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Singular solutions of semilinear elliptic and parabolic equations

Qi S. Zhang, Z. Zhao

Department of Mathematics, University of Missouri, Columbia MO 65211, USA

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1. Introduction

We shall study the existence of singular positive solutions for the following semilinear elliptic and parabolic problems.

$$
(1.1)\quad \begin{cases} \Delta u(x) + V(x)\ u^p(x) = 0, & x \in D - \{0\}, \\ u(x) > 0, & x \in D - \{0\}, \\ u(x) \sim \frac{c}{|x|^{n-2}}, & \text{near} \quad x = 0, \quad \text{for any sufficiently small} \quad c > 0, \\ u(x) = 0, & x \in \partial D. \end{cases}
$$

Here and throughout the paper $D \subset \mathbf{R}^\mathbf{n}$, $n \geq 3$, is a bounded Lipschitz domain containing 0, Δ is the Laplacian and $p > 1$.

$$
(1.2)\quad \begin{cases} \Delta u(x) + V(x)\ u^p(x) = 0, & x \in \mathbf{R}^n - \{0\}, \\ u(x) > 0, & x \in \mathbf{R}^n - \{0\}, \\ u(x) \sim \frac{c}{|x|^{n-2}}, & \text{near} \quad x = 0,\ \infty, \quad \text{for any sufficiently small} \quad c > 0. \end{cases}
$$

In (1.1) and (1.2), the notion $u(x) \sim \frac{c}{|x|^{n-2}}$ near 0 or ∞ means that for some $C_1, C_2 > 0$, $\frac{cC_1}{|x|^{n-2}} \leq u(x) \leq \frac{cC_2}{|x|^{n-2}}$ when x is near 0 or ∞.

$$
(1.3)\quad \begin{cases} \Delta u(x,t) + V(x)\ u^p(x,t) - u_t(x,t) = 0, & x \in D - \{0\}, 0 < t \leq T, \\ \lim_{x \to 0} u(x,t) = \infty, & 0 < t < T, \\ u(x,t) = 0, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = u_0(x). \end{cases}
$$

Solutions of these problems are understood as distributional solutions in $D - \{0\}$ for (1.1) and (1.2), in $D - \{0\} \times (0,T)$ for (1.3). Under our conditions to be specified later, these solutions are continuous except at $x = 0$.

Equations in (1.1) and (1.2) contain several well-known types which have been studied extensively. For instance, when $V = 1$, the equation in (1.1) is the Lane-Emden equation. When $V = \frac{1}{1+|x|^2}$, the equation in (1.2) becomes the Matukuma equation. Since the 1960's, many interesting and important results concerning the existence and non-existence of positive singular solutions of (1.1) and (1.2) have appeared, which include, among others, the papers [NSa], [BM], [Se1, 2], [GV], [GS], [CGS], [N], [SEF], [A], [BO] and [L]. More references can be found in a recent paper [LS] by Li and Santanilla. However, to our knowledge, these existing results on problem (1.1) share a striking common condition that the domain D is a ball. It is a natural question to ask: "What happens if D is no longer a ball?"

The first purpose of this paper to address this question. Needless to say that many well-known symmetry results can no longer be expected when D is a general bounded domain. Nevertheless we shall establish, for the case of bounded Lipschitz domains and under a natural condition on V, an existence theorem on (1.1), which matches the existing ones when D is a ball.

The second purpose is to discuss the parabolic problem (1.3), which in addition to being the parabolic counter part of (1.1), is also a model of nonlinear reaction diffusion equation with a point source at the origin. It is interesting to compare problem (1.3) with the L^p problem studied first by Weissler [W] and later by others. In the paper [W], Weissler considered the problem in L^p spaces.

$$(1.3') \qquad \begin{cases} \Delta u(x,t) + u^p(x,t) - u_t(x,t) = 0, & (x,t) \in D \times (0,T], \\ u(x,t) = 0, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = u_0(x), & 0 < T < \infty. \end{cases}$$

It was shown that if $u_0 \in L^q(D)$, $q > n(p-1)/2$ and $q \geq p$, the the above problem has a unique solution in $C([0,T]; L^q(D))$, which is continuous in $D \times (0,T]$, for some $T > 0$. As to be specified in Theorem C and Remark 1.4 below, problem (1.3) contains a special case that apparently falls into the category studied in [W] except that D in (1.3') is replaced by $D - \{0\}$. However this time we obtain a solution that is singular as soon as $t > 0$. Another point worth mentioning is that this paper seems to be the first in studying nonlinear heat equations on non-smooth domains by using some of the most up to date linear results such as [ACS] and [FS].

Let us introduce the conditions on the potential function V. It turns out that these conditions are related to the next two functional classes which are widely used in the study of Schrödinger equations. More properties pertaining to these classes can be found in Sect. 2 and the references [AS], [Si] and [Zhao1].

Definition 1.1. *(see [AS]) A Borel measurable function U belongs to the Kato class K_n if* $\lim_{r\to 0}[\sup_x \int_{|x-y|\leq r} \frac{|U(y)|}{|x-y|^{n-2}} dy] = 0$,

Definition 1.2. *([Zhao1]) A Borel measurable function U is called a Green tight function in $\mathbf{R^n}$ if $U \in K_n$ and* $\lim_{M\to\infty}[\sup_x \int_{|y|\geq M} \frac{|U(y)|}{|x-y|^{n-2}} dy] = 0.$

The basic assumptions on V are the following. For problems (1.1) and (1.3) we require the function $h \equiv V(x)/|x|^{(n-2)(p-1)}$ is in the Kato class K_n and for problem (1.2) we need h to be a Green tight function in $\mathbf{R^n}$. It is important to remember that the Kato class properly contains L^q_{loc} class with $q > n/2$ (see [AS]).

As will be seen in the remarks after the theorems and in Sect. 2, the reasons to impose these conditions are threefold.

One: They are more general than those in the current literature.

Two: They have the minimum requirements on the smoothness of V and make no assumption on the sign of V.

Three: In case that V is radial and D is a ball, our conditions reduce to those in [LS], which are essentially the optimal ones dealing with (1.1) and (1.2) when D is a ball.

Now we are ready to present the main elliptic results of the paper.

Theorem A. *Suppose the function $V(.)/|.|^{(n-2)(p-1)}$ is in the class K_n, then problem (1.1) has infinitely many solutions. More specifically, there exists a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$ and $\rho \in (0, 1)$, there exists a solution of (1.1) such that*

$$\lambda(1-\rho)G(x,0) \leq u(x) \leq \lambda(1+\rho)G(x,0),$$

where $G(x, y)$ is the Green's function of the Laplacian in D.

By Proposition 2.1 in Sect. 2, Theorem A implies the next result which is quite well-known (see, e.g. [N] and [GS]).

Corollary A. *When D is the unit ball, $V = |x|^{-l}$, $l < 2$ and $1 < p < (n-l)/(n-2)$, (1.1) has positive solutions u such that, for some positive constants C_1 and C_2, $\frac{C_1}{|x|^{n-2}} \leq u(x) \leq \frac{C_2}{|x|^{n-2}}$ near $x = 0$.*

Remark 1.1. Solutions given in Theorem A are solutions of (1.1). This is clear from the well known property of the Green's function G (see [K] Theorem 1.2.8). When $x \in D$ approaches ∂D in a nontangential manner, $G(x, 0) \to 0$ and $G(x, 0) \sim \frac{1}{|x|^{n-2}}$ when x is near 0. We also remark that solutions for (1.1) are not unique in general. The following is an example. By Theorem A, there are $\lambda > 0$ and $\rho \in (0, 1)$ such that there exists a solution of (1.1) satisfying

$$\lambda(1-\rho)G(x,0) \leq u(x) \leq \lambda(1+\rho)G(x,0).$$

Now choose a positive $\lambda_1 < \lambda$ such that $\lambda_1(1+\rho) < \lambda(1-\rho)$. By Theorem A again, there exists a solution u_1 of (1.1) satisfying

$$\lambda_1(1-\rho)G(x,0) \leq u_1(x) \leq \lambda_1(1+\rho)G(x,0).$$

By our choice of λ_1 and the fact that $G(x, 0) > 0$ in the interior of D, we know that $u_1(x) < u(x)$ in the interior of D. Therefore they are two different solutions of (1.1).

Theorem A is not restricted to the special nonlinearity u^p. In fact if the equation in (1.1) is replaced by $\Delta u + f(x, u) = 0$ and the function $f(x, G(x, 0))/G(x, 0)$ is in the Kato class, then the conclusion of Theorem A still holds. This will be clear from the proof.

Theorem B. *Suppose the function $V(.)/|.|^{(n-2)(p-1)}$ is in the class of Green tight functions in $\mathbf{R^n}$, then problem (1.2) has infinitely many solutions. More specifically, there exists a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$ and $\rho \in (0, 1)$, there exists a solution of (1.2) such that*

$$\lambda(1-\rho)G_0(x,0) \le u(x) \le \lambda(1+\rho)G_0(x,0),$$

where $G_0(x, y)$ is the Green's function of the Laplacian in $\mathbf{R^n}$.

Remark 1.2. By Proposition 2.1 (b), Theorem B contains Theorem 2.1 in [LS], which states that (1.2) has a solution provided that $V = V(|x|)$ is Hölder continuous and satisfies the condition $\int_0^\infty r^{n-1-p(n-2)}V(r)dr < \infty$, which is sharp in the radial case (see Theorem 2.3 in [LS]). Since $G_0(x, 0) = \frac{c_n}{|x|^{n-2}}$ for a dimensional constant c_n, we know that solutions given in Theorem B are solutions of (1.2).

Remark 1.3. Theorem A and B still hold if one replaces Δ by a uniformly elliptic operator in divergence form with bounded measurable coefficients. Only minor modifications are needed in the proof.

Our main result about the parabolic problem (1.3) is

Theorem C. *Let D be a bounded Lipschitz domain. Suppose the function $V(.)/|.|^{(n-2)(p-1)}$ is in the class K_n, then the following conclusions hold.*

(a). If u_0 and V are non-negative, there exists a $M > 0$ such that for any u_0 satisfying $u_0(x) \le MG(x, 0)$, problem (1.3) has a global positive solution in $(D - \{0\}) \times (0, \infty)$ such that for all $t > 0$

$$u(x,t) \le CG(x,0) \quad \text{and} \quad \lim_{x\to 0} u(x,t) = \infty.$$

(b). Under the same the assumptions in (a), there exists a sequence $t_k \to \infty$, $k = 1, 2, ...$, such that for $x \neq 0$, $u(x, t_k)$ converges pointwise when $k \to \infty$.

Remark 1.4. Due to Proposition 2.1, when $V = |x|^{-l}$, $l < 2$ and $1 < p < (n-l)/(n-2)$, the result in Theorem C holds. If we replace $D - \{0\}$ by D and take $l = 0$, we will reach problem (1.3') which was studied in [W] and others. By [W], for every $u_0 \in L^q(D)$, $q > n(p-1)/2$ and $q \ge p$, solutions of (1.3') are continuous as soon as $t > 0$. In contrast, solutions of (1.3) are singular as soon as $t > 0$.

We list a number of notations to be used frequently. $G(x, y)$ will be the Green's function of Δ in D and $G_0(x, y)$ will be he fundamental solution of Δ in $\mathbf{R^n}$. $\Gamma(x, t; y, s)$ with $t > s$ denotes the heat kernel with Dirichlet boundary conditions on $D \times (0, \infty)$. For any domain Ω and a function f, $K_\Omega(f) \equiv \sup_{x\in\Omega} \int_\Omega \frac{|f(y)|}{|x-y|^{n-2}}dy$. The function h is reserved for

$$h(x) = V(x)/|x|^{(n-2)(p-1)} \tag{1.4}$$

We shall prove Theorem A and B in Sect. 3 and Theorem C in Sect. 4. To prove the theorems, We shall convert the problems into suitable integral equations and use Schauder fixed point theorem to establish existence. To achieve this, some delicate and original estimates will be presented.

2. Preliminaries

In the next proposition we prove that in the case V is radial and D is a ball, our conditions on V become those in the paper [LS].

Proposition 2.1. *(a). Suppose that* $V = V(|x|)$ *and satisfies*

$$\int_0^{r_0} r^{n-1-p(n-2)} V(r)dr < \infty \tag{2.1}$$

for some $r_0 > 0$, *then the function* $h = V(.)/|.|^{(n-2)(p-1)}$ *defined on* $B(0, r_0)$ *is in the Kato class* K_n. *In particular, if*

$$V = |x|^{-l}, l < 2, 1 < p < (n-l)/(n-2), \tag{2.1'}$$

then (2.1) is satisfied.

(b). Suppose that $V = V(|x|)$ *and satisfies*

$$\int_0^{\infty} r^{n-1-p(n-2)} V(r)dr < \infty, \tag{2.2}$$

then the function $h = V(.)/|.|^{(n-2)(p-1)}$ *is a Green tight function in* $\mathbf{R^n}$.

Proof. (a). By Proposition 4.10 in [AS], a radial function $U = U(|x|)$ in $B(0, r_0)$ belongs to the Kato class K_n if and only if $\int_0^{r_0} r|U(r)|dr < \infty$. Taking $U = h = V(r)/r^{(n-2)(p-1)}$, we immediately reach the conclusion that (2.1) implies that h is in the Kato class. In the special case that (2.1') holds, $r^{n-1-p(n-2}V(r) = rn - 1 - p(n-2) - l$ and $n - 1 - p(n-2) - l > -2$. Hence (2.1) holds.

(b). By Proposition 1 in [Zhao1], a radial function $U = U(|x|) \in K_n$, which satisfies, for a $L > 0$, $\int_L^{\infty} r|U(r)|dr < \infty$, is a Green tight function in $\mathbf{R^n}$. We can finish the proof by taking $U = h = V(r)/r^{(n-2)(p-1)}$. q.e.d.

We shall use the following three-G theorem in a substantial way. We refer the reader to the paper [CFZ] for a proof.

Three-G Theorem. *For a bounded Lipschitz domain* D, *there exists a constant* C *depending on* D *such that*

$$\frac{G(x,y)G(y,z)}{G(x,z)} \le C[\frac{1}{|x-y|^{n-2}} + \frac{1}{|y-z|^{n-2}}], \tag{2.3}$$

for all x, y *and* $z \in D$. *(2.3) still holds if* G *is replaced by* G_0 *which is the fundamental solution of the Laplacian in* $\mathbf{R^n}$. An immediate consequence of the three-G theorem is the

Corollary 2.1. *Suppose* $U \in L^1(D)$ *belongs to the Kato class, then*

$$\frac{1}{G(x,0)} \int_D G(x,y)|U(y)|G(y,0)dy \le CK_D(U). \tag{2.4}$$

Remark 2.1. It is easy to see that if D and G in (2.4) are replaced by $\mathbf{R^n}$ and G_0 respectively, then

$$\frac{1}{G_0(x,0)}\int_D G_0(x,y)|U(y)|G_0(y,0)dy \le CK_\infty(U), \tag{2.5}$$

where $K_\infty(U) \equiv \sup_{x\in\mathbf{R^n}}\int_{\mathbf{R^n}} \frac{|U(y)|}{|x-y|^{n-2}}dy$. By [Zhao1], if U is Green tight in $\mathbf{R^n}$, then $K_\infty(U) < \infty$.

3. Proof of Theorem A and B: the elliptic case

Proof of Theorem A. We would like to show that there exists a $\lambda_0 > 0$ such that for all $\lambda \in (0,\lambda_0]$ and $\rho \in (0,1)$, the following integral equation (3.1) has a solution u such that

(i) u is continuous in $D - \{0\}$;
(ii) $\lambda(1-\rho)G(x,0) \le u(x) \le \lambda(1+\rho)G(x,0),\ x \in D - \{0\}$.

$$u(x) = \lambda G(x,0) + \int_D G(x,y)V(y)u^p(y)dy. \tag{3.1}$$

To achieve this, it is enough to show that there exists a continuous function w such that $\lambda(1-\rho) \le w(x) \le \lambda(1+\rho)$, $x \in D$ and that

$$w(x) = \lambda + \frac{1}{G(x,0)}\int_D G(x,y)V(y)G^p(y,0)w^p(y)dy. \tag{3.2}$$

Indeed for w satisfying (3.2) then $u = w(x)G(x,0)$ satisfies (3.1). We shall use the Schauder fixed point theorem. More specifically, let

$$S = \{w \in C(D)|\lambda(1-\rho) \le w(x) \le \lambda(1+\rho)\}$$

and T be the integral operator on S, which is defined as

$$Tw(x) = \lambda + \frac{1}{G(x,0)}\int_D G(x,y)V(y)G^p(y,0)w^p(y)dy,$$

we will show that T has a fixed point in S whenever λ is sufficiently small. To this end we need to check that the $TS \subset S$ and TS is compact in S.

Let $w \in S$, then $w \le \lambda(1+\rho)$ and

$$|Tw(x) - \lambda| \le \frac{[\lambda(1+\rho)]^p}{G(x,0)}\int_D G(x,y)|V(y)|G^p(y,0)dy.$$

Since

$$|V(y)|G^{p-1}(y,0) \le C|V(y)|/|y|^{(n-2)(p-1)} = C|h(y)|$$

and h belongs to the Kato class by assumption, the three-G theorem (see Corollary 2.1) implies

$$\begin{aligned}\frac{1}{G(x,0)}\int_D & G(x,y)|V(y)|G^p(y,0)dy \\ &= \frac{1}{G(x,0)}\int_D G(x,y)|V(y)|G^{p-1}(y,0)G(y,0)dy \le CK_D(h).\end{aligned}$$

Therefore $|Tw(x)-\lambda| \le CK_D(h)\lambda^p$, which implies, when λ is sufficiently small,

$$\lambda(1-\rho) \le Tw(x) \le \lambda(1+\rho). \tag{3.3}$$

Next we intend to show that $w(x)$ belongs to $C(\bar{D})$. For simplicity we write

$$I(x) \equiv Tw(x) - \lambda = \frac{1}{G(x,0)}\int_D G(x,y)V(y)G^p(y,0)w^p(y)dy.$$

We need to consider two cases. First, let x_0 be an interior point of D. For any small $\delta > 0$, we write

$$\begin{aligned} I(x) &= \frac{1}{G(x,0)}\int_{|y|\le\delta} G(x,y)V(y)G^p(y,0)w^p(y)dy \\ &\quad + \frac{1}{G(x,0)}\int_{|y|\ge\delta} G(x,y)V(y)G^p(y,0)w^p(y)dy \\ &\equiv I_1(x) + I_2(x).\end{aligned}$$

For any $\epsilon > 0$, by the three-G theorem again, we have

$$I_1(x) \le CK_{B(0,\delta)}(h) \le \epsilon/4,$$

when δ is sufficiently small. The last inequality is due to Corollary 2.1 and the fact that $h = V(x)G^{p-1}(x,0)$ is in the Kato class.

Since x_0 is an interior point, can choose δ so small that $B(x_0,\delta) \subset D$. We write

$$\begin{aligned} I_2(x) &= \frac{1}{G(x,0)}\int_{B(x_0,\delta)\cap B(0,\delta)^c} G(x,y)V(y)G^p(y,0)w^p(y)dy \\ &\quad + \frac{1}{G(x,0)}\int_{B(x_0,\delta)^c\cap B(0,\delta)^c} G(x,y)V(y)G^p(y,0)w^p(y)dy \\ &\equiv I_{21}(x) + I_{22}(x).\end{aligned}$$

As in the last paragraph, we can use the three-G theorem to show, for $x \in B(x_0,\delta)$, that

$$|I_{21}(x)| \le \epsilon/4$$

when δ is sufficiently small.

Note thet $G(x,0) > 0$ in the interior of D. Moreover $G(x,y)$ and $G(y,0)$ have no singularities when $x \in B(x_0, 3\delta/4)$ and $y \in B(x_0,\delta)^c \cap B(0,\delta)^c$, hence $I_{22}(x)$ is a continuous function in $B(x_0,\delta/2)$. Let x_1, $x_2 \subset B(x_0,\delta/2)$ and choose δ sufficiently small, then

$$|I(x_1) - I(x_2)| \le |I_1(x_1) - I_1(x_2)| + |I_{21}(x_1) - I_{21}(x_2)| + |I_{22}(x_1) - I_{22}(x_2)| < 2\epsilon.$$

This shows that $I(x)$ is continuous in the interior of D.

Secondly, let $x_0 \in \partial D$. In this case we write

$$\begin{aligned} I(x) &= \tfrac{1}{G(x,0)} \int_{D-B(x_0,\delta)} G(x,y)V(y)G^p(y,0)w^p(y)dy \\ &+ \tfrac{1}{G(x,0)} \int_{B(x_0,\delta)\cap D} G(x,y)V(y)G^p(y,0)w^p(y)dy \\ &= I_3(x) + I_4(x). \end{aligned}$$

It is clear that for $x \in D \cap B(x_0, 3\delta/4)$, the functions

$$J^+(x) \equiv \int_{D-B(x_0,\delta)} G(x,y)V^+(y)G^p(y,0)w^p(y)dy$$

and

$$J^-(x) \equiv \int_{D-B(x_0,\delta)} G(x,y)V^-(y)G^p(y,0)w^p(y)dy$$

are non-negative solutions of the Laplacian, so is the function $G(x,0)$. By the well-known theorem 7.9 in [JK], we know that $I_3 = \frac{J^+(x)}{G(x,0)} - \frac{J^-(x)}{G(x,0)}$ is a continuous function in $\bar{D} \cap B(x_0, \delta/2)$. Next we have , by the three-G theorem $I_4(x) \le CK_{B(x_0,\delta)}(h)$, $x \in B(x_0,\delta) \cap D$. Given any $\epsilon > 0$, we can then choose δ small so that

$$|I(x_1) - I(x_2)| \le |I_3(x_1) - I_3(x_2)| + |I_4(x_1) - I_4(x_2)| < \epsilon,$$

for all x_1, x_2 in $B(x_0, \delta/2)$. This shows that $I(x)$ is continuous up to the boundary. Therefore we have proved that $TS \subset S$ when λ is small. From the above argument, we also know that TS is compact, since the functions in Tw are equicontinuous for all $w \in S$. Now the Schauder fixed point theorem implies the existence of a fixed point of T in S. q.e.d.

Proof of Theorem B. The proof is similar to that of Theorem A. We would like to show that there exists a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$ and $\rho \in (0,1)$, the following integral equation (3.4) has a solution u such that

(i) u is continuous in $\mathbf{R^n} - \{\mathbf{0}\}$;

(ii) $\lambda(1-\rho)G_0(x,0) \le u(x) \le \lambda(1+\rho)G_0(x,0)$, $x \in \mathbf{R^n}$ $\quad G_0(x,y) = c_n/|x-y|^{n-2}$.

$$u(x) = \lambda G_0(x,0) + \int_{\mathbf{R^n}} G_0(x,y)V(y)u^p(y)dy. \tag{3.4}$$

To achieve this, it is enough to show that there exists a continuous function w such that $\lambda(1-\rho) \le w(x) \le \lambda(1+\rho)$, $x \in \mathbf{R^n}$ and that

$$w(x) = \lambda + \frac{1}{G_0(x,0)} \int_{\mathbf{R^n}} G_0(x,y)V(y)G_0^p(y,0)w^p(y)dy. \tag{3.5}$$

We shall use the Schauder fixed point theorem. More specifically, let

$$C_{\lambda,\rho} = \{w \in C(\mathbf{R^n}) \mid \lambda(\mathbf{1}-\rho) \le \mathbf{w(x)} \le \lambda(\mathbf{1}+\rho), \}$$

and T be the integral operator on $C_{\lambda,\rho}$, which is defined as

$$Tw(x) = \lambda + \frac{1}{G_0(x,0)} \int_{\mathbf{R^n}} G_0(x,y)V(y)G_0^p(y,0)w^p(y)dy,$$

we will show that T has a fixed point in $C_{\lambda,\rho}$ whenever ρ is sufficiently small. To this end we need to check that the $TC_{\lambda,\rho} \subset C_{\lambda,\rho}$ and $TC_{\lambda,\rho}$ is compact in $C_{\lambda,\rho}$.

Let $w \in C_{\lambda,\rho}$, then

$$|Tw(x) - \lambda| \leq \frac{[\lambda(1+\rho)]^p}{G_0(x,0)} \int_{\mathbf{R^n}} G_0(x,y)|V(y)|G_0^p(y,0)dy.$$

Clearly

$$|V(y)|G_0^{p-1}(y,0) \leq C|V(y)|/|y|^{(n-2)(p-1)} = C|h(y)|.$$

Since h is Green tight by assumption, by (2.5), we have

$$\begin{aligned} \tfrac{1}{G_0(x,0)} \int_{\mathbf{R^n}} & G_0(x,y)|V(y)|G_0^p(y,0)dy \\ &= \tfrac{1}{G_0(x,0)} \int_{\mathbf{R^n}} G_0(x,y)|V(y)|G_0^{p-1}(y,0)G(y,0)dy \leq CK_\infty(h). \end{aligned}$$

Therefore $|Tw(x) - \lambda| \leq CK_\infty(h)\lambda^p$, which implies, when λ is sufficiently small,

$$\lambda(1-\rho) \leq Tw(x) \leq \lambda(1+\rho). \tag{3.6}$$

To establish compactness we need to show that

$$\lim_{|x|\to\infty} Tw(x) = \lambda + \int_{\mathbf{R^n}} V(y)G_0^p(y,0)w^p(y)dy \tag{3.7}$$

uniformly for all $w \in C_{\lambda,\rho}$. We remark that the righthand side of (3.7) is a finite number since $|VG_0^{p-1}| \leq C|h|$ and h is a Green tight function. By the definition of Green tight functions, for any $\epsilon > 0$, there is a $M > 0$ such that

$$\sup_{x\in\mathbf{R^n}} \int_{|y|\geq M} \frac{h(y)}{|x-y|^{n-2}}dy < \epsilon/2. \tag{3.8}$$

Now

$$\begin{aligned} &|Tw(x) - \lambda - \textstyle\int_{\mathbf{R^n}} V(y)G_0^p(y,0)w^p(y)dy| \\ &\leq \tfrac{1}{G_0(x,0)} \textstyle\int_{|y|\geq M} G_0(x,y)|V(y)|G_0^p(y,0)w^p(y)dy \\ &\quad + |\tfrac{1}{G_0(x,0)} \textstyle\int_{|y|\leq M} G_0(x,y)V(y)G_0^p(y,0)w^p(y)dy \\ &\qquad - \textstyle\int_{|y|\leq M} V(y)G_0^p(y,0)w^p(y)dy| \\ &\quad + \textstyle\int_{|y|\geq M} V(y)G_0^p(y,0)w^p(y)dy \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

Applying the three-G theorem to I_1 and using (3.8) we have

$$I_1 \leq C \sup_{x\in\mathbf{R^n}} \int_{|y|\geq M} \frac{h(y)}{|x-y|^{n-2}}dy < C\epsilon/2.$$

When $|y| \leq M$ and let $|x|$ be sufficiently large we know that $I_2 < \epsilon$ since

$$\lim_{|x|\to\infty} G_0(x,y)/G_0(x,0) = \lim_{|x|\to\infty} |x|^{n-2}/|x-y|^{n-2} = 1$$

uniformly for $|y| \le M$. It is also clear that $I_3 < \epsilon$ when M is large. Therefore

$$|Tw(x) - \lambda - \int_{\mathbf{R^n}} V(y) G_0^p(y,0) w^p(y) dy| < (C+2)\epsilon,$$

when $|x|$ is sufficiently large. This proves (3.7).

Following the argument in the proof of Theorem A, we find that Tw is equi-continuous. Now that we know $TC_{\lambda,\rho}$ is a convex, compact subset of $C_{\lambda,\rho}$, by Schauder fixed point theorem, we can find a $w \in C_{\lambda,\rho}$ such that (3.5) holds. q.e.d.

4. Proof of Theorem C: the parabolic case

Before proving the theorem we need two preliminary results.

Proposition 4.1. *Suppose the function $h = V(x)/|x|^{(n-2)(p-1)}$ is in the Kato class K_n. If a bounded function $w = w(x,t)$ satisfies the next integral relation (4.1), then the function $u = u(x,t) \equiv w(x,t)G(x,0)$ is a distributional solution of the equation (1.3) in the region $(D - \{0\}) \times (0,T)$.*

$$\begin{aligned} w(x,t) = \; & \tfrac{1}{G(x,0)} \int_D \Gamma(x,t;y,0) u_0(y) dy + \tfrac{m}{G(x,0)} \int_0^t \Gamma(x,t;0,s) ds \\ & + \tfrac{1}{G(x,0)} \int_0^t \int_D \Gamma(x,t;y,s) V(y) [w(y,s)G(y,0)]^p dyds. \end{aligned} \tag{4.1}$$

Here $m > 0$.

Proof. Let $\phi \in C^\infty(D \times (0,T))$ be such that the closure of supp ϕ is a subset of $(D - \{0\}) \times (0,T)$. Using ϕ as a test function, it is easy to see that

$$\begin{aligned} u = \; & w(x,t)G(x,0) = \int_D \Gamma(x,t;y,0) u_0(y) dy + m \int_0^t \Gamma(x,t;0,s) ds \\ & + \int_0^t \int_D \Gamma(x,t;y,s) V(y) u^p dyds. \end{aligned} \tag{4.1'}$$

is a distributional solution of $\Delta u - u_t + Vu^p = 0$ in $(D - \{0\}) \times (0,T)$. Here Γ is the Dirichlet heat kernel in $D \times (0,\infty)$. q.e.d.

Remark 4.1. We want to underline the role played by $m \int_0^t \Gamma(x,t;0,s) ds$ on the right hand side of (4.1'). It is easy to check that this function is a solution of the heat equation in $(D - \{0\}) \times (0,\infty)$. More importantly, as we shall see in the proof of Theorem C below, it is this function that provides the singularity for u.

Proposition 4.2. *Suppose the function $h = V(x)/|x|^{(n-2)(p-1)}$ is in the Kato class K_n. Let $\delta > 0$ and $w = w(x,t)$ be a bounded function in $D \times [0,\infty)$, then we have*

(a). the function

$$T(x,t) \equiv \int_0^t \int_{D - B(0,\delta)} \Gamma(x,t;y,s) V(y) [w(y,s)G(y,0)]^p dyds$$

is continuous in $\bar{D} \times [0,\infty)$.

(b). As functions of x, $T(.,t)$ are equi-continuous for all $t \ge 1$.

Proof. (a). Since D is bounded, $|V(x)| \leq C|h(x)|$, therefore V is also in the Kato class K_n. For simplicity let h_1 be the function such that $h_1(y,s) = V(y)[w(y,s)G(y,0)]^p$ if $|y| > \delta$ and $h_1(y,s) = 0$ if $|y| \leq \delta$. Then clearly

$$|h_1(y,s)| \leq C(\delta)|V(y)|,$$

which is in the Kato class K_n. Now we can write $T(x,t)$ as

$$T(x,t) = \int_0^t \int_D \Gamma(x,t;y,s)h_1(y,s)dyds$$

For any $\eta > 0$, we have

$$\begin{aligned} T(x,t) &= \int_0^t \int_{D\cap B(x,\eta)} \Gamma(x,t;y,s)h_1(y,s)dyds \\ &\quad + \int_0^t \int_{D-B(x,\eta)} \Gamma(x,t;y,s)h_1(y,s)dyds \equiv T_1(x,t) + T_2(x,t). \end{aligned}$$

Using the equality $\int_0^\infty \Gamma(x,t;y,0)dt = G(x,y)$, we have

$$|T_1(x,t)| \leq C(\delta)||w||_{L^\infty} \int_{|x-y|\leq\eta} G(x,y)|V(y)|dy.$$

Therefore, for any $\epsilon > 0$, when η is small, we have, by the fact that V is in Kato class,

$$|T_1(x,t)| < \epsilon.$$

It is clear that T_2 is a continuous function since the kernel of the integral is a bounded function. Given any $P_i = (x_i, t_i) \in D \times [0,\infty)$, $i = 1,2$,

$$|T(x_1,t_1) - T(x_2,t_2)| \leq |T_1(x_1,t_1) - T_1(x_2,t_2)| + |T_2(x_1,t_1) - T_2(x_2,t_2)| < C\epsilon,$$

when $|P_1 - P_2|$ is sufficiently small. This proves the continuity of $T(.,.)$ and finishes the proof of part (a).

(b). From the proof of (a), it is clear that we only need to prove that $T_2(.,t)$ is equi-continuous for all $t > 1$. For $x_1, x_2 \in D$ and $t > 1$,

$$\begin{aligned} &|T_2(x_1,t) - T_2(x_2,t)| \\ &\leq |\int_0^t \int_{D-B(x_1,\eta)} \Gamma(x_1,t;y,s)h_1dyds - \int_0^t \int_{D-B(x_2,\eta)} \Gamma(x_1,t;y,s)h_1dyds| \\ &\quad + \int_0^t \int_{D-B(x_2,\eta)} |\Gamma(x_2,t;y,s) - \Gamma(x_1,t;y,s)||h_1(y,s)|dyds \\ &\leq \int_0^t \int_{(B(x_1,\eta)-B(x_2,\eta))\cup(B(x_1,\eta)-B(x_2,\eta))} \Gamma(x_1,t;y,s)|h_1(y,s)|dyds \\ &\quad + \int_0^t \int_{D-B(x_2,\eta)} |\Gamma(x_2,t;y,s) - \Gamma(x_1,t;y,s)||h_1(y,s)|dyds \\ &\leq C(\delta) \int_{(B(x_1,\eta)-B(x_2,\eta))\cup(B(x_1,\eta)-B(x_2,\eta))} G(x_1,y)|V(y)|dyds \\ &\quad +C(\delta) \int_0^t \int_{D-B(x_2,\eta)} |\Gamma(x_2,t;y,s) - \Gamma(x_1,t;y,s)||V(y)|dyds \\ &\equiv T_3 + T_4. \end{aligned}$$

For any $\epsilon > 0$, since V is in the Kato class, we know that $T_3 < \epsilon$ when $|x_1 - x_2|$ is sufficiently small. Choose η such that $B(x_2, \eta) \subset D$. Next we notice that for $y \in D - B(x_2, \eta)$, $\Gamma(.,.;y,s)$ is a solution of the heat equation in $B(x_2, \eta/2) \times (0, \infty)$. By the standard parabolic theory (see [A]), we can find positive constants $C(\eta)$ and $\alpha < 1$ such that, for $x_1 \in B(x_2, \eta/4)$,

$$|\Gamma(x_2, t; y, s) - \Gamma(x_1, t; y, s)| \leq C(\eta)|x_2 - x_1|^\alpha \sup_{(z,\tau)\in Q_{\eta/2}} \Gamma(z, \tau; y, s),$$

where $Q_{\eta/2} \equiv B(x_2, \eta/2) \times [t - \eta^2/4, t + \eta^2/4]$. By Harnack inequality and the Gaussian bound in [A]

$$\sup_{(z,\tau)\in Q_{\eta/2}} \Gamma(z, \tau; y, s) \leq C\Gamma(x_2, t + 4\eta^2; y, s) \leq \frac{C}{(t + 4\eta^2 - s)^{n/2}} e^{-c\frac{|x_2 - y|^2}{t+4\eta^2 - s}}.$$

Since $|x_2 - y| \geq \eta$, we have

$$\sup_{(z,\tau)\in Q_{\eta/2}} \Gamma(z, \tau; y, s) \leq \frac{C}{(t + 4\eta^2 - s)^{n/2}} e^{-c\frac{\eta^2}{t+4\eta^2 - s}}.$$

Now

$$T_4 \leq C(\eta)|x_2 - x_1|^\alpha \int_0^t \int_{D-B(x_2,\eta)} \frac{C}{(t+4\eta^2-s)^{n/2}} e^{-c\frac{\eta^2}{t+4\eta^2-s}} |V(y)|dyds$$
$$\leq C(\eta) \int_{D-B(x_2,\eta)} |V(y)|dy \ |x_2 - x_1|^\alpha.$$

This proves that, when $|x_1 - x_2|$ is small

$$|T_2(x_1, t) - T_2(x_2, t)| < C\epsilon,$$

and hence the modulus of continuity of T_2 and $T(., t)$ is independent of $t > 1$. q.e.d.

Now we are ready to give the

Proof of Theorem C, part (a).
We divide the proof into several steps.

Step 1. Let Λ be the integral operator

(4.2)
$$\begin{aligned}\Lambda w(x, t) = \ & \frac{1}{G(x,0)} \int_D \Gamma(x, t; y, 0)u_0(y)dy + \frac{m}{G(x,0)} \int_0^t \Gamma(x, t; 0, s)ds \\ & + \frac{1}{G(x,0)} \int_0^t \int_D \Gamma(x, t; y, s)V(y)[w(y, s)G(y, 0)]^p dyds \\ & \equiv f_1 + f_2 + \Lambda_3 w,\end{aligned}$$

which is defined on

(4.3) $$S_\alpha = \{w \in C(D \times (0, \infty))| 0 \leq w(x, t) \leq \alpha\}.$$

By Proposition 4.1, we only need to prove that Λ has a fixed point w in S_α provided that $0 \leq u_0(x) \leq MG(x, 0)$ and that α, M and m are sufficiently small. This is so because $u = w(x, t)G(x, 0)$ will be a solution of (1.3). By standard continuation argument, it is sufficient to show that under the same condition the operator Λ has a fixed point in

$$S_{\alpha,T} = \{w \in C(D \times (0,T)) | 0 \le w(x,t) \le \alpha\}, \tag{4.4}$$

for all $T > 0$. At the same time all parameters such as α, M and m must be chosen independently of T

To this end we need to check, when α, M and m are sufficiently small, that $\Lambda S_{\alpha,T} \subset S_{\alpha,T}$ and $\Lambda S_{\alpha,T}$ is compact in $S_{\alpha,T}$ for all $T > 0$.

Step 2. We want to show, under the assumptions in step 1, that

$$0 \le \Lambda w \le \alpha \tag{4.5}$$

when $w \in S_{\alpha,T}$.

For any $w \in S_{\alpha,T}$, we know that $w(x,t) \le \alpha$. By the inequality

$$\int_0^t \Gamma(x,t;y,s)ds \le G(x,y) \tag{4.6}$$

and the assumption $u_0(x) \le MG(x,0)$, we have

$$\begin{aligned} \Lambda w(x,t) \le \quad & \tfrac{M}{G(x,0)} \int_D \Gamma(x,t;y,0)G(y,0)dy + \tfrac{m}{G(x,0)} G(x,0) \\ & + \tfrac{\alpha^p}{G(x,0)} \int_D G(x,y)V(y)G^p(y,0)dy. \end{aligned} \tag{4.7}$$

Since $|V(y)|G^{p-1}(y,0) \le Ch(y)$ and h is in the Kato class K_n, we have, as in Sect. 3,

$$\int_D G(x,y)V(y)G^p(y,0)dy \le CK_D(h)G(x,0), \tag{4.8}$$

which implies, via (4.7),

$$\Lambda w(x,t) \le \frac{M}{G(x,0)} \int_D \Gamma(x,t;y,0)G(y,0)dy + m + C\alpha^p K_D(h). \tag{4.9}$$

To control the first term on the righthand side of (4.9) we observe that

$$\begin{aligned} & \int_D \Gamma(x,t;y,0)G(y,0)dy \\ & = \int_D \Gamma(x,t;y,0) \int_0^\infty \Gamma(y,s;0,0)dsdy = \int_0^\infty \int_D \Gamma(x,t;y,0)\Gamma(y,s;0,0)dyds \\ & = \int_0^\infty \Gamma(x,t+s;0,0)ds \le \int_0^\infty \Gamma(x,s;0,0)ds \\ & = G(x,0), \end{aligned}$$

where we have used the well-known equality $\int_0^\infty \Gamma(y,s;0,0)ds = G(x,y)$ and the reproducing property of the heat kernel. From (4.9) we then get

$$\Lambda w(x,t) \le M + m + C\alpha^p K_D(h).$$

Since $p > 1$, we can take α, m and M sufficiently small so that

$$\Lambda w(x,t) \le \alpha,$$

for all $t > 0$. This proves (4.5).

Step 3. We need to show that $\Lambda S_{\alpha,T}$ is compact in $S_{\alpha,T}$ for any $T > 0$. According to (4.2), $\Lambda w = f_1 + f_2 + \Lambda_2 w$. Since f_1 and f_2 are two fixed and bounded functions, it is enough to prove that

$$\Lambda_3 S_{\alpha,T} \quad \text{is compact in} \quad S_{\alpha,T}$$

for any $T > 0$. In this step, we shall prove that

$$\Lambda_3 w = \frac{1}{G(x,0)} \int_0^t \int_D \Gamma(x,t;y,s) V(y) [w(y,s)G(y,0)]^p dyds \tag{4.10}$$

is equi-continuous for all $w \in S_{\alpha,T}$. We consider two cases separately.

Case 1. Let x_0 be an interior point, we need to show that $\Lambda_3 w$ is continuous in a neighborhood of (x_0, t_0), $t_0 > 0$. For a small $\delta > 0$ we write

$$\begin{aligned} \Lambda_3 w &= \tfrac{1}{G(x,0)} \int_0^t \int_{|y|\le\delta} \Gamma(x,t;y,s) V(y) [w(y,s)G(y,0)]^p dyds \\ &\quad + \tfrac{1}{G(x,0)} \int_0^t \int_{|y|\ge\delta} \Gamma(x,t;y,s) V(y) [w(y,s)G(y,0)]^p dyds \\ &\equiv I_1(x,t) + I_2(x,t). \end{aligned} \tag{4.11}$$

Using (4.6) and the fact that $w \in S_{\alpha,T}$, we know that

$$I_1(x,t) \le \frac{\alpha^p}{G(x,0)} \int_{|y|\le\delta} G(x,y) V(y) G^p(y,0) dy.$$

By the three-G theorem and following the argument in the proof of Theorem A, we have

$$I_1(x,t) \le C \alpha^p K_{B(0,\delta)}(h), \tag{4.12}$$

where $h = V(x)/|x|^{(n-2)(p-1)}$ again. Since $G(x,0) > 0$ in the interior of D, $I_2(x,t)$ is, by Proposition 4.2, a continuous function in a small neighborhood of (x_0, t_0). Let (x_i, t_i) be two points in $B(x_0,\delta) \times [x_0 - \delta^2, x_0 + \delta^2]$, then, for any $\epsilon > 0$, we can choose δ so small that

$$|\Lambda_3 w(x_1,t_1) - \Lambda_3 w(x_2,t_2)| \le |I_1(x_1,t_1) - I_1(x_2,t_2)| + |I_2(x_1,t_1) - I_2(x_2,t_2)| < \epsilon.$$

This shows that $\Lambda_3 w$ is continuous in the interior of $D \times [0,\infty)$. It is important to note that the modulus of continuity of $\Lambda_3 w$ is independent of the choice of w in $S_{\alpha,T}$.

Case 2. $x_0 \in \partial D$. We need to prove that $\Lambda_3 w$ is continuous at (x_0, t_0).

For a small $\delta > 0$ we write

(4.13)

$$\begin{aligned} \Lambda_3 w &= \tfrac{1}{G(x,0)} \int_0^t \int_{D\cap B(x_0,2\delta)} \Gamma(x,t;y,s) V(y) [w(y,s)G(y,0)]^p dyds \\ &\quad + \tfrac{1}{G(x,0)} \int_0^t \int_{D-B(x_0,2\delta)} \Gamma(x,t;y,s) V(y) [w(y,s)G(y,0)]^p dyds \\ &\equiv I_3(x,t) + I_4(x,t). \end{aligned}$$

For $x \in B(x_0, \delta)$ we have, as in case one,
(4.14)

$$|I_3(x,t)| \leq \frac{\alpha^p}{G(x,0)} \int_{D\cap B(x_0,2\delta)} G(x,y)V(y)G^p(y,0)dy \leq C\alpha^p K_{B(x_0,2\delta)}(h).$$

Next we turn our attention to $I_4(x,t)$. When δ is sufficiently small, it is clear that the function

$$G(x,0)I_4(x,t) = \int_0^t \int_{D-B(x_0,2\delta)} \Gamma(x,t;y,s)V(y)[w(y,s)G(y,0)]^p dyds$$

is a non-negative solution of the heat equation in the cylinder $D \cap B(x_0, 3\delta/2) \times (0, \infty)$, so is the function $G(x, 0)$ itself. It is also not hard to see that

$$G(x,0)I_4(x,0) = 0.$$

Therefore the function

$$f(x,t) = \begin{cases} G(x,0)I_4(x,t), & t > 0 \\ 0, & t \leq 0 \end{cases} \tag{4.15}$$

is a non-negative solution of the heat equation in $D \cap B(x_0, 3\delta/2) \times (-\infty, \infty)$. Therefore both $G(x, 0)$ and $G(x, 0) + f(x, t)$ are positive solutions of the heat equation in $D \cap B(x_0, 3\delta/2) \times (-\infty, \infty)$. By Theorem1 and Corollary 1 in [ACS] or by [FS], we know that $\frac{G(x,0)+f(x,t)}{G(x,0)}$ is a continuous function in $\bar{D} \cap B(x_0, \delta) \times [-T, T]$, $T > 0$. In particular, $I_4(x, t)$ is continuous in $\bar{D} \cap B(x_0, \delta) \times [0, T)$, $T > 0$.

Let (x_i, t_i), $i = 1, 2$, be two points in $D \times B(x_0, \delta) \times [t_0 - \delta^2, t_0 + \delta^2]$, then, for any $\epsilon > 0$, we can choose δ so small that

$$\begin{aligned} |\Lambda_3 \quad & w(x_1,t_1) - \Lambda_3 w(x_2,t_2)| \leq |I_3(x_1,t_1) - I_3(x_2,t_2)| + |I_4(x_1,t_1) - I_4(x_2,t_2)| \\ & \leq 2C\alpha^p K_{B(x_0,2\delta)}(h) + |I_4(x_1,t_1) - I_4(x_2,t_2)| \\ & < \epsilon. \end{aligned}$$

This shows that $\Lambda_3 w$ is continuous in $\bar{D} \times [0, \infty)$. Again it is important to note that the modulus of continuity of $\Lambda_3 w$ is independent of the choice of w in $S_{\alpha,T}$.

Step 4. Next we show that Λ is a continuous operator. Let $w_1, w_2 \in S_{\alpha,T}$, then

$$\begin{aligned} & ||\Lambda w_1 - \Lambda w_2||_{L^\infty} \\ & \leq C||w_1^p - w_2^p||_{L^\infty} \tfrac{1}{G(x,0)} \textstyle\int_0^t \int_D \Gamma(x,t;y,s)V(y)[G(y,0)]^p dyds \\ & \leq ||w_1^p - w_2^p||_{L^\infty} \tfrac{1}{G(x,0)} \textstyle\int_D G(x,y)V(y)[G(y,0)]^p dy \\ & \leq C||w_1^p - w_2^p||_{L^\infty} K_D(h). \end{aligned}$$

Here we used (4.8) to reach the last inequality. Hence Λ is continuous.

By this time it is clear from the Ascoli-Arzela theorem that $\Lambda_3 S_{\alpha,T}$ is a compact subset of $S_{\alpha,T}$. By the observation at the beginning of step 3,

$$\Lambda S_{\alpha,T} = f_1 + f_2 + \Lambda_3 S_{\alpha,T}.$$

Since $\Lambda S_{\alpha,T} \subset S_{\alpha,T}$ by step 2, we know that $\Lambda S_{\alpha,T}$ is a compact subset of $S_{\alpha,T}$, $T > 0$. By the Schauder fixed point theorem, for any $T > 0$, Λ has a fixed point in $S_{\alpha,T}$. Therefore Proposition 4.1 implies that $u = G(x,0)w(x,t)$ is a solution of the equation in (1.3) and $0 \le u(x,t) \le \alpha G(x,0)$. Finally, since u_0 and V are non-negative, we have

$$u(x,t) \ge m \int_0^t \Gamma(x,t;0,s)ds = mG(x,0) - m\int_t^\infty \Gamma(x,s,0,0)ds \tag{4.16}$$

Since $\int_t^\infty \Gamma(x,s,0,0)ds$ is finite when $t > 0$, we know that u is singular at $x = 0$ as soon as $t > 0$. Part (a) of the theorem is proven.

Proof of Theorem C, part (b). Let u be a solution of (1.3) obtained in the proof of part (a), then $u = w(x,t)G(x,0)$ satisfies

$$\begin{aligned} u(x,t) = {} & \int_D \Gamma(x,t;y,0)u_0(y)dy + m\int_0^t \Gamma(x,t;0,s)ds \\ & + \int_0^t \int_D \Gamma(x,t;y,s)V(y)[w(y,s)G(y,0)]^p dyds \end{aligned}$$

where $0 \le w(x,t) \le \alpha$. Therefore

(4.17)

$$\begin{aligned} |x|^{n-2}u(x,t) = {} & |x|^{n-2}\int_D \Gamma(x,t;y,0)u_0(y)dy + m|x|^{n-2}\int_0^t \Gamma(x,t;0,s)ds \\ & + |x|^{n-2}\int_0^t \int_D \Gamma(x,t;y,s)V(y)[w(y,s)G(y,0)]^p dyds. \end{aligned}$$

By the Gaussian bounds in [A],

$$\lim_{t\to\infty} |x|^{n-2}\int_D \Gamma(x,t;y,0)u_0(y)dy = 0.$$

It is also well known that

$$\lim_{t\to\infty} |x|^{n-2}\int_0^t \Gamma(x,t;0,s)ds = |x|^{n-2}\lim_{t\to\infty}\int_0^t \Gamma(x,t;0,s)ds = |x|^{n-2}G(x,0),$$

when $x \ne 0$. Therefore we only need to show that the third term

$$J(x,t) \equiv |x|^{n-2}\int_0^t \int_D \Gamma(x,t;y,s)V(y)[w(y,s)G(y,0)]^p dyds$$

on the righthand side of (4.17) has a convergent subsequence when $t \to \infty$. We claim that for any $t > 1$ the functions $J(.,t)$ are equi-continuous. Suppose, for the moment, that we take the claim for granted, then the set $\{J(.,t)\ |t > 1\}$ is compact under the maximum norm. Therefore there exists a sequence $t_k \to \infty$ so that the functions $u_k \equiv |x|^{n-2}u(x,t_k)$ converge uniformly to a function $|x|^{n-2}u_\infty$ when $k \to \infty$.

Now it only remains to prove the above claim. This is done by following the lines of case 1 in step 3 of part (a). Let $x_0 \in D$ and for a small $\delta > 0$ we write

$$\begin{aligned} J(x,t) &= |x|^{n-2} \int_0^t \int_{|y|\leq\delta} \Gamma(x,t;y,s)V(y)[w(y,s)G(y,0)]^p dyds \\ &\quad + |x|^{n-2} \int_0^t \int_{|y|\geq\delta} \Gamma(x,t;y,s)V(y)[w(y,s)G(y,0)]^p dyds \\ &\equiv J_1(x,t) + J_2(x,t). \end{aligned}$$

Using (4.6) again and the fact that $0 \leq w \leq \alpha$, we know that

$$J_1(x,t) \leq \frac{\alpha^p}{G(x,0)} \int_{|y|\leq\delta} G(x,y)V(y)G^p(y,0)dy.$$

Since $G(x,0) \leq C/|x|^{n-2}$, by the three-G theorem and following the argument in the proof of Theorem A, we have

$$J_1(x,t) \leq C\alpha^p K_{B(0,\delta)}(h),$$

where $h = V(x)/|x|^{(n-2)(p-1)}$ again. By Proposition 4.2 (b), $J_2(.,t)$ is equicontinuous for $t > 1$. Let x_i, $i = 1,2$, be two points in $B(x_0,\delta) \cap D$, then, for any $\epsilon > 0$, we can choose δ small that when $|x_1 - x_2|$ is small

$$|J(x_1,t) - J(x_2,t)| \leq |J_1(x_1,t) - J_1(x_2,t)| + |J_2(x_1,t) - J_2(x_2,t)| < \epsilon.$$

This shows that $J(.,t)$ is continuous in D, $t > 1$. It is important to note that the modulus of continuity of J is independent of the choice of t when $t > 1$. This proves the claim and the theorem.

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