Math. Ann. 316, 401–418 (2000)

Mathematische Annalen

A presentation of mapping class groups in terms of Artin groups and geometric monodromy of singularities ⋆

Makoto Matsumoto

Received: 22 January 1998 / in final form: 16 February 1999

Abstract. From Wajnryb's presentation, we extract a simple presentation of the mapping class group of the genus g surface as a quotient of an Artin group by simple relations among the centers of sub-Artin groups.

Topological meanings are given by using deformation of simple singularities.

Mathematics Subject Classification (1991): 57N05, 20F36, 14B07, 20F05, 57M05

1. A presentation of mapping class groups by Artin groups

1.1. Artin groups

Let n be a positive integer and let I be $\{1, 2, \ldots, n\}$. By a *Coxeter matrix*, we mean a symmetric $n \times n$ matrix $M = (m_{ij})$ with m_{ij} being an integer ≥ 2 or ∞ for $1 \leq i \neq j \leq n$, and $m_{ii} = 1$ for $1 \leq i \leq n$. Its *Artin group* is defined by generators $a_1, \cdots, a_n$ and relations $a_i a_j a_i \cdots = a_j a_i a_j \cdots$, where both sides are words of length m_{ij}, for each $m_{ij} < \infty$, $1 \leq i \neq j \leq n$. If we add the relation $a_i^2 = 1$ for each i, then we get the *Coxeter group* of M. In the following, we consider only the case where all m_{ij} are finite.

The Coxeter matrix can be conveniently described by a graph Γ, where the vertex set is I and two distinct vertices i, j are joined by $(m_{ij} - 2)$ edges (hence no loops but multiple edges are allowed; from now on only such graphs are considered). This coincides with the classical notation of Dynkin diagrams.

Conversely, any graph Γ yields a Coxeter matrix. Its Artin group is denoted by $A(\Gamma)$. For example, if we denote by P_n a straight path consisting of n vertices with $n-1$ edges, then $A(P_n)$ is isomorphic to the braid group of $n+1$ strings. Let Γ be any connected simple (i.e. no multiple edges) graph. It is easy to see that the abelianization of $A(\Gamma)$ induces a natural surjection

$$\deg : A(\Gamma) \to \mathbb{Z},$$

M. Matsumoto
Department of Mathematics, Kyushu University, 6-10-1 Hakozaki Higashi-ku Fukuoka 812-8251 Japan (e-mail: matumoto@math.kyushu-u.ac.jp)

⋆ A part of this research was done during the author's visit at Max-Planck-Institut für Mathematik.

which we call the *degree* of an element of $A(\Gamma)$.

For an induced (i.e., every edge in Γ with its ends in H is also an edge of H, sometimes also called *full*) subgraph H of Γ, there exists an obvious homomorphism $A(H) \to A(\Gamma)$. Since this is proved to be injective by Van der Lek[10], we call $A(H)$ the *sub-Artin group* associated with subgraph H.

For general Γ, the structure of $A(\Gamma)$ is not well understood. However, if Γ is a Dynkin diagram of classical root systems, namely, if Γ is one of A_n, B_n, D_n E_6, E_7, E_8, F_4, G_2, H_3, H_4, $I_2(p)$, then the group structure is well analyzed by Brieskorn-Saito [8] and P. Deligne[9] independently.

For the proof of the next theorem, see Sections 4–7 in [8], Satz 7.1, Satz 7.2 and Zusatz below Satz 7.2. (Note that their proofs are rather group theoretic. P. Deligne's arguments are more geometric.) We say a word w of finite (possibly empty) product of a_i $(i \subset I)$ is *positive*, if the word contains no a_j^{-1}.

Theorem 1.1 (Brieskorn-Saito, Deligne). *Let Γ be a Dynkin diagram. Consider the following properties of an element $w \in A(\Gamma)$.*

- *w has a presentation as a positive word of a_j.*
- *For any a_i, $a_i^{-1}w$ has a positive word presentation of a_j's.*

Then, there exists a unique element $\Delta(\Gamma)$ satisfying these properties, which is minimal in the sense that if w satisfies the above properties, then $\Delta(\Gamma)^{-1}w$ has a positive word presentation by a_j's. $\Delta(\Gamma)$ has the following properties.

(i) *$\Delta(\Gamma)$ is mapped to the longest length element w_0 in the Coxeter group and its degree is the same as the length of w_0 (i.e. the number of positive roots).*
(ii) *The center of $A(\Gamma)$ is free cyclic with a generator $c(\Gamma)$, defined by $c(\Gamma) = \Delta(\Gamma)^2 = \Pi^h$ if $w_0 \neq -1$ and $c(\Gamma) = \Delta(\Gamma) = \Pi^{h/2}$ if $w_0 = -1$, where h is the Coxeter number and Π is a product of all a_j's with an arbitrary order.*

They also obtained a Garside-type[11] normal-form theorem. See Section 6, Satz 6.6 in [8].

1.2. Mapping class groups

Let $\Sigma_g^{n,<b>}$ denote a compact oriented genus g surface with n ordered points specified and with b boundary components.

Its *mapping class group* $M_g^{n,<b>}$ is defined to be the group of isotopy classes of orientation preserving self-diffeomorphisms of $\Sigma_g^{n,<b>}$ which fix the n points pointwise, and are identity on the boundary.

We denote $\Sigma_g^{<b>} := \Sigma_g^{0,<b>}$, $\Sigma_g^n := \Sigma_g^{n,<0>}$, $\Sigma_g := \Sigma_g^{0,<0>}$, and similarly for corresponding mapping class groups. These groups were proved to be finitely presented[20]. The explicit presentations of $M_g^{<1>}$ and M_g were given by Hatcher-Thurston[13], Harer[12], and finally by Wajnryb[22] (with an error

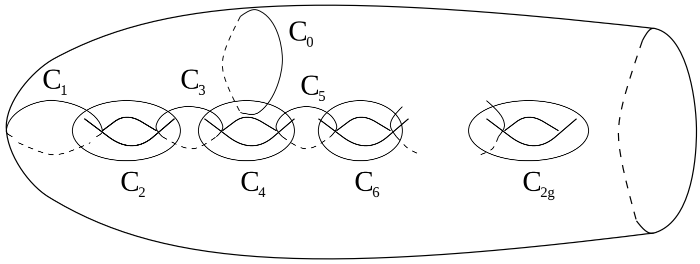

Fig. 1. Dehn-Lickorish-Humphries generators of mapping class groups

corrected by Birman-Wajnryb [4]). The result is the following theorem [22] (see the notation in [2]). Let $C_0, C_1, \ldots, C_{2g}$ be the simple closed curves on $\Sigma_g^{<1>}$ as in Figure 1. Let $c_0, c_1, \ldots, c_{2g}$ denote the corresponding Dehn twists in $M_g^{<1>}$.

Theorem 1.2 (Wajnryb). *The mapping class group $M_g^{<1>}$ admits a presentation with generators $a_0, a_1, \ldots, a_{2g}$, which are mapped to $c_0, c_1, \ldots, c_{2g}$, and relations:*

(A) $a_i a_j = a_j a_i$ *if* $C_i \cap C_j = \emptyset$, *and* $a_i a_j a_i = a_j a_i a_j$ *if* $C_i \cap C_j$ *consists of one point.*
(B) (*For* $g \geq 2$.) $(a_1 a_2 a_3)^4 = a_0 (a_4 a_3 a_2 a_1 a_1 a_2 a_3 a_4)^{-1} a_0 (a_4 a_3 a_2 a_1 a_1 a_2 a_3 a_4)$.
(C) (*For* $g \geq 3$.)

$$a_0 b_1 b_2 = a_1 a_3 a_5 b_3,$$

where

$$b_1 := (a_4 a_3 a_5 a_4) a_0 (a_4 a_3 a_5 a_4)^{-1},$$

$$b_2 := (a_2 a_1 a_3 a_2) b_1 (a_2 a_1 a_3 a_2)^{-1},$$

$$b_3 := (a_4^{-1} a_3^{-1} a_2^{-1} a_1^{-1} u a_2 a_3 a_4 a_5 a_6)^{-1} a_0 (a_4^{-1} a_3^{-1} a_2^{-1} a_1^{-1} u a_2 a_3 a_4 a_5 a_6),$$

and

$$u := (a_5 a_6) b_1 (a_5 a_6)^{-1}.$$

Here, a_i is mapped to c_i in the mapping class group.

Moreover, we obtain the presentation of M_g by adding a relation (D) (*omitted here, see [22][4]. Note that the error corrected in [4] lies in this relation*).

For the topological meanings of these relations, see Birman's survey [2]. Because of the pictorial descriptions, the relations (A), (B), (C), (D) are called *braid*, *chain*, *lantern*, *hyperelliptic* relations, respectively. (Note that (D) in her survey has again a small mistake in (7) on P.20., compare with the relation in [22][4].)

Let T_g be the graph shown in Figure 2. It is clear from the relation (A) that we have a surjective homomorphism $A(T_g) \to M_g^{<1>}$. It is natural to consider how

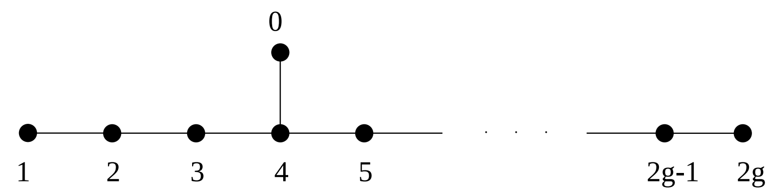

Fig. 2. The graph T_g

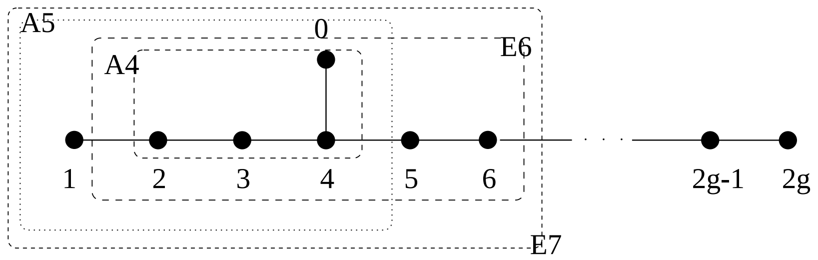

Fig. 3. Sub-Artin groups whose centers give (b) and (c)

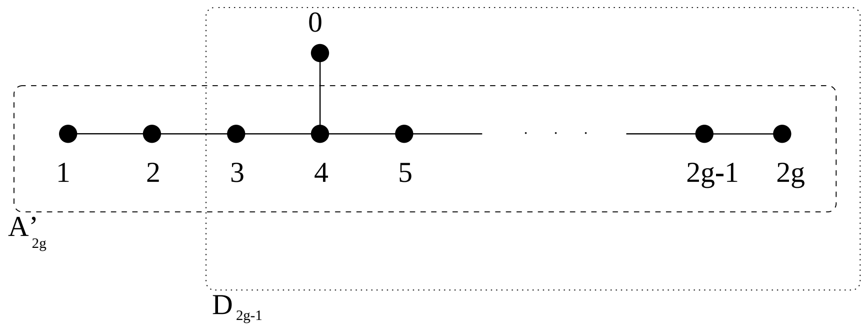

Fig. 4. Sub-Artin groups whose centers give (d) and (e)

the other relations can be interpreted in terms of Artin groups of finite types. It turns out that the relations (B), (C), (D) are equivalent to very simple relations among $c(\Gamma)$ for several Γ. Let H be an induced subgraph of T_g of Dynkin type. Then, we denote by the same symbol $c(H)$ the image of $c(H) \in A(H) \to A(T_g)$ in $A(T_g)$. We consider only the following subgraphs: $H =$ one of A_4, A_5, E_6 and E_7 as shown in Figure 3, or $H = A'_{2g}$, D_{2g-1} as in Figure 4.

Theorem 1.3. *Under the relation* (A)*, the relation* (B) *in Theorem 1.2 is equivalent to*

$$\text{(b)} \qquad c(A_5) = c(A_4)^2.$$

Under the relations (A) and (b), *the relation* (C) *is equivalent to*

$$\text{(c)} \qquad c(E_7) = c(E_6).$$

Under the relations (A), (b) *and* (c), *the relation* (D) *is equivalent to*

$$\text{(d)} \qquad a_1^{2g-2} = c(D_{2g-1}).$$

A presentation of M_g^1 *is obtained by the relations* (A), (b), (c), *and*

$$\text{(e)} \qquad c(A'_{2g})^2 = 1.$$

For the case $g = 3$, *we may replace* (d) *with*

$$\text{(f)} \qquad c(A'_6)\Delta(E_6) = \Delta(E_6)c(A'_6),$$

and (e) *with*

$$\text{(g)} \qquad c(E_7) = 1.$$

Thus, we have presentations $A(T_g)/[\text{(b)}, \text{(c)}] \cong M_g^{<1>}$, $A(T_g)/[\text{(b)}, \text{(c)}, \text{(e)}] \cong M_g^1$, and $A(T_g)/[\text{(b)}, \text{(c)}, \text{(d)}] \cong M_g$.

Remark 1.1. Explicitly, we have the following.

$$\begin{aligned}
c(A_4) &:= (a_0a_2a_3a_4)^5 \\
c(A_5) &:= (a_0a_1a_2a_3a_4)^6 \\
c(E_6) &:= (a_0a_2a_3a_4a_5a_6)^{12} \\
c(E_7) &:= (a_0a_1a_2a_3a_4a_5a_6)^9 \\
c(A'_{2g}) &:= (a_1a_2a_3a_4 \cdots a_{2g})^{2g+1} \\
c(D_{2g-1}) &:= (a_0a_3a_4a_5 \cdots a_{2g})^{4g-4} \\
\Delta(E_6) &:= (a_0a_2a_3a_4a_5a_6)^4(a_0a_2a_3a_4a_5)(a_0a_2a_3a_4)(a_2a_3a_4)
\end{aligned}$$

Here, $c(\Gamma)$ does not depend on the order of the elements in the parenthesis.

Remark 1.2. For genus three, the new presentation by (b), (c), (f), (g) is useful to realize the Hecke algebra representation of the mapping class group (§10 in [17], where Jones succeeded in the genus two case). K. Nishiyama and the author are preparing an article for this.

Remark 1.3. It would be an interesting future work to obtain a purely algebraic geometric proof (independent of Hatcher-Thurston) of the presentation in Theorem 1.3 for M_3^1. Remarkably, E. Looijenga gave another beautiful presentation of the mapping class group of genus 3 surface using the affine Artin group $\hat{E}_7$ and the orbifold fundamental group of the moduli space of plane quartic curves[18][19].

The topological meaning of these relations are as follows.

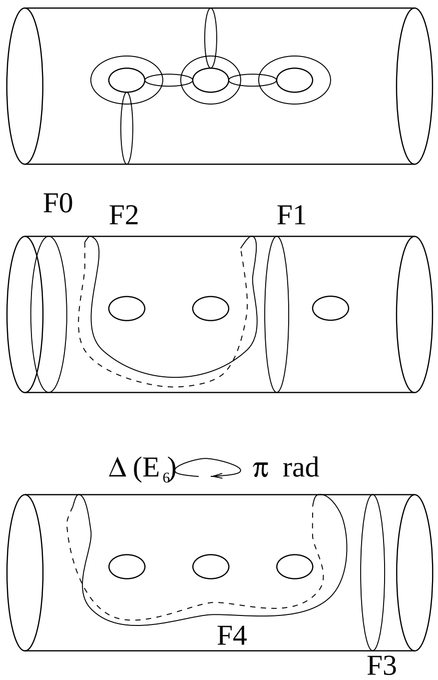

Fig. 5. The image of centers of sub-Artin groups

Theorem 1.4. *The image of* $c(A_4)^2$, $c(A_5)$, $c(E_6)$, $c(E_7)$ *in* $M_3^{<2>}$ *is, respectively,* f_2, $f_0 f_1$, f_4, $f_0^2 f_3$, *where* f_i *is the Dehn twists along* F_i *shown in Figure 5. The image of* $\Delta(E_6)$ *is the half twist along* F_4, *that is, rotate the genus three surface bounded by* F_4 *anticlockwise* π *radian so that the left and right holes exchange positions.*

Theorem 1.5. *The image of* $c(A'_{2g})$ *is the hyperelliptic involution, and that of* $c(D_{2g-1})$ *is* $f_5^{2g-3} f_6$, *shown in Figure 6.*

These theorems may be shown by direct calculation of Dehn twists, but here we shall use geometry of deformation of singularities, see Sect. 2.

1.3. Chain and lantern relations by Artin groups

The equivalence of (B) and (b), (C) and (c) in Theorem 1.3 is shown by a computer program implementing the Brieskorn-Saito Algorithm [8] of Garside type[11] to obtain their *normal form*. Here the transformation is briefly explained so that the reader may reproduce the calculation by using a computer.

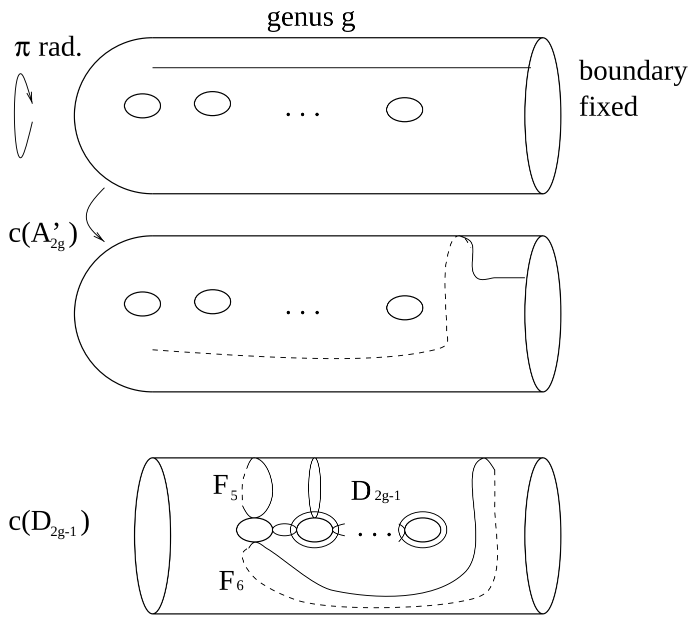

Fig. 6. The image of centers of sub-Artin groups

Computational result for (B) The relation (B) can be written as

$$(a_1a_2a_3)^4(a_4a_3a_2a_1a_1a_2a_3a_4)^{-1}a_0^{-1}(a_4a_3a_2a_1a_1a_2a_3a_4)a_0^{-1} = 1.$$

The computer program showed that the normal form of the conjugate of the left hand side

$$(a_1a_2a_3a_4)[(a_1a_2a_3)^4(a_4a_3a_2a_1a_1a_2a_3a_4)^{-1}a_0^{-1}(a_4a_3a_2a_1a_1a_2a_3a_4)a_0^{-1}] (a_1a_2a_3a_4)^{-1}$$

is

$$\Delta_{12340}^{-2}\Delta_{2340}^{4},$$

(here $\Delta_{ijk\cdots}$ denotes $\Delta(H)$ with H the induced subgraph by the vertices $i, j, k, \ldots$,) i.e., this is equivalent to

$$c(A_5) = c(A_4)^2.$$

Computational result for (C) This is the hard part. First, note that (B) is equivalent to $\Delta_{12340}^2 = \Delta_{2340}^4$, which can be rewritten as

$$a_1a_2a_3a_4a_0a_0a_4a_3a_2a_1 = \Delta_{2340}^2,$$

since $\Delta_{12340}^2 = a_1a_2a_3a_4a_0a_0a_4a_3a_2a_1\Delta_{2340}^2$ (easy to check by drawing braids).

After eliminating b_1, b_2, b_3, u from (C), put

$$L := [\text{the left hand side of (C)}][\text{the right hand side of (C)}]^{-1}.$$

Let W be the positive word which is a product of the following 132 a_j's, whose suffix is in the order

53042 33425 30645 30642
53042 53106 42534 00453
64253 10425 30642 53064
53042 33425 31064 25306
42531 06425 31064 25306
45306 42313 42530 45306
42531 42304 56.

This word was found by trial and error. In the following, we shall denote simply j instead of a_j, for such long words. Then,

$$WLW^{-1}$$

has its normal form

$$\Delta_{0123456}^{-1}K,$$

where K turns out to be $T_1T_2T_3$ with

$$T_1 = (5304232342530645306425304253),$$

$$T_2 = (06456123400432156453),$$

$$T_3 = (04253064253064).$$

Then we replace 1234004321 in T_2 by Δ_{2340}^2 by using (B), to obtain T_2'. Then, the normal form of $T_1T_2'T_3$ turns out to be Δ_{023456}^2. Thus, under the relation (b), it is proved that (C) is equivalent to $\Delta_{0123456}^{-1}\Delta_{023456}^2 = 1$, i.e., to (c).

1.4. Other relations

Since it is proven that $A(T_g)/[(\text{b}), (\text{c})] \cong M_g^{<1>}$, we may use the topological relations among Dehn twists for other relations under (b) and (c). The next is well known.

Lemma 1.1. *Let c be the Dehn twist along a curve C, and let $\sigma \in M_g^{n,<b>}$. Then, The conjugate $\sigma c \sigma^{-1}$ is the Dehn twist along $\sigma(C)$.*

Lemma 1.2. *The kernel of $M_g^{<1>} \to M_g^1$ is generated by the Dehn twist f_∞, and the kernel of $M_g^{<1>} \to M_g$ is normally generated by $d_g^{-1}d_g'$ shown in Figure 7. Thus, the relation* (D) *is equivalent to $d_g = d_g'$.*

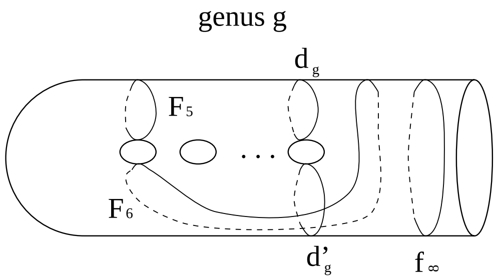

Fig. 7. Curves for the relation (D)

Proof. If σ is in the kernel of the first map, then σ can be isotopically deformed to be identity on the left side of f_∞. Then, σ is a mapping class group of the annulus, which is known to be generated by the Dehn twist f_∞. The latter statement follows from that the kernel of $M_g^1 \to M_g$ is $\pi_1(\Sigma_g)$, and is generated by the move of the marked point, or the difference of a pair of Dehn twists which separates this point. See Wajnryb[22] for the details.

Proof of the rest of the Main Theorem. Note that (f_5, f_6) in Figure 7 is, as a pair of curves, the image of (d_g, d_g') by a mapping class, so by Lemma 1.1 it suffices to prove that (d) is equivalent to $f_5 = f_6$. If we plug the last picture in Figure 6 into Figure 7, then the left boundary is filled and $c_1 = f_5$. Now (d) is equivalent to $f_5 = f_6$, and by Theorem 1.5 it is equivalent to $f_5^{2g-2} = f_5^{2g-3} f_6 = c(D_{2g-1})$. This proves the equivalence between (D) and (d).

Since $c(A_{2g}')^2$ is the Dehn twist f_∞, the relation (e) gives M_g^1 by Lemma 1.2.

Suppose $g = 3$. Then $c(E_7) = c(A_6')^2$ by Theorems 1.4 and 1.5. It is obvious that $\Delta(E_6)$ and $c(A_6')$ commute in M_3, being horizontal and vertical involutions. It suffices to prove that $d_3 = d_3'$ is derived from this commutativity. For $g = 3$, d_3 is $\Delta(E_6)a_1\Delta(E_6)^{-1}$ and hence $d_3' = c(A_6')d_3c(A_6')^{-1}$, by Lemma 1.1. Thus the problem is reduced to the commutativity of $c(A_6')$ with $d_3 = \Delta(E_6)a_1\Delta(E_6)^{-1}$, which is a direct consequence of the assumption and the commutativity of $c(A_6')$ and a_1.

Now, Theorem 1.3 is proved, assuming Theorems 1.4 and 1.5. In the next section, we prove these theorems.

2. Geometric monodromy of simple singularities

2.1. Topology

An Artin group with $2 \le m_{ij} \le 3$ $(i \ne j)$ is called *small*. We shall show that a homomorphism from a small Artin group to $M_g^{n,<b>}$ that maps each generator to a Dehn twist has a simple geometric configuration. This topological result is interesting by itself, and will be used to prove that the geometric monodromy action on the Milnor fiber coincides with those given by the Dehn twists along the vanishing cycles. This is classically known as the action on cohomology, and could be directly proved by using A'Campo's real cut theory[1], but here we give an elementary independent proof, considering a possible future application for higher genus cases.

A key lemma is the following topological statement by Ishida[15].

Theorem 2.1. *Let C, C' be two closed curves in a surface $\Sigma_g^{b,<n>}$, homotopically shifted so that the number of intersections is minimized. Let c, c' be the corresponding two Dehn twists. Then,*

(i) *c and c' commute if and only if $C \cap C' = \emptyset$,*
(ii) *c and c' satisfy the braid relation $cc'c = c'cc'$ if and only if $C \cap C' =$ one point,*
(iii) *c and c' satisfy no relation, i.e., generate a free group, otherwise.*

Then, the following proposition immediately follows.

Proposition 2.1. *Let Γ be a simple graph. Suppose that there exists a group homomorphism*

$$A(\Gamma) \to M_g^{n,<b>},$$

which maps each generator a_i to a Dehn twist c_i. Take a simple closed curve C_i which represents c_i for each i, so that the number of the geometric intersections is minimal. (*For example, take the geodesics.*)

Then, $C_i \cap C_j = \emptyset$, $=$ one point, respectively holds, for a_i and a_j are nonadjacent, adjacent, respectively.

There are several homeomorphic types for the tubular neighborhood N_Γ of the union of C_j. However, if Γ is acyclic, then N_Γ is unique up to homeomorphism. This can be shown as follows. Take one curve, say C_1. Then its tubular neighborhood is a ribbon $\tilde{C}_1$. Since it is orientable, its homeomorphic type is uniquely determined. Take another curve which crosses with C_1 at one point, say C_2. A part of C_2 appears on $\tilde{C}_1$. There is a unique way to fatten C_2 such that its orientation is compatible with C_1. This can be iterated until all curves are fattened, since there is no cycle in Γ and every time when a curve is fattened, there is only one point intersecting with already fattened curves. Thus, we obtain

Proposition 2.2. *Suppose Γ is acyclic in Proposition 2.1. Then, the tubular neighborhood N_Γ of the union of C_i's is uniquely determined up to homeomorphism.*

2.2. Geometric monodromy of simple singularities

Brieskorn[6] gave an explicit construction of the miniversal deformation of a simple singularity on a curve.

Theorem 2.2. *Let Γ be a Dynkin diagram of type A_n, D_n, E_n, with rank n. Let $T = \mathrm{Spec}\mathbb{R}[z_1, \dots, z_n]$ be the associated n-dimensional affine space, $W \subset GL(T)$ the Weyl group (i.e. the Coxeter group). Let $P_1, \dots, P_n$ be the polynomial generator of the invariant ring $\mathbb{R}[z_1, \dots, z_n]^W$. Then, there exists a miniversal deformation*

$$F \to T^{\mathbb{C}}/W = \mathrm{Spec}\mathbb{C}[P_1, \dots, P_n],$$

defined by a weighted homogeneous polynomial

$$F(x, y, P_1, \dots, P_l).$$

Brieskorn also showed that the fundamental group of the smooth-fiber locus in $T^{\mathbb{C}}/W$ is the corresponding Artin group [7].

Theorem 2.3. *Let $\Phi \subset T$ be the set of roots. For a root $\alpha \in \Phi$, let $\tau_\alpha \in W$ be the reflection with respect to α, and H_α $(\alpha \in \Phi)$ the reflection hyperplane. Fix a fundamental root system Π and let C be the corresponding Weyl chamber. Since C is contractible, we can take C as a base point of the fundamental group. Then,*

$$A(\Gamma) \cong \pi_1((T - \bigcup_{\alpha\in\Phi} H_\alpha)^{\mathbb{C}}/W, C)$$

holds, if each a_α, $\alpha \in \Pi$, is mapped to the path from C to $\tau_\alpha(C)$, which goes π radian around the hyperplane $H_\alpha^{\mathbb{C}}$ adjacent to C. (I.e., if we take the quotient of $(T - H_\alpha)^{\mathbb{C}}$ by the translation along H_α, then we get a projection to $\mathbb{C} - \{0\}$. Under this projection, C is projected to the positive real line and $\tau_\alpha(C)$ is projected to the negative real line. Take a path from the positive line to the negative line given by $e^{\pi\sqrt{-1}t}$, $t : 0 \to 1$. Lift this path, then divide by W to obtain a closed path.)

Let us take a *Milnor fiber* F_x, with $x \in C$. (Here we say by a Milnor fiber the intersection of the fiber on x with a ball centered at the origin in the total space.) Then, the fundamental group of the smooth-fiber locus with base point x, i.e., $A(\Gamma)$, acts on F_x modulo isotopy. This yields a morphism called *geometric monodromy* $\rho_\Gamma : A(\Gamma) \to M_g^{<b>}$, where (g, b) is the homeomorphic type of the Milnor fiber. It is well known that a_α is a Dehn twist. Thus, by Proposition 2.2, we can prove the following.

Theorem 2.4. *The geometric monodromy ρ_Γ is obtained by mapping each generator of $A(\Gamma)$ to the Dehn twist along the corresponding vanishing cycle.*

If we take the abelianization

$$A(\Gamma) \to M_g^{<b>} \to M_g \to \mathrm{Aut}H^1(\Sigma_g),$$

then we get the well-known Picard-Lefschez formula.

Proof. By Proposition 2.2, from the Dynkin diagram one can reproduce the tubular neighborhood of the union of the C_i. Assume $\Gamma = A_k$. If k is odd, then it is well known that the Milnor fiber F_Γ is homeomorphic to $\Sigma_g^{<2>}$, where $g = (k-1)/2$, and if k is even, then homeomorphic to $\Sigma_g^{<1>}$, where $g = k/2$. It is easy to check by induction that the tubular neighborhood N_Γ of $C_1, \dots, C_k$ given in Proposition 2.2 is homeomorphic to F_Γ. If $k = 1$, then obviously the neighborhood is $\Sigma_0^{<2>}$, and for $k = 2$, then we add one ribbon to this, yielding $\Sigma_1^{<1>}$, and so on.

Thus, the monodromy ρ_Γ is determined if we know how N_Γ is embedded in F_Γ. Since the genus is the same, we may consider only the boundary of N_Γ. If one boundary component of N_Γ is contractible in F_Γ, then the other boundary component of N_Γ cuts off the two boundary components of F_Γ. This means that any set of vanishing cycles does not separate the two boundary components of the Milnor fiber, which is a contradiction. (C.f. The calculation of the monodromy of the center, independently given in §2.4, manifests this.)

Thus, each boundary component of N_Γ cuts off one boundary component of F_Γ, i.e., the embedding is homotopically isomorphic. This proves Theorem 2.4 for the A_k case, since obviously the theorem is true if $F_\Gamma = N_\Gamma$.

Other cases follow by the same argument, by checking the following two points: (1) Tubular neighborhood is homeomorphic to the Milnor fiber. (2) The boundary components of the Milnor fiber are separated in the complement of N_Γ. □

2.3. Calculation of the monodromy of the center

Here we follow the notation in Theorems 2.2 and 2.3. The following is well known (c.f. [5]).

Lemma 2.1. *Let $f_\alpha(z_1, \dots, z_n)$ be the linear polynomial corresponding to the root α, and put*

$$D := \Big(\prod_{\alpha \in \Phi^+} f_\alpha\Big)^2,$$

where Φ^+ is the set of positive roots. Then, D defines

$$(T - \bigcup H_\alpha)^{\mathbb{C}}/W \to \mathbb{C} - \{0\},$$

and by taking π_1 we get the degree map.

Proof. Obviously D is invariant by W. By the definition of τ_α, its image by D is a circle around the origin. □

Lemma 2.2. *Let w_0 be the longest length element in W, i.e., w_0 maps C to $-C$. Let Δ' be a path from C to $-C$ in $T^{\mathbb{C}}$, given by*

$$t \in [0,1] \to (e^{\pi\sqrt{-1}t} z_1, \ldots, e^{\pi\sqrt{-1}t} z_n) \in T^{\mathbb{C}}.$$

Then,

$$\Delta' \in \pi_1((T - \bigcup H_\alpha)^{\mathbb{C}}/W, C)$$

satisfies

$$\Delta' a_\alpha = a_{w_0(\alpha)} \Delta'.$$

Proof. Since $T^{\mathbb{C}}$, $H_\alpha^{\mathbb{C}}$ has $\mathbb{C}^*$-action, we have a homotopy

$$e^{\pi\sqrt{-1}t} a_\alpha(s) : [0,1] \times [0,1] \to (T - \bigcup H_\alpha)^{\mathbb{C}}.$$

Since $-a_\alpha(s)$ defines a path from $-C$ to $-\tau_\alpha(C)$ along the root $w_0(\alpha)$, after divided by W, we obtain the desired formula. □

Proposition 2.3.

$$\Delta' = \Delta(\Gamma) \in A(\Gamma).$$

Proof. By [8],

$$\Delta(\Gamma) a_\alpha = a_{w_0(\alpha)} \Delta(\Gamma)$$

holds. Thus, $\Delta' \Delta(\Gamma)^{-1}$ lies in the center. Since $(\Delta')^2$ is given by $t : 0 \mapsto 2$ in Lemma 2.2, its degree is the same as the degree of D as a homogeneous polynomial, i.e., twice the number of positive roots. Thus, the degree of Δ' is the number of positive roots, the same as $\Delta(\Gamma)$. Since the degree map restricts to an injection on the center, we have the result. □

2.4. The relations between the center and the boundary Dehn twists

Brieskorn gave the weight of the variables of $F(x, y, P(z))$ as the homogeneous polynomial, when the weight of each z_i is considered to be one (see the table on P.281 and the second corollary on the next page in [6]).

Let w_1, w_2 be the weight of x, y, respectively. Each fiber is naturally compactified by filling one, two, or three points at infinity. We can choose a local parameter u_j of the form $x^{n_j} y^{m_j}$, $n_j, m_j \in \mathbb{Z}$, at each point of infinity ∞_j $(j = 1, 2, 3)$. This is a homogeneous polynomial with weight $w_{\infty j} = n_j w_1 + m_j w_2$.

Let $\mathcal{F} \to B := (T - \bigcup H_\alpha)^{\mathbb{C}}$ be the smooth fibration of curves defined by $F(x, y, P(z)) \to z := (z_1, \ldots, z_n)$. Let $b = (b_1, \ldots, b_n)$ be a point in the Weyl chamber $C \subset B$, and let $\mathcal{F}_b$ be the fiber above b, i.e., the curve defined by $F(x, y, P(b)) = 0$. We lift the path Δ' to a homotopy

$$\begin{aligned} \mathcal{F}_b \times [0,1] &\to \qquad\qquad \mathcal{F} \\ (x, y) \times t &\mapsto (\xi(x,t), \eta(y,t), P(e^{\pi\sqrt{-1}t} b)) \end{aligned}$$

so that each boundary component corresponding to each point of infinity is fixed pointwise. This homotopy yields at $t = 1$ a diffeomorphism $\mathcal{F}_b \to \mathcal{F}_{-b}$. By taking pullback by w_0 we obtain a selfdiffeomorphism of $\mathcal{F}_b$, which is by definition the monodromy corresponding to Δ.

If we forget about the boundary, then

$$\xi(x,t) = e^{w_1\pi\sqrt{-1}t}x, \quad \eta(x,t) = e^{w_2\pi\sqrt{-1}t}y$$

will give such a homotopy lift. This lifting gives the isomorphism given by $x \mapsto e^{w_1\pi\sqrt{-1}}x,\ y \mapsto e^{w_2\pi\sqrt{-1}}y),\ b \mapsto -b$. Then pullback by w_0. Note that the given automorphism $\varphi : F_b \to F_b$ is an automorphism as an algebraic curve, with each point at infinity fixed. Algebraicity is obvious. Preservation of the infinity is because $\mathcal{F} \to B$ is a pullback of a family over B/W, and the monodromy of the equation $F(x, y, P(z)) = 0$ with $P(z) \in B/W$ near the origin does not exchange the points at infinity.

During the homotopy lift, the algebraic structure of F_b does not change. So we may use the hyperbolic structure of F_b.

Since each fundamental root is pullbacked by w_0 to $w_0(\alpha)$, so is the corresponding hyperplane, and so is each Dehn twist. From this, the algebraic automorphism φ is determined uniquely except for A_n as follows. Take the geodesics for the representative of the vanishing cycles, cut these, then take a model in the upper half plane. Being homotopic to the tubular neighborhood of the cycles, the model of each connected component is a geodesic polygon with one point of infinity inside. This polygon must have a symmetry corresponding to w_0, where the map on edges of this polygon is strongly restricted from the action of w_0 on the diagram. That is, each edge of the polygon on a vanishing cycle C must be mapped to an edge on $w_0(C)$.

The action of w_0 on the Dynkin diagram is known to be the following (c.f. [5]): the identity for $w_0 = -1$ (cases of D_n, n: even, E_7 and E_8), the horizontal exchange of the two ends of the longest path for A_n and E_6, and the vertical involution (it's unique) for D_n with n odd.

If $w_0 = -1$, it acts as the identity on each polygon (easily checked for each case). For the cases of A_n and D_n (both with odd n) and E_6, then w_0 acts by π radian rotation on each components of the model (it is determined by the move of edges without ambiguity). In the case of A_n (n: even), C_1 is mapped to C_{2g} in Figure 1. The upper half plane model is a $(4 \times 4g - 2)$-gon (each curve gives four edges, except for C_1 and C_{2g}, each of which gives two edges). There is an ambiguity of $\pi/2$ radian right rotation or left. This ambiguity is eliminated by the consideration of boundary in the following.

Now we shall see the boundary. The local coordinate u_j for each point at infinity has weight w_j, and the corresponding monodromy around the boundary $|u_j| = \varepsilon$ ($\varepsilon > 0$) is the $w_j \times \pi$ radian Dehn twist. So, to fix the boundary, we need to compose the inverse Dehn twists along the boundary. For the case of A_n

(n: even), the weight at infinity is $-1/2$ and hence the boundary is twisted by $\pi/2$ radian clockwise if we do not modify. Thus, w_0 must be right rotation. So the monodromy is to rotate the model by $\pi/2$ radian clockwise, then twist the boundary anticlockwise with same radian.

Thus, we get the following

Proposition 2.4. *The geometric monodromy of $\Delta(\Gamma)$ on the curve $F(x, y, P(b))$ with open disks around points at infinity removed is obtained as follows. Let $\Sigma_g^{<b>}$ denote the obtained real two-dimensional surface.*

(i) *Cut off $\Sigma_g^{<b>}$ along the vanishing cycles. Then one gets a disjoint union of polygons each of which has a unique boundary component inside. Identify each polygon with a regular polygon, with an open disk at the center of the polygon removed.*

(ii) *Rotate each polygon by α radian clockwise, where*

$$\alpha = \begin{cases} 0 & \textit{for } D_n \textit{ with } n \textit{ even, } E_7 \textit{ and } E_8 \\ \pi & \textit{for } A_n \textit{ and } D_n \textit{ (both with } n \textit{ odd), and } E_6 \\ \pi/2 & \textit{for } A_n \textit{ with } n \textit{ even} \end{cases}$$

(iii) *Twist the j-th boundary by $-w_{\infty j}\pi$ radian counter-clockwise, where this value is seen in the following table.*

In the last step, we have to specify which polygon contains a point at infinity of weight $w_{\infty j}$. The set of $w_{\infty j}$ can be easily obtained by looking at the ramification over x-line (see an argument below). The ambiguity occurs if $w_{\infty j}$ depends on j, i.e., in the cases of D_n and E_7. Here we chose the numbering of j so that the first polygon (i.e. $j = 1$), after gluing the edges, constitutes pants (i.e. a sphere minus three disks). Then we have Table I for $w_{\infty j}$. Note that for the other cases the choice of numbering does not matter.

The explicit monodromy is easily obtained from the following table. The weights are copied from the table in [6], and the other values follow by a simple calculation of the ramification indices. (See an example below.) The row "automorphism" denotes the automorphism on the upper half plane model.

In particular, this table gives the proof of Theorems 1.4 and 1.5. Here we shall prove the latter theorem (i.e. the case of D_n, n odd). It shows how to determine $w_{\infty j}$ in this case. The other cases are similarly proved.

Consider D_n with n odd. The corresponding singular curve is the union of the smooth line $x = 0$ and a cusped line $x^{n-2} + y^2 = 0$. The picture of Dehn twists is as in Figure 8. Now we see the boundary component 1 came from the point at infinity in the line $x = 0$, by observing that two vanishing cycles cut out one pants at the left. (More precisely, if we deform only the cusped line to one-punctured hyperelliptic curve $y^2 + x^{n-2} + c = 0$, then the one-punctured projective line $x = 0$ intersects with this curve normally at two points. This shows

Table 1. Weight at infinities of simple singularities

Type	A_n		D_n	
equation	$x^{n+1}+y^2$		$x(x^{n-2}+y^2)$	
w_1, w_2	$(1, (n+1)/2)$		$(2, n-2)$	
	n=even	n=odd	n=even	n=odd
# of components	1	2	3	2
$-(w_{\infty 1}, w_{\infty 2}, w_{\infty 3})$	$(w_1/2)$ $(1/2)$	(w_1, w_1) $(1, 1)$	(w_2, w_1, w_1) $(n-2, 2, 2)$	$(w_2, w_1/2)$ $(n-2, 1)$
Description of Δ Boundary twists automorphism	$\pi/2$ rad. $-\frac{\pi}{2}$ rotation	π, π π rotation	$(n-2)\pi, 2\pi, 2\pi$ identity	$(n-2)\pi, \pi$ π rotation for each
$c(\Gamma)$	hyp. ell. inv.	(1,1) twists	as above	(n-2,1) twists

Type	E_6	E_7	E_8
equation	x^4+y^3	$y(x^3+y^2)$	x^5+y^3
w_1, w_2	(3,4)	(4,6)	(6,10)
# of components	1	2	1
$(w_{\infty 1}, w_{\infty 2}, w_{\infty 3})$	$(w_1/3)$ (1)	$(w_1, w_1/2)$ (4, 2)	$(w_1/3)$ 2
Description of Δ Boundary twists automorphism	π π rotation	$4\pi, 2\pi$ identity	2π identity
$c(\Gamma)$	1 twist	as above	as above

that two vanishing cycles cut out one pants containing the boundary component numbered 1.) The local parameter corresponding to the component 1 is y^{-1} (since $x = 0$ is the y-line), hence its weight is $-w_2 = -(n-2)$. For the component 2, the local parameter is ramified with index two above x^{-1}, since $n-2$ is odd. Thus it has weight $-w_1/2 = -1$. Since $\Delta(\Gamma)$ acts on the Dynkin diagram by a vertical involution, its monodromy is the vertical involution of the surface with twists at boundaries by $(n-2)\pi$, π radian, anticlockwise respectively, so that the boundary is fixed. Then, $c(\Gamma)$ corresponds to the products of the Dehn twists on the boundary with these weights, i.e., $D_1^{n-2}D_2$. By putting $n = 2g-1$, we get Theorem 1.5.

Remark 2.1. It is interesting to observe that some of these center-boundary relations also played important roles in classical literatures. For example, the chain relation (c.f. [2]) is exactly same with the relation for $c(A_3)$. Humphries' relation[14] used to get the minimal generating set is essentially equivalent to the relation about $\Delta(E_6)$. See the end of §1.4, where $\Delta(E_6)$ maps a_1 to d_3.

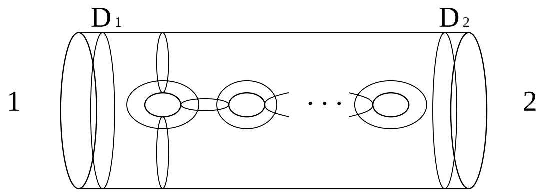

Fig. 8. Configuration of D_n, n odd

Acknowledgements. I would like to show my sincere gratitude to the following persons, who kindly helped this research: K. Saito for informing me of the Artin group and its relation to deformation of singularities, E. Brieskorn and P. Slodowy for fundamental facts on singularities, E. Looijenga for the relation between the moduli of genus three curves and E_7 Artin groups, Takayuki Oda for the motivation and the basics on mapping class groups. This research is an extension of the first section of [21], which was motivated by a joint work with Ihara[16], to whom I am grateful. I am thankful to the anonymous referees, who gave valuable comments.

References

1. N. A'Campo, Le groupe de monodromie du déploiement des singularités isolée de courbes planes I, Math. Ann. 213 (1) 1–32 (1975)
2. J.S. Birman, Mapping class groups of surfaces, Contemporary Mathematics **78** 13–43 (1988)
3. J.S. Birman and H. Hilden, Isotopies of homeomorphisms of Riemann surfaces and a theorem about Artin's braid group, Ann. of Math. **97**, 424–439 (1973)
4. Joan S. Birman and B. Wajnryb, Errata: Presentations of the mapping class group, Israel. J. Math. **88**, 425-427 (1994)
5. N. Bourbaki, "Groupes et algèbres de Lie, IV, V, VI" Éléments de Mathématique 1972
6. E. Brieskorn, Singular elements of semisimple algebraic groups, in Actes Congrès Intern. Math. 1970, t. 2, 279–284
7. E. Brieskorn, Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe, Inv. Math. **12**, 57–61 (1971)
8. E. Brieskorn and K. Saito, Artin Gruppen und Coxeter Gruppen, Invent. Math. **17**, 245–271 (1972)
9. P. Deligne, Les immeubles des groupes de tresses generalisées, Invent. Math. **17**, 273–302 (1972)
10. H. van der Lek, Extended Artin groups, Proc. Symp. Pure Math. 40–Part 2, 117–121 (1983)
11. F.A. Garside, The braid group and other groups, Quart. J. Math. Oxford, 2.Ser. **20**, 235–254 (1969)
12. J. Harer, The second homology group of the mapping class group of an orientable surface, Invent. Math. **72**, 221–239 (1983)
13. A. Hatcher and W. Thurston, A presentation for the mapping class group of a closed orientable surface, Topology **19**, 221–237 (1980)
14. S.P. Humphries, Generators for the mapping class group. Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977) Lecture Notes in Math. 722, 44–47, 1979
15. A. Ishida, The structure of subgroup of mapping class groups generated by two Dehn twists, Proceedings of the Japan Academy **72**, Ser. A, No.10 240–241 (1996)

16. Y. Ihara and M. Matsumoto, On Galois Actions on Profinite Completions of Braid Groups, in AMS Contemporary Math. **186** "Recent Developments in the Inverse Galois Problem." 173–200 (1994)
17. V.F.R. Jones, Hecke algebra representations of braid groups and link polynomials, Annals of Mathematics, **126** (1987), 355–388
18. E. Looijenga, Cohomology of $\mathcal{M}_3$ and $\mathcal{M}_3^1$, Contemporary Mathematics **150** 205–228 (1993)
19. E. Looijenga, Affine Artin groups and the fundamental groups of some moduli spaces, math. AG/9801117
20. J. McCool, Some finitely presented subgroups of the automorphism group of a free group, J. Algebra **35**, 205–213 (1975)
21. M. Matsumoto, Galois group $G_{\mathbb{Q}}$, Singularity E_7, and Moduli $\mathcal{M}_3$, London Math. Soc. Lecture Note Series **243** Geometric Galois Actions 2. The Inverse Galois Problem, Moduli Spaces and Mapping class Groups. 1997 (179–218)
22. B. Wajnryb, A simple presentation for the mapping class group of an orientable surface, Israel J. Math. **45**, 157–174 (1983)