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Monopole moduli spaces for compact 3-manifolds

Marc Pauly[*]

Département de Mathématiques (CP 218), Université Libre de Bruxelles, B-1050 Bruxelles, Belgium
(e-mail: pauly@ulb.ac.be)

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1 Introduction

It is a well-known fact in gauge theory that solutions (∇, Φ) of the (non-linear) Bogomolny monopole equation $F_\nabla = *d_\nabla \Phi$ on Euclidean 3-space can be reinterpreted as time-invariant anti-self-dual connections on Euclidean 4-space: If $P : \mathbb{R}^4 \to \mathbb{R}^3$ denotes the projection which 'forgets' the time coordinate t, then the connection $\nabla^4 := P^*\nabla + dt \otimes P^*\Phi$ is easily seen to have an anti-self-dual curvature 2-form. This construction can be paralleled for a different projection $\pi : B^4 \to B^3$, namely a radial extension of the Hopf fibration $S^3 \to S^2$. In that case the group preserving our anti-self-dual connection will no longer be the time-translation group $\mathbb{R}$, but the compact group S^1 acting on the four-ball B^4. To be somewhat more precise, there exists a circle-invariant 1-form ξ on B^4 such that $\pi^*\nabla + \xi \otimes \pi^*\Phi$ is anti-self-dual with respect to the Euclidean metric. It should be noted that the S^1-action on the pull-back bundle acts trivially on the fibre over the origin of B^4. This is somewhat restrictive, since one would also like to work with non-trivial S^1-actions on this fibre. It is therefore preferable to define the pull-back bundle only over $B^4 \setminus 0$, which allows us to define a particularly interesting type of singular monopoles. Indeed we can look at monopoles that are only defined on $B^3 \setminus 0$, but whose corresponding circle-invariant connection extends smoothly in some gauge of the pull-back bundle over $B^4 \setminus 0$, along with the S^1-action on the bundle. In such a situation, the S^1-action may act non-trivially on the fibre over the origin of B^4. This definition of singular monopoles is quite satisfactory, as it is based on a re-interpretation in terms of regular anti-self-dual (or ASD) connections on the four-ball.

[*] Aspirant du Fonds National Belge de la Recherche Scientifique

The above construction, linking singular objects on the 3-ball to regular ones on the 4-ball, can be used in the framework of gauge theory on compact 3-manifolds. As a matter of fact, smooth monopoles (∇, Φ) over any compact 3-manifold necessarily satisfy the much stronger equations $F_\nabla = 0$, $d_\nabla \Phi = 0$; one has therefore to look for singular solutions. In our setup, we fix a finite number of points $\{p_1, \ldots, p_n\}$ on a compact, Riemannian, oriented 3-manifold X and we consider $SU(2)$ monopoles that are singular (in the sense defined hereabove) at each point p_i. To any such solution, we can associate n nonnegative integers $k_1, \ldots, k_n$, where $(k_i, -k_i)$ are the weights of the representation $S^1 \to SU(2)$ in the fibre over the origin of B^4 (the 4-ball is projected via the Hopf fibration to a small 3-ball around the singularity p_i). The integer k_i is called the *charge* of the monopole at the singularity p_i.

Actually, the reinterpretation of our monopoles in terms of ASD connections can be performed only over small 3-balls around the singularities. As we show in Sect. 2.3, it is indeed impossible (for topological reasons) to extend these n local correspondences to one global correspondence intertwining singular monopoles on a compact 3-manifold and regular ASD connections on a 4-manifold with an S^1-action.

It should be pointed out that the monopoles we consider have genuine singularities at the points p_i:

Proposition 6 *If (∇, Φ) is a monopole with a singularity of charge k at p, then the pointwise norm of Φ satisfies $|\Phi| = k/2r + \underline{o}(1/r)$, where r is the geodesic distance from p.*

We should also make a remark about the 3- and 4-dimensional metrics. As a matter of fact, the local construction we have discussed so far works only for the Euclidean metric on B^3. However (cf. proposition 4), it is true that for any metric on B^3, one can modify the 1-form ξ and the metric on B^4 (without losing too much regularity) so that the correspondence between the Bogomolny and the ASD equation still holds.

For our purposes, we fix the charge at each singularity, i.e. we only consider monopoles whose charge at the singularity p_i equals a prescribed integer k_i. The aim of this paper is to work out a formula for the virtual dimension of the moduli space $\mathscr{M}$ of singular monopoles on X with charges $(k_1, \ldots, k_n)$ at the singularities $(p_1, \ldots, p_n)$. By moduli space we mean the space of all singular monopoles with charges $(k_1, \ldots, k_n)$, modulo the action of the group of bundle automorphisms (or gauge group). The gauge group does indeed preserve the Bogomolny equation, so that the moduli space parametrizes the essentially different singular monopoles.

The techniques used for the study of this moduli space are fairly standard and have been applied to the study of instanton moduli spaces ([3], [4]). In the affine space of all pairs (∇, Φ), we construct through each pair a 'slice' transverse to its orbit under the gauge group action, thus obtaining a fairly adequate local model for the space of gauge equivalence classes of pairs. This *'gauge-fixing result'* (Proposition 9) is very similar to the one in instanton theory, and their proofs are identical (with one minor exception due to the fact that here we are working with singular objects; however, this is not a problem, since it suffices to view our singular pairs as regular anti-self-dual connections over the 4-ball). It will follow from the gauge-fixing result that the moduli space around a point $[(\nabla, \Phi)]$ can be identified with the set of all monopoles in the slice through (∇, Φ), modulo the (finite-dimensional) stabilizer of (∇, Φ) in the gauge group. As in standard instanton theory, the virtual dimension of our moduli space is given by the index of a Fredholm operator L (which is however not elliptic, again due to the singularity at p). The explicit computation of this index leads to our main result:

Theorem 1 *The virtual dimension of the moduli space $\mathscr{M}$ of singular monopoles on X with singularities at $(p_1, \ldots, p_n)$ and charges $(k_1, \ldots, k_n)$ equals $4\sum_{i=1}^{n} k_i$.*

To obtain this index formula,we use the excision property for indices of elliptic operators (one of the many ingredients of the Atiyah-Singer Index Theorem), and we show in Proposition 11 that the problem of computing the index of L boils down to computing the index of the standard elliptic operator from instanton theory (namely $d^*_{\nabla^4} \oplus d^+_{\nabla^4}$), acting on S^1-invariant 1-forms over S^4 (in this particular case, the S^1-action on S^4 fixes two points and acts freely everywhere else).

The computation of this new index is carried out in Proposition 12. Our result is a simple application of the Atiyah-Singer Fixed Point Theorem for G-invariant elliptic operators. The combination of Propositions 11 and 12 yields the main result.

Finally, we exhibit (in Sect. 5) a 1-parameter family of $SO(3)$-invariant, singular monopoles with $k_1 = 1$ (i.e. with one singularity of charge 1) on the 3-sphere with its round metric. We also show (via a Weitzenböck formula) that the moduli spaces of singular monopoles on the 3-sphere are regular, which means in particular that in the $k_1 = 1$ case, the moduli space is given locally by a 4-parameter family of solutions (as predicted by the virtual dimension formula).

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2 The Bogomolny and the ASD equation

Let (X, g) be a compact, oriented Riemannian 3-manifold, and $\eta \to X$ an $SU(2)$ vector bundle. Let ∇ be an $SU(2)$ connection in η, and Φ a section of $\operatorname{ad}\eta$ (i.e. a trace-free, skew-hermitian endomorphism of η).

We say that the pair (∇, Φ) is a **monopole** if it satisfies the Bogomolny equation

$$F_\nabla = *d_\nabla \Phi.$$

Here F_∇ denotes the curvature of ∇, and $*$ the Hodge star operator induced by the metric g. Now let u be a gauge transformation, i.e. an $SU(2)$ bundle automorphism of η. Gauge transformations act on the space of all pairs (∇, Φ) by the rule $u \cdot (\nabla, \Phi) = (\nabla', \Phi')$, where

$$\begin{aligned} \nabla' &= u \nabla u^{-1} \\ \Phi' &= u \Phi u^{-1}. \end{aligned}$$

If (∇, Φ) is a monopole, then clearly (∇', Φ') is a monopole. It means that the Bogomolny equation is invariant under gauge transformations.

A basic question about the Bogomolny equation on a compact manifold would be to examine the moduli space of monopoles for given (X, g, η), i.e. the space of all monopoles modulo gauge equivalence. Whereas the moduli spaces of the 4-dimensional anti-self-duality (ASD) equation are highly interesting objects, the next proposition shows that the moduli spaces for the monopole equation on a compact 3-manifold are too constrained to give rise to new information about the 3-manifold X.

Proposition 1 *If (∇, Φ) is a monopole on a compact 3-manifold X, then $F_\nabla = *d_\nabla \Phi = 0$.*

Proof. Combine the Bianchi identity $d_\nabla F_\nabla = 0$ and the monopole equation to deduce that $d_\nabla^* d_\nabla \Phi = 0$, where $d_\nabla^* : \Omega^1(\mathrm{ad}\,\eta) \to \Omega^0(\mathrm{ad}\,\eta)$ is the formal adjoint of d_∇. The result follows. □

2.1 The Euclidean case

We define an S^1-action on B^4 (viewed as subset of $\mathbb{C}^2 \cong \mathbb{R}^4$) by

$$\theta \cdot (z_1, z_2) := (e^{i\theta} z_1, e^{i\theta} z_2).$$

This action fixes the origin of B^4, whereas it is free on $B^4 \setminus 0$. Now define coordinates $(y_i)_{i=1,\dots,4}$ on B^4 by $z_1 = y_1 + iy_2$, $z_2 = y_3 + iy_4$, and let (x_1, x_2, x_3) be coordinates on B^3. Then the polynomial map $\pi : B^4 \to B^3$ defined by

$$\begin{aligned} x_1 &= 2(y_1 y_3 + y_2 y_4) \\ x_2 &= 2(y_2 y_3 - y_1 y_4) \\ x_3 &= y_1^2 + y_2^2 - y_3^2 - y_4^2 \end{aligned}$$

exhibits $B^4 \setminus 0$ as an S^1-principal bundle over $B^3 \setminus 0$.

Note that if r_n denotes the ‘distance from the origin’-function in $\mathbb{R}^n$, then $\pi^* r_3 = r_4^2$.

We denote by $\frac{\partial}{\partial\theta}$ the smooth vector field on B^4 given by

$$\left(\frac{\partial}{\partial\theta}\right)_y := \left.\frac{d}{d\theta}\right|_{\theta=0} (\theta \cdot y).$$

Let ε be an $SU(2)$ vector bundle over $B^3 \setminus 0$ and consider its pull-back $\pi^*\varepsilon$ to $B^4 \setminus 0$. This pull-back bundle comes with a natural S^1-action projecting onto the S^1-action on $B^4 \setminus 0$. Given any S^1-invariant 1-form ξ on $B^4 \setminus 0$, satisfying $\xi(\frac{\partial}{\partial\theta}) \not\equiv 0$, we can define a one-to-one correspondence between

(a) pairs (∇, Φ) of an $SU(2)$ connection in ε and a section of $\mathrm{ad}\,\varepsilon$, and

(b) S^1-invariant $SU(2)$ connections ∇^4 in $\pi^*\varepsilon$,

by letting $\nabla^4 = \pi^*\nabla + \xi \otimes \pi^*\Phi$. One shows easily that ∇ and Φ can be unambiguously recovered from ∇^4.

This correspondence has (at least for a particular choice of ξ and the metrics) the following nice property.

Proposition 2 *Endow B^4 (resp. B^3) with the Euclidean metric and the orientation form $-dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4$ (resp. $-dx_1 \wedge dx_2 \wedge dx_3$), and let $\xi = 2(-y_2 dy_1 + y_1 dy_2 - y_4 dy_3 + y_3 dy_4)$. Then (∇, Φ) is a monopole if and only if $\nabla^4 = \pi^*\nabla + \xi \otimes \pi^*\Phi$ satisfies the anti-self-dualtity (or ASD) equation $*F_{\nabla^4} = -F_{\nabla^4}$.*

Proof.

$$\begin{aligned} F_{\nabla^4} &= F_{\pi^*\nabla + \xi \otimes \pi^*\Phi} \\ &= \pi^* F_\nabla + d_{\pi^*\nabla}(\xi \otimes \pi^*\Phi) + (\xi \otimes \pi^*\Phi) \wedge (\xi \otimes \pi^*\Phi) \\ &= \pi^* F_\nabla + d\xi \otimes \pi^*\Phi - \xi \wedge \pi^*(d_\nabla \Phi). \end{aligned}$$

Now, for the given orientation of B^4, $d\xi = 4(dy_1 \wedge dy_2 + dy_3 \wedge dy_4)$ is an anti-self-dual 2-form. Therefore

$$F^+_{\nabla^4} = (\pi^* F_\nabla)^+ - (\xi \wedge \pi^*(d_\nabla \Phi))^+,$$

where the superscript denotes the self-dual part of a (bundle-valued) 2-form. One checks that for any 1-form ω on B^3, one has

$$\xi \wedge \pi^*\omega = *\pi^*(*\omega), \qquad \text{and therefore}$$

$$\begin{aligned} F^+_{\nabla^4} &= (\pi^* F_\nabla)^+ - (\pi^*(*d_\nabla \Phi))^+ \\ &= \Big(\pi^*(F_\nabla - *d_\nabla \Phi)\Big)^+. \end{aligned}$$

This establishes the only-if part. As for the converse, just notice that for any 2-form α on $B^3 \setminus 0$, $(\pi^*\alpha)^+ = 0$ implies that $\alpha = 0$. □

2.2 Generalization for arbitrary metrics

The next proposition shows that the correspondence between the monopole and the ASD equation in the Euclidean case is no accident.

Proposition 3 *Suppose B^4 and B^3 carry arbitrary metrics g^4 and g^3, and let ξ be an S^1-invariant 1-form on $B^4 \setminus 0$ with $\xi(\frac{\partial}{\partial\theta}) \neq 0$. Then the equivalence*

$$(\nabla, \Phi) \text{ is a monopole on } B^3 \setminus 0 \iff \nabla^4 := \pi^*\nabla + \xi \otimes \pi^*\Phi \text{ is ASD on } B^4 \setminus 0$$

holds exactly when
(i) the metric g^4 on $B^4 \setminus 0$ is conformal to $\pi^* g^3 + \xi^2$,
(ii) $\xi = \pi^*(1/f) \cdot \omega$, where f is a smooth non-zero function on $B^3 \setminus 0$ satisfying $\triangle f = 0$ and $\frac{1}{2\pi}[*df] = 1 \in H^2(B^3 \setminus 0, \mathbb{R})$, and ω is an S^1-invariant 1-form on $B^4 \setminus 0$ satisfying $\omega(\frac{\partial}{\partial\theta}) = 1$ and $d\omega = -\pi^*(*df)$.

Proof. Re-reading the proof of Proposition 2, one sees immediately that the equivalence holds exactly when
(i) $\xi \wedge \pi^*\alpha = *\pi^*(*\alpha)$ for any 1-form α on $B^3 \setminus 0$,
(ii) $d\xi$ is an anti-self-dual 2-form.

It is easy to check that condition (i) is satisfied if and only if the metric on $B^4 \setminus 0$ is conformal to $\pi^* g^3 + \xi^2$. One can write $\xi = \pi^*(f^{-1}) \cdot \omega$, where f is a smooth non-zero function on $B^3 \setminus 0$, and ω a connection 1-form of the principal S^1-bundle $B^4 \setminus 0 \to B^3 \setminus 0$ (f is non-singular, since $\xi(\frac{\partial}{\partial\theta}) \neq 0$). Hence $d\omega = \pi^* B$, where B is the curvature 2-form of the bundle. The exterior derivative of ξ equals

$$d\xi = \pi^*\left(B/f\right) - \pi^*\left(df/f\right) \wedge \xi.$$

Using conditions (i) and (ii), we get

$$\begin{aligned} 0 &= d\xi + *d\xi \\ &= \pi^*\left(B/f\right) + \pi^*\left(*df/f\right) + \xi \wedge \pi^*\left(df/f + *B/f\right), \end{aligned}$$

and hence

$$B = - * df.$$

As B is closed, f must be harmonic on $B^3 \setminus 0$. Note that a given harmonic function determines B, and hence ω (up to gauge equivalence) and ξ. However, there is an extra topological condition coming from the fact that ω is a connection 1-form: the curvature 2-form B has to lie in the cohomology class $2\pi c_1(P)$, where P stands for the degree one S^1-bundle $B^4 \setminus 0 \to B^3 \setminus 0$. This gives the condition $\frac{1}{2\pi}[*df] = 1$. □

In the Euclidean case (the model case investigated in Sect. 2.1), we have

$$\begin{aligned} f &= \frac{1}{2r_3} \\ \omega &= \frac{1}{r_4^2}(-y_2 dy_1 + y_1 dy_2 - y_4 dy_3 + y_3 dy_4) \\ \pi^* g^3 + \xi^2 &= 4r_4^2(dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2). \end{aligned}$$

(r_n is the radius in Euclidean n-space; recall that $\pi^* r_3 = r_4^2$.)

Suppose we are given a 1-form ξ as in Proposition 3, as well as a 3-dimensional metric g^3. Then there is still not a unique metric g^4 on $B^4 \setminus 0$

for which the correspondence between the Bogomolny and ASD equation holds. Indeed, we have to choose the conformal factor by which $\pi^* g^3 + \xi^2$ is to be multiplied. Naturally we want to choose this factor in a way to obtain some regularity of the metric coefficients at the origin of B^4. The Euclidean example suggests that $\pi^* f$ is a reasonable choice for our conformal factor. We therefore define our **4-dimensional metric** g^4 on $B^4 \setminus 0$ to be given by

$$g^4 := \pi^* f \cdot (\pi^* g^3 + \xi^2) = \pi^*(f g^3) + \pi^*(1/f)\omega^2.$$

Proposition 4 *Let g^3 be an arbitrary metric on B^3. Then there exists a smooth harmonic function f as in Proposition 3 of the form $f = \frac{1}{2r}$ + higher order terms (r is the geodesic distance from the origin). Moreover, one can choose a connection 1-form ω on $B^4 \setminus 0$ with curvature $- * df$ in such a way that the 1-form $\xi = \pi^*(1/f)\omega$ is in $L^2_{3,loc}(B^4)$ and the metric $g^4 = \pi^* f(\pi^* g^3 + \xi^2)$ is in $L^2_{5,loc}(B^4)$.*

We say that a function is in $L^2_{l,loc}(B^4)$ if all its derivatives up to order l have finite L^2-norm over any compact subset of B^4.

We skip the rather technical proof of this proposition, since it is focused on an analytic construction of a function f of the desired form, from which the regularity results about ξ and g^4 follow easily.

2.3 Non-existence of a global correspondence

We have established the existence of a local correspondence between the Bogomolny equation and the ASD equation. One might wonder why we are considering a correspondence on B^3 rather than one which reinterprets monopoles on a general compact 3-manifold X as circle-invariant ASD connections on a 4-manifold. The precise question we are going to answer is as follows.

Given a compact, oriented, Riemannian 3-manifold (X, g) with marked points $p_1, \dots, p_n$, can one construct an oriented, Riemannian 4-manifold (Y, g^4) with an S^1-action such that
(i) the S^1-action is free everywhere on Y except at n fixed points $q_1, \dots, q_n$,
(ii) there is a smooth map $\pi : Y \to X$ such that $\pi(q_i) = p_i$ and $Y \setminus \{q_i\} \xrightarrow{\pi} X \setminus \{p_i\}$ is a principal S^1-bundle,
(iii) there is a smooth S^1-invariant 1-form ξ on $Y \setminus \{q_i\}$ such that a pair (∇, Φ) on $X \setminus \{p_i\}$ is a monopole if and only if $\pi^*\nabla + \xi \otimes \pi^*\Phi$ is an ASD connection on $Y \setminus \{q_i\}$?

The answer is always negative (except in the trivial case $n = 0$). To understand this, we cut out small open balls around the p_i's and the q_i's, thus providing us with a principal S^1-bundle $\tilde{Y} \xrightarrow{\pi} \tilde{X}$ such that $\partial\tilde{X}$ is a union of n two-spheres and $\partial\tilde{Y}$ is a union of n three-spheres. Take a 2-form e representing the Euler class of this principal bundle. As e is closed, its integral over $\partial\tilde{X}$ vanishes. On the other hand, it is equal to the sum of the integrals of e over each of the n two-spheres. One shows that the principal S^1-bundle over any such two-sphere is isomorphic to the Hopf fibration $S^3 \to S^2$, which has degree 1. Hence, for each p_i, there is a contribution of ± 1 to the integral of e over $\partial\tilde{X}$. Call p_i a (+)-point if this contribution is +1, and a ($-$)-point otherwise. As the sum of the contributions is zero, there must be as many (+)-points as ($-$)-points and consequently n must be even.

It is possible to construct, for any 3-manifold X with an even number of marked points, a 4-manifold Y satisfying conditions (i) and (ii). The construction is obvious for $n = 0$. Furthermore, if the construction of Y has been achieved for X with n marked points, then it can also be achieved for X with $n + 2$ marked points. This is done by gluing a copy of S^4 into Y. Indeed there is an S^1-action on S^4 with 2 fixed points and quotient S^3, so this gluing adds two more marked points to X.

However, condition (iii) will imply a contradiction. By Proposition 3, one obtains a smooth, harmonic, nowhere-zero function f on $X \setminus \{p_i\}$. Moreover, the topological condition $\frac{1}{2\pi}[*df] = 1$ implies that f goes to $+\infty$ at ($-$)-points, whereas it goes to $-\infty$ at (+)-points. This contradicts the non-vanishing of f.

3 The moduli space of good monopoles

3.1 Good monopoles

Let p be a point on a compact Riemannian 3-manifold X. We wish to study pairs (∇, Φ) consisting of a smooth connection ∇ in an $SU(2)$ vector bundle $\eta \to X \setminus p$ and a smooth section Φ of $\mathrm{ad}\,\eta$, and satisfying the Bogomolny equation $F_\nabla = *d_\nabla \Phi$. To impose some regularity at p, we shall use the **upstairs connection** ∇^4 over $B^4 \setminus 0$ constructed in the previous section. Using exponential coordinates around p, we obtain a map $\pi : B^4 \to U$ from the 4-ball to the 3-ball of radius 1 around p. We also know from Proposition 4 that we can find a 1-form ξ and a metric on B^4 such that the upstairs connection ∇^4, obtained from the restriction of (∇, Φ) to the 3-ball U, is ASD. Our regularity condition will require the upstairs connection to be regular at the origin of the four-ball. Thus, by Proposition 2 and its subsequent generalization in Sect. 2.2, a monopole (∇, Φ) on $X \setminus p$ satisfying our regularity condition can be interpreted (at least near the singularity p) as a regular S^1-invariant ASD connection on the four-ball B^4. To be more precise, the S^1-invariant connection

$$\nabla^4 = \pi^*(\nabla|_{U\setminus p}) + \xi \otimes \pi^*(\Phi|_{U\setminus p})$$

is defined on the S^1-equivariant bundle $\pi^*(\eta|_{U\setminus p}) \to B^4 \setminus 0$. There are smooth trivializations of $\pi^*(\eta|_{U\setminus p})$ in which the S^1-action on the bundle extends to a smooth action on B^4. We call such a trivialization a k**-gauge** of $\pi^*(\eta|_{U\setminus p})$, where $k \in \{0, 1, 2, \ldots\}$ denotes the weight of the representation $S^1 \to SU(2)$ given by the action of S^1 on the fibre at the origin of B^4. Note that any k-gauge can be transformed, via some smooth gauge transformation over B^4, to a k-gauge in which the S^1 action is given by

$$\theta \cdot (y, a_1, a_2) = (\theta \cdot y, e^{ki\theta} a_1, e^{-ki\theta} a_2) \quad (\theta \in S^1, y \in B^4, (a_1, a_2) \in \mathbb{C}^2).$$

Definition 1 *A* **good smooth pair** (∇, Φ) **of charge** k *is a pair consisting of a smooth connection* ∇ *in* $\eta \to X \setminus p$ *and a smooth section* Φ *of* $\mathrm{ad}\,\eta$ *such that there exists a* k*-gauge* τ *of* $\pi^*(\eta|_{U\setminus p})$ *in which the connection*

$$\pi^*(\nabla|_{U\setminus p}) + \xi \otimes \pi^*(\Phi|_{U\setminus p})$$

is represented by a smooth connection matrix over B^4*. We say that* (∇, Φ) *is a good smooth pair of charge* k *with respect to* τ*.*

Remark. In this definition, as well as in our whole study of the moduli space, we assume without loss of generality that we are dealing only with one singularity p.

Requiring smoothness of the upstairs connection on B^4 is too strong, since the 4-dimensional metric g^4 is not necessarily smooth at the origin. For this reason, we now introduce a larger class of singular monopoles.

Definition 2 *A* **good monopole of charge** k *is a pair* (∇, Φ)*, where* ∇ *is a smooth connection in* $\eta \to X \setminus p$*, and* Φ *a smooth section of* $\mathrm{ad}\,\eta$*, with the following properties:*
(i) $F_\nabla = *d_\nabla \Phi$ *(the Bogomolny equation),*
(ii) The upstairs connection ∇^4 *has finite energy over any compact subset of* B^4*,*
(iii) $\frac{|\Phi|}{f} \to k$ *at* p *(*f *is the harmonic function on* $U \setminus p$ *introduced in Proposition 3).*

A priori, this definition does not seem to have much to do with good smooth monopoles. But as we shall see, any good smooth monopole of charge k is a good monopole of charge k (Proposition 6). Furthermore, good monopoles of charge k have a nice interpretation in terms of k-gauges (Proposition 5).

Definition 3 *The* **moduli space** $\mathscr{M}_k^\infty$ **of good monopoles of charge** k *is the set of all good monopoles of charge* k*, divided by the group of all smooth gauge transformations of the bundle* η*.*

The 'infinity' superscript signifies the smooth nature of the objects we are considering. Note that the moduli space $\mathscr{M}_k^\infty$ is defined without the use of k-gauges.

3.2 Good L_3^2 pairs and good L_3^2 monopoles

Proposition 5 *Let* (∇, Φ) *be a good monopole of charge* k*. Then there exists a* k*-gauge of* $\pi^*(\eta|_{U\setminus p})$ *in which* ∇^4 *is represented by an* $L^2_{3,loc}(B^4)$ *connection matrix.*

Proof. ∇^4 is a finite-energy, anti-self-dual connection on $B^4 \setminus 0$ (with respect to the metric g^4). The coefficients of the metric are in $L^2_{5,loc}(B^4)$ (cf. Proposition 4). In that case Uhlenbeck's Removable Singularity Theorem [10] guarantees the existence of a gauge whose corresponding connection matrix A lies in $L^2_{6,loc}(B^4)$.

One then constructs an $L^2_{4,loc}(B^4)$ gauge transformation to change this gauge to a k'-gauge, namely a gauge where the S^1 bundle action is given by $\theta \cdot (y, a_1, a_2) = (\theta y, e^{k'i\theta}a_1, e^{-k'i\theta}a_2)$. We will be done if we can show $k' = k$, which clearly follows from the following result. $\square$

Proposition 6 *Suppose that the upstairs connection ∇^4 corresponding to a pair (∇, Φ) is represented by an $L^2_{3,loc}(B^4)$ connection matrix in a k'-gauge τ. Then $\frac{|\Phi|}{f} \to k'$ at p, or equivalently $2r|\Phi| \to k'$ at p (r is the geodesic distance from p).*

Proof. In the k'-gauge τ, the S^1 bundle action can be seen as the map

$$u : B^4 \times S^1 \to SU(2) : (y, \theta) \to \begin{pmatrix} e^{k'i\theta} & 0 \\ 0 & e^{-k'i\theta} \end{pmatrix}.$$

In the same gauge τ,

$$(\pi^*\nabla)_{\frac{\partial}{\partial\theta}} = \frac{\partial}{\partial\theta} - \frac{\partial w}{\partial\theta}w^{-1},$$

where $w : B^4 \setminus 0 \to SU(2)$ is the gauge transformation from any S^1-invariant gauge of $\pi^*\eta$ to τ. Hence

$$\frac{\partial w}{\partial\theta}w^{-1} = \begin{pmatrix} k'i & 0 \\ 0 & -k'i \end{pmatrix}.$$

We know that $\nabla^4 = d + B$, where B is an $L^2_{3,loc}(B^4)$ connection matrix. As $\nabla^4 = \pi^*\nabla + \xi \otimes \pi^*\Phi$, we get

$$\pi^*(\Phi/f) = \begin{pmatrix} k'i & 0 \\ 0 & -k'i \end{pmatrix} + B\left(\frac{\partial}{\partial\theta}\right).$$

The matrix-valued 1-form B is in $L^2_{3,loc}(B^4)$, so by the Sobolev Imbedding Theorem in dimension 4, B is bounded in a neighbourhood of the origin. Thus $B\left(\frac{\partial}{\partial\theta}\right) \to 0$ at the origin of B^4. The result follows. $\square$

Proposition 5 motivates our next definition.

Definition 4 *Let τ be a k-gauge. Then a* **good L^2_3 pair of charge k with respect to** *τ is a pair consisting of an $L^2_{3,loc}$ connection ∇ in $\eta \to X \setminus p$ and an $L^2_{3,loc}$ section Φ of* ad η *such that the connection*

$$\nabla^4 = \pi^*(\nabla|_{U\setminus p}) + \xi \otimes \pi^*(\Phi|_{U\setminus p})$$

is represented (in the k-gauge τ) by an $L^2_{3,loc}(B^4)$ connection matrix. In other words, we use the k-gauge τ and the standard trivialization $(dy_1, \ldots, dy_4)$ of T^*B^4 to represent ∇^4 by a collection of functions. We say that (∇, Φ) is a good L^2_3 pair of charge k if each of these functions has finite L^2_3-norm over any compact subset of B^4.

We suppose from now on that we have fixed a k-gauge τ. All good L^2_3 pairs are supposed to be good with respect to this gauge.

Definition 5 *The* **configuration space** $\mathscr{C}_k$ *is the set of all good* L^2_3 *pairs of charge* k*. The* **gauge group** $\mathscr{G}_k$ *is the set of all* $L^2_{4,loc}$ *gauge transformations* u *of* η *such that* $\pi^*(u|_{U\setminus p})$ *is represented (in the* k*-gauge* τ*) by an* $L^2_{4,loc}$ *matrix over* B^4*. The* **moduli space** $\mathscr{M}_k$ *of good* L^2_3 *monopoles is the quotient* $\{\text{monopoles in } \mathscr{C}_k\}/\mathscr{G}_k$.

One can show that there is a natural bijection between the moduli spaces $\mathscr{M}_k$ and $\mathscr{M}_k^\infty$. We can also restate our definition of $\mathscr{M}_k^\infty$ in a form which is independent of any choices of f and ξ. This shows that our moduli space is a natural object.

Proposition 7 *Let* ∇ *be a smooth connection in* $\eta \to X \setminus p$*, and* Φ *a smooth section of* $\operatorname{ad}\eta$*. Suppose that the pair* (∇, Φ) *satisfies the monopole equation. Then* (∇, Φ) *is a good monopole of charge* k *if and only if*
(i) $2r|\Phi| \to k$ *at* p*,*
(ii) $d(r^2|\Phi|^2)$ *is bounded near* p*.*
(here r *is the geodesic distance from* p*.)*

Proof. The result follows from a straightforward re-writing of the finite-energy condition for ∇^4 in terms of ∇ and Φ. □

3.3 The local structure of the moduli space

We now begin the analysis of the moduli space $\mathscr{M}_k$. Our approach is very close to the standard theory on instanton moduli spaces. These moduli spaces have been studied intensively in [3], [4] and [6].

Notation. Henceforth we shall frequently drop the reference to the charge k in notations like $\mathscr{C}_k$ or $\mathscr{G}_k$.

The configuration space $\mathscr{C}$ is an affine space modelled on a vector space denoted by $T^{\mathscr{C}}$. The space $T^{\mathscr{C}}$ is given a Banach space structure via the following norm:

$$||(a,\varphi)||^2_{L^2_3} := ||a||^2_{L^2_3(X\setminus\frac{1}{3}U)} + ||\varphi||^2_{L^2_3(X\setminus\frac{1}{3}U)} + ||\pi^*a + \xi\otimes\pi^*\varphi||^2_{L^2_3(B^4_{2/3})}.$$

Let us fix a pair $(\nabla, \Phi) \in \mathscr{C}$. Then the tangent space to its orbit (under the gauge group action) at the point (∇, Φ) is given by the image of a linear map $D_{(\nabla,\Phi)}$ from the Lie algebra of $\mathscr{G}$ (denoted by $T^{\mathscr{G}}$) into $T^{\mathscr{C}}$. (Like $T^{\mathscr{C}}$, the vector space $T^{\mathscr{G}}$ can also be endowed with a Banach norm). For $v \in T^{\mathscr{G}}$, one has

$$D_{(\nabla,\Phi)}v = (-d_\nabla v, [v, \Phi]).$$

We introduce the following L^2-inner products on $T^{\mathscr{G}}$ and $T^{\mathscr{C}}$:

$$\begin{aligned}
\langle v_1, v_2\rangle_{T^{\mathscr{G}}} &= \int_X \gamma\langle v_1, v_2\rangle \,\mathrm{vol} \\
\langle (a_1,\varphi_1),(a_2,\varphi_2)\rangle_{T^{\mathscr{C}}} &= \int_X (\langle a_1, a_2\rangle + \langle \varphi_1, \varphi_2\rangle)\,\mathrm{vol},
\end{aligned}$$

where γ is a smooth positive function on $X \setminus p$ satisfying $\gamma = \begin{cases} 1 \text{ on } X \setminus \frac{2}{3}U \\ f \text{ on } \frac{1}{3}U \end{cases}$.
The formal adjoint $D^*_{(\nabla,\Phi)}$ of $D_{(\nabla,\Phi)}$ with respect to these inner products maps $T^{\mathscr{C}}$ to $T_2^{\mathscr{G}} := \{v \in L^2_{2,loc}(\mathrm{ad}\,\eta) | \pi^* v \in L^2_{2,loc}(B^4)\}$ (the target space would be more complicated without the function γ) and is given by the formula

$$D^*_{(\nabla,\Phi)}(a,\varphi) = \gamma^{-1}(-d^*_\nabla a + [\Phi,\varphi]).$$

The main difficulty in the study of our moduli space comes obviously from the singularity of the objects we consider. In our context, we cannot talk about elliptic operators in the usual sense, since these are defined on Sobolev spaces over compact manifolds. The Banach spaces we introduced in the previous subsection are clearly not of this type. However some of our differential operators behave exactly like elliptic operators.

Proposition 8 *The operator $L = D^*_{(\nabla,\Phi)} D_{(\nabla,\Phi)} : T^{\mathscr{G}} \to T_2^{\mathscr{G}}$ admits a left and a right parametrix. As a consequence,* Ker L *is finite-dimensional and* Im L *is closed and has finite codimension.*

Proof. We can modify L over $\frac{1}{3}U$ to obtain an elliptic differential operator L_X over X. This operator L_X admits a left and a right parametrix. Moreover, the restriction of L to U can be viewed (by definition of our singularity) as the regular elliptic differential operator $d^*_{\nabla^4} d_{\nabla^4}$ over the four-ball B^4 (again due to the presence of the function γ). This 4-dimensional operator can be extended to an elliptic operator L_{S^4} over the four-sphere. The operator L_{S^4} also admits a left and a right parametrix.

From the left (right) parametrices for L_X and L_{S^4}, one constructs (using suitable multiplications by cutoff functions) a left (right) parametrix for L. □

Proposition 8 allows us to carry out the local description of the moduli space in the standard way, i.e. by constructing a 'slice' through a monopole (∇, Φ) transverse to the gauge group orbit, and viewing the moduli space as the zero set (modulo the finite-dimensional stabilizer of (∇, Φ)) of a smooth Fredholm map defined on the slice. Standard theory leads to the following gauge-fixing result:

Proposition 9 *Let $(\nabla, \Phi) \in \mathscr{C}$. Then*
i) there exists $\epsilon_1 > 0$ such that for any $(a,\varphi) \in T^{\mathscr{C}}$ with $||(a,\varphi)||_{L^2_3} < \epsilon_1$, the pair $(\nabla,\Phi)+(a,\varphi)$ is gauge equivalent to a pair $(\nabla,\Phi)+(a',\varphi')$ with $D^*_{(\nabla,\Phi)}(a',\varphi') = 0$,
ii) there exists $\epsilon_2 > 0$ with the following property:
If $(a,\varphi), (a',\varphi') \in$ Ker *$D^*_{(\nabla,\Phi)}$ satisfy $||(a,\varphi)||_{L^2_3}, ||(a',\varphi')||_{L^2_3} < \epsilon_2$, and $(\nabla + a', \Phi + \varphi') = u(\nabla + a, \Phi + \varphi)$ for some $u \in \mathscr{G}$, then $u \in \mathscr{G}_{(\nabla,\Phi)}$, where $\mathscr{G}_{(\nabla,\Phi)}$ is the stabilizer of (∇,Φ) in $\mathscr{G}$.*

We now bring in the Bogomolny equation. We start with the following easy lemma.

Lemma 1 *If $(\nabla,\Phi) \in \mathscr{C}$, then the self-dual 2-form $\Big(\pi^*(F_\nabla - *d_\nabla \Phi)\Big)^+$ lies in $L^2_{2,loc}(B^4)$.*

Proof. Let $\nabla^4 = \pi^*\nabla + \xi \otimes \pi^*\Phi$. The proof of Proposition 2 gives

$$\Big(\pi^*(F_\nabla - *d_\nabla\Phi)\Big)^+ = F^+_{\nabla^4}. \qquad \square$$

Thus, if we define $\Omega^2 := \{\lambda \in L^2_{2,loc}(\Lambda^2 T^*(X\setminus p)\otimes \mathrm{ad}\,\eta) | (\pi^*\lambda)^+ \in L^2_{2,loc}(B^4)\}$, then the lemma implies that for each $(\nabla,\Phi) \in \mathscr{C}$, the bundle-valued 2-form $F_\nabla - *d_\nabla\Phi$ lies in Ω^2. As we did already with $T^{\mathscr{C}}$ and $T^{\mathscr{G}}$, we endow this new vector space with a Banach space structure.

Let $(\nabla,\Phi) \in \mathscr{C}$ be a monopole, and let $\mu_{(\nabla,\Phi)}$ be the map

$$\mu_{(\nabla,\Phi)} : \mathrm{Ker}\, D^*_{(\nabla,\Phi)} \to \Omega^2 : (a,\varphi) \mapsto F_{\nabla+a} - *d_{\nabla+a}(\Phi+\varphi).$$

The gauge-fixing result implies that there is a neighbourhood S of 0 in $\mathrm{Ker}\, D^*_{(\nabla,\Phi)}$ such that $(\mu^{-1}_{(\nabla,\Phi)}(0)\cap S)/\mathscr{G}_{(\nabla,\Phi)}$ is homeomorphic to a neighbourhood of $[(\nabla,\Phi)]$ in $\mathscr{M}$.

Using a variant of Proposition 8, one can show that $\mu_{(\nabla,\Phi)}$ is a smooth Fredholm map, i.e. its derivative at each point is a Fredholm operator. It follows that the **virtual dimension of** $\mathscr{M}$ equals the index of $d\mu$ minus the dimension of $\mathscr{G}_{(\nabla,\Phi)}$ (here $d\mu$ stands for the derivative of $\mu_{(\nabla,\Phi)}$ at the origin). Now it is easy to see that

$$\mathrm{Ind}\, d\mu - \dim \mathscr{G}_{(\nabla,\Phi)} = \mathrm{Ind}(D^*_{(\nabla,\Phi)} \oplus dB_{(\nabla,\Phi)}),$$

where $dB_{(\nabla,\Phi)} : T^{\mathscr{C}} \to \Omega^2 : (a,\varphi) \mapsto d_\nabla a - *d_\nabla\varphi - *[a,\Phi]$ is the linearization of the Bogomolny equation. The operator $D^*_{(\nabla,\Phi)} \oplus dB_{(\nabla,\Phi)}$ is easily seen to be Fredholm (again by adapting the proof of Proposition 8). Hence its index is well-defined. In Sect. 4, we shall prove the following index formula.

Proposition 10 *Let $(\nabla,\Phi) \in \mathscr{C}_k$ (recall that k is the charge at the singularity p). Then the Fredholm operator $D^*_{(\nabla,\Phi)} \oplus dB_{(\nabla,\Phi)}$ has index $4k$.*

As a corollary, we have

Theorem 1 *Let X be a compact, oriented and connected 3-manifold. Fix n points $p_1,\dots,p_n \in X$ and n non-negative integers $k_1,\dots,k_n$. Then the virtual dimension of the moduli space $\mathscr{M}_{(k_1,\dots,k_n)}$ of good monopoles having charge k_i at the singularity p_i equals $4\sum_{i=1}^n k_i$.*

4 Proof of the index formula

We now come to the explicit computation of the index of $\delta_{(\nabla,\Phi)} := D^*_{(\nabla,\Phi)} \oplus dB_{(\nabla,\Phi)}$. It is easy to see that
(a) for fixed k, $\mathrm{Ind}\,\delta_{(\nabla,\Phi)}$ does not depend on the choice of (∇,Φ) in $\mathscr{C}_k$,
(b) for the trivial pair $(\nabla,\Phi) = (d,0) \in \mathscr{C}_0$ (and hence for all $(\nabla,\Phi) \in \mathscr{C}_0$), $\mathrm{Ind}\,\delta_{(\nabla,\Phi)} = 0$. This is because $\delta_{(\nabla,\Phi)}$ is essentially $d \oplus d^* : \Omega^1 \oplus \Omega^3 \to \Omega^0 \oplus \Omega^2$, which has index zero.

What remains to see is how the index changes as the charge k increases. To deal with this question, we use the excision property for indices.

4.1 The excision property for indices

We formulate the excision property for indices (Atiyah-Singer, [1] p.522, [3] p.264) in some generality. Suppose that
(i) Z is a compact manifold decomposed as a union of two open sets $Z = U \cup V$,
(ii) $L : \Gamma(E) \to \Gamma(F)$ and $L' : \Gamma(E') \to \Gamma(F')$ are a pair of elliptic differential operators over Z,
(iii) there are bundle isomorphisms $\alpha : E|_V \to E'|_V$, $\beta : F|_V \to F'|_V$ such that $L = \beta^{-1}L'\alpha$ over V.
We obtain thus a set of data $\mathscr{D} := (Z, U, V, L, L', \alpha, \beta)$. Now suppose we have another set of data $\tilde{\mathscr{D}} := (\tilde{Z}, \tilde{U}, \tilde{V}, \tilde{L}, \tilde{L}', \tilde{\alpha}, \tilde{\beta})$ verifying the same conditions. Suppose moreover that there is a diffeomorphism $i : U \to \tilde{U}$ and four bundle maps $e : E|_U \to \tilde{E}|_{\tilde{U}}$, $e' : E'|_U \to \tilde{E}'|_{\tilde{U}}$, $f : F|_U \to \tilde{F}|_{\tilde{U}}$, $f' : F'|_U \to \tilde{F}'|_{\tilde{U}}$ covering i and such that

$$\begin{cases} L = f^{-1}\tilde{L}e \\ L' = f'^{-1}\tilde{L}'e' \end{cases} \qquad \text{over } U.$$

Then $\operatorname{Ind} L - \operatorname{Ind} L' = \operatorname{Ind} \tilde{L} - \operatorname{Ind} \tilde{L}'$.

The idea of the proof consists in constructing (from L and L') a pseudodifferential operator P over the manifold Z such that
(i) $\operatorname{Ind} P = \operatorname{Ind} L - \operatorname{Ind} L'$,
(ii) the elements of $\operatorname{Ker} P$ and $\operatorname{Ker} P^*$ are supported in U.
Doing the same construction for the set of data $\tilde{\mathscr{D}}$, we can arrange that $\operatorname{Ker} P \overset{i}{\cong} \operatorname{Ker} \tilde{P}$ and $\operatorname{Ker} P^* \overset{i}{\cong} \operatorname{Ker} \tilde{P}^*$ (since all their elements are supported in U (resp. $\tilde{U}$)). Hence $\operatorname{Ind} P = \operatorname{Ind} \tilde{P}$ and the result follows. To construct P, one considers first the differential operator $D = L \oplus (L')^*$, so that $\operatorname{Ind} D = \operatorname{Ind} L - \operatorname{Ind} L'$. One proceeds to define the pseudodifferential zeroth order operator

$$P_0 := (1 + DD^*)^{-1/2} D,$$

whose index equals $\operatorname{Ind} D$. Finally, one deforms the symbol of P_0 over V to obtain the identity, thus producing another pseudodifferential operator P with the same index as P_0. This P can be chosen in such a way that $\operatorname{Ker} P$ and $\operatorname{Ker} P^*$ are supported in U.

We want to use the excision property for $Z = X$, $\tilde{Z} = S^4$. Let us explain first how we proceed for $X = U \cup V$, where U is the standard 3-ball around p, and $V := X \setminus \frac{1}{2}\overline{U}$. Fix an integer $k \geq 0$ and pick $(\nabla, \Phi) \in \mathscr{C}_k$ and $(\nabla', \Phi') \in \mathscr{C}_{k+1}$ such that $(\nabla, \Phi)|_V = (\nabla', \Phi')|_V$ (this can always be done). It follows that the operators $\delta_{(\nabla,\Phi)}$ and $\delta_{(\nabla',\Phi')}$ agree over V. At this stage we are no longer in the theoretical setup described above. Indeed, the operators $\delta_{(\nabla,\Phi)}$ and $\delta_{(\nabla',\Phi')}$ are not elliptic operators over X, since they are not defined at p. However, they are elliptic over V, and this is actually all we need to construct a pseudodifferential operator P satisfying
(i) $\operatorname{Ind} P = \operatorname{Ind} \delta_{(\nabla,\Phi)} - \operatorname{Ind} \delta_{(\nabla',\Phi')}$,
(ii) $\operatorname{Ker} P$ and $\operatorname{Ker} P^*$ are supported in U.

Indeed, checking this construction step by step, one sees that the singularity at $p \in U$ does not affect the construction, since the deformations of the various pseudodifferential operators are taking place away from p.

Consider now the 4-sphere S^4. Choosing a 4-ball B^4 in S^4 gives us a projection $\pi : B^4 \to U$ down to the 3-dimensional U. We define $\tilde{U} := B^4$, $\tilde{V} := S^4 \setminus \pi^{-1}(\frac{1}{2}\overline{U})$, so that $S^4 = \tilde{U} \cup \tilde{V}$. The standard S^1-action on $B^4 \cong \tilde{U}$ extends to an S^1-action on S^4 with two antipodal fixed points (denoted by $0 \in \tilde{U}$ and $\infty \in \tilde{V}$). In the same way, the four-dimensional metric $g^4 = \pi^* f \cdot (\pi^* g + \xi^2)$ on $\tilde{U}$ extends to an S^1-invariant metric on S^4. By definition of $\mathscr{C}_k$, the S^1-invariant connection $\pi^*\nabla + \xi \otimes \pi^*\Phi$ in the pull-back bundle over $\tilde{U} \setminus 0$ can be extended across 0 in a gauge where S^1 acts with weights $(k, -k)$. One can also extend this connection to an S^1-invariant connection ∇^4 in an equivariant vector bundle $N \to S^4$. This bundle N can be chosen in such a way that that S^1 acts with weights $(0, 0)$ on the fibre over ∞. Similarly we define an S^1-invariant connection $\nabla^{4\prime}$ in an equivariant vector bundle N'. Observe that N and N' are not isomorphic as vector bundles over S^4. However, one can arrange that their restrictions to $\tilde{V}$ are, and as ∇^4 and $\nabla^{4\prime}$ agree over $\tilde{U} \cap \tilde{V}$, one can even arrange $\nabla^4|_{\tilde{V}} = \nabla^{4\prime}|_{\tilde{V}}$.

The operator $(\delta_{\nabla^4})^{S^1}$ is the restriction of the elliptic operator $\delta_{\nabla^4} := d^*_{\nabla^4} \oplus d^+_{\nabla^4}$ to S^1-invariant sections. Instead of considering the whole operator

$$d^*_{\nabla^4} \oplus d^+_{\nabla^4} : \Gamma(T^*S^4 \otimes \operatorname{ad} N) \to \Gamma((\underline{\mathbb{R}} \oplus \Lambda^2_+ T^*S^4) \otimes \operatorname{ad} N),$$

we restrict ourselves to the subspaces of S^1-*invariant* sections of these two vector bundles. So once again we violate the standard conditions required in the statement of the excision property. However, $(\delta_{\nabla^4})^{S^1}$ and $(\delta_{\nabla^{4\prime}})^{S^1}$ are restrictions of genuine elliptic operators over S^4, and $(\delta_{\nabla^4})^{S^1}|_{\tilde{V}} = (\delta_{\nabla^{4\prime}})^{S^1}|_{\tilde{V}}$. It is therefore still possible to construct a pseudodifferential operator $\tilde{P}$ such that

(i) $\operatorname{Ind} \tilde{P} = \operatorname{Ind}(\delta_{\nabla^4})^{S^1} - \operatorname{Ind}(\delta_{\nabla^{4\prime}})^{S^1}$,

(ii) $\operatorname{Ker} \tilde{P}$ and $\operatorname{Ker} \tilde{P}^*$ are supported in $\tilde{U}$.

Furthermore one can arrange that $\tilde{P}$ maps S^1-invariant sections to S^1-invariant sections. Using the correspondences $\delta_{(\nabla,\Phi)}|_U \leftrightarrow (\delta_{\nabla^4})^{S^1}|_{\tilde{U}}$ and $\delta_{(\nabla',\Phi')}|_U \leftrightarrow (\delta_{\nabla^{4\prime}})^{S^1}|_{\tilde{U}}$ given by the projection $\pi : \tilde{U} \to U$, we see that $\operatorname{Ker} P \cong \operatorname{Ker} \tilde{P}$ and $\operatorname{Ker} P^* \cong \operatorname{Ker} \tilde{P}^*$.

We have thus proved

Proposition 11 *Let* $(\nabla, \Phi) \in \mathscr{C}_k$, $(\nabla', \Phi') \in \mathscr{C}_{k+1}$ *be such that* $(\nabla, \Phi) = (\nabla', \Phi')$ *outside* $\frac{1}{2}U$. *Then*

$$\operatorname{Ind} \delta_{(\nabla,\Phi)} - \operatorname{Ind} \delta_{(\nabla',\Phi')} = \operatorname{Ind}(\delta_{\nabla^4})^{S^1} - \operatorname{Ind}(\delta_{\nabla^{4\prime}})^{S^1}.$$

Introducing the notation $\nabla^4_{(k,0)} := \nabla^4$ in order to keep track of the weights of the S^1-action over the fixed points 0 and ∞, we obtain the following corollary.

Corollary 1 *If* $(\nabla, \Phi) \in \mathscr{C}_k$, *then* $\operatorname{Ind} \delta_{(\nabla,\Phi)} = \operatorname{Ind}(\delta_{\nabla^4_{(k,0)}})^{S^1} - \operatorname{Ind}(\delta_{\nabla^4_{(0,0)}})^{S^1}$.

At this stage, it appears already that $\operatorname{Ind} \delta_{(\nabla,\Phi)}$ does not depend on the manifold X.

4.2 The index of $(d^*_{\nabla^4} \oplus d^+_{\nabla^4})^{S^1}$

This subsection deals with the explicit computation of $(d^*_{\nabla^4} \oplus d^+_{\nabla^4})^{S^1}$, which is carried out in greater generality than what is really needed (namely the index corresponding to a connection on S^4, endowed with an S^1-action fixing two points). Let Y be a compact, oriented, Riemannian 4-manifold with an S^1-action that is free everywhere except at n fixed points $q_1, \dots, q_n$ (recall from Sect. 2.3 that n must be even). Suppose the metric on Y is S^1-invariant. Consider an $SU(2)$ vector bundle $E \to Y$ with an S^1-equivariant action. Let ∇^4 be an S^1-invariant connection. Then we want to compute the index of the operator

$$(\delta_{\nabla^4})^{S^1} := (d^*_{\nabla^4})^{S^1} \oplus (d^+_{\nabla^4})^{S^1}.$$

This index calculation has already appeared in various papers (e.g. [5]). Hereafter, we describe a rather direct way to work out the index.

The operator $(\delta_{\nabla^4})^{S^1}$ is obtained by restricting the standard elliptic operator in instanton theory ([3], p.137), namely

$$\delta_{\nabla^4} := d^*_{\nabla^4} \oplus d^+_{\nabla^4} : \Omega^1(\operatorname{ad} E) \to \Omega^0(\operatorname{ad} E) \oplus \Omega^2_+(\operatorname{ad} E),$$

to the subspaces of S^1-invariant elements in Ω^0, Ω^1 and Ω^2_+. Let us note the following as well.

(a) Any $\theta \in S^1$ preserves the fibre $E_{q_j} \cong \mathbb{C}^2\,(j = 1, \dots, n)$. Hence there is a basis of E_{q_j} in which $\theta \in S^1$ acts by $\begin{pmatrix} e^{k_j i\theta} & 0 \\ 0 & e^{-k_j i\theta} \end{pmatrix}$ for some non-negative integer k_j.

(b) For each q_j, there is a coordinate chart (y_1, y_2, y_3, y_4) around q_j in which the S^1-action becomes

$$\theta \cdot (y_1, y_2, y_3, y_4) = (y_1, y_2, y_3, y_4) \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

We say that q_j is a (+)-point (resp. a (−)-point) if the orientation on Y is given by $dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4$ (resp. $-dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4$). There are $n/2$ (+)-points and $n/2$ (−)-points on Y.

The main ingredient of the index computation will be an application of the Atiyah-Singer Fixed Point Theorem for G-invariant elliptic operators. Let us rewrite our elliptic operator as $\delta_{\nabla^4} : \Gamma(F) \to \Gamma(F')$ with

$$\begin{aligned} F &= \big(T^*Y \otimes \operatorname{ad} E\big) \otimes \underline{\mathbb{C}} \\ F' &= \big((\underline{\mathbb{R}} \oplus \Lambda^2_+ T^*Y) \otimes \operatorname{ad} E\big) \otimes \underline{\mathbb{C}}. \end{aligned}$$

We complexified our vector bundles, since the Atiyah-Singer Fixed Point Theorem only holds for complex vector bundles. This operation clearly leaves the index of $(\delta_{\nabla^4})^{S^1}$ unchanged.

Theorem 2 *(Atiyah-Singer, [2] p.560, [9] p.123).*

$$\mathrm{Ind}(\delta_{\nabla^4})^{S^1} = \frac{1}{2\pi}\int_0^{2\pi} \ell(\theta)d\theta,$$

where $\ell : S^1 \to \mathbb{R}$ is the function defined by

$$\ell(\theta) = \sum_{i=1}^{n} \frac{\mathrm{tr}_\theta(F_{q_i}) - \mathrm{tr}_\theta(F'_{q_i})}{\mathrm{tr}_\theta(\Lambda^{-1}(TY_{q_i} \otimes \mathbb{C}))}.$$

Notation: F_{q_i} (resp. F'_{q_i}, TY_{q_i}) is the fibre of the vector bundle F (resp. F', TY) over the fixed point $q_i \in Y$. For a complex vector space V with a linear S^1-action, $\mathrm{tr}_\theta(V)$ denotes the trace of the isomorphism of V associated to θ. Moreover, if V is an m-dimensional complex vector space, then $\Lambda^{-1}(V)$ is the virtual vector space $\sum_{i=0}^{m}(-1)^i \Lambda^i V$.

We compute the different terms in the formula for $\ell(\theta)$:

$$\begin{aligned}
\mathrm{tr}_\theta(F_{q_i}) &= 4\cos\theta(1 + 2\cos 2k_i\theta).\\
\mathrm{tr}_\theta(F'_{q_i}) &= \begin{cases} 4(1 + 2\cos 2k_i\theta) & \text{if } q_i \text{ is a } (-)\text{-point,}\\ 2(1+\cos 2\theta)(1 + 2\cos 2k_i\theta) & \text{if } q_i \text{ is a } (+)\text{-point.}\end{cases}\\
\mathrm{tr}_\theta(\Lambda^{-1}(TY_{q_i} \otimes \mathbb{C})) &= 4(1-\cos\theta)^2.
\end{aligned}$$

It now follows from the Fixed Point Theorem that if the weights at the (+)-points are labelled by $k_1^+, \ldots, k_{n/2}^+$ (and similarly for the (−)-points), we have:

$$\ell(\theta) = \sum_{i=1}^{n/2}\left(\frac{2(\cos 2k_i^+\theta - \cos 2k_i^-\theta)}{1-\cos\theta} - (1 + 2\cos 2k_i^+\theta)\right).$$

From the equality

$$\mathrm{Ind}(\delta_{\nabla^4})^{S^1} = \frac{1}{2\pi}\int_0^{2\pi} \ell(\theta)d\theta,$$

it is clear that we are done if we can compute the numbers

$$R(m, m') := \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{2(\cos 2m\theta - \cos 2m'\theta)}{1-\cos\theta} - (1 + 2\cos 2m\theta)\right)d\theta.$$

Indeed, $\mathrm{Ind}(\delta_{\nabla^4})^{S^1} = \sum_{i=1}^{n/2} R(k_i^+, k_i^-)$. Observe that

$$R(m, m) = \begin{cases} -1 \text{ if } m \neq 0,\\ -3 \text{ if } m = 0.\end{cases}$$

Using elementary trigonometric identities, one gets

$$R(m, m'+1) - R(m, m') = \frac{2}{\pi}\int_0^{2\pi} \frac{\sin\left((2m'+1)\theta\right)\sin\theta}{1-\cos\theta}d\theta = 4.$$

We have thus proved

Proposition 12

$$\mathrm{Ind}(\delta_{\nabla^4})^{S^1} = 4\left(\sum_{i=1}^{n/2} k_i^- - k_i^+\right) - \frac{n}{2} - 2\kappa,$$

where κ is the number of (+)*-points q_i^+ such that $k_i^+ = 0$.*

Corollary 2 *(= Proposition 10). If $(\nabla, \Phi) \in \mathscr{C}_k$, then* $\mathrm{Ind}\,\delta_{(\nabla,\Phi)} = 4k$.

Proof. For the S^1-action on S^4 considered in 4.1, there are $n = 2$ fixed points. Moreover, the fixed point 0 is easily seen to be a $(-)$-point by considering the orientations chosen in Proposition 2. Hence $\mathrm{Ind}(\delta_{\nabla^4_{(k,0)}})^{S^1} = 4k - 3$ and $\mathrm{Ind}(\delta_{\nabla^4_{(0,0)}})^{S^1} = -3$. Applying corollary 1 yields the result. □

5 Singular monopoles on the three-sphere

5.1 Explicit solutions for $k_1 = 1$

Let us try and find explicit examples of good monopoles with one singularity on the 3-sphere of radius 1 with its standard round metric. Let $p \in S^3$ be the singularity and $\eta \to S^3 \setminus p$ an $SU(2)$ vector bundle (necessarily trivial). We want to construct our monopoles on $S^3 \setminus p$ by modelling them on the Bogomolny-Prasad-Sommerfield monopole on Euclidean 3-space [8]. Let q be the point opposite p in S^3. Then the stereographic projection from q onto the tangent space of S^3 at p gives us as identification $S^3 \setminus \{p, q\} \cong \mathbb{R}^3 \setminus 0$. Pick any gauge σ of η over $S^3 \setminus p$. We shall be looking for $SO(3)$-invariant monopoles (∇, Φ) of the form

$$\begin{aligned}
\nabla^\sigma &= t(r)\cdot\Big((x_2dx_3 - x_3dx_2)\underline{\sigma}_1 + (x_3dx_1 - x_1dx_3)\underline{\sigma}_2 + (x_1dx_2 - x_2dx_1)\underline{\sigma}_3\Big),\\
\Phi^\sigma &= s(r)\cdot(x_1\underline{\sigma}_1 + x_2\underline{\sigma}_2 + x_3\underline{\sigma}_3),
\end{aligned}$$

where $(\underline{\sigma}_1, \underline{\sigma}_2, \underline{\sigma}_3)$ denotes the standard basis of $su(2)$, (x_1, x_2, x_3) are standard coordinates on $\mathbb{R}^3$, r is the radius function on $\mathbb{R}^3$, and s, t are smooth functions $(0, +\infty) \to \mathbb{R}$. We compute

$$(F_\nabla)^\sigma = \left(-\frac{t'}{r} + 2t^2\right)\beta\cdot\sum_{i=1}^{3} x_i\underline{\sigma}_i + (rt' + 2t)\sum_{i=1}^{3}(dx_{i+1}\wedge dx_{i+2})\underline{\sigma}_i,$$

where $\beta = x_1\,dx_2\wedge dx_3 + x_2\,dx_3\wedge dx_1 + x_3\,dx_1\wedge dx_2$. On the other hand

$$(d_\nabla\Phi)^\sigma = \left(\frac{s'}{r} - 2st\right)\alpha\cdot\sum_{i=1}^{3} x_i\underline{\sigma}_i + (s + 2r^2st)\sum_{i=1}^{3} dx_i\,\underline{\sigma}_i,$$

where $\alpha = x_1dx_1 + x_2dx_2 + x_3dx_3$. Hence a pair (∇, Φ) of the above form is a monopole on $S^3 \setminus \{p, q\}$ if and only if

$$-\frac{t'}{r}+2t^2 = -\left(1+r^2/4\right)^{-1}\left(\frac{s'}{r}-2st\right) \tag{1}$$

$$rt'+2t = -\left(1+r^2/4\right)^{-1}(s+2r^2st) \tag{2}$$

Putting $S = sr$, $T = 1+2r^2t$, (2) becomes

$$T'(1+r^2/4) = -2ST, \tag{3}$$

and (1) + $\frac{1}{r^2}$(2) becomes

$$\frac{T^2-1}{2r^2}(1+r^2/4) = -S'. \tag{4}$$

Letting $r = 2\tan u$ ($u \in (0, \frac{\pi}{2})$), the system formed by (3) and (4) can be written as

$$-4ST = \dot{T} \tag{5}$$

$$-4\dot{S}\cos^2 u \sin^2 u = T^2-1, \tag{6}$$

where $\dot{}$ denotes differentiation with respect to the variable u.

As the gauge σ of η is non-singular at q, ∇^σ must be bounded at q. Observe that $|\nabla^\sigma|^2 = C_0 t^2 r^2 (1+r^2/4)^2$ for some positive constant C_0. Therefore $t^2r^6 = r^2(tr^2)^2$ must be bounded at $r = +\infty$ and consequently $tr^2 \to 0$ as $r \to +\infty$. It follows that we only have to consider solutions with $T(u) \to 1$ as $u \to \frac{\pi}{2}$.

On $\mathscr{D} = \{u \in (0, \frac{\pi}{2}) | T(u) \neq 0\}$, we can define

$$M(u) := \frac{\cos u \sin u}{T(u)}.$$

Note that in case T is smooth and $T(u) \to 1$ as $u \to \frac{\pi}{2}$, one has $M(u) \neq 0$ and $M(u) \to 0^+$ as $u \to \frac{\pi}{2}$. Now the ODEs (5) and (6) imply

$$\dot{M}^2 - M\ddot{M} = 1, \qquad \text{and hence}$$

$$\frac{d}{du}\left(\frac{\dot{M}^2-1}{M^2}\right) = \frac{2M\dot{M}(M\ddot{M}-\dot{M}^2+1)}{M^4} = 0.$$

The solutions of the first-order ODE $\dot{M}^2 = 1 + AM^2$ (where A is a constant) are as follows:

$$\text{If } A > 0, \quad M(u) = \pm\frac{1}{\sqrt{A}}\sinh(\sqrt{A}u+B).$$

$$\text{If } A = 0, \quad M(u) = \pm(u+B).$$

$$\text{If } A < 0, \quad M(u) = \pm\frac{1}{\sqrt{-A}}\sin(\sqrt{-A}u+B).$$

In all three cases, B is a constant. Observe that each of these functions is smooth on $\mathbb{R}$, and hence $\mathscr{D} = (0, \frac{\pi}{2})$. Among the above solutions, only the following satisfy the conditions $M(u) \neq 0$ and $M(u) \to 0^+$ as $u \to \frac{\pi}{2}$:

$$\begin{array}{rcll} M(u) & = & \frac{1}{C}\sinh\Big(C(\pi/2-u)\Big), & C \in (0,+\infty) \\ M(u) & = & \pi/2-u & \\ M(u) & = & \frac{1}{C}\sin\Big(C(\pi/2-u)\Big), & C \in (0,2] \end{array}$$

From (5) one deduces that

$$S(u) = -\frac{1}{2}\cot(2u) + \frac{1}{4}\frac{\dot{M}}{M}.$$

Hence, for the explicit expressions of L above, one obtains

$$\begin{array}{rcll} S(u) & = & -\frac{1}{2}\cot(2u) - \frac{C}{4}\coth\Big(C(\pi/2-u)\Big) < 0, & C \in (0,+\infty) \\ S(u) & = & -\frac{1}{2}\cot(2u) - \frac{1}{4(\pi/2-u)} < 0 & \\ S(u) & = & -\frac{1}{2}\cot(2u) - \frac{C}{4}\cot\Big(C(\pi/2-u)\Big) \leq 0, & C \in (0,2] \end{array}$$

Note that in the third case, the solution corresponding to $C = 2$ yields $S = 0$ and $T = 1$, i.e. $s = t = 0$. This is a trivial monopole, i.e. ∇ is a product connection and $\Phi = 0$. It is a good monopole of charge 0.

Each one of the above solutions has $S(u) \to 0$ as $u \to \frac{\pi}{2}$. It is straightforward to check that the corresponding monopoles can be extended smoothly across q to give a smooth pair (∇, Φ) on $S^3 \setminus p$.

Now $|\Phi| = -S$, and as $\dot{S} > 0$ for any non-trivial monopole, one sees that the norm of the Higgs field Φ decreases as one approaches the point opposite the singularity, where it eventually vanishes. As for a neighbourhood of the singularity, one checks that the non-trivial solutions have

$$2r_3|\Phi| = -4uS(u) = \frac{2u}{\tan 2u} + R(u),$$

where $R(u) \to 0$ as $u \to 0$ and dR is bounded near $u = 0$. Proposition 7 therefore allows us to conclude that all our non-trivial solutions are good monopoles of charge 1.

5.2 Regularity of the moduli spaces

We wish to show that the actual dimension of the moduli space $\mathscr{M}$ of good monopoles of charge $(k_1, \ldots, k_n)$ on S^3 equals its virtual dimension $4\sum k_i$ (provided $\mathscr{M} \neq \emptyset$). To do this, it is sufficient to show that the linearization of the Bogomolny equation, i.e.

$$dB_{(\nabla,\Phi)} : T^{\mathscr{C}} \to \Omega^2 : (a,\varphi) \mapsto d_\nabla a - *d_\nabla\varphi - *[a,\Phi]$$

is surjective. We define an L^2-inner product on Ω^2 as follows:

$$\langle \lambda, \lambda' \rangle_{\Omega^2} := \int_X \gamma^{-1} \langle \lambda, \lambda' \rangle \,\mathrm{vol}\,.$$

The function γ was introduced in Sect. 3.3. Multiplication by γ^{-1} ensures convergence of the integral defining the inner product on Ω^2. Recall that $T^{\mathscr{C}}$ carries a natural inner product (also defined in 3.3). One computes that the formal adjoint $dB^*_{(\nabla,\Phi)}$ of $dB_{(\nabla,\Phi)}$ with respect to the inner products on $T^{\mathscr{C}}$ and Ω^2 is given by the formula

$$dB^*_{(\nabla,\Phi)}\lambda = (d^*_\nabla(\gamma^{-1}\lambda) - [\Phi, *\gamma^{-1}\lambda], -d^*_\nabla(*\gamma^{-1}\lambda)).$$

To prove that $dB_{(\nabla,\Phi)}$ is surjective, it is enough to prove that $\langle dB_{(\nabla,\Phi)} dB^*_{(\nabla,\Phi)}\lambda, \lambda\rangle_{\Omega^2} > 0$ for all non-zero $\lambda \in \Omega^2$. It turns out that

$$dB_{(\nabla,\Phi)} dB^*_{(\nabla,\Phi)}\lambda = (d_\nabla d^*_\nabla + d^*_\nabla d_\nabla)(\gamma^{-1}\lambda) - [d_\nabla \Phi, *\gamma^{-1}\lambda] + [[\Phi, \gamma^{-1}\lambda], \Phi].$$

Let $\psi := \gamma^{-1}\lambda$. There is a Weitzenböck formula for the Hodge Laplacian $d_\nabla d^*_\nabla + d^*_\nabla d_\nabla$ acting on ψ ([6], p.96), namely

$$(d_\nabla d^*_\nabla + d^*_\nabla d_\nabla)\psi = \nabla^*\nabla\psi + \psi \circ \mathrm{Ric} + \mathscr{F}^g(\psi) + \mathscr{F}^\nabla(\psi),$$

where $\nabla^*\nabla$ is the trace Laplacian, and the remaining three terms are defined as follows: for any $V, W \in T_xX$, and (e_1, e_2, e_3) any orthonormal frame of T_xX,

$$(\psi \circ \mathrm{Ric})(V, W) := \psi(\mathrm{Ric}(V), W) + \psi(V, \mathrm{Ric}(W)),$$

(Ric : $T_xX \to T_xX$ is the Ricci tensor)

$$(\mathscr{F}^g(\psi))(V, W) := \sum_{j=1}^{3} \psi(e_j, R(V, W)e_j),$$

(R is the curvature tensor of the metric g)

$$(\mathscr{F}^\nabla(\psi))(V, W) := \sum_{j=1}^{3} \{[F_\nabla(e_j, V), \psi(e_j, W)] - [F_\nabla(e_j, W), \psi(e_j, V)]\}.$$

One checks that $\mathscr{F}^\nabla(\psi) = [*F_\nabla, *\psi]$, so that for monopoles (∇, Φ), we obtain

$$dB_{(\nabla,\Phi)} dB^*_{(\nabla,\Phi)}\lambda = \nabla^*\nabla\psi + \psi \circ \mathrm{Ric} + \mathscr{F}^g(\psi) + [[\Phi, \psi], \Phi].$$

For the metric of (positive) constant curvature on S^3, the 2-forms $\psi \circ \mathrm{Ric}$ and $\mathscr{F}^g(\psi)$ are both positive scalar multiples of ψ, and hence

$$dB_{(\nabla,\Phi)} dB^*_{(\nabla,\Phi)}\lambda = \nabla^*\nabla\psi + K \cdot \psi + [[\Phi, \psi], \Phi]$$

for some positive constant K. It follows that

$$\begin{aligned}
\langle dB_{(\nabla,\Phi)} dB^*_{(\nabla,\Phi)}\lambda, \lambda\rangle_{\Omega^2} &= \int_X (\langle \nabla^*\nabla\psi, \psi\rangle + \langle K\psi, \psi\rangle + \langle [[\Phi, \psi], \Phi], \psi\rangle)\,\mathrm{vol} \\
&= \int_X (|\nabla\psi|^2 + K|\psi|^2 + |[\Phi, \psi]|^2)\,\mathrm{vol} \\
&> 0 \qquad \text{for any non-zero } \lambda \in \Omega^2.
\end{aligned}$$

As an obvious corollary, we obtain that the moduli space of good monopoles of charge $k_1 = 1$ is given locally by a 4-parameter family of monopole solutions.

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