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On a nonlinear Schrödinger equation with periodic potential

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1. Introduction and statement of results

We consider the nonlinear stationary Schrödinger equation

$$\text{(NS)} \qquad \begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for} \quad x \in \mathbb{R}^N; \\ u(x) \to 0 & \text{as} \quad |x| \to \infty\,. \end{cases}$$

This equation appears in several applications from mathematical physics. For instance, standing waves or traveling waves of nonlinear time dependent equations of Schrödinger or Klein-Gordon type correspond to solutions of (NS). Solutions of (NS) can also be interpreted as stationary states of the corresponding reaction-diffusion equation $u_t = \Delta u - V(x)u + g(x,u)$ which models phenomena from chemical dynamics.

Depending on the potential V, the spectrum of the Schrödinger operator $S := -\Delta + V$ on $L^2(\mathbb{R}^N)$ can be quite complicated. In this paper we deal with the case where

(V_1) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_i, $i = 1, \ldots, N$.

In this case the spectrum $\sigma(S)$ is purely absolutely continuous and bounded below; cf. [16], section XIII.16, in particular Theorem XIII.100. In recent years this case has found considerable interest. In [7] Coti-Zelati and Rabinowitz proved the existence of infinitely many solutions of (NS) for $0 < \min \sigma(S)$, provided g satisfies various growth conditions, of course.

If 0 lies in a gap of $\sigma(S)$ and if the primitive of g is strictly convex Alama and Li [2], [3], Buffoni et al. [5] and Jeanjean [11] found solutions using variational methods. Without the convexity condition the problem becomes more complicated because one has to deal with a strongly indefinite functional whose gradient is not of the form Fredholm $+$ compact. With the help of a special degree theory Troestler and Willem [19] found at least one solution of (NS). Their result has been improved by Kryszewski and Szulkin [12] who found one solution under weaker conditions on g, and infinitely many if g is odd in u. Also interesting is the work of Heinz, Küpper and Stuart who considered a parameter dependent situation with $V(x)$ replaced by $V(x)-\lambda$. For $\lambda \notin \sigma(S)$ they found solutions u_λ converging towards the trivial solution 0 as λ approaches a boundary point of $\sigma(S)$; cf. [10] and the references therein.

The goal of this paper is to prove the existence of nontrivial solutions of (NS) when 0 is a boundary point of the continuous spectrum of $S=-\Delta+V$. This seems to be the first result dealing with the case $0 \in \sigma_{cont}(S)$. Let us state this assumption precisely.

(V_2) $0 \in \sigma(S)$ and there exists $\beta > 0$ such that $(0,\beta] \cap \sigma(S) = \emptyset$.

This implies in particular that V cannot be constant because for $V \equiv$ const one has $\sigma(-\Delta+V) = [V,\infty)$. The nonlinearity should satisfy the conditions:

(g_1) $g \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is 1-periodic in x_i, $\quad i=1,\dots,N$.
(g_2) There are constants $a_1 > 0$ and $2 < \gamma \le \mu < 2^*$ such that

$$a_1|u|^\mu \le \gamma G(x,u) \le g(x,u)u \quad \text{for all} \quad x \in \mathbb{R}^N, \quad u \in \mathbb{R}.$$

(g_3) There are constants $a_2 > 0$ and $2 < p \le q < 2^*$ such that

$$|g(x,u)| \le a_2\left(|u|^{p-1} + |u|^{q-1}\right) \quad \text{for all} \quad x \in \mathbb{R}^N, \quad u \in \mathbb{R}.$$

Here $2^* = 2N/(N-2)$ if $N \ge 3$, and $2^* = \infty$ if $N = 1,2$. Our first result is

Theorem 1.1. *Suppose* (V_1), (V_2) *and* (g_1), (g_2), (g_3) *hold. Then* (NS) *has a nontrivial (weak) solution* $u \in H^2_{\text{loc}}(\mathbb{R}^N)$. *Moreover,* u *lies in* $L^t(\mathbb{R}^N)$ *for* $\mu \le t \le 2^*$.

In contrast to the papers mentioned above we do not know whether or not u lies in $H^1(\mathbb{R}^N)$. It is an interesting problem whether (NS) has infinitely many geometrically distinct solutions, that is, solutions which do not just differ by a translation. So far this is only known for $0 < \min\sigma(S)$; cf. [7]. We shall show the existence of infinitely many solutions under additional conditions:

(g_4) There are constants $a_3, \varepsilon > 0$ such that for all x, u, v

$$|g(x,u+v) - g(x,u)| \le a_3(|u|^{p-2} + |v|^{p-2} + |u|^{q-1})|v| \quad \text{if } |v| \le \varepsilon.$$

(g_5) g is odd in u: $g(x,-u) = -g(x,u)$ for all x, u.

Theorem 1.2. *Suppose* (V_1), (V_2) *and* (g_1) – (g_5) *hold with* $p = \mu$. *Then problem* (NS) *has infinitely many geometrically distinct solutions which lie in* $H^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^t(\mathbb{R}^N)$, $\mu \le t \le 2^*$.

The proofs of the theorems are based on variational methods applied to the functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)\, dx - \int_{\mathbb{R}^N} G(x,u)\, dx$$

where $G(x,u) := \int_0^u g(x,t)\, dt$ is the primitive of g. It is well known that $\Phi\colon H^1(\mathbb{R}^N) \to \mathbb{R}$ is of class $\mathcal{C}^1$ and that critical points of Φ are solutions of (NS). In fact, in the papers mentioned above the authors find critical points of Φ in $H^1(\mathbb{R}^N)$. This does not seem to work in our case where $0 \in \sigma(S)$. By assumption (V_2) we have a splitting $X = H^1(\mathbb{R}^N) = X^- \oplus X^+$ corresponding to the decomposition of $\sigma(S)$ into $\sigma(S) \cap (-\infty, 0]$ and $\sigma(S) \cap [\beta, \infty)$. We can define a new norm $\|\cdot\|_E$ on $X^\pm$ by setting

$$\|u^\pm\|_E^2 := \pm \int_{\mathbb{R}^N} \left(|\nabla u^\pm|^2 + V(x)|u^\pm|^2\right)\, dx \quad \text{for } u^\pm \in X^\pm.$$

Now Φ can be written as

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|_E^2 - \|u^-\|_E^2\right) - \int_{\mathbb{R}^N} G(x,u)\, dx$$

where $u = u^- + u^+ \in X^- \oplus X^+$.

However, $\|\cdot\|_E$ is not equivalent to the H^1-norm since $0 \in \sigma(S)$. Thus it is reasonable to work with the completion E of $H^1(\mathbb{R}^N)$ with respect to $\|\cdot\|_E$. Unfortunately, $\Psi(u) = \int_{\mathbb{R}^N} G(x,u)\, dx$ is not defined on E. The main idea is to use the geometry of Φ on $H^1(\mathbb{R}^N)$ in order to construct some kind of Palais-Smale sequence and to show that after translations a subsequence converges in a certain sense to a weak solution u of (NS). More precisely, let E_μ be the completion of $H^1(\mathbb{R}^N)$ with respect to $\|\cdot\|_\mu = \left(\|\cdot\|_E^2 + |\cdot|_\mu^2\right)^{1/2}$, so $H^1(\mathbb{R}^N) \subset E_\mu \subset E$. Then $u \in E_\mu$ is the limit of a (PS)*-sequence of Φ with respect to the weak topology on E_μ. The proof of Theorem 1.1 concludes with showing that $u \ne 0$ and $u(x) \to 0$ as $|x| \to \infty$. A major step in this argument is to show that E_μ embeds continuously into $L^t(\mathbb{R}^N)$ for $\mu \le t \le 2^*$ and that E_μ^- embeds continuously into $H^2_{\mathrm{loc}}(\mathbb{R}^N)$.

Here $E_\mu = E_\mu^- \oplus E_\mu^+$ again corresponds to the above splitting of $\sigma(S)$. It is worthwhile to mention that under the conditions of Theorem 1.1 the functional Ψ is not defined on E_μ.

The more rigorous growth conditions required in Theorem 1.2 imply that Ψ and Φ are defined on E_μ. The existence of infinitely many critical points of $\Phi \in \mathcal{C}^1(E_\mu)$ follows from an indirect argument. We first prove an abstract critical point theorem which yields the existence of an unbounded sequence of critical values of Φ provided Φ satisfies certain mountain pass type assumptions. In order to prove an intersection property (a linking) we do not need to introduce a new degree theory as in [19] and [12]. Instead we find a reduction to a finite-dimensional situation where the classical Brouwer degree applies. In our opinion this approach is simpler and more direct than those in [19], [12]. The Palais-Smale condition is replaced essentially by requiring that there exists a discrete subset B of E_μ^+ such that an arbitrary ε-neighborhood of $E_\mu^- \times B$ contains all but finitely many elements of an arbitrary Palais-Smale sequence. We then show that this holds for our Φ provided Φ has only finitely many critical points (up to translations). A similar indirect argument can be found in the papers [6] by Coti-Zelati, Ekeland, Séré and [17], [18] by Séré who were interested in homoclinic orbits of time periodic Hamiltonian systems. The Palais-Smale condition used in these papers is slightly weaker than the version we use.

At the end of this introduction we state two results dealing with the case where 0 is a left end point of $\sigma(S)$, i.e. we replace (V_2) by

(V_3) $0 \in \sigma(S)$ and there exists $\beta > 0$ such that $[-\beta, 0) \cap \sigma(S) = \emptyset$.

Theorem 1.3. *Suppose* (V_1), (V_3) *hold and* $-g$ *satisfies* (g_1) – (g_3). *Then* (NS) *has a nontrivial solution in* $H^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^t(\mathbb{R}^N)$, $\mu \le t \le 2^*$.

Theorem 1.4. *Suppose* (V_1), (V_3) *hold and* $-g$ *satisfies* (g_1) – (g_5) *with* $p = \mu$. *Then* (NS) *has infinitely many geometrically distinct solutions in* $H^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^t(\mathbb{R}^N)$, $\mu \le t \le 2^*$.

Except for the superlinearity condition (g_2) all other conditions are the same for g or $-g$. Thus if 0 is a left endpoint of $\sigma(S)$ we need that g decays superlinearly. The proofs of Theorems 1.3 and 1.4 are analogous to those of 1.1 and 1.2 working with $-\Phi$ instead of Φ.

The paper is organized as follows. In Sect. 2 we discuss the space E_μ and prove the essential embedding $E_\mu^- \subset H^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^t(\mathbb{R}^N)$, $\mu \le t \le 2^*$. We also prove that a weak solution $u \in E_\mu$ of (NS) satisfies $u(x) \to 0$, $|x| \to \infty$. In Sect. 3 we prove Theorem 1.1. The abstract critical point theorem for even functionals is the content of Sect. 4. Finally, in Sect. 5 we deduce Theorem 1.2 from the abstract critical point theorem.

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for the suggestion to simplify the proof of 2.3 and the reference to the book [1].

2. Preliminaries

Let $-\alpha$ be a lower bound for $\sigma(-\Delta + V)$ so that

$$0 \in \sigma(-\Delta + V) \subset (-\alpha, 0] \cup (\beta, \infty).$$

Set $H = L^2(\mathbb{R}^N)$ with inner product $\langle \cdot, \cdot \rangle$ and let $(P_\lambda : H \to H)_{\lambda \in \mathbb{R}}$ denote the spectral family of $S = -\Delta + V$. Setting $H^- := P_0 H$ and $H^+ := (Id - P_0)H$ we have the decomposition $H = H^- \oplus H^+$. The domain of S and $|S|$ is $\mathcal{D}(S) = \mathcal{D}(|S|) = H^2(\mathbb{R}^N)$ and

$$|S|u = \begin{cases} Su & \text{for } u \in \mathcal{D}(S) \cap H^+; \\ -Su & \text{for } u \in \mathcal{D}(S) \cap H^-. \end{cases}$$

Observe that $H^- \subset \mathcal{D}(S)$ because the spectrum of S is bounded below. The domain of $|S|^{1/2}$ is the Hilbert space $H^1(\mathbb{R}^N)$ with the usual scalar product and associated norm $\left(|\nabla u|_2^2 + |u|_2^2\right)^{1/2}$. Here and in the sequel we write $|\cdot|_p$ for the L^p-norm. Let E be the completion of $H^1(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_E := \left| |S|^{1/2} u \right|_2 = \left(\int_{-\infty}^{\infty} |\nu| \, d\langle P_\nu u, u \rangle \right)^{1/2}.$$

Clearly E is a Hilbert space with inner product $\langle u, v \rangle_E = \langle |S|^{1/2} u, |S|^{1/2} v \rangle$. We have the orthogonal decomposition $E = E^- \oplus E^+$ corresponding to the decomposition of $\sigma(S)$. We shall write $u = u^- + u^+$ with $u^\pm \in E^\pm$ for $u \in E$. Since the spectrum of S restricted to H^+ is contained in (β, ∞) it is bounded away from 0, hence the norm $\|\cdot\|_E$ is equivalent to the H^1-norm on E^+:

$$\|\cdot\|_E \sim \|\cdot\|_{H^1} \quad \text{on} \quad E^+ \tag{2.1}$$

so $E^+ = H^1(\mathbb{R}^N) \cap H^+$. However, on the subspace $H^1(\mathbb{R}^N) \cap H^-$ the norm $\|\cdot\|_E$ is weaker than $\|\cdot\|_{H^1}$ and $H^1(\mathbb{R}^N) \cap H^- = H^-$ is not complete with respect to $\|\cdot\|_E$. Indeed, since $0 \in \sigma(S)$ is a continuous spectrum point there is a sequence (u_k) in $\mathcal{D}(S)$ such that $|u_k|_2 = 1$ and $Su_k \to 0$, hence $\|u_k\|_E \to 0$. Since $H^- \subset \mathcal{D}(S)$ we have for $u \in H^-$

$$0 \le \|u\|_E^2 = -\langle Su, u \rangle = -\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx.$$

Therefore $|\nabla u|_2 \leq c|u|_2$ for $u \in H^-$ and by the Sobolev and Hölder inequalities

$$|u|_t \leq c_1 |\nabla u|_2^{1-\gamma} |u|_2^{\gamma} \leq c_2 |u|_2 \quad \text{for} \quad u \in H^-$$

where $2 \leq t \leq 2^*$, $\gamma = \frac{2}{t} \cdot \frac{2^*-t}{2^*-2}$, with c, c_1, c_2 positive constants.

For each $n \in \mathbb{N}$ we set

$$E_n^- := P_{-1/n} H = P_{-1/n} H^- \subset H^- \subset E^-$$

and

$$E_n := E_n^- \oplus E^+ \subset E.$$

Since the spectrum of S restricted to E_n is bounded away from 0 we have

(2.2) $$\|\cdot\|_E \sim \|\cdot\|_{H^1} \quad \text{on} \quad E_n.$$

Let

$$Q_n := P_{-1/n} + (Id - P_0) : E \to E_n$$

denote the orthogonal projection. Then we have for any $u \in H^1(\mathbb{R}^N)$:

(2.3)
$$Q_n u \to u \text{ as } n \to \infty, \text{ with respect to } \|\cdot\|_E \text{ and } |\cdot|_t, \quad 2 \leq t < 2^*.$$

Next we recall the $\mathbb{Z}^N$-action on H given by the formula

$$(a * u)(x) := u(a + x) \quad \text{for} \quad a \in \mathbb{Z}^N, u \in H, x \in \mathbb{R}^N.$$

Clearly the norms $\|\cdot\|_{H^1}$ and $|\cdot|_t$, $2 \leq t \leq 2^*$, are invariant with respect to this action. Moreover, S commutes with this action by (V_1) and so does P_λ for each $\lambda \in \mathbb{R}$. Hence $\|\cdot\|_E$ is invariant, the Q_n are equivariant and the subspaces E_n and $E^\pm$ are closed under this action.

We need to introduce yet another norm on E defined by

$$\|u\|_\mu := \left(\|u\|_E^2 + |u|_\mu^2\right)^{1/2}.$$

Let E_μ^- be the completion of H^- with respect to $\|\cdot\|_\mu$ and set $E_\mu := E_\mu^- \oplus E^+$. Then E_μ is the completion of $H^1(\mathbb{R}^N)$ with respect to $\|\cdot\|_\mu$ due to (2.1). Clearly $(E_\mu, \|\cdot\|_\mu)$ is a Banach space, $H^1(\mathbb{R}^N) \subset E_\mu \subset E$ and all norms $\|\cdot\|_E$, $\|\cdot\|_{H^1}$, $\|\cdot\|_\mu$ are equivalent on E^+. It is not difficult to check that $\|\cdot\|_\mu$ is uniformly convex so E_μ is reflexive, hence bounded sets in E_μ are weakly compact.

Lemma 2.1. *E_μ^- embeds continuously into $H^2_{\text{loc}}(\mathbb{R}^N)$ hence compactly into $L^t_{\text{loc}}(\mathbb{R}^N)$ for $2 \leq t < 2^*$. Moreover, it embeds continuously into $L^t(\mathbb{R}^N)$ for $\mu \leq t \leq 2^*$. Finally $Su \in L^2$ for $u \in E_\mu^-$.*

Proof. For $u \in E_\mu^-$ let $(u_n)_{n\in\mathbb{N}}$ be a sequence in H^- with $\|u_n - u\|_\mu \to 0$, $n \to \infty$. We first show that $u \in H^1_{\text{loc}}(\mathbb{R}^N)$. Given a bounded domain $\Omega \subset \mathbb{R}^N$ we take a function $\eta \in C_0^\infty(\mathbb{R}^N)$ with $\eta \equiv 1$ in Ω. Since for $v \in H^- \subset H^2(\mathbb{R}^N)$

$$-\Delta(\eta v)\eta v = \eta^2 \cdot (-\Delta v) \cdot v + v^2 \cdot (-\Delta\eta)\eta - 2\eta v \nabla v \cdot \nabla\eta$$

we get

$$|\nabla(\eta v)|_2^2 \le \langle Sv, \eta^2 v\rangle + \frac{1}{2}|\nabla(\eta v)|_2^2 + c|v|_\mu^2$$

where c is here and below a generic constant depending on Ω. This implies

$$\frac{1}{2}|\nabla(\eta v)|_2^2 \le c\left(\|v\|_\mu + |v|_\mu + |v|_\mu^2\right)$$

and it follows that $(u_n)_n$ is a Cauchy sequence in $H^1(\Omega)$, so $u \in H^1(\Omega)$.

Next we show that $Su \in L^2$. Since $\inf\sigma(S) > -\alpha > -\infty$ we have

$$\begin{aligned}
|S(u_n - u_m)|_2^2 &= \int_{-\alpha}^{0} \lambda^2 d\,|P_\lambda(u_n - u_m)|_2^2 \\
&\le -\alpha \int_{-\alpha}^{0} \lambda d\,|P_\lambda(u_n - u_m)|_2^2 \\
&= \alpha \left||S|^{1/2}(u_n - u_m)\right|_2^2 \\
&= \alpha\|u_n - u_m\|_E^2.
\end{aligned}$$

Therefore $(Su_n)_n$ is a Cauchy sequence in L^2 and it follows that $Su_n \to Su$ in L^2.

In order to see $u \in H^2_{\text{loc}}(\mathbb{R}^N)$ we use the Calderon-Zygmund inequality (cf. [9], Theorem 9.11). For $r > 0$, $\varepsilon > 0$, and $y \in \mathbb{R}^N$ we obtain

$$\begin{aligned}
&\|u_n - u_m\|_{H^2(B(y,r))} \\
&\quad \le c_{r,\varepsilon}\left(|u_n - u_m|_{L^2(B(y,r+\varepsilon))} + |S(u_n - u_m)|_{L^2(B(y,r+\varepsilon))}\right).
\end{aligned}$$

This implies $u \in H^2_{\text{loc}}(\mathbb{R}^N)$.

Finally we show $u \in L^t(\mathbb{R}^N)$ for $\mu \le t \le 2^*$. This is clear for $t = \mu$. For $r > 0$, $\varepsilon > 0$ and $y \in \mathbb{R}^N$ we have

$$\begin{aligned}
|u|_{L^{2^*}(B(y,r))} &\le c\|u\|_{H^1(B(y,r))} \\
&\le c_{r,\varepsilon}\left(|Su|_{L^2(B(y,r+\varepsilon))} + |u|_{L^\mu(B(y,r+\varepsilon))}\right)
\end{aligned}$$

hence,

$$\int_{B(y,r)} |u|^{2^*}\,dx$$

$$\le c_{r,\varepsilon}\left(|Su|_2^{2^*-2}\int_{B(y,r+\varepsilon)} |Su|^2\,dx + |u|_\mu^{2^*-\mu}\int_{B(y,r+\varepsilon)} |u|^\mu\,dx\right).$$

We fix $r>0$ and cover $\mathbb{R}^N$ by balls $B(y,r),\ y\in Y\subset\mathbb{R}^N$, such that for $\varepsilon>0$ small, at most $N+1$ balls $B(y,r+\varepsilon),\ y\in Y$, intersect nontrivially. It follows that

$$\int_{\mathbb{R}^N} |u|^{2^*}dx \le c\left(|Su|_2^{2^*} + |u|_\mu^{2^*}\right)$$

so $u\in L^{2^*}$. By interpolation we get $u\in L^t$ for any $t\in[\mu,2^*]$. □

Corollary 2.2. *Any bounded sequence (u_k) in E_μ has a subsequence which converges weakly in E_μ and strongly in $L^t_{\mathrm{loc}}(\mathbb{R}^N)$ for any $2\le t<2^*$.*

In the proofs of the results from §1 we obtain weak solutions $u\in E_\mu$ of

(2.4) $$-\Delta u + V(x)u = g(x,u) \qquad \text{for } x\in\mathbb{R}^N.$$

By Lemma 2.1 we have $u\in H^1_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, from our assumptions on V and g it follows that $a(x)\equiv -V(x)+g(x,u)/u\in L^{N/2}_{\mathrm{loc}}(\mathbb{R}^N)$. This implies $u\in L^t_{\mathrm{loc}}(\mathbb{R}^N)$ for any $t<\infty$. In addition, using L^p-theory and the Gagliardo-Nirenberg inequality one can further show that $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$; see e.g. [15], Proposition 2.15. If g is of class $\mathcal{C}^1$ then a classical bootstrap argument and Schauder estimates instead of L^p-theory show that weak solutions of (2.4) are in fact classical solutions. Now we shall show that a weak solution $u\in E_\mu$ of (2.4) satisfies also $u(x)\to 0$ as $|x|\to\infty$.

Corollary 2.3. *If $u\in E_\mu$ solves* (2.4) *then $u(x)\to 0$ as $|x|\to\infty$.*

Proof. By the above arguments and (5.5) of [1] which we may clearly apply for any ball $B(y,r)$ we have

$$\operatorname*{ess\,sup}_{x\in B(y,1)} |u(x)| \le K_1\cdot\|u\|_{L^2(B(y,2))}\,.$$

Hence, the Hölder inequality yields

(2.5) $$\|u\|_{L^\infty(B(y,1))} \le K_2\cdot\|u\|_{L^\mu(B(y,2))}$$

for all $y \in \mathbb{R}^N$, where K_1 and K_2 are constants independent of $y \in \mathbb{R}^N$. Now we fix $\varepsilon > 0$ arbitrarily. Since $u \in L^\mu(\mathbb{R}^N)$ we have $\lim_{R\to\infty} \int_{|x|\geq R} |u|^\mu \, dx = 0$. We may take $R > 0$ so large that $\|u\|_{L^\mu(\{|x|\geq R\})} < \varepsilon$. Then for $y \in \mathbb{R}^N$ with $|y| \geq R + 2$ we have by (2.5)

$$\|u\|_{L^\infty(B(y,1))} \leq K_2 \cdot \varepsilon \tag{2.6}$$

Since ε is arbitrary (2.6) shows that $u(x) \to 0$ as $|x| \to \infty$. □

3. One nontrivial solution

In this section we prove Theorem 1.1. Thus we assume that (V_1), (V_2), (g_1), (g_2) and (g_3) are satisfied. Let $\Psi\colon H^1(\mathbb{R}^N) \to \mathbb{R}$ be given by $\Psi(u) = \int_{\mathbb{R}^N} G(x,u)\, dx$. Then

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx - \int_{\mathbb{R}^N} G(x,u)\, dx \\ &= \frac{1}{2}\left(\|u^+\|_E^2 - \|u^-\|_E^2\right) - \Psi(u)\end{aligned}$$

where $u = u^- + u^+$ according to the splitting $E = E^- \oplus E^+$. Observe that (g_2) and (g_3) imply $p \leq \mu \leq q$. If $p < \mu$ then $L^p(\mathbb{R}^N)$ does not embed into $L^\mu(\mathbb{R}^N)$. Therefore, Ψ is not defined on E_μ except when $p = \mu$. Therefore we shall use an approximation argument.

For each $n \in \mathbb{N}$ we set $\Phi_n := \Phi|E_n, \Psi_n := \Psi|E_n$ where $E_n = E_n^- \oplus E^+$, $E_n^- = P_{-1/n}H$, is as in Sect. 2. Clearly $\Phi_n, \Psi_n \in \mathcal{C}^1(E_n, \mathbb{R})$ and

$$\begin{aligned}D\Psi_n(u)v &= \int_{\mathbb{R}^N} g(x,u)v\, dx \\ D\Phi_n(u)v &= \langle Lu, v\rangle_E - \int_{\mathbb{R}^N} g(x,u)v\, dx\end{aligned}$$

where $Lu = u^+ - u^-$.

Definition 3.1. *A sequence $(u_j)_{j\in\mathbb{N}}$ is said to be a (PS)$_c^*$-sequence for Φ with respect to $(E_n, \|\cdot\|_E)$, some $c \in \mathbb{R}$, if*

- *$u_j \in E_{n_j}$ with $n_j \to \infty$ as $j \to \infty$;*
- *$\Phi(u_j) \to c$ as $j \to \infty$;*
- *$\|D\Phi_{n_j}(u_j)\|_E \to 0$ as $j \to \infty$.*

Lemma 3.2. *If (u_j) is a (PS)$_c^*$-sequence for Φ then $\|u_j\|_E$ and $|u_j|_\mu$ are bounded or equivalently, $\|u_j\|_\mu$ is bounded. Moreover $c \geq 0$ and $c = 0$ if and only if $\|u_j\|_\mu \to 0$.*

Proof. As a consequence of (g_2) we obtain

$$
\begin{aligned}
\Phi(u_j) - \frac{1}{2}D\Phi(u_j)u_j &= \int_{\mathbb{R}^N} \left(\frac{1}{2}g(x,u_j)u_j - G(x,u_j)\right)\,dx \\
&\geq \frac{\gamma-2}{2\gamma}\int_{\mathbb{R}^N} g(x,u_j)u_j\,dx \\
&\geq \frac{a_1(\gamma-2)}{2\gamma}|u_j|_\mu^\mu
\end{aligned}
\tag{3.1}
$$

Setting $\varepsilon_j := \|D\Phi_{n_j}(u_j)\|_E$ this implies

$$|u_j|_\mu^\mu \leq d(1+\varepsilon_j\|u_j\|_E) \tag{3.2}$$

where d denotes a generic constant independent of j. Let $\theta \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ be such that $0 \leq \theta(t) \leq 1$ and $\theta(t) = 0$ if $|t| \leq 1$, $\theta(t) = 1$ if $|t| \geq 2$. We set $g_1(x,t) := \theta(t)g(x,t)$ and $g_2(x,t) = g(x,t) - g_1(x,t) = (1-\theta(t))g(x,t)$. Then by (g_2) and (g_3) we obtain with $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$

$$d\cdot|g_1(x,t)|^{q'} \leq g_1(x,t)t \quad \text{and} \quad d\cdot|g_2(x,t)|^{p'} \leq g_2(x,t)t. \tag{3.3}$$

Using the first inquality in (3.1) we see

$$d\cdot(\Phi(u_j)+\varepsilon_j\|u_j\|_E) \geq |g_1(\cdot,u_j)|_{q'}^{q'} + |g_2(\cdot,u_j)|_{p'}^{p'}.$$

Moreover, the Hölder inequality yields

$$
\begin{aligned}
\left|\int_{\mathbb{R}^N} g(x,u_j)u_j^+\,dx\right| &\leq d\left(|g_1(\cdot,u_j)|_{q'}|u_j^+|_q + |g_2(\cdot,u_j)|_{p'}|u_j^+|_p\right) \\
&\leq d\left(\Phi(u_j)+\varepsilon_j\|u_j\|_E\right)^{1/p'}|u_j^+|_p \\
&\quad + d\left(\Phi(u_j)+\varepsilon_j\|u_j\|_E\right)^{1/q'}|u_j^+|_q.
\end{aligned}
\tag{3.4}
$$

By the form of Φ we have

$$
\begin{aligned}
\|u_j^+\|_E^2 &= D\Phi(u_j)u_j^+ + \int_{\mathbb{R}^N} g(x,u_j)u_j^+\,dx \\
&\leq d\left(1+\|u_j\|_E^{1/q'}+\|u_j\|_E^{1/p'}\right)\cdot\|u_j^+\|_E.
\end{aligned}
\tag{3.5}
$$

and

$$\|u_j^-\|_E^2 \leq 2\Phi(u_j) + \|u_j^+\|_E^2. \tag{3.6}$$

Since $1/p' < 1$ and $1/q' < 1$ it follows that $\|u_j\|_E$ is bounded, hence, applying (3.2) once more $|u_j|_\mu$ is bounded.

Next, letting $j \to \infty$ in (3.1) yields $c \geq 0$. Clearly $c = 0$ if $\|u_j\|_\mu \to 0$. Now suppose $c = 0$.

$$\begin{aligned}\|u_j\|_E^2 &= \|u_j^+ + u_j^-\|_E^2 \\ &\leq -2\Phi(u_j) + 2\|u_j^+\|_E^2 \\ &= -2\Phi(u_j) + 2D\Phi(u_j)u_j^+ + 2\int\limits_{\mathbb{R}^N} g(x,u_j)u_j^+\,dx\end{aligned}$$

From $\Phi(u_j) \to c = 0$ and $\varepsilon_j \to 0$ we now deduce $\|u_j\|_E \to 0$. Since $|u_j|_\mu \to 0$ follows from (3.1) we have $\|u_j\|_\mu \to 0$ as claimed. □

Next we recall a lemma due to P.L. Lions.

Lemma 3.3. *Fix $r > 0$ and $s \in [2, 2^*)$. If (u_n) is bounded in $H^1(\mathbb{R}^N)$ and if*

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |u_n|^s\,dx \to 0 \quad \textit{as } n \to \infty$$

then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for any $t \in (2, 2^*)$.

A proof of this lemma can be found in [13].

Lemma 3.4. *Each* (PS)$_c^*$*-sequence with $c > 0$ gives rise to a nontrivial solution of* (NS) *which lies in E_μ.*

Proof. Let (u_j) be a (PS)$_c^*$-sequence. By Lemma 3.2 the sequence is bounded with respect to $\|\cdot\|_\mu$, hence, $\|u_j^+\|_{H^1}$ is bounded because of (2.1). We claim that for $r > 0$ arbitrary there exists a sequence (y_j) in $\mathbb{R}^N$ and $\eta > 0$ such that

$$\liminf_{j\to\infty} \int_{B(y_j,r)} |u_j^+|^2\,dx \geq \eta. \tag{3.7}$$

Indeed, if not then $u_j^+ \to 0$ in $L^t(\mathbb{R}^N)$ by Lemma 3.3, for any $t \in (2, 2^*)$. Moreover, from (3.4) and the Hölder inequality we get

$$\begin{aligned}&\Phi(u_j) - \frac{1}{2}D\Phi(u_j)u_j^+ \\ &\quad= -\frac{1}{2}\|u_j^-\|_E^2 + \frac{1}{2}\int_{\mathbb{R}^N} g(x,u_j)u_j^+\,dx - \int_{\mathbb{R}^N} G(x,u_j)\,dx \\ &\quad\leq \frac{1}{2}\int_{\mathbb{R}^N} g(x,u_j)u_j^+\,dx \\ &\quad\leq d\left(|u_j^+|_p + |u_j^+|_q\right)\end{aligned}$$

This yields $c = \lim_{j\to\infty} \Phi(u_j) \le 0$, a contradiction, thus proving (3.7).

Now we choose $a_j \in \mathbb{Z}^N$ such that $|a_j - y_j| = \min\{|a - y_j| : a \in \mathbb{Z}^N\}$ and set $v_j := a_j * u_j = u_j(\cdot + a_j)$. Using (3.7) and the invariance of E_{n_j}, $E^{\pm}$ under the action of $\mathbb{Z}^N$ we see that $v_j \in E_{n_j}$, $v_j^+ \in E^+$ and

$$\|v_j^+\|_{L^2(B(0,r+\sqrt{N}/2))} \ge \frac{\eta}{2}. \tag{3.8}$$

Moreover, $\|v_j\|_E = \|u_j\|_E$ and $|v_j|_\mu = |u_j|_\mu$, hence $\|v_j\|_\mu$ is bounded. Corollary 2.2 yields the existence of a subsequence (which we continue to denote by (v_j)) such that $v_j \to u$ weakly in E_μ and $v_j \to u$ strongly in $L^t_{\text{loc}}(\mathbb{R}^N)$, any $t \in [2, 2^*)$. Clearly (3.8) implies $\|u^+\|_{L^2(B(0,r+\sqrt{N}/2))} \ge \frac{\eta}{2}$, so $u \ne 0$.

Let $v \in C_0^\infty(\mathbb{R}^N)$ be any test function. As in the proof of Lemma 3.2 we see that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} g(x, v_j)(Id\ - Q_{n_j})v\,dx\right| \\
&\qquad \le |g_1(\cdot, v_j)|_{q'}|(id - Q_{n_j})v|_q + |g_2(\cdot, v_j)|_{p'}|(Id - Q_{n_j}v|_p \\
&\qquad \le d(|(Id - Q_{n_j})v|_q + |(Id - Q_{n_j})v|_p)
\end{aligned}$$

The right hand side converges to 0 as $j \to \infty$. Now

$$\begin{aligned}
\langle Lv_j, v\rangle_E &= \langle Lv_j, Q_{n_j}v\rangle_E \\
&= D\Phi(v_j)Q_{n_j}v + \int_{\mathbb{R}^N} g(x, v_j)v\,dx - \int_{\mathbb{R}^N} g(x, v_j)(Id - Q_{n_j})v\,dx
\end{aligned}$$

and therefore, letting $j \to \infty$, we have

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)\,dx = \langle Lu, v\rangle_E = \int_{\mathbb{R}^N} g(x, u)v\,dx.$$

This shows that $u \in E_\mu$ solves $-\Delta u + V(x)u = g(x, u)$ in the weak sense. The results of Sect. 2 then show that u lies in $H^2_{\text{loc}}(\mathbb{R}^N) \cap L^t(\mathbb{R}^N)$, $\mu \le t \le 2^*$, and u satisfies $u(x) \to 0$ as $|x| \to \infty$. □

In order to conclude the proof of Theorem 1.1 it suffices to find a $(\mathrm{PS})^*_c$-sequence for some $c > 0$. This will be done with the help of a linking theorem due to Kryszewski and Szulkin [12], generalizing a theorem of Benci and Rabinowitz [4].

Theorem 3.5. *Let X be a real Hilbert space and suppose $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ satisfies the hypotheses:*

(i) *There exists a bounded selfadjoint linear operator $L : X \to X$ and a functional $\varPsi \in \mathcal{C}^1(X,\mathbb{R})$ which is bounded below, weakly sequentially lower semicontinuous with $\nabla\varPsi : X \to X$ weakly sequentially continuous and such that*

$$\varPhi(u) = \frac{1}{2}\langle Lu, u\rangle - \varPsi(u).$$

(i) *There exists a closed separable L-invariant subspace Y of X and a positive constant α such that*

$$\langle Lu, u\rangle \le -\alpha\|u\|^2 \qquad \text{for } u \in Y$$

and

$$\langle Lu, u\rangle \ge \alpha\|u\|^2 \qquad \text{for } u \in Z := Y^{\perp}.$$

(iii) *There are constants $\kappa, \rho > 0$ such that $\varPhi(u) \ge \kappa$ for all $u \in Z$ with $\|u\| = \rho$.*

(iv) *There exists $z_0 \in Z$, $\|z_0\| = 1$, and $R > \rho$ such that $\varPhi(u) \le 0$ for $u \in \partial M$ where $M = \{u = y + \zeta z_0 : y \in Y, \|u\| < R, \zeta > 0\}$.*

Then there exists a sequence (u_k) such that $\nabla\varPhi(u_k) \to 0$ and $\varPhi(u_k) \to c$ for some $c \in [\kappa, \sup\varPhi(\overline{M})]$.

A proof of Theorem 3.5 can be found in [12], Theorem 3.4. Since $\nabla\varPsi$ is not compact Kryszewski and Szulkin contruct a degree theory which applies to special pseudo-gradient vector fields for $\varPhi$. A somewhat simpler proof using only the Brouwer degree is possible with the method from Sect. 4 below.

Lemma 3.6. *There exists $\rho > 0$ such that*

$$\kappa := \inf\left\{\varPhi(u)\colon u \in E^+, \|u\|_E = \rho\right\} > 0.$$

Proof. It follows easily from (g_3) that for $u \in E^+$

$$\begin{aligned}\varPhi(u) &= \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} G(x,u)\,dx \\ &\ge \frac{1}{2}\|u\|_E^2 - d\left(|u|_p^p + |u|_q^q\right).\end{aligned}$$

□

Lemma 3.7. *Fix $e \in E^+$ with $\|e\|_E = 1$. Then there exist $\sigma > 0$ and $R > \rho$ such that for every $n \in \mathbb{N}$*

$$\sup\varPhi|\overline{M}_n \le \sigma \quad \text{and} \quad \varPhi(u) \le 0 \quad \text{for } u \in \partial M_n$$

where $M_n = \{u = u^- + \zeta e\colon u^- \in E_n^-, \|u\|_E < R, \zeta > 0\}$.

Proof. Hypothesis (g_2) implies for $u = u^- + \zeta e$

$$\begin{aligned}\Phi(u) &= \frac{\zeta^2}{2} - \frac{1}{2}\|u^-\|_E^2 - \int_{\mathbb{R}^N} G(x,u)\,dx \\ &\le \frac{\zeta^2}{2} - \frac{1}{2}\|u^-\|_E^2 - \frac{a_1}{\gamma}|u^- + \zeta e|_\mu^\mu \\ &\le \frac{\zeta^2}{2} - \frac{1}{2}\|u^-\|_E^2 - d\,\zeta^\mu\end{aligned}$$

where $d > 0$ is independent of n and u. The lemma follows because $\mu > 2$. □

Lemma 3.8. *$\Phi_n \in \mathcal{C}^1(E_n, \mathbb{R})$ has the form $\Phi_n(u) = \frac{1}{2}\langle Lu, u\rangle_E - \Psi(u)$ where $\Psi \in \mathcal{C}^1(E_n, \mathbb{R})$ is bounded below, weakly sequentially lower semicontinuous and $\nabla_E\Psi : E_n \to E_n$ is weakly sequentially continuous.*

Proof. This follows from the fact (2.2) that $\|\cdot\|_E$ and $\|\cdot\|_{H^1}$ are equivalent on E_n. □

Setting $X := E_n$, $Y := E_n^-$ and $Z := E^+$ we have proved that Φ_n satisfies all hypotheses of Theorem 3.5. Consequently there exists a sequence $(v_m)_{m\in\mathbb{N}}$ in E_n such that $D\Phi_n(v_m) \to 0$ and $\Phi_n(v_m) \to c_n \in [\kappa, \sigma]$ as $m \to \infty$. For $m(n)$ large we therefore have

$$\|D\Phi_n(v_{m(n)})\|_E + |c_n - \Phi_n(v_{m(n)})| < \frac{1}{n}.$$

Thus along a subsequence $c_{n_j} \to c \in [\kappa, \sigma]$ and $u_j := v_{m(n_j)}$ is a $(\mathrm{PS})_c^*$-sequence as required. This finishes the proof of Theorem 1.1.

4. An abstract critical point theorem

Throughout this section, let X be a reflexive Banach space with the direct sum decomposition $X = X^- \oplus X^+$, $u = u^- + u^+$ for $u \in X$, and suppose that X^- is separable. Let $P^\pm$ denote the projection onto $X^\pm$. For a functional Φ on X we set $\Phi_a = \{u \in X : \Phi(u) \ge a\}$, $\Phi^b = \{u \in X : \Phi(u) \le b\}$ and $\Phi_a^b = \Phi_a \cap \Phi^b$. Finally we write $\mathcal{K} = \{u \in X : \Phi'(u) = 0\}$ for the set of critical points.

We consider a functional Φ satisfying the hypotheses:

(Φ_1) $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ is even and $\Phi(0) = 0$;
(Φ_2) there exist $\kappa, \rho > 0$ such that $\Phi(u) \ge \kappa$ for every $u \in X^+$ with $\|u\| = \rho$;

(Φ_3) there exists a strictly increasing sequence of finite dimensional subspaces $Y_n \subset X^+$ such that

$$\sup \Phi(X_n) < \infty \qquad \text{where } X_n := X^- \oplus Y_n,$$

and an increasing sequence of real numbers $R_n > 0$ with

$$\sup \Phi(X_n \setminus B_n) < \inf \Phi(B_\rho X)$$

where $B_n := \{u \in X_n : ||u|| \le R_n\}$ and $B_\rho X := \{u \in X : \|u\| \le \rho\}$.

Thus Φ has the typical mountain pass geometry. If the (PS)-condition would hold then Φ would have an unbounded sequence of positive critical values. However, this is not the case in our application. In order to formulate the hypotheses which do hold we introduce a new notion.

Definition 4.1. *Fix an interval $I \subset \mathbb{R}$. A set $\mathcal{A} \subset X$ is a* (PS)$_I$*-attractor if for any* (PS)$_c$*-sequence $(u_n)_{n\in\mathbb{N}}$ with $c \in I$, and any $\varepsilon, \delta > 0$ one has $u_n \in U_\varepsilon(\mathcal{A} \cap \Phi_{c-\delta}^{c+\delta})$ provided n is large enough.*

Clearly, a (PS)$_I$-attractor contains all critical points with levels in I. If $\mathcal{A}$ is a (PS)$_I$-attractor so is any set containing $\mathcal{A}$. In applications it is important to find small (PS)$_I$-attractors. In general there need not exist a smallest (PS)$_I$-attractor or minimal ones. Of course, if the Palais-Smale condition holds the set of critical points with values in I is the smallest (PS)$_I$-attractor.

In the sequel we shall write X_w for the space X with the weak topology and similarly X_w^-. It will be convenient to work with $X_\tau := X_w^- \times X^+$, that is, X_τ is the vector space X with the product topology of $X_w^- \times X^+$. Then notions like open, w-open or τ-open refer to the norm topology, the weak topology or the τ-topology, respectively.

Now we can state the hypotheses which replace the Palais-Smale condition:

(Φ_4) $\Phi'\colon X_\tau \to X_w^*$ is continuous, and $\Phi\colon X_\tau \to \mathbb{R}$ is upper semicontinuous.

(Φ_5) for any compact interval $I \subset (0, \infty)$ there exists a (PS)$_I$-attractor $\mathcal{A}$ such that

$$\inf\{\|u^+ - v^+\| : u, v \in \mathcal{A},\ u \ne v\} > 0.$$

Theorem 4.2. *If Φ satisfies* (Φ_1) – (Φ_5) *then there exists an unbounded sequence (c_n) of positive critical values.*

The proof of Theorem 4.2 will occupy the rest of this section. For a symmetric subset $A = -A \subset X$ we need the class $\mathcal{M}(A)$ of maps $g\colon A \to X$ with the properties

(4.1) $g\colon A_\tau \to X_\tau$ is τ-continuous and odd;
(4.2) $\Phi\big(g(u)\big) \le \Phi(u)$ for every $u \in A$;
(4.3) each $u \in A$ has a τ-neigborhood $W_u \subset X$ such that $(id - g)(W_u)$ is contained in a finite-dimensional subspace of X.

We write $\mathrm{gen(A)} \in \mathbb{N}_0 \cup \{\infty\}$ for the Krasnoselski genus of a symmetric subset A of X, that is, $\mathrm{gen(A)}$ is the least integer k such that there exists an odd continuous map $A \to S^{k-1}$. If no such map exists then $\mathrm{gen(A)} := \infty$. Now we define a kind of pseudo-index for the topology of sublevel sets Φ^c by setting

$$\psi(c) := \min\{\mathrm{gen(g(\Phi^c) \cap S_\rho X^+)} : \mathrm{g} \in \mathcal{M}(\Phi^c)\} \in \mathbb{N}_0 \cup \{\infty\}$$

where ρ is from (Φ_2) and $S_\rho X^+ = \{u \in X^+ : \|u\| = \rho\}$. From ($\Phi_2$) it follows that $\psi(c) = 0$ for $c < \kappa$ since then $\Phi^c \cap S_\rho X^+ = \emptyset$ and $\mathrm{gen}(\emptyset) = 0$. Therefore Theorem 4.2 is a consequence of the next three lemmas.

Lemma 4.3. *If* $c \ge \sup \Phi(X_n)$ *then* $\psi(c) \ge n$*.*

Lemma 4.4. *If there are no critical values in the interval* (a, b)*,* $0 < a < b$*, then* ψ *is constant on* (a, b)*.*

Lemma 4.5. $\psi\colon [0, \infty) \to \mathbb{N}_0$ *assumes only finite values.*

Proof of Lemma 4.3. Set $B_n := \{u \in X^- \oplus Y_n : \|u\| \le R_n\}$ and fix $c \ge \sup \Phi(X_n) = \sup \Phi(B_n)$. We shall show that $\mathrm{gen(g(B_n) \cap S_\rho X^+) \ge n}$ for any $g \in \mathcal{M}(\Phi^c)$. Then $\psi(c) \ge n$ because $B_n \subset \Phi^c$ and because the genus is monotone. Fix $g \in \mathcal{M}(\Phi^c)$. Since B_n is τ-compact it follows from (4.3) that $(id - g)(B_n)$ is contained in a finite-dimensional subspace F of X. We may assume that $F^+ := P^+F \supset Y_n$ and $F = F^- \oplus F^+$ with $F^- := P^-F \subset X^-$. Consider the set

$$\mathcal{O} := \{u \in B_n \cap F : \|g(u)\| < \rho\} \subset F$$

and the map

$$h : \partial\mathcal{O} \to F^-, \quad h(u) := P^- \circ g(u).$$

We observe that $g(B_n \cap F) \subset F$ because $(id - g)(B_n) \subset F$. Thus h is well defined. Moreover, $g\colon B_n \cap F \to F$ is continuous by (4.1) since F is finite-dimensional. In addition, (4.2) implies that $0 \in \mathcal{O}$ and $\overline{\mathcal{O}} \subset int(B_n \cap F)$. Therefore $\mathcal{O}$ is a bounded open neighborhood of 0 in $F_n := F \cap (X^- \oplus Y_n)$, hence, $\mathrm{gen}(\partial\mathcal{O}) = \dim \mathrm{F_n}$. From the monotonicity of the genus we obtain

$$\mathrm{gen}\left(\partial\mathcal{O} \setminus \mathrm{h}^{-1}(0)\right) \le \mathrm{gen}(\mathrm{F_n^-} \setminus \{0\}) = \dim \mathrm{F_n^-}\,.$$

The continuity and the subadditivity yield

$$\mathrm{gen}(\partial\mathcal{O}) \le \mathrm{gen}\left((\mathrm{h}^{-1}(0)\right) + \mathrm{gen}(\partial\mathcal{O}\backslash \mathrm{h}^{-1}(0))$$

It follows that

$$\mathrm{gen}(\mathrm{h}^{-1}(0)) \ge \dim \mathrm{F_n} - \dim \mathrm{F_n^-} = \dim \mathrm{Y_n} \ge \mathrm{n}.$$

Finally, $h(u) = 0$ implies $g(u) \in X^+$ and $u \in \partial\mathcal{O}$ implies $\|g(u)\| = \rho$, thus $g(h^{-1}(0)) \subset g(B_n) \cap S_\rho X^+$. Therefore, using the monotonicity of the genus once more we obtain the desired inequality

$$\mathrm{gen}(\mathrm{g}(\mathrm{B_n}) \cap \mathrm{S}_\rho \mathrm{X}^+) \ge \mathrm{gen}(\mathrm{g}(\mathrm{h}^{-1}(0))) \ge \mathrm{gen}(\mathrm{h}^{-1}(0)) \ge \mathrm{n}.$$

□

Proof of Lemma 4.4. Given positive numbers $a < b$ such that Φ has no critical values in (a, b) we want to show that ψ is constant on (a, b). We may assume that there are no critical values in $I := [a, b]$ and fix $c < d$ in (a, b). By the monotonicity of the genus we have $\psi(c) \le \psi(d)$. In order to prove $\psi(d) \le \psi(c)$ we shall construct a map $g \in \mathcal{M}(\Phi^d)$ with $g(\Phi^d) \subset \Phi^c$. Then $h\circ g \in \mathcal{M}(\Phi^d)$ for any $h \in \mathcal{M}(\Phi^c)$ because $id - h\circ g = id - g + (id - h)\circ g$ is τ-locally finite-dimensional as in (4.3) if $id - g$ and $id - h$ have this property. This implies

$$\begin{aligned}\psi(c) &= \inf\{\mathrm{gen}(\mathrm{h}(\Phi^\mathrm{c}) \cap \mathrm{S}_\rho \mathrm{X}^+) : \mathrm{h} \in \mathcal{M}(\Phi^\mathrm{c})\} \\ &\ge \inf\{\mathrm{gen}(\mathrm{h}(\mathrm{g}(\Phi^\mathrm{d})) \cap \mathrm{S}_\rho \mathrm{X}^+) : \mathrm{h} \in \mathcal{M}(\Phi^\mathrm{c})\} \\ &\ge \inf\{\mathrm{gen}(\mathrm{h}(\Phi^\mathrm{d}) \cap \mathrm{S}_\rho \mathrm{X}^+) : \mathrm{h} \in \mathcal{M}(\Phi^\mathrm{d})\} \\ &= \psi(d)\end{aligned}$$

as required. Here we used the monotonicity of the genus in the second line. In order to construct $g \in \mathcal{M}(\Phi^d)$ with $g(\Phi^d) \subset \Phi^c$ we choose a (PS)$_I$-attractor $\mathcal{A}$ and $\sigma > 0$ such that

(4.4) $$\|u^+ - v^+\| > 2\sigma \quad \text{for } u, v \in \mathcal{A},\ u \ne v.$$

This exists according to (Φ_5). We set

$$B := P^+(\mathcal{A}) = \{u^+ : u \in \mathcal{A}\} \subset X^+$$

and consider the τ-open set

$$\begin{aligned}U_\sigma &:= \{u \in X : \|u^+ - v^+\| < \sigma \quad \text{for some } v \in \mathcal{A}\} \\ &= X^- \times U_\sigma(B)\end{aligned}$$

Since $\mathcal{A}$ is a (PS)$_I$-attractor and $U_\sigma(\mathcal{A}) \subset U_\sigma$ there exists $\alpha > 0$ such that

(4.5) $$\|\Phi'(u)\| \geq 2\alpha \qquad \text{for } u \in \Phi_c^d \setminus U_\sigma.$$

For $u \in \Phi_a^b$ we choose a pseudo-gradient vector $w(u) \in X$ satisfying $\|w(u)\| \leq 2$ and $\Phi'(u)w(u) > \|\Phi'(u)\|$. If $u \in \Phi_c^d \backslash U_\sigma$ we therefore have $\Phi'(u)w(u) \geq 2\alpha$. Therefore there exists a τ-open neighborhood N_u of u such that

(4.6) $$\Phi'(v)w(u) > \alpha \quad \text{for } v \in N_u, u \in \Phi_c^d \backslash U_\sigma.$$

Here we used the hypothesis (Φ_4) that $\Phi' \colon X_\tau \to X_w^*$ is continuous. Similarly, every $u \in \Phi_c^d \cap U$, hence by (4.4), $u \in X^- \times U_\sigma(v^+)$ for some $v \in \mathcal{A}$, has a τ-open neighborhood $N_u \subset X^- \times U_\sigma(v^+)$ such that

(4.7) $$\Phi'(v)w(u) \geq \|\Phi'(u)\| \quad \text{for } v \in N_u,\ u \in \Phi_c^d \cap U_\sigma.$$

Finally, if $\Phi(u) < c$ we set $N_u := X \backslash \Phi_c$ and $w(u) := 0$. Since $\Phi \colon X_\tau \to \mathbb{R}$ is τ-upper semicontinuous, N_u is τ-open. It follows from results of Dowker [8] and Michael [14] that X_τ and every subset of X_τ are paracompact. Thus there exists a τ-locally finite partition of unity $(\pi_j)_{j \in J}$ subordinate to the covering $(N_u : u \in \Phi^d)$ of Φ^d. Here $\pi_j \colon \Phi^d \to [0,1]$ is continuous with respect to the τ-topology on Φ^d, hence it is continuous with the norm topology on Φ^d. It is not difficult to see that one may construct the maps π_j such that π_j is also locally Lipschitz continuous with respect to the norm in Φ^d.

For $j \in J$ we choose $u_j \in \Phi^d$ with $\operatorname{supp} \pi_j \subset N_{u_j}$ and define

$$V_0(u) := \sum_{j \in J} \pi_j(u) w(u_j)$$

and

$$V \colon \Phi^d \to X, \quad V(u) := \frac{1}{2}(V_0(u) - V_0(-u)).$$

Then V is odd, locally Lipschitz continuous and, in addition, continuous with the τ-topology on Φ^d and on X. Moreover, for every $u \in \Phi^d$ there exists a τ-neigborhood W_u such that $(id - V)(W_u)$ is contained in a finite-dimensional subspace of X. We also have

(4.8) $$\|V(u)\| \leq 2 \quad \text{for all } u \in \Phi^d;$$

(4.9) $$\Phi'(u)V(u) \geq 0 \quad \text{for all } u \in \Phi^d;$$

(4.10) $$\Phi'(u)V(u) > \alpha \quad \text{for all } u \in \Phi_c^d \setminus U_\sigma;$$

(4.11) $$\Phi'(u)V(u) > 0 \quad \text{for all } u \in \Phi_c^d \cap U_\sigma.$$

Let $\varphi\colon \Phi^d \times [0,\infty) \to \Phi^d$, $\varphi(x,t) = \varphi^t(x)$, be the semiflow associated to $-V$, that is $d\varphi^t/dt = -V \circ \varphi^t$ for $t > 0$ and $\varphi^0 = id$. For every $u \in \Phi^d$ and every $t > 0$ there exists a τ-neighborhood W_u and an $\varepsilon > 0$ such that $(id - \varphi)(W_u \times (t-\varepsilon, t+\varepsilon))$ is contained in a finite-dimensional subspace of X. Since the vector field $V : (\Phi^d)_\tau \to X_\tau$ is τ-continuous also $\varphi\colon (\Phi^d)_\tau \times [0,\infty) \to (\Phi^d)_\tau$ is τ-continuous. Now we claim that for every $u \in \Phi^d$ there exists a time $T_1(u) > 0$ such that $\Phi(\varphi(u, T_1(u))) < c$. If this has been proved then there also exists a τ-open neighborhood W_u of u such that $\Phi(\varphi(v, T_1(u))) < c$ for $v \in W_u$. As above we choose a partition of unity $(\pi_j\colon \Phi^d \to [0,1])_{j\in J}$ subordinate to $(W_u : u \in \Phi^d)$ and define $T(u) := \sum_{j\in J} \pi_j(u) T_1(u_j)$ where u_j is chosen so that $\operatorname{supp} \pi_j \subset W_{u_j}$. It is not difficult to check that the map

$$g\colon \Phi^d \to \Phi^c, \quad g(u) := \varphi(u, T(u))$$

is well defined and lies in $\mathcal{M}(\Phi^d)$. Thus the proof of Lemma 4.4 is finished once the existence of $T_1(u)$ is established.

We fix $u \in \Phi^d$ and suppose $\lim\limits_{t\to\infty} \Phi(\varphi^t(u)) \geq c$. Since $\mathcal{A}$ is a (PS)$_I$-attractor $\|\Phi'(v)\|$ is bounded away from 0 for v outside an arbitrarily small neighborhood of $\mathcal{A}$ in Φ_a^b. This implies that there exists a time $T > 0$ such that $\varphi^t(u) \in U_\sigma$ for all $t \geq T$. By (4.4) there exists $v \in \mathcal{A}$ such that $\varphi^t(u) \in X^- \times U_\sigma(v^+)$ for all $t \geq T$. By the construction of the pseudo-gradient vector field V it follows for $t \geq T$ that

$$\begin{aligned} \frac{d}{dt}\Phi\left(\varphi^t(u)\right) &\leq -\inf\left\{\|\Phi'(u_j)\| : \pi_j\left(\varphi^t(u)\right) \neq 0\right\} \\ &\leq -\inf\left\{\|\Phi'(u_j)\| : u_j \in \Phi_c^d \cap X^- \times U_\sigma(v^+)\right\} \end{aligned}$$

This cannot be bounded away from 0 because $\lim\limits_{t\to\infty} \Phi(\varphi^t(u)) \geq c$. So there exists a sequence $(u_{j_k})_k$ in $\Phi_c^d \cap X^- \times U_\sigma(v^+)$ with $\|\Phi'(u_{j_k})\| \to 0$. Then u_{j_k} lies in arbitrarily small (norm) neighborhoods of $\mathcal{A}$ for k large, hence, $u_{j_k} \to v$ as $k \to \infty$. Therefore $\Phi'(v) = 0$ and $\Phi(v) \in [c,d]$ which is a contradiction to the assumption that there are no critical values in $[a,b]$. □

Proof of Lemma 4.5. We work with a comparison function $\psi_d\colon [0,d] \to \mathbb{N}_0$ in order to show the finiteness of ψ. For $d > 0$ fixed set

$$\mathcal{M}_0(\Phi^d) := \{g \in \mathcal{M}(\Phi^d) : g \text{ is a homeomorphism from } \Phi^d \text{ to } g(\Phi^d)\}.$$

Then we define for $c \in [0,d]$

$$\psi_d(c) := \min\left\{\mathrm{gen}(\mathrm{g}(\Phi^{\mathrm{c}}) \cap \mathrm{S}_\rho \mathrm{X}^+) : \mathrm{g} \in \mathcal{M}_0(\Phi^{\mathrm{d}})\right\}.$$

Since $\mathcal{M}_0(\Phi^d) \subset \mathcal{M}(\Phi^d) \hookrightarrow \mathcal{M}(\Phi^c)$ via restriction $g \mapsto g|\Phi^c$ we have $\psi(c) \le \psi_d(c)$. Thus it suffices to show $\psi_d(c) < \infty$ for $c < d$. Clearly $\psi_d(c) = 0$ for $c < \kappa$ by (Φ_3) because $id \in \mathcal{M}_0(\Phi^d)$. We claim that for any $c \in [\kappa, d)$ there exists $\delta > 0$ such that $\psi_d(c+\delta) \le \psi_d(c-\delta) + 1$. This implies the finiteness of $\psi_d(c)$ for $c \in [0, d)$. We proceed as in the proof of Lemma 4.4. For $I := [\kappa/2, d]$ there exists a (PS)$_I$-attractor $\mathcal{A}$ and $\sigma > 0$ such that

$$\|u^+ - v^+\| > 6\sigma \quad \text{for } u, v \in \mathcal{A}, u \ne v. \tag{4.12}$$

Setting $B := P^+(\mathcal{A})$ and $U_\sigma := X^- \times U_\sigma(B)$ there exists $\alpha > 0$ such that

$$\|\Phi'(u)\| \ge 2\alpha \quad \text{for } u \in \Phi^d_{\kappa/2} \setminus U_\sigma. \tag{4.13}$$

Next we construct a pseudo-gradient vector field $V\colon \Phi^d \to X$. For $u \in \Phi^d_{\kappa/2} \setminus U_\sigma$ we choose $w(u) \in X$ with $\|w(u)\| \le 2$ and $\Phi'(u)w(u) \ge \|\Phi'(u)\| \ge 2\alpha > \alpha$. This implies $\Phi'(v)w(u) > \alpha$ for v in some τ-neighborhood N_u of u. If $\Phi(u) < \kappa/2$ then we set $N_u := X \setminus \Phi_{\kappa/2}$ and $w(u) = 0$. If $u \in \Phi^d_{\kappa/2} \cap U_\sigma$ we set $N_u := U_\sigma$ and $w(u) := 0$. Let $(\pi_j)_{j\in J}$ be a τ-locally finite partition of unity subordinated to the τ-open covering $(N_u : u \in \Phi^d)$ of Φ^d. As before the maps $\pi_j\colon \Phi^d \to [0,1]$ are Lipschitz continuous with the norm on Φ^d and continuous with the τ-topology on Φ^d. Now we define $V_0(u) := \sum_{j\in J} \pi_j(u)w(u_j)$ and $V(u) := \frac{1}{2}(V_0(u) - V_0(-u))$ and let $\varphi^t\colon \Phi^d \to \Phi^d$, $t \ge 0$, be the semiflow associated to $-V$. We claim that there exists $\delta > 0$ such that

$$\varphi^1(\Phi^{c+\delta}) \subset \Phi^{c-\delta} \cup U_{3\sigma} \tag{4.14}$$

where $U_{3\sigma} := X^- \times U_{3\sigma}(B)$. Postponing the proof of (4.14) we first deduce $\psi_d(c+\delta) \le \psi_d(c-\delta) + 1$. Choose $g \in \mathcal{M}_0(\Phi^d)$ such that $\psi_d(c) = \mathrm{gen}(\mathrm{g}(\Phi^{\mathrm{c}-\delta}) \cap \mathrm{S}_\rho \mathrm{X}^+)$. Then $g \circ \varphi^1 \in \mathcal{M}_0(\Phi^d)$ so that

$$\begin{aligned}
\psi_d(c+\delta) &\le \mathrm{gen}(\mathrm{g} \circ \varphi^1(\Phi^{\mathrm{c}+\delta}) \cap \mathrm{S}_\rho \mathrm{X}^+) \\
&\le \mathrm{gen}(\mathrm{g}(\Phi^{\mathrm{c}-\delta} \cup \mathrm{U}_{3\sigma}) \cap \mathrm{S}_\rho \mathrm{X}^+) \\
&\le \mathrm{gen}(\mathrm{g}(\Phi^{\mathrm{c}-\delta}) \cap \mathrm{S}_\rho \mathrm{X}^+) + \mathrm{gen}(\mathrm{g}(\mathrm{U}_{3\sigma})) \\
&\le \psi^d(c-\delta) + 1.
\end{aligned}$$

Here we used the standard properties of the genus and in addition that $g(U_{3\sigma})$ is homeomorphic to $U_{3\sigma}$ which in turn is homotopy equivalent to the discrete set B by (4.12). The homotopy equivalence $g(U_{3\sigma}) \to B$ is odd hence $\mathrm{gen}(\mathrm{g}(\mathrm{U}_{3\sigma})) \le \mathrm{gen}(\mathrm{B}) \le 1$.

It remains to prove (4.14). We argue indirectly and suppose there exists a sequence $u_n \in \Phi^{c+1/n}$ with $\varphi^1(u_n) \notin \Phi^{c-1/n} \cup U_{3\sigma}$. For $n > 2/\alpha$ with α from (4.13) there exists $t_n \in (0,1)$ such that $\varphi^{t_n}(u_n) \in U_\sigma$. Thus there exists $0 \le r_n < s_n \le 1$ with $\varphi^{r_n}(u_n) \in \partial U_\sigma$, $\varphi^{s_n}(u_n) \in \partial U_{3\sigma}$ and $\varphi^t(u_n) \in U_{3\sigma} \setminus U_\sigma$ for $t \in (r_n, s_n)$. This implies $\|\varphi^{r_n}(u_n) - \varphi^{s_n}(u_n)\| \ge 2\sigma$ hence, $s_n - r_n \ge \sigma$ because $\|V(u)\| \le 2$. Now (4.13) yields

$$\begin{aligned} c - \frac{1}{n} &< \Phi(\varphi^{s_n}(u_n)) \\ &< \Phi(\varphi^{r_n}(u_n)) - \sigma\alpha \\ &< c + \frac{1}{n} - \sigma\alpha \end{aligned}$$

for any $n \in \mathbb{N}$. This contradiction finishes the proof of (4.14) hence the proof of Lemma 4.5. □

5. Proof of Theorem 1.2

As in Sect. 3 the solutions of (NS) will be obtained as critical points of the functional

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)\, dx - \int_{\mathbb{R}^N} G(x,u)\, dx \\ &= \frac{1}{2}(||u^+||_E^2 - ||u^-||_E^2) - \Psi(u). \end{aligned}$$

From (g_4) it follows that Φ is defined on the Banach space $X := E_\mu$. Let $\mathcal{K}$ be the set of critical points of Φ and observe that for $u \in \mathcal{K}\setminus\{0\}$

$$\Phi(u) - \frac{1}{2}\Phi'(u)u = \int_{\mathbb{R}^N} \big(g(x,u)u - G(x,u)\big) > 0$$

by (g_2), hence

$$\mathcal{K} \subset \Phi_0 \text{ and } \mathcal{K} \cap X^- = \{0\}. \tag{5.1}$$

Let $\mathcal{F} \subset \mathcal{K}$ consist of arbitrarily chosen representatives of the orbits of $\mathcal{K}$ under the action of $\mathbb{Z}^N$. Since g is odd by assumption (g_5) we may assume that $\mathcal{F} = -\mathcal{F}$. As a consequence of the invariance of Φ under the group action $*$ we obtain

$$(\mathbb{Z}^N * u_1) \cap (\mathbb{Z}^N * u_2) = \emptyset \quad \text{if} \quad u_1, u_2 \in \mathcal{K} \quad \text{with} \quad \Phi(u_1) \ne \Phi(u_2). \tag{5.2}$$

It is not difficult to verify that Φ satisfies (Φ_1) – (Φ_4). In order to apply Theorem 4.2 we need to check (Φ_5). Let $[r]$ denote the integer part of r for any $r \in \mathbb{R}$. Along the lines of the proof of [12], Proposition 4.2 (see also [7]), one can easily establish the following lemma.

Lemma 5.1. *Let the assumptions of Theorem 1.2 be satisfied and assume that*

(5.3) $$\inf_{\mathcal{K}\setminus\{0\}} \Phi > \alpha > 0.$$

Let $(u_n) \subset E_\mu$ be a $(PS)_c$-sequence. Then either $u_n \to 0$ (corresponding to $c = 0$); or $c \geq \alpha$ and there are $l \leq [c/\alpha],\ v_i \in \mathcal{F}\setminus\{0\},\ i = 1, \cdots, l$, a subsequence denoted again by (u_n), and l sequences $(a_{in})_n$ in $\mathbb{Z},\ i = 1, \cdots, l$ such that

$$||u_n - \Sigma_{i=1}^l a_{in} * v_i||_\mu \to 0, \quad \text{as } n \to \infty,$$

$$|a_{in} - a_{jn}| \to \infty \quad \text{as } n \to \infty, \text{ if } i \neq j,$$

and

$$\Sigma_{i=1}^l \Phi(v_i) = c.$$

Now suppose (NS) has only finitely many geometrically distinct solutions in E_μ, that is, $\mathcal{F}$ is finite. It follows from (5.1) that $\alpha := \frac{1}{2} \min \Phi(\mathcal{K}\setminus\{0\}) > 0$. Given a compact intervall $I \subset (0, \infty)$ with $d := \max I$ we set $l := [d/\alpha]$ and

$$[\mathcal{F}, l] := \{\Sigma_{i=1}^j k_i * v_i;\ 1 \leq j \leq l, k_i \in \mathbb{Z}^N, v_i \in \mathcal{F}\}.$$

As a consequence of Lemma 5.1 we see that $[\mathcal{F}, l]$ is a $(PS)_I$-attractor. It is easy to check that

(5.4) $$\inf\{||u^+ - v^+|| :\ u, v \in [\mathcal{F}, l], u \neq v\} > 0$$

(see e.g. [7]). Therefore (Φ_5) is also satisfied and Theorem 4.2 yields the existence of an unbounded sequence of critical values of Φ. Hence $\mathcal{F}$ cannot be finite and Theorem 1.2 is proved.

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