(ii) $H(1) = (\Lambda''_1, ..., \Lambda''_n)$, *where the configuration* $(\Lambda''_1, ..., \Lambda''_n)$ *is an arbitrary permutation of* $(\Lambda'_1, ..., \Lambda'_n)$,
(iii) for all t *of* $[0, 1]$, *the set* $H(t)$ *is an admissible configuration.*

Remark that two equivalent configurations give diffeomorphic manifolds M_1 and diffeomorphic manifolds N. Indeed, by definition, $H(t)$ is an admissible configuration, therefore $M_1(t)$ (respectively $N(t)$) is a differentiable manifold for all t. The union of these manifolds fibers over the interval, and this fibration is a submersion at each point of the interval. Therefore it is locally trivial by Ehresmann's lemma (see [M-K] p.19-21 for a direct proof), so $M_1(0)$ and $M_1(1)$ (respectively $N(0)$ and $N(1)$) are diffeomorphic.

However, the converse is false. There are diffeomorphic manifolds M_1 coming from non equivalent admissible configurations (see [LdM1] p. 242 for an example).

If two admissible configurations can be obtained one from the other by an affine transformation of $\mathbb{C}^m$ and are equivalent, the two manifolds N are biholomorphic. In fact, there is a biholomorphism between the open sets S which sends leaf onto leaf. If $d > 2$, the converse is true. It is a generalization of Theorem 5 of [LdM-Ve] (the proof is the same).

Theorem 10. *Let* $(\Lambda_1, ..., \Lambda_n)$ *and* $(\Lambda'_1, ..., \Lambda'_n)$ *be two equivalent admissible configurations, and let* N *and* N' *be the corresponding compact complex manifolds. Suppose* $d > 2$. *Then* N *and* N' *are biholomorphic if and only if the two configurations can be obtained one from the other by an affine transformation of* $\mathbb{C}^m$, *i.e. if and only if there exists* $(\Lambda''_1, \dots, \Lambda''_n)$ *an arbitrary permutation of* $(\Lambda'_1, ..., \Lambda'_n)$ *and*

$$A \in GL_m(\mathbb{C}) \text{ and } B \in \mathbb{C}^m \text{ such that}$$
$$(\Lambda''_1, \dots, \Lambda''_n) = (A\Lambda_1 + B, \dots, A\Lambda_n + B)\,.$$

Now let $(\Lambda_1, ..., \Lambda_n)$ be an admissible configuration and let E be the set of admissible configurations equivalent to $(\Lambda_1, ..., \Lambda_n)$, quotiented by the affine equivalence relation of Theorem 10.

Lemma VI.1. *The set* E *is a finite quotient or an open set of* $(\mathbb{C}^{n-m-1})^m$.

Proof. By Lemma I.1, we may assume that $(\Lambda_1, ..., \Lambda_m)$, the m first vectors of the base point of E, have complex rank m and that there exists $A \in GL_m(\mathbb{C})$ and $B \in \mathbb{C}^m$ mapping the $m+1$ first vectors of the configuration onto

$$(A\Lambda_1 + B, \dots, A\Lambda_{m+1} + B) = (e_1, \dots, e_m, 0) ,$$

where $(e_1, \dots, e_m)$ is the canonical basis of $\mathbb{C}^m$. The vectors $(\Lambda_{m+2}, \dots, \Lambda_n)$ are sent by this transformation onto a set $M = (M_1, \dots, M_{n-m-1})$ of vectors of $\mathbb{C}^m$.

Let $\Lambda' = (\Lambda'_1, \dots, \Lambda'_n)$ represent a class in E. As Λ' is equivalent to Λ, there exists a permutation Λ'' such that Λ and Λ'' are homotopic (that is to say, according to the definition of equivalent configurations, that there is a continuous path of admissible configurations joining Λ to Λ''). Now, as above, there is a unique affine transformation mapping $(\Lambda'_1, \dots, \Lambda'_{m+1})$ onto $(e_1, \dots, e_m, 0)$. By this transformation the vectors $(\Lambda'_{m+2}, \dots, \Lambda'_n)$ are sent onto $n-m-1$ vectors of $\mathbb{C}^m$, let us call them $M' = (M'_1, \dots, M'_{n-m-1})$.

It is clear that the matrix M' does not depend on the choice of the representant and that two different classes in E give different sets M'. Therefore we have identified E with a finite quotient of a set P of $(\mathbb{C}^{n-m-1})^m$ up to permutation.

Finally, observe that if Λ' is close enough to Λ (for the product topology on $\mathbb{C}^m \times \dots \times \mathbb{C}^m$), it is equivalent to Λ. So it defines a class in E whose corresponding point in P is close to M. Therefore, P is open. □

Thus, for each manifold N, we have constructed an analytic deformation space of dimension $m(n-m-1)$. For $d > 2$, two different points of E correspond to different complex structures on the same manifold N, up to diffeomorphism. From this, E is a reduced moduli space, and there is a holomorphic injection of E into the Kuranishi space of N.

Theorem 11 (see [LdM-Ve], Theorem 6). *Let $(\Lambda_1, \dots, \Lambda_n)$ be an admissible configuration, let N be the corresponding compact complex manifold, and let E be defined as above. Suppose $d > 3$ and suppose that the Λ_i are all different. Then the open set E is a universal deformation space (a moduli space) of N.*

Proof. As in [LdM-Ve], we shall prove that the dimension of E and that of $H^1(N, \Theta)$ are the same. Since the dimension of the Kuranishi space of N, let us call K, is smaller than or equal to that of $H^1(N, \Theta)$ (because of the Kodaira-Spencer map, see [Su] p. 160), this implies that E and K have the same dimension. Now, as E injects holomorphically in K, in a regular point, the Kodaira-Spencer map is an isomorphism. This is sufficient to prove, by a theorem of Kodaira-Spencer, that E is a versal deformation space of N, like K (see [Su] p. 160). Finally, as two points of E correspond to two different complex structures on N, it follows from all this that E is a universal deformation space of N. To compute the dimension of E, we consider, in addition to the exact sequence of Theorem 3, the two following short exact sequences on $V = S/\mathbb{C}^*$

$$
\begin{array}{llll}
0 \to \Theta^{inv} \to \Theta_V & \xrightarrow{[\tilde{\xi}_1,-]\oplus\ldots\oplus[\tilde{\xi}_m,-]} & \Theta_{etr} \to 0 \ , \\
0 \to \mathcal{O}^{inv}.\tilde{\xi}_1 \oplus \ldots \oplus \mathcal{O}^{inv}.\tilde{\xi}_m \to \Theta^{inv} & \xrightarrow{\pi} & \Theta_b \to 0 \ ,
\end{array}
$$

where the sheaf Θ_V is the sheaf of germs of holomorphic vector fields on V, and the sheaf $\Theta_{etr} \subset (\Theta_V)^{\oplus m}$ is the image sheaf of Θ_V by the linear operators corresponding with Lie brackets $[\tilde{\xi}_1, -] \oplus \ldots \oplus [\tilde{\xi}_m, -]$. Lastly, the sheaf Θ^{inv} is the sheaf of germs of holomorphic vector fields on V which commute with $\tilde{\xi}_1, \ldots, \tilde{\xi}_m$ and Θ_b is defined by these sequences.

Then we have

$$H^i(V, \Theta_b) \simeq H^i(N, \Theta) \ \text{ for all i.}$$

As $d > 3$, by Scheja's results [Sc] we have the following identifications

$$
\begin{aligned}
H^0(V, \Theta_V) &= H^0(\mathbb{C}P^{n-1}, \Theta_{\mathbb{C}P^{n-1}}) \simeq sl_n(\mathbb{C}) \\
&= \{\text{matrices of } M_n(\mathbb{C}) \text{ of trace } 0\} \\
\text{and} \qquad & H^1(V, \mathcal{O}_V) = H^2(V, \mathcal{O}_V) = H^1(V, \Theta_V) = H^2(V, \Theta_V) = 0 \ .
\end{aligned}
$$

The exact sequence of Theorem 3 gives the long exact sequence in cohomology

$$
\left\{
\begin{array}{l}
0 \to \mathbb{C}^m \to H^1(V, \mathcal{O}^{inv}) \to 0 \\
0 \to H^1(V, \mathcal{O}_{etr}) \to H^2(V, \mathcal{O}^{inv}) \to 0 \ ,
\end{array}
\right.
$$

and therefore $H^1(V, \mathcal{O}^{inv}) = \mathbb{C}^m$ and $H^2(V, \mathcal{O}^{inv}) = H^1(V, \mathcal{O}_{etr})$.

As the vectors Λ_i are all different, $H^0(V, \Theta^{inv})$ can be identified with the diagonal matrices modulo the scalar ones, and, by Theorem 6, we have $H^0(V, \Theta_b) = \mathbb{C}^{n-m-1}$.

The two long exact sequences in cohomology are thus

$$
\begin{array}{l}
0 \to \mathbb{C}^{n-1} \to sl_n(\mathbb{C}) \to H^0(V, \Theta_{etr}) \to H^1(V, \Theta^{inv}) \to 0 \ , \\
0 \to \mathbb{C}^m \oplus \ldots \oplus \mathbb{C}^m \to H^1(V, \Theta^{inv}) \to H^1(V, \Theta_b) \xrightarrow{p} H^2(V, (\mathcal{O}^{inv})^{\oplus m}) \\
\quad \to \ldots \ ,
\end{array}
$$

Using the previous isomorphism between $H^1(V, \mathcal{O}_{etr})$ and $H^2(V, \mathcal{O}^{inv})$ and Čech cocycles, it is straightforward to verify that the map p has 0 as image. The second long exact sequence turns to be

$$0 \to \mathbb{C}^m \oplus \ldots \oplus \mathbb{C}^m \to H^1(V, \Theta^{inv}) \to H^1(V, \Theta_b) \to 0 \ .$$

As the tangent space of E injects into $H^1(V, \Theta_b)$, the dimension of this space is greater than or equal to $m(n-m-1)$. We have to know the dimension of $H^0(V, \Theta_{etr})$ in order to say more. An element of this space is a collection of m linear vector fields $(M_1, \ldots, M_m)$ of $\mathbb{C}^n$ verifying locally that there exists a germ of vector field χ over $\mathbb{C}^n$ such that

$$[\chi, R] = 0,\ [\chi, \xi_i] = M_i, \quad 1 \leq i \leq m\,.$$

In particular, by the Jacobi identity, one has

$$[M_k, \xi_l] = [M_l, \xi_k] \ \text{ for } \ 1 \leq k,\ l \leq m\,.$$

Let us denote by $(m_{ij}^k)_{i,j=1}^n$ the coefficients of the matrix which characterize the linear vector field M_k. We have

$$m_{ji}^l(\lambda_j^k - \lambda_i^k) = m_{ji}^k(\lambda_j^l - \lambda_i^l) \ \text{ for } \ 1 \leq k,\ l \leq m\,.$$

As the Λ_i are all different, we thus obtain $n(n-1)(m-1)$ non trivial relations between the coefficients of the vector fields M_1, ..., M_m. Therefore

$$\dim_{\mathbb{C}}\ H^0(V, \Theta_{etr}) \leq (n-1)(n+m)\,.$$

But the exact sequences imply then

$$\dim_{\mathbb{C}}\ H^0(V, \Theta_{etr}) = (n-1)(n+m) \ \text{ and } \ \dim_{\mathbb{C}}\ H^1(V, \Theta_b) = m(n-m-1)$$

which achieves the proof. □

Remark. Some manifolds N have a disconnected moduli space. Let $(\Lambda_1, ..., \Lambda_n)$ and $(\Lambda'_1, ..., \Lambda'_n)$ be two admissible configurations, non equivalent but giving diffeomorphic manifolds M_1. Suppose $d > 2$. Then, by Theorem 10, we have two reduced moduli spaces E and E', whose union is not connected: it is not possible to go continuously from a complex structure of E on N to a complex structure of E'.

Remark. As in Theorem 3, we have an identification between Θ_{etr} and a subsheaf of $\mathcal{C}^1(\mathfrak{G}, \Theta)$, the set of 1-cochains of the Lie algebra $\mathfrak{G}$ with values in Θ. Moreover, the map $\Theta \to \Theta_{etr}$ is the Koszul differential (see [Le3]).

VII. Some elements of classification of M_1

In this section, we give some results on the classification of M_1, up to diffeomorphism. We choose to classify M_1 rather than N, because we have a powerful classification tool for M_1: the explicit smooth action of the real torus on it. We begin this section by a reduction theorem.

Theorem 12. *Let $(\Lambda_1, \dots, \Lambda_n)$ be an admissible configuration. If $k > 1$ and $m > 1$, then there is an admissible configuration $(\Lambda'_1, \dots, \Lambda'_{n-2})$ such that M_1 is diffeomorphic to $M'_1 \times S^1 \times S^1$, where M_1 and M'_1 are the manifolds corresponding respectively to $(\Lambda_1, \dots, \Lambda_n)$ and $(\Lambda'_1, \dots, \Lambda'_{n-2})$.*

Remark. The configuration $(\Lambda_1, \dots, \Lambda_n)$ corresponds to an action of $\mathbb{C}^m$, but the configuration $(\Lambda'_1, \dots, \Lambda'_{n-2})$ to an action of $\mathbb{C}^{m-1}$.

This theorem is a reduction theorem in the sense that it allows us to restrict the configurations to study in order to give a classification of M_1, up to diffeomorphism. The following immediate corollary specifies this.

Corollary. *To establish the classification of the manifolds M_1 up to diffeomorphism, it is sufficient to use the configurations with one or without an indispensable point.*

This corollary is the motive for the following definition.

Definition. *We call reduced admissible configuration an admissible configuration with only one or without an indispensable point.*

Proof of the theorem. Let $(\Lambda_1, \dots, \Lambda_n)$ be an admissible configuration with $k > 1$. We then have

$$M \simeq \left\{ w \in \mathbb{C}^n \ \mid \ \sum_{i=1}^{n} \Lambda_i |w_i|^2 = 0 \right\}.$$

In order to simplify, we shall suppose that z_1 and z_2 are indispensable points. As in the proof of Theorem 2, we may write the following diffeomorphism

$$\begin{aligned} M \simeq\ & S^1 \times S^1 \\ & \times \left\{ w \in \mathbb{C}^{n-2},\ (r_1, r_2) \in (\mathbb{R}^+_*)^2 \ \mid \ \Lambda_1 r_1 + \Lambda_2 r_2 + \sum_{i=3}^{n} \Lambda_i |w_{i-2}|^2 = 0 \right\}. \end{aligned}$$

Let us call M' the manifold on the righthand side of this expression. By Lemma I.1, the system which defines M' has maximal rank, and, using the last two equations for example, is equivalent to

$$|w_1|^2\Lambda'_1 + \ldots + |w_{n-2}|^2\Lambda'_{n-2} = 0\ ,$$

where the Λ'_i are vectors of $\mathbb{C}^{m-1}$. We must now verify that $(\Lambda'_1, \ldots, \Lambda'_{n-2})$ is an admissible configuration.

To achieve that, remark that the associate polytopes of M and M' are combinatorially equivalent (there is a bijection between them which maps a face onto a face and respects the inclusion of faces - see [B-L] or [Gr]) and can be identified with

$$K = \left\{ r \in (\mathbb{R}^+)^n \ \mid\ \sum_{i=1}^{n} r_i\Lambda_i = 0,\ \sum_{i=1}^{n} r_i = 1 \right\}.$$

Equality of the associate polytope means that $S = S' \times (\mathbb{C}^*)^2$. Now, let us suppose that the weak hyperbolicity condition is not satisfied for the configuration $(\Lambda'_1, \ldots, \Lambda'_{n-2})$, for example let us suppose that 0 belongs to the convex hull of $(\Lambda'_1, \ldots, \Lambda'_{2m-2})$. Then

$$P = \{w_{2m-1} = \ldots = w_n = 0,\ w_j \neq 0 \text{ for } 1 \leq j \leq 2m-2\}$$

is included in S', and therefore $P \times (\mathbb{C}^*)^2$ is included in S. Thus 0 belongs to the convex hull of $(\Lambda_1, \ldots, \Lambda_{2m})$, which is absurd. $\square$

Corollary. *Let $(\Lambda_1, \ldots, \Lambda_n)$ be an admissible configuration. If $k > 2$ and $m > 1$, then there is an admissible configuration $(\Lambda'_1, \ldots, \Lambda'_{n-2})$ such that N is diffeomorphic to $N' \times S^1 \times S^1$, where N and N' are the manifolds corresponding respectively to $(\Lambda_1, \ldots, \Lambda_n)$ and $(\Lambda'_1, \ldots, \Lambda'_{n-2})$.*

Remark. Theorem 12 is a reduction theorem only for the classification up to diffeomorphism. It is completely different for the classification of N up to biholomorphism. To have for example N diffeomorphic to $S^1 \times S^1 \times N'$ may have an interest, because the complex structure obtained may not respect this decomposition. In particular, as a consequence of Theorem 5, we have

Proposition VII.1. *Let $(\Lambda_1, \ldots, \Lambda_n)$ be an admissible configuration satisfying condition (H). Suppose*

$$3 \leq k \leq m-1\ .$$

Then, N is diffeomorphic but not biholomorphic to $T_1^{\mathbb{C}} \times N'$, where $T_1^{\mathbb{C}}$ is a 1-dimensional complex torus and N' is defined as in the previous corollary.

The proof of Theorem 12 uses the associate polytope. In fact this polytope plays, as we shall now see, a fundamental rôle in the classification of M_1. To achieve that, remark that this polytope has the following property: its dimension is $n - 2m - 1$ and from each of its vertices come exactly d edges (this is a direct consequence of the fact that the different facets of the associate polytope describe the different components of S); therefore it is a simple polytope (see [Gr]). There is a natural map from the set of admissible configurations into the set of simple convex polytopes: the one that to an admissible configuration assigns its associate polytope.

Two equivalent admissible configurations give two diffeomorphic manifolds M_1, therefore two diffeomorphic open sets S, and lastly two combinatorially equivalent associate polytopes, so that the previous map projects onto a map J from the set $\mathcal{C}$ of admissible configurations modulo equivalence into the set $\mathcal{P}$ of simple polytopes modulo combinatorial equivalence

$$J\,:\,[(\Lambda_1,\ldots,\Lambda_n)]\in\mathcal{C}\mapsto\left[\left\{r\in(\mathbb{R}^+)^n\ \mid\ \sum_{i=1}^n r_i\Lambda_i=0,\ \sum_{i=1}^n r_i=1\right\}\right]\in\mathcal{P}\,,$$

where brackets are used to denote the equivalence classes. This map can, in some way, be compared with the momentum map in symplectic geometry, because it reflects the action of the real torus on M_1 and its image is a convex polytope (see [De]); it has also to be compared with the abstract moment map defined in [Ka]. Remark now that the equivalence class of a reduced admissible configuration contains only reduced configurations, so that the map J can be restricted to the set $\mathcal{C}_{\text{reduced}}$ of reduced admissible configurations modulo equivalence. Let $\tilde{J}$ be this restriction.

Theorem 13. *The map $\tilde{J}\,:\,\mathcal{C}_{reduced}\to\mathcal{P}$ is a bijection: to every simple convex polytope (modulo combinatorial equivalence), we may associate a unique reduced admissible configuration (modulo equivalence) having this polytope as associate polytope.*

Remark. This result emphasizes the closeness between the map $\tilde{J}$ and the momentum map in symplectic geometry (the reader can compare with the results of [De], for example).

Remark. In order to make this theorem work for simplexes, it is necessary to consider the case $m=0$ too. This case can be thought in the following way: we have $S=\mathbb{C}^n-\{0\}$ and we consider only the action of $\mathbb{C}^*$ on S, so we have $M_1=S^{2n-1}$ and $N=\mathbb{C}P^{n-1}$.

Proof. We need a definition of convex geometry. Let $V=\{v_1,\ldots,v_n\}$ be a set of n points in $\mathbb{R}^q$, and $W=\{w_1,\ldots,w_n\}$ a set of n points in $\mathbb{R}^{n-q-1}$. Then consider $\mathcal{H}(v_1,\ldots,v_n)$ and $\mathcal{H}(w_1,\ldots,w_n)$ the convex polytopes formed by the convex hulls of the elements of V and W respectively. For a set $I=\{i_1,\ldots,i_p\}$ of integer indexes between 1 and n, we shall denote by I^C the complementary indexes set, that is to say

$$I^C=\{1\leq j\leq n\ \mid\ j\notin I\}\,.$$

Lastly, we denote by $Relint\ \mathcal{H}(v_1, \ldots, v_n)$ the relative interior of $\mathcal{H}(v_1, \ldots, v_n)$.

Definition (see [B-L], p.511). *The set W is a Gale diagram of V if and only if, for all I set of indices, we have*

$$0 \in Relint\ \mathcal{H}(w_i)_{i \in I^C} \iff \mathcal{H}(v_i)_{i \in I} \text{ is a face of the polytope } \mathcal{H}(v_1, \ldots, v_n)\ .$$

The notion of Gale diagram allows us to bind the associate polytope to the convex hull $\mathcal{H}(\Lambda_1, \ldots, \Lambda_n)$.

Lemma VII.2. *Let $(\Lambda_1, \ldots, \Lambda_n)$ be an admissible configuration. Then the convex polytope $\mathcal{H}(\Lambda_1, \ldots, \Lambda_n)$ is a Gale diagram of the dual of the associate polytope.*

Proof. First, let us remark that the dimension of the associate polytope is adequate. Let now I be a set of indexes of dimension p. If $0 \in Relint\ \mathcal{H}(\Lambda_i)_{i \in I}$, then, by definition, the subspace

$$\{z_i = 0,\ i \in I^C,\ z_j \neq 0,\ j \in I\}$$

is included in S. Besides, as the different faces of the associate polytope describe the different components of S, there is a numbering of the facets of the associate polytope such that the intersection of the facets indexed by $i \in I^C$ is a face of this polytope. Dualizing, the facets transform into $n - k$ vertices $v_1, \ldots, v_{n-k}$ (where k is the number of indispensable points), and the previous property means that $\mathcal{H}(v_i)_{i \in I^C}$ is a face of the dual of the associate polytope. It is then sufficient to add the vertices $v_{n-k+1} = \ldots = v_n = 0$ (corresponding to k indispensable points) in order to verify the definition of a Gale diagram.

Conversely, the demonstration is still valid. Finally, the weak hyperbolicity condition implies that if $0 \in \mathcal{H}(\Lambda_i)_{i \in I}$, then in fact $0 \in Relint\ \mathcal{H}(\Lambda_i)_{i \in I}$. □

Given a set V, a Gale diagram of V is obtained by a Gale transform of V (see [B-L], p. 511, [Gr] p. 85): consider $(v_1, \ldots, v_n)$ as a matrix, to which you add a column of one in order to have a $(d + 1) \times n$ matrix. The kernel of this matrix viewed as a linear map is the space of affine relations between the v_i. A basis of this space is called a Gale transform of V. This shall allow us to construct $\tilde{J}^{-1}$.

Let P be a simple polytope of dimension p with q facets. We have

$$n - 2m - 1 = p \quad \text{and} \quad n - k = q\ .$$

As we want a reduced configuration, k is equal to 0 or 1. This gives two possible values for n, but only one of them gives an integer value for m. The dimensions (m, n, k) are thus determined completely. Like in Lemma VII.2, if $k = 1$, we add 0 as a vertex of P to represent the indispensable point and have the required dimension.

A Gale transform of the dual of this polytope, extended if necessary, shall give a configuration $(\Lambda_1, \dots, \Lambda_n)$ in $\mathbb{C}^m$. We now have to verify that it is admissible.

By definition of a Gale transform, $0 \in \mathcal{H}(\Lambda_1, \dots, \Lambda_n)$ so the Siegel condition is fulfilled. Suppose that the weak hyperbolicity condition is not fulfilled, for example suppose that $0 \in \mathcal{H}(\Lambda_1, \dots, \Lambda_{2m})$. Moreover, as the Λ_i are not zero, we may assume that $0 \in \mathit{Relint}\ \mathcal{H}(\Lambda_1, \dots, \Lambda_{2m})$, restricting ourselves to a subconfiguration if necessary. Therefore, by the definition of a Gale diagram, there is a face of the dual of the polytope P corresponding to the complementary set of indexes, therefore a face of dimension $n-2m-2$ with $n-2m$ vertices. Such a face cannot be a simplex. But, as P is simple, its dual is simplicial, so there is a contradiction. The configuration is admissible.

Finally, if we consider two combinatorially equivalent polytopes, the Gale transforms $(v_1, \dots, v_p)$ and $(v'_1, \dots, v'_p)$ of these polytopes are isomorphic (see [Gr]) in the following way

$$\text{for all } I, \text{ we have } 0 \in \mathcal{H}((v_i)_{i\in I}) \iff 0 \in \mathcal{H}((v'_i)_{i\in I})\ .$$

Now, the equivalence between two admissible configurations means that you may homotope $(\Lambda_1, \dots, \Lambda_n)$ to (a permutation of) $(\Lambda'_1, \dots, \Lambda'_n)$ without, at any step, making $0 \in \mathbb{C}^m$ pass through an internal facet of the convex hull of the vectors, therefore such as 0 always stays in the same "chamber" of the convex hull, which implies that the two configurations are isomorphic in the previous way. $\square$

From $\tilde{J}^{-1}$, we may construct a map which sends a simple polytope into a manifold M_1, but this map, according to the remark at the beginning of section VI, is not injective.

In the particular case where the associate polytope is a polygon and where the manifold M_1 is simply connected, a complete classification up to diffeomorphism is given in [McG]. One obtains connected sums of products of spheres (see next section). The proof by induction on the number of vertices of Mac Gavran cannot be generalized, even in the case of polyhedrons. However, we conjecture

Conjecture. *Let $(\Lambda_1, \dots, \Lambda_n)$ be an admissible configuration, and let M_1 be the corresponding manifold. Then M_1 is diffeomorphic to $P \times C$, where P is a product of odd dimensional spheres and C a 2-connected connected sum of products of spheres.*

This conjecture is true in the case of a single vector field. A direct computation of the homology classes is done in [LdM2] (see [LdM1] and [LdM-Ve] p. 263), but it cannot be generalized. In the general case, we have the following result.

Lemma VII.3. *The manifold M_1 is diffeomorphic to $P \times C$, where P is a product of odd-dimensional spheres and C a 2-connected manifold.*

Proof. Let us decompose the open set S as

$$S = (\mathbb{C}^*)^k \times (\mathbb{C}^2 - \{0\})^{p_1} \times \ldots \times S_0 \, .$$

with S_0 a 2-connected open set. By a slight adaptation of the diffeomorphisms used in the proof of Theorem 2, this decomposition gives the following splitting of M_1

$$M_1 \simeq (S^1)^k \times (S^3)^{p_1} \times \ldots \times C$$

with C a 2-connected manifold and the lemma is proved. □

Therefore the conjecture consists in proving that C is a connected sum of products of spheres.

Remark. As the bundle $S \to M_1$ has a contractible fiber, M_1 and S have the same homology. As S is a subspace arrangement, it is theoretically possible to compute its homology by the formula of Goresky-Mac Pherson (see [G-McP], and [J-O-S] too). However, as there are subspaces of different codimensions, it is not clear that the homology is even only free (see [Je] for an example of a subspace arrangement with homology containing torsion terms).

VIII. Examples of compact complex manifolds

First, recall that the case $m = 1$ is the construction of [LdM-Ve], so we thus obtain all their examples, including in particular those of Hopf, Calabi-Eckmann, Haefliger (linear case), Loeb-Nicolau (linear case).

We now give the examples corresponding to an action of $\mathbb{C}^2$ on $\mathbb{C}^6$ and $\mathbb{C}^7$. Following Theorem 13, the classification is made using the number of indispensable points k and the combinatorial type of the associate polytope.

action on	***k***	**associate polytope**	**manifold *N***
$\mathbb{C}^6$	4	segment	$S^3 \times T^3$
$\mathbb{C}^7$	4	triangle	$S^5 \times T^3$
$\mathbb{C}^7$	3	square	$S^3 \times S^3 \times T^2$
$\mathbb{C}^7$	2	pentagon	$\left(\mathop{\sharp}\limits_{i=1}^{5} S^3 \times S^4\right) \times S^1$
$\mathbb{C}^7$	1	hexagon	$\mathop{\sharp}\limits_{i=1}^{8} S^4 \times S^4 \mathop{\sharp}\limits_{i=1}^{9} S^3 \times S^5$

where T^p is the real torus of dimension p. When $k = 0$, we obtain

$$M_1 \simeq \mathop{\sharp}_{i=1}^{14} S^3 \times S^6 \mathop{\sharp}_{i=1}^{35} S^4 \times S^5 ,$$

and N is the basis of a non trivial circle bundle, whose total space is this manifold. In the same way, we have, for an action of $\mathbb{C}^2$ on $\mathbb{C}^8$ (the classification is not complete)

k	**associate polytope**	**manifold N**
4	tetrahedron	$S^7 \times T^3$
3	prism (triangular basis)	$S^5 \times S^3 \times T^2$
2	cube	$S^3 \times S^3 \times S^3 \times S^1$
2	pentagonal book	$\left(\mathop{\sharp}_{i=1}^{2} S^4 \times S^5 \mathop{\sharp}_{i=1}^{3} S^3 \times S^6\right) \times S^1$
1	prism (pentagonal basis)	$\left(\mathop{\sharp}_{i=1}^{5} S^3 \times S^4\right) \times S^3$
1	hexagonal book	$\mathop{\sharp}_{i=1}^{6} S^3 \times S^7 \mathop{\sharp}_{i=1}^{8} S^4 \times S^6 \mathop{\sharp}_{i=1}^{3} S^5 \times S^5$ (?)

where the pentagonal (respectively hexagonal) book denotes the polyhedron obtained from the prism with pentagonal (respectively hexagonal) basis by contracting a rectangular face to a segment.

The expressions of N in these two arrays are obtained by combining Theorem 12 and Theorem 13 and the complete classification for the case $m = 1$ of [LdM-Ve] (see p. 257), except for the case of the hexagonal book. In this last case, we have computed the homology by the formula of [G-McP] and given the class of N up to diffeomorphism, according to the conjecture of Sect. VII.

Remark. In some cases (for example in the case of the hexagon in the first array), the manifold N is a connected sum of products of spheres, including even dimensional spheres. This has to be stressed, for there does not exist any almost complex structure on products of even dimensional spheres, except for $S^2 \times S^4$ and products using S^2 and S^6 (see [D-S]).

Combining Theorem 12 and Theorem 13 and the results of [LdM-Ve] (Theorem 1; see [LdM1] and [LdM2] too), we obtain:

Theorem 14. *Let $n \in \mathbb{N}$ and $p = 2l + 1 \leq n$. Let $n = n_1 + \ldots + n_p$ be any decomposition of n into integers. Lastly, let $d_1 = n_1 + \ldots + n_l$, ..., $d_p = n_p + n_1 + \ldots + n_{l-1}$.*

Then, there exists a complex structure on the manifold

$$\left(\mathop{\sharp}_{i=1}^{p} S^{2d_i-1} \times S^{2n-2-2d_i}\right) \times S^1 .$$

Let us denote by $(\alpha)S^l \times S^m$ the connected sum of α copies of $S^l \times S^m$. Using the results of [McG] and Theorem 13 for polygons, we prove

Theorem 15. *Let $p > 3$. Then there exists a complex structure on*

(i) the manifold $\left(\mathop{\sharp}_{j=1}^{p-3} \left(j \binom{p-2}{j+1}\right) S^{2+j} \times S^{p-j}\right) \times S^1$ *for every odd p,*

(ii) the manifold $\mathop{\sharp}_{j=1}^{p-3} \left(j \binom{p-2}{j+1}\right) S^{2+j} \times S^{p-j}$ *for every even p.*

Therefore, this theorem gives a family of compact complex manifolds N, which are diffeomorphic to connected sums of products of spheres. Notice that, if the conjecture of part VII is true, every admissible configuration with only one indispensable point and such that the open set S does not have components of type $\mathbb{C}^p - \{0\}$ (for $p > 1$) will produce a compact complex manifold N which is a connected sum of products of spheres. Besides, it is possible to elaborate a computer program which gives the manifold M_1 once the associate polytope is known, using first a numbering of the facets of the polytope to compute S, then computing the homology of S and M_1 by the formula of Goresky-Mac Pherson and finally describing the diffeomorphic type of M_1 according to the conjecture.

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