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Singular integrals on star-shaped Lipschitz surfaces in the quaternionic space

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Introduction

In [McQ1], [McQ2], [LMcS], [GLQ] and [LMcQ] the singular integral theory associated with the bounded holomorphic functional calculi of Dirac operators on one- and higher- dimensional Lipschitz graphs was established. In [Q1-3] and [GQW] the theory was further extended to periodic cases. This paper is devoted to an analogous theory in the quaternionic space **Q**. The theory is closely related to some recent development of operator theory and harmonic analysis. Even restricted to the unit sphere, the study of the paper is new: It provides a large class of singular integral operators, each is analogous to the Hilbert transform in the context, that constitutes a bounded holomorphic functional calculus of the spherical Dirac operator. The theory proves identifications between the three forms: Fourier multipliers, singular integrals and Cauchy-Dunford's integrals for functional calculi on both the unit sphere and star-shaped Lipschitz surfaces. It also provides explicit formulas to obtain the singular integral kernels from the Fourier multipliers and vice versa.

The study of the paper restricted to the sphere does not fall into the scope of the well studied *Calderón-Zygmund spherical convolution operator theory*, for a survey of that we refer the reader to [Sa] and [CW]. The operators studied there are multiplier operators on spherical Laplace-Beltrami eigenspace expansions, or alternatively, Fourier-Laplace expansions, of L^2-functions on the sphere. The present theory, however, is about Fourier multipliers on spherical Dirac operator eigenspace expansions of the L^2-functions .

The nature of the theory is different from what is developed in [QR] either, in which Möbius transforms are used to transfer, using change of variables, the singular integral theory established in [LMcQ] to certain surfaces which may not be Lipschitz.

There is some recent development on function theory of quaternionic variables which would be worthwhile mentioning. In [ABLSS1] regular functions of several quaternionic variables and the Cauchy-Fueter complex of differential operators are studied. In [ABLSS2] regular functions of one quaternionic variable which satisfy a large class of differential equations are studied. As a consequence they proved that the functions under consideration cannot have compact singularities. Some more advanced results along this line are given in [ALPS]. In a more recent paper [CLSS] the authors extend the work by K. Imaeda, that gives rise to Maxwell's equations, to define a notion of regularity for functions of one and several biquaternionic variables. The function theory developed in the present paper has no overlap with the above mentioned function theories. It is enlightened by the latest development of harmonic analysis dealing with Fourier theory in conjunction with functional calculi of Dirac operators on surfaces in the quaternionic space.

In [Q4] and [Q5] we establish the analogous theory in $\mathbf{R}^n$. Although $\mathbf{R}^n$ is no longer an algebra and more complicated to deal with, the methods we use are suggested by the present paper.

The applications of the theory include the kind of boundary value problems discussed in [LMcQ] and [Mc3], but on closed Lipschitz surfaces. Partition of unity has been used in order to make use of singular integral theory developed on infinite graphs to boundary value problems on closed curves and surfaces (see e.g. [V], [K1] and [K2]). The study of the paper forms part of our efforts in providing effective operator algebras right on closed curves and surfaces so to allow the inverse operator problems to be solved directly. A detailed study concerning the application aspect will appear elsewhere.

The paper is arranged into four sections. Section 0 contains preliminaries. In Sect. 1 we construct a class of regular functions that will act as singular integral kernels in the later sections. Whilst the theory for the periodic cases is built up from periodisation based on Poisson summation formulas ([Q1], [GQW], [Q3]), there is no analogous method available for the unit sphere. As a substitution, Fueter's result provides a method to construct regular functions of a quaternionic variable from holomorphic functions of a complex variable (see, e.g. [Su], [De]). This enables us to transfer the theory established in [Q1], [Q2] and [GQW] for a complex variable to the present case. Theorems 1, 2 and 3 can be understood as concerning Fourier series, in particular Fourier and inverse Fourier transforms between the kernel functions and the bounded holomorphic multipliers; they can also be understood as regular continuations of power and principal series (see Remark 6). Not only for their connections to singular (Theorem 1) and fractional (Theorem 3) integrals on the surfaces, but also the results themselves would be of interest in the Laurent series theory in quaternions (also see Proposition 4). Theorems 1 and 3 are the main technical results to the theory developed in the following sections.

In Sect. 2 we establish a singular integral theory (Theorem 4) using the kernel functions obtained in Theorem 1 of Sect. 1. The theory includes two aspects: the identification between the singular integral expressions and the Fourier multiplier expressions; and the L^2-boundedness of the operators. If we restrict ourselves to the sphere, then the boundedness can be easily deduced by making use of the Plancherel theorem on the sphere. In our new context, i.e. on starlike Lipschitz surfaces, however, there does not exist Plancherel's theorem. The nature of the boundedness problem turns to be of the same kind as that of Coifman-McIntosh-Meyer's (CMcM's) theorem on the L^2-boundedness of the Cauchy integral operator on Lipschitz graphs (cf. [CMcM], [Mc1], [LMcS], [GLQ], [CM], [GQW], [Ta]). The proof presented here is an adaptation of a proof of [GQW] based on Littlewood-Paley theory. The identification result in particular implies Parseval's identity on the sphere linking the kernel functions to the bounded holomorphic multipliers (Corollary 3)

In Sect. 3 we provide a third version of the operators: Cauchy-Dunford's integrals of functional calculus. We show that the spherical Dirac operator can be expressed as a sum of two type-ω operators. The Dirac operator therefore enjoys all the basic properties possessed by type-ω operators studied in [Mc2] and [CDMcY].

Notations C, C_ν, etc. will be used for constants which may vary from one occurrence to the next. Subscripts, such as ν in C_ν, etc. are used to stress dependence of constants. In most cases, if a paragraph contains a piece of argument or statement, and if the notation $\pm$ appears in both the condition and the conclusion parts, then the argument or statement is meant to be valid for two symmetric cases: one is for all the $\pm$ being replaced by +; and the other is for all the $\pm$ being replaced by $-$. Similarly, when we introduce a new notation, if $\pm$ appears in both its name part and its definition parts, then we are simultaneously defining two notations: one is for all the $\pm$ being +; and the other is for all the $\pm$ being $-$. According to the convention, we will need to write $\mp$ as $-(\pm)$ in the sequel.

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0 Preliminaries

Let **Q** and $\mathbf{Q^c}$ denote the algebras of Hamilton's quaternions over **R**, the real number field, and **C**, the complex number field, respectively, with the usual canonical basis, $\mathbf{i}_0, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ ($\mathbf{i}_0$ being the identity of $\mathbf{Q}$ which will henceforth be identified with the identity 1 of $\mathbf{R}$), where

$$\mathbf{i}_1\mathbf{i}_2 = -\mathbf{i}_2\mathbf{i}_1 = \mathbf{i}_3, \quad \mathbf{i}_2\mathbf{i}_3 = -\mathbf{i}_3\mathbf{i}_2 = \mathbf{i}_1, \quad \mathbf{i}_3\mathbf{i}_1 = -\mathbf{i}_1\mathbf{i}_3 = \mathbf{i}_2,$$

and

$$\mathbf{i}_1^2 = \mathbf{i}_2^2 = \mathbf{i}_3^2 = -1.$$

A general quaternion is of the form $q = \sum_{l=0}^{3} q_l \mathbf{i}_l = q_0 + \underline{q}$, $q_l \in \mathbf{R}$ or $q_l \in \mathbf{C}$, depending on $q \in \mathbf{Q}$ or $q \in \mathbf{Q^c}$, respectively, where q_0 and $\underline{q} = q_1\mathbf{i}_1 + q_2\mathbf{i}_2 + q_3\mathbf{i}_3$ are called the real and the imaginary part of q, respectively. Denote by $\mathscr{O}$ the real vector space $\{q_1\mathbf{i}_1 + q_2\mathbf{i}_2 + q_3\mathbf{i}_3 : q_l \in \mathbf{R}, l = 1, 2, 3\}$. The quaternionic conjugate of q, denoted by $\overline{q}$, is defined to be $\overline{q} = q_0 - \underline{q}$. We have $\overline{qq'} = \overline{q'}\,\overline{q}$. For any non-zero element $q \in \mathbf{Q}$ there exists an inverse $q^{-1} \in \mathbf{Q} : q^{-1} = \frac{\overline{q}}{|q|^2}$ such that $q^{-1}q = qq^{-1} = 1$. The natural inner product between q and q' in $\mathbf{Q^c}$, denoted by $< q, q' >$, is the number $\sum_l q_l q_l'$, and the norm of q associated with this inner product is $|q| = (\sum_l |q_l|^2)^{\frac{1}{2}}$. We have $|qq'| = |q||q'|$. The angle between q and q' in $\mathbf{Q}$, denoted by $\arg(q, q')$, is defined to be $\arccos \frac{<q,q'>}{|q||q'|}$, where the inverse function arccos takes values in $[0, \pi)$. By *the unit sphere of quaternions* we mean the set $\{q \in \mathbf{Q} : |q| = 1\}$, denoted by $\mathbf{S}$. We will use the terminology *the real axis* in both the complex and the quaternionic spaces with the obvious meanings.

Denote, by

$$D = \frac{1}{2}\sum_{l=0}^{3} \frac{\partial}{\partial q_l}\mathbf{i}_l,$$

the *Dirac* operator. Functions to be studied in this paper will be $\mathbf{Q^c}$-valued, but defined in sets of $\mathbf{Q}$. We will assume, whenever they are involved in the context, the existence of the partial derivatives defined in the same region in which the function itself is defined. The operator D can be applied to such a function $f = f_0\mathbf{i}_0 + f_1\mathbf{i}_1 + f_1\mathbf{i}_2 + f_3\mathbf{i}_3$ from the left- and the right-hand side in the following manners:

$$Df(q) = \frac{1}{2}\sum_l \sum_k \frac{\partial f_k}{\partial q_l}\mathbf{i}_l\mathbf{i}_k, \qquad fD(x) = \frac{1}{2}\sum_l \sum_k \frac{\partial f_k}{\partial q_l}\mathbf{i}_k\mathbf{i}_l,$$

respectively.

If $Df = 0$ or $fD = 0$, then f is said to be a *left-regular* or a *right-regular* function, respectively. A function which is both left- and right- regular is called a *regular* function. For left- and right-regular functions the following versions of Cauchy-Fueter's theorem and Cauchy-Fueter's formula hold ([Su] or [DSS]):

Assume that Ω is a bounded open domain with a Lipschitz boundary, and f, g are respectively left- and right-regular functions defined in an open neighborhood of the closure $\Omega \cup \partial\Omega$. Then

$$\int_{\partial\Omega} g(q)n(q)f(q)d\sigma(q) = 0,$$

where $d\sigma$ is the surface area measure and $n(q)$ the outward pointing unit normal to $\partial\Omega$ at $q \in \partial\Omega$.

Under the above assumption, if $q \in \Omega$, then

$$f(q) = \frac{1}{2\pi^2} \int_{\partial\Omega} E(q' - q)n(q')f(q')d\sigma(q'),$$

where $E(q) = \frac{\overline{q}}{|q|^4}$ is the Cauchy-Fueter kernel.

The relation $E(qq') = E(q')E(q)$ will be frequently used.

We will also use the operators

$$\overline{D} = \frac{1}{2}\left(\frac{\partial}{\partial q_0} - \sum_{l=1}^{3} \frac{\partial}{\partial q_l}\mathbf{i}_l\right),$$

$$\underline{D} = \frac{1}{2}\sum_{l=1}^{3} \frac{\partial}{\partial q_l}\mathbf{i}_l \qquad \text{and} \qquad \Delta = 4D\overline{D} = \sum_{l=0}^{3} \frac{\partial^2}{\partial q_l^2}.$$

We note that for any regular function f we have

$$\overline{D}f = \frac{\partial}{\partial q_0} f = -\underline{D}f. \tag{1}$$

1 Laurent series of Kernel functions

Denote by I the Kelvin inversion defined by

$$I(f)(q) = E(q)f(q^{-1}).$$

It is obvious that I^2=identity. We recall that I maps a left-regular function inside (outside) the unit sphere to a left-regular function outside (inside) the unit sphere; and a right-regular function inside (outside) the unit sphere to a right-regular function outside (inside) the unit sphere ([DSS]). This is a special case of Bojarski's result, for a proof of which we refer the reader to [PQ].

Define, for $k \in \mathbf{Z}_+$, the set of the positive integers, $P^{(-k)}(q) = \frac{(-1)^{k-1}}{(k-1)!}\overline{D}^{k-1} \cdot E(q)$; and $P^{(k-1)} = I(P^{(-k)})$. Since E is regular, owing to the relations indicated in (1), $P^{(-k)}$ has alternative expressions: $P^{(-k)}(q) = \frac{(-1)^{k-1}}{(k-1)!}(\frac{\partial}{\partial q_0})^{k-1}E(q) = \frac{1}{(k-1)!}\underline{D}^{k-1}E(q)$.

Lemma 1. *For $k \in \mathbf{Z}_+$, $P^{(-k)}$ is regular away from the origin and homogeneous of degree $-2-k$; $P^{(k-1)}$ is a polynomial of $q_l, l = 0,1,2,3$, regular and homogeneous of degree $k-1$. Moreover,*

$$P^{(k-1)}(q) = \overline{q}|q|^{2k}P^{(-k)}(\overline{q}). \tag{2}$$

Proof. The regularity of $P^{(-k)}$ follows from the regularity of the Cauchy kernel E and the commutativity of $\frac{\partial}{\partial q_0}$ with the operator D. The homogeneity of $P^{(-k)}$ is from taking derivatives to the homogeneous function E. The regularity and the homogeneity of $P^{(k-1)}$ come from the corresponding properties of $P^{(-k)}$ and the property of the Kelvin inversion. Since $\overline{D}^{k-1}E$ is homogeneous of degree $-2-k$, it follows that $(\overline{D}^{k-1}E)(\frac{\overline{q}}{|q|^2}) = |q|^{4+2k}\frac{(k-1)!}{(-1)^{k-1}}P^{(-k)}(\overline{q})$. Multiplying $E(q)\frac{(-1)^{k-1}}{(k-1)!}$ to the both sides, we obtain (2). The fact that $P^{(k-1)}$ is a polynomial is a consequence of the last equality in Remark 2.

We will use a Fueter's result ([Su], [De]) which says that if f^0 is holomorphic, defined in an open set O in the upper half plane of $\mathbf{C}$, $f^0(z) = u(x,y)+\mathbf{i}v(x,y), z = x+\mathbf{i}y$, where u and v are real-valued functions, then $\Delta\mathbf{f^0}$ is a regular function in the open set $\mathbf{O} = \{q = q_0 + \underline{q} \in \mathbf{Q} : (q_0, |\underline{q}|) \in O\}$, where

$$\mathbf{f^0}(q) = u(q_0, |\underline{q}|) + \mathbf{e}v(q_0, |\underline{q}|),$$

and $\mathbf{e} = \frac{\underline{q}}{|\underline{q}|}$. We will call $\mathbf{f^0}$ the *induced function* from f^0, denoted by $\mathbf{f}^0 = \vec{f}^0$, and $\mathbf{O}$ the *induced set* from O, denoted by $\mathbf{O} = \vec{O}$. In the sequel we will use the notation $\mathbf{A} = \{q = q_0 + \underline{q} \in \mathbf{Q} : (q_0, |\underline{q}|) \in A\}$ for any set A in the complex plane, denoted by $\mathbf{A} = \vec{A}$, regardless whether or not it is open or in the upper half plane. We will frequently use the relations

$$\overline{D}\Delta\mathbf{f^0} = \Delta(\overrightarrow{\frac{df^0}{dz}}) \tag{3}$$

and

$$\Delta f = \frac{2}{|\underline{q}|}\frac{\partial u}{\partial y}(q_0, |\underline{q}|) + 2\mathbf{e}(\frac{1}{|\underline{q}|}\frac{\partial v}{\partial y}(q_0, |\underline{q}|) - \frac{1}{|\underline{q}|^2}v(q_0, |\underline{q}|)). \tag{4}$$

We refer the reader to [De] for proofs of (3), (4) and the two-sided regularity of $\Delta\mathbf{f^0}$.

Fueter's method is naturally related to the concept *intrinsic functions* of a complex variable and that of a quaternionic variable. Rinehart ([Ri]) introduced and motivated the study of the class of intrinsic functions on a linear associative algebra, say $\mathscr{U}$, with identity, over a field $\mathscr{F}$. Let G be the group of all automorphisms and antiautomorphisms of $\mathscr{U}$ that leaves $\mathscr{F}$ element-wise invariant.

Definition 1. *A subset $\mathscr{D}$ of $\mathscr{U}$ is called an intrinsic set of $\mathscr{U}$ if $\Omega\mathscr{D} = \mathscr{D}$ for every Ω in G.*

Definition 2. *The single-valued function F, with domain $\mathscr{D}$ and range in $\mathscr{U}$, is said to be an intrinsic function if $\mathscr{D}$ is an intrinsic set of $\mathscr{U}$ and if $Z \in \mathscr{D}$ implies $F(\Omega Z) = \Omega F(Z)$ for all Ω in G.*

It is well known that in the complex field the only nonidentical automorphism or antiautomorphism is the complex conjugate mapping. Accordingly, the intrinsic sets are those which are symmetric with respect to the real axis and the intrinsic functions on $\mathbf{C}$ are those f^0 satisfying $\overline{f^0(z)} = f^0(\overline{z})$ ([Ri],[Tu]). If $f^0 = u + \mathbf{i}v$, where u, v are real-valued, then the above equality is equivalent to $u(x,-y) = u(x,y), v(x,-y) = -v(x,y)$. In particular, $v(x,0) = 0$, i.e. f^0 is real-valued if restricted on the real line in its domain.

It is also well known that in $\mathbf{Q}$ the group G of the automorphisms and antiautomorphisms consists of all the linear transformations which leave 1 fixed and effect orthogonal transformations on the vector space $\mathscr{O}$. For each unit element $\mathbf{e}$ in $\mathscr{O}$ the linear span of 1 and $\mathbf{e}$ over $\mathbf{R}$ is called the *complex plane in* $\mathbf{Q}$ *induced by* $\mathbf{e}$, denoted by $\mathbf{C^e}$. The intrinsic sets in $\mathbf{Q}$ are those which are symmetric with respect to the real axis in every induced complex plane $\mathbf{C^e}$ in $\mathbf{Q}$. It is proved that if f^0 is an intrinsic function defined in an intrinsic set O of $\mathbf{C}$, then the induced function $\mathbf{f^0}$ is an intrinsic function defined in the intrinsic set $\mathbf{O}$; conversely, all intrinsic functions on $\mathbf{Q}$ are formed in this way ([Ri]).

In this case $\mathbf{f^0}$ is identical with the *primary function* of f^0, and f^0 the *stem function* of $\mathbf{f^0}$, the terminology related to the Hermite interpolation extending functions defined in $\mathbf{C}$ to functions defined in finite dimensional associative algebras (see e.g. [Ri]). In the sequel we will be using the terminology "stem" functions only for intrinsic holomorphic functions defined in open, simply-connected intrinsic sets in the complex plane. In this language Fueter's result says that the Laplacian of the primary function of a stem function is regular.

Denote by τ^0 the mapping

$$\tau^0 : f^0 \to -\frac{1}{4}\Delta\mathbf{f^0}.$$

It is noted that τ^0 is linear with respect to addition and real-scalar multiplication. As shown in [Su],

$$E(q) = \tau^0((\cdot)^{-1})(q),$$

which is

$$P^{(-1)}(q) = \tau^0((\cdot)^{-1})(q)$$

in our notation. In general, we have

Lemma 2. $P^{(-k)} = \tau^0((\cdot)^{-k}), k \in \mathbf{Z}_+$.

Proof. Denote $\psi^0(z) = \frac{1}{z^k}$. Owing to the relations $\psi^0(z) = \frac{(-1)^{k-1}}{(k-1)!}(\frac{d}{dz})^{k-1}(\frac{1}{z})$ and $\bar{D}\tau^0(f^0) = \tau^0(\frac{df^0}{dz})$, which is from (3), we have

$$\tau^0(\psi^0) = \frac{(-1)^{k-1}}{(k-1)!}\bar{D}^{k-1}\tau^0(\frac{1}{(\cdot)}) = \frac{(-1)^{k-1}}{(k-1)!}\bar{D}^{k-1}E = P^{(-k)}.$$

From $\tau^0(\frac{1}{(\cdot)}) = E(\cdot)$ we have $\tau^0(\frac{1}{1-\cdot}) = E(1-\cdot)$. Thus the mapping τ^0 formally maps the series

$$\frac{1}{1-z} = -\frac{1}{z} - \frac{1}{z^2} - \cdots - \frac{1}{z^{k+1}} - \cdots, \qquad |z| > 1,$$

term by term, to the series

$$E(1-q) = \sum_{k=1}^{\infty} -P^{(-k)}(q), \qquad |q| > 1. \tag{5}$$

Applying the Kelvin inversion to both sides of (5), since $E(q)E(1-q^{-1}) = -E(1-q)$, we obtain, formally,

$$E(1-q) = \sum_{k=0}^{\infty} P^{(k)}(q), \qquad |q| < 1, \tag{6}$$

corresponding to

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^k + \cdots, \qquad |z| < 1.$$

The actual equality of (5) and (6) is justified by the estimates

$$|P^{(-k)}(q)| \leq C(1+k^3)|q|^{-2-k}, \qquad |q| > 1, \tag{7}$$

and

$$|P^{(k)}(q)| \leq C(1+k^3)|q|^{k}, \qquad |q| < 1, \tag{8}$$

deduced from the estimates (9) on page 431 and (2) on page 429 of [So]. (5) and (6) can also be deduced from Laurent series theory of monogenic functions (see Ch.II of [DSS]).

The following relation is anticipated which could be used to alternatively define $P^{(k)}$ for $k = 0, 1, 2, \ldots$

Lemma 3. *$I(\tau^0((\cdot)^{-k})) = \tau^0((\cdot)^{k+1})$, and thus $P^{(k-1)} = \tau^0((\cdot)^{k+1}), k \in \mathbf{Z}_+$.*

The proof is crumble-some for which we refer the interested reader to [Q4].

In the sequel, a series of the form $\sum_{k=0}^{\infty} c_k z^k$, or $\sum_{k=-\infty}^{-1} c_k z^k$, or $\sum_{k=-\infty}^{\infty} c_k z^k$ is called a *Taylor series* (*power series*), or a *principal series*, or *Laurent* series. Series $\sum c_k P^{(k)}$ and $\sum c_k z^k$ will be said to be *associated* to each other. Owing to the observation made in the last paragraph, τ^0 maps $\sum_{k=-\infty}^{-1} c_k z^k$ to its associated series; but does not maps $\sum_{k=1}^{\infty} c_k z^k$ to its associated series.

Remark 1. *It is easy to prove, using more direct methods than those in the proofs of (5) and (6), that in general the convergence radii of a pair of associated series are the same: To get the same radius of convergence after taking partial derivatives on induced functions, one can use the same methods as in the real case, i.e. extend into several complex variables and then use the multiple Cauchy integral formula over polydiscs ([Ry1]). Some results concerning radii of convergence along some of these lines can be found in [Ry2].*

In this paper the *domain* of a power series $\phi^0(z) = \sum_{k=0}^{\infty} c_k z^k$ will be meant to be the largest simply-connected region which the power function, originally defined in its convergence disc, can be holomorphically extended to. The same convention applies to principal series. The domain of $\phi^0(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ is defined to be the intersection set of the domains of $\phi^{0,+}$ and $\phi^{0,-}$, where $\phi^{0,+}(z) = \sum_{k=0}^{\infty} c_k z^k$, and $\phi^{0,-}(z) = \sum_{k=-\infty}^{-1} c_k z^k$, and $\phi^0(z) = \phi^{0,+} + \phi^{0,-}$ in the intersection set. Using this convention, for instance, the series $\sum_{k=1}^{\infty} z^k + \sum_{-\infty}^{-1} -z^k = \frac{1+z}{1-z}$, a function holomorphic in $\mathbf{C} \setminus \{1\}$. The convention also applies to series $\sum c_k P^{(k)}$, but using "regular" in place of "holomorphic". An example is

$$
\begin{aligned}
\sum_{k=1}^{\infty} P^{(k)}(q) + \sum_{-\infty}^{-1} -P^{(k)}(q) &= \tau^0(\sum_{k=1}^{\infty} z^k + \sum_{-\infty}^{-1} -z^k) \\
&= \tau^0(\frac{1+z}{1-z}) \\
&= \tau^0(-1 + \frac{2}{1-z}) \\
&= \tau^0(\frac{2}{1-z}) = 2E(1-q),
\end{aligned}
$$

a function regularly defined everywhere except $q = 1$. Be notice that if $c_k \in \mathbf{R}$ for all k, then $\sum c_k z^k$ is an intrinsic function defined in an intrinsic set O, and $\sum c_k P^{(k)}$ is defined in the intrinsic set $\mathbf{O}$.

Remark 2. *The relation between $P^{(k)}$ and the entries of general Laurent series of regular functions may be easily established. The following setting is standard (see [Su], [DSS]). Let $U_k, k = \ldots -5, -4, -3, 0, 1, 2, 3, \ldots,$ be the set of the functions $f : \mathbf{Q} \setminus \{0\} \to \mathbf{Q^c}$ which are regular and homogeneous of degree k (note that U_{-2} and U_{-1} are empty sets). Let α be an unordered set of n integers $\{i_1, \ldots, i_n\}$ with $1 \le i_k \le 3$; α can also be specified by three integers n_1, n_2, n_3 with $n_1+n_2+n_3 = n$, where n_1 is the number of 1's in α, n_2 the number of 2's and n_3 the number of 3's. There are $\frac{1}{2}(n+1)(n+2)$ such sets α and the set being consisted of which is denoted by σ_n. When $n = 0$, let $\sigma_0 = \{\emptyset\}$. We write ∂_α for the nth order differential operator*

$$\partial_\alpha = \frac{\partial^n}{\partial q_{i_1} \cdots \partial q_{i_n}} = \frac{\partial^n}{\partial q_1^{n_1} \partial q_2^{n_2} \partial q_3^{n_3}}.$$

Set

$$E_\alpha = \partial_\alpha E$$

and

$$P_\alpha(q) = \frac{1}{n!} \sum (q_0 \mathbf{i}_{i_1} - q_{i_1}) \cdots (q_0 \mathbf{i}_{i_n} - q_{i_n}),$$

where the sum is over all $n!/(n_1!n_2!n_3!)$ different orderings of $n_1$1's, $n_2$2's and $n_3$3's. It is proved that $\{E_\alpha : \alpha \in \sigma_n\}$ forms a basis of U_{-3-n} and $\{P_\alpha : \alpha \in \sigma_n\}$ forms a basis of U_n and

$$
\begin{aligned}
E(p-q) &= \sum_{n=0}^{\infty} \sum_{\alpha \in \sigma_n} P_\alpha(q) E_\alpha(p) \\
&= \sum_{n=0}^{\infty} \sum_{\alpha \in \sigma_n} E_\alpha(p) P_\alpha(q), \qquad |q| < |p|. \qquad (9)
\end{aligned}
$$

Let $q = 1, |p| > 1$, we obtain, for $k \ge 0$,

$$P^{(-k-1)} = \sum_{\alpha \in \sigma_k} P_\alpha(1) E_\alpha = \frac{1}{k!} D^k E \in U_{-3-k}.$$

Let $p = 1, |q| < 1$, we have

$$P^{(k)} = \sum_{\alpha\in\sigma_k} E_\alpha(1)P_\alpha \in U_k.$$

We will come back with more relations between P_α, E_α and $P^{(k)}$ in Sect. 2 ((16) and (17) in the proof of Proposition 4).

We will be using the following sets in the complex plane. Set, for $\omega \in (0, \frac{\pi}{2})$,

$$\mathbf{S}^{\mathbf{c}}_{\omega,\pm} = \{z \in \mathbf{C} : |\arg(\pm z)| < \omega\},$$

where the angle $\arg(z)$ of the complex number z takes values in $(-\pi, \pi]$,

$$\mathbf{S}^{\mathbf{c}}_{\omega,\pm}(\pi) = \{z \in \mathbf{C} : |\mathrm{Re}(z)| \le \pi, z \in \mathbf{S}^{\mathbf{c}}_{\omega,\pm}\},$$
$$\mathbf{S}^{\mathbf{c}}_{\omega} = \mathbf{S}^{\mathbf{c}}_{\omega,+} \cup \mathbf{S}^{\mathbf{c}}_{\omega,-},$$
$$\mathbf{S}^{\mathbf{c}}_{\omega}(\pi) = \mathbf{S}^{\mathbf{c}}_{\omega,+}(\pi) \cup \mathbf{S}^{\mathbf{c}}_{\omega,-}(\pi),$$
$$\mathbf{W}^{\mathbf{c}}_{\omega,\pm}(\pi) = \{z \in \mathbf{C} : |\mathrm{Re}(z)| \le \pi \text{ and } \mathrm{Im}(\pm z) > 0\} \cup \mathbf{S}^{\mathbf{c}}_{\omega}(\pi),$$
$$\mathbf{H}^{\mathbf{c}}_{\omega,\pm} = \{z = \exp(i\eta) \in \mathbf{C} : \eta \in \mathbf{W}^{\mathbf{c}}_{\omega,\pm}(\pi)\},$$

and

$$\mathbf{H}^{\mathbf{c}}_{\omega} = \mathbf{H}^{\mathbf{c}}_{\omega,+} \cap \mathbf{H}^{\mathbf{c}}_{\omega,-}.$$

These sets are illustrated in the diagram below. $\mathbf{W}^{\mathbf{c}}_{\omega,+}(\pi)$ and $\mathbf{W}^{\mathbf{c}}_{\omega,-}(\pi)$ are “W”- and “M”-shaped regions, respectively. $\mathbf{H}^{\mathbf{c}}_{\omega,+}$ is a heart-shaped region, and the complement of $\mathbf{H}^{\mathbf{c}}_{\omega,-}$ is a heart-shaped region. With the obvious meaning we sometimes write $\mathbf{H}^{\mathbf{c}}_{\omega,\pm} = e^{i\mathbf{W}^{\mathbf{c}}_{\omega,\pm}(\pi)}$, etc.

Remark 3. *The above introduced sets naturally arise from our integral operator theory. Star-shaped Lipschitz curves in the complex plane have the parameterization $\gamma = \gamma(\theta) = \exp \mathbf{i}(\theta + \mathbf{i}A(\theta))$, where A is a 2π-periodic Lipschitz function. Let the Lipschitz constant $\|A'\|_\infty = \tan(\omega_0), \omega_0 \in (0, \frac{\pi}{2})$. The integrals under study are convolutions using the multiplicative structure of the complex field and of the form $p.v. \int_\gamma \phi^0(z\eta^{-1})f(\eta)\frac{d\eta}{\eta}, z \in \gamma$, where ϕ^0 is a kernel function. A simple computation shows that the condition $z, \eta \in \gamma$ implies $z\eta^{-1} \in \mathbf{H}^{\mathbf{c}}_{\omega}, \omega \in (\omega_0, \frac{\pi}{2})$, and thus the domains of the kernel functions need to contain the sets $\mathbf{H}^{\mathbf{c}}_{\omega}$ (also see the explanation made before Lemma 4).*

The following function spaces are used in the theory:

$$K(\mathbf{H}^{\mathbf{c}}_{\omega,\pm}) = \{\phi^0 : \mathbf{H}^{\mathbf{c}}_{\omega,\pm} \to \mathbf{C} \ : \phi^0 \text{ is holomorphic and satisfies } |\phi^0(z)| \le \frac{C_\nu}{|1-z|}$$
$$\text{in every } \ \mathbf{H}^{\mathbf{c}}_{\nu,\pm}, 0 < \nu < \omega\}, \tag{10}$$
$$K(\mathbf{H}^{\mathbf{c}}_{\omega}) = \{\phi^0 : \mathbf{H}^{\mathbf{c}}_{\omega} \to \mathbf{C} : \phi^0 = \phi^{0,+} + \phi^{0,-}, \phi^{0,\pm} \in K(\mathbf{H}^{\mathbf{c}}_{\omega,\pm})\},$$
$$H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega,\pm}) \quad = \{b : \mathbf{S}^{\mathbf{c}}_{\omega,\pm} \to \mathbf{C} : b \text{ is holomorphic and satisfies}$$
$$|b(z)| \le C_\nu \text{ in every } \mathbf{S}^{\mathbf{c}}_{\nu,\pm}, 0 < \nu < \omega\},$$

and

$$H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega}) = \{b : \mathbf{S}^{\mathbf{c}}_{\omega} \to \mathbf{C} : b_\pm = b\chi_{\{z\in\mathbf{C}:\pm Rez>0\}} \in H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})\}.$$

The study of the paper will be based on the main results of [Q1] that will now be recalled for the reader’s convenience. The results address the relation between the classes $K(\mathbf{H}^{\mathbf{c}}_{\omega,\pm})$ and $H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$.

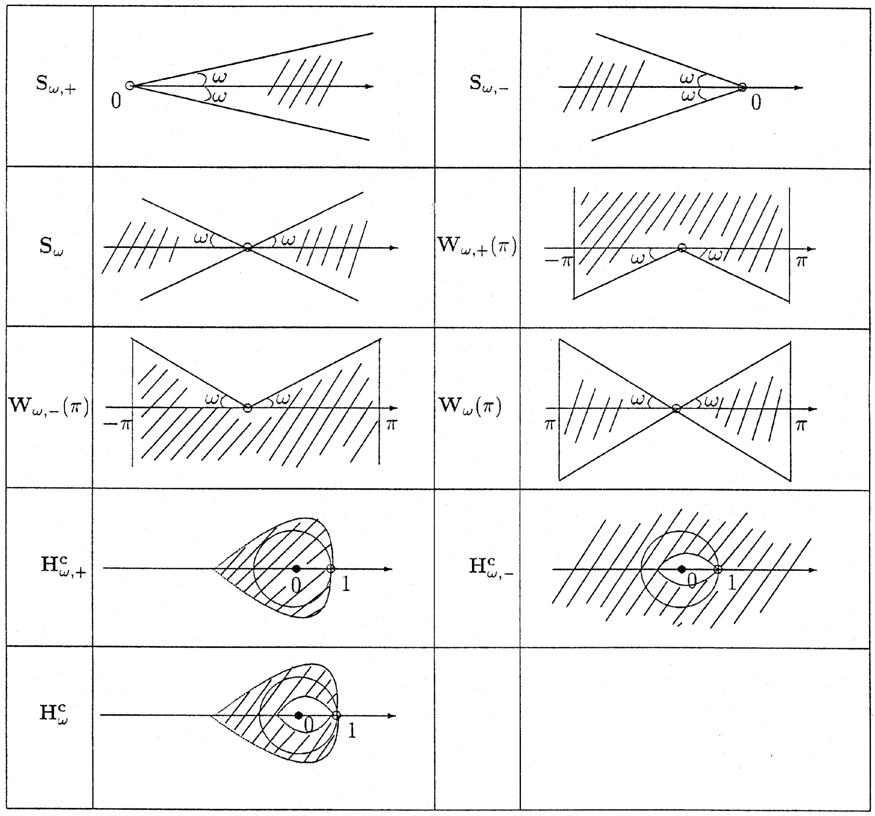

Fig. 1.

Theorem A. *If b belongs to $H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$ or $H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega})$, then $\phi^0(z) = \sum_{n=\pm 1}^{\pm\infty} b(n)z^n$ or $\phi^0(z) = {\sum_{n=-\infty}^{\infty}}' b(n)z^n$ belongs to $K(\mathbf{H}^{\mathbf{c}}_{\omega,\pm})$ or $K(\mathbf{H}^{\mathbf{c}}_{\omega})$, respectively.*

Theorem B. *If $\phi^0 \in K(\mathbf{H}^{\mathbf{c}}_{\omega,\pm})$, then for every $\nu \in (0,\omega)$, there exists a function $b^{\nu} \in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\nu,\pm})$ such that $\phi^0 = \sum_{n=\pm 1}^{\pm\infty} b^{\nu}(n)z^n$. Moreover,*

$$b^{\nu}(z) = \lim_{\delta\to 0+} \frac{1}{2\pi} \int_{l^{\pm}(\nu)} \exp(-\mathbf{i}\eta z)\phi^0(\exp(\mathbf{i}(\eta \pm \delta\mathbf{i})))d\eta,$$

where the path $l^{\pm}(\nu) = \{z \in \mathbf{C} : z = r\exp(\mathbf{i}(\pi \pm \nu)), r$ is from $\pi\sec(\nu)$to 0; and then $z = r\exp(-(\pm\mathbf{i}\nu)), r$ is from 0 to $\pi\sec(\nu)\}$.

If $\phi^0 \in K(\mathbf{H}^{\mathbf{c}}_{\omega})$, then its Taylor series part $\phi^{0,+}$ belongs to $K(\mathbf{H}^{\mathbf{c}}_{\omega,+})$ and its principal series part $\phi^{0,-}$ belongs to $K(\mathbf{H}^{\mathbf{c}}_{\omega,-})$, respectively. By invoking Theorem B we have two functions $b^{+,\nu}$ and $b^{-,\nu}, 0 < \nu < \omega$, associated with $\phi^{0,+}$ and $\phi^{0,-}$, respectively. Adding up, we obtain the correspondence $\phi^0 \to b^{\nu} = b^{+,\nu} + b^{-,\nu} \in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\nu})$.

In $\mathbf{Q}$ we will be working on the heart-shaped regions

$$\mathbf{H}_{\omega,\pm} = \{q \in \mathbf{Q} : \frac{-(\pm \ln|q|)}{\arg(q,1)} > -\tan(\omega)\}$$

and

$$\mathbf{H}_{\omega} = \mathbf{H}_{\omega,+} \cap \mathbf{H}_{\omega,-}.$$

That is

$$\mathbf{H}_{\omega} = \{q \in \mathbf{Q} : \frac{|\ln|q||}{\arg(q,1)} < \tan(\omega)\}.$$

The reason of using these sets is the same as what is described in Remark 3 in relation to convolution integrals using the multiplicative structure of the underlying space. Precisely, we will be working on convolution singular integrals on star-shaped Lipschitz surfaces and the kernel functions ought to be defined in $\mathbf{H}_{\omega}$. The following observation for the complex plane case has motivated the definition of $\mathbf{H}_{\omega}$: Let $A = A(x)$ be a 2π-periodic Lipschitz curve whose Lipschitz constant is less than $\tan(\omega)$, then for $z = \exp i(x + \mathbf{i}A(x)), \eta = \exp \mathbf{i}(y + \mathbf{i}A(y))$, we have $z\eta^{-1} = \exp \mathbf{i}((x-y) + \mathbf{i}(A(x) - A(y)))$. This implies that $\frac{|\ln|z\eta^{-1}||}{|x-y|} = \frac{|A(x)-A(y)|}{|x-y|} < \tan(\omega)$.

Denote by $\mathbf{H}^{\mathbf{e}}_{\omega,\pm}$ and $\mathbf{H}^{\mathbf{e}}_{\omega}$ the images on $\mathbf{C}^{\mathbf{e}} \subset \mathbf{Q}$ of the sets $\mathbf{H}^{\mathbf{c}}_{\omega,\pm}$ and $\mathbf{H}^{\mathbf{c}}_{\omega}$ in $\mathbf{C}$, respectively, under the mapping $i_{\mathbf{e}} : a + b\mathbf{i} \to a + b\mathbf{e}$. We have

Lemma 4.

$$\mathbf{H}_{\omega,\pm} = \cup_{\mathbf{e}\in J} \mathbf{H}^{\mathbf{e}}_{\omega,\pm}$$

and

$$\mathbf{H}_{\omega} = \cup_{\mathbf{e}\in J} \mathbf{H}^{\mathbf{e}}_{\omega},$$

where the index set J is the set of all the unit vectors in $\mathscr{O}$.

Proof. We will only prove the lemma for the case "+." The remaining cases can be dealt with similarly. Let $q \in \mathbf{H}_{\omega,+}$, then $q \in \mathbf{C}^{\mathbf{e}}$, with $\mathbf{e} = \frac{\underline{q}}{|\underline{q}|}$. Denoting $\frac{q_0}{|q|} = \cos\theta$, we have, in the complex plane $\mathbf{C}^{\mathbf{e}}$,

$$q = |q|(\frac{q_0}{|q|} + \frac{|\underline{q}|}{|q|}\mathbf{e}) = |q|e^{\mathbf{e}\theta} = e^{\mathbf{e}(\theta+\mathbf{e}\rho)},$$

where $\rho = -\ln|q|$. The condition $\frac{-\ln|q|}{\theta} > -\tan(\omega)$ thus becomes $\frac{\rho}{\theta} > -\tan(\omega)$, a condition characterising a points $\theta + \mathbf{e}\rho \in \mathbf{W}^{\mathbf{e}}_{\omega,+}(\pi)$ by which we denote the image on $\mathbf{C}^{\mathbf{e}}$ of the set $\mathbf{W}^{\mathbf{c}}_{\omega,+}(\pi)$ under the mapping $i_{\mathbf{e}}$. Since $\mathbf{H}^{\mathbf{e}}_{\omega,+} = e^{i\mathbf{W}^{\mathbf{e}}_{\omega,+}(\pi)}$, we thus have $q \in \mathbf{H}^{\mathbf{e}}_{\omega,+}$, and so

$$\mathbf{H}_{\omega,\pm} \subset \cup_{\mathbf{e}} \mathbf{H}^{\mathbf{e}}_{\omega,\pm}.$$

Since the argument is reversible, the proof is complete.

Set

$$K(\mathbf{H}_{\omega,\pm}) = \{ \phi : \mathbf{H}_{\omega,\pm} \to \mathbf{Q}^{\mathbf{c}} : \phi = \sum_{i=\pm 1}^{\pm\infty} c_i P^{(i)}, c_i \in \mathbf{C}, \text{is regular and satisfies } |\phi(q)| \le C_\nu \frac{1}{|1-q|^3} \text{ in every } \mathbf{H}_{\nu,\pm}, 0 < \nu < \omega\} \tag{11}$$

and

$$K(\mathbf{H}_{\omega}) = \{\phi : \mathbf{H}_{\omega} \to \mathbf{Q}^{\mathbf{c}} : \phi = \phi^{+} + \phi^{-}, \phi^{\pm} \in K(\mathbf{H}_{\omega,\pm})\}.$$

Theorem 1. *If* $b \in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$ *and* $\phi(q) = \sum_{k=\pm 1}^{\pm\infty} b(k)P^{(k)}(q)$, *then* $\phi \in K(\mathbf{H}_{\omega,\pm})$.

Proof. We will first prove the theorem for b in $H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$, where

$$H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm}) = \{b \in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm}) : b|_{\mathbf{R}\cap\mathbf{S}^{\mathbf{c}}_{\omega,\pm}} \text{ is real valued}\},$$

and then prove the theorem for the general case $b \in H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$.

We will first consider the case "-". For $b \in H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,-})$, let $\phi^0(z) = \sum_{k=-1}^{-\infty} b(k)z^k$. Theorem A asserts that ϕ^0 is in $K(\mathbf{H}^{\mathbf{c}}_{\omega,-})$. Since $b(k) \in \mathbf{R}$, ϕ^0 is a stem function defined in the intrinsic set $\mathbf{H}^{\mathbf{c}}_{\omega,-}$. The associated $\phi(q)$ therefore is regularly defined in $\mathbf{H}_{\omega,-}$ away from the q_0-axis. It is easy to show, using Morera's theorem for functions of a quaternionic variable, that the associated function ϕ is regularly defined across the q_0-axis in $\mathbf{H}_{\omega,-}$.

Now we prove the estimate for ϕ. What is interested here is the behavior of ϕ at $q \approx 1$, its sole singular point. This implies that $|\underline{q}| \approx 0$.

Writing $\phi^0 = u + \mathbf{i}v$ as usual and using the relation (4), we have

$$\begin{aligned}\tau^0(\phi^0)(q) &= \tfrac{2}{|\underline{q}|}\tfrac{\partial u}{\partial y}(q_0,|\underline{q}|) + 2\frac{\underline{q}}{|\underline{q}|}\big(\tfrac{1}{|\underline{q}|}\tfrac{\partial v}{\partial y}(q_0,|\underline{q}|) - \tfrac{1}{|\underline{q}|^2}v(q_0,|\underline{q}|)\big)\\ &= I_1 + \frac{\underline{q}}{|\underline{q}|}I_2. \end{aligned} \tag{12}$$

In order to estimate I_1 and I_2 we will need the following lemma.

Lemma 6. *For* $\phi^0 \in K(\mathbf{H}^{\mathbf{c}}_{\omega,\pm})$, *we have, for any* $0 < \nu < \omega$,

$$|\phi^{0(n)}(z)| \le \frac{2n!C_\nu}{\delta^n(\nu)}\frac{1}{|1-z|^{1+n}}, \quad z \in \mathbf{H}^{\mathbf{c}}_{\nu,\pm},$$

where C_ν *is the constants in (11) and* $\delta(\nu) = \min\{\frac{1}{2}, \tan(\omega-\nu)\}$.

Proof. We first notice that at the local $z \approx 1$ the set $\mathbf{H}^{\mathbf{c}}_{\nu,\pm}$ can be approximated by the cone of the angle $\pi \pm 2\nu$ and vertex $(1,0)$ pointing to the $\pm$ direction of the x-axis. This claim can be justified from the relation $e^{\eta} - 1 \approx \eta, 0 \approx \eta \in \mathbf{C}$. Then for any point $1 \approx z \in \mathbf{H}^{\mathbf{c}}_{\nu,\pm}$ the disc $S_r(z)$ of radius $r = \delta(\nu)|1-z|$ centered at z is contained in $\mathbf{H}^{\mathbf{c}}_{\nu,\pm}$. Using Cauchy's formula, we have

$$\phi^{0(n)}(z) = \frac{n!}{2\pi i}\int_{S_r(z)} \frac{\phi^0(\eta)}{(\eta - z)^{1+n}}d\eta.$$

Therefore,

$$|\phi^{0(n)}(z)| \le \frac{n!}{2\pi}\int_0^{2\pi} \frac{C_\nu}{|1-\eta|}\frac{1}{r^n}d\theta \le \frac{2n!C_\nu}{\delta^n(\nu)}\frac{1}{|1-z|^{1+n}},$$

where we have used the relation $|1-\eta| \ge |1-z| - |z-\eta| = |1-z| - r \ge |1-z| - \frac{1}{2}|1-z| = \frac{1}{2}|1-z|$. The proof is complete.

Continuing the proof of the theorem, we first consider the case $|\underline{q}| > \frac{\delta(\nu)}{4}|1-q|$. Using the estimates in Lemma 6 for $n = 1$, from the definitions of I_1, I_2 in (12), we have

$$|I_1| + |I_2| \leq \frac{C_\nu}{|1-q|^3}.$$

Since the above case covers all the points $q_0 \leq 1, q \approx 1$ in the region, we need now only consider the case $q_0 \geq 1$ and $|\underline{q}| \leq \frac{\delta(\nu)}{4}|1-q|$. First study I_1. Since $u(x,y)$ is an even function with respect to its second argument y, $\frac{\partial u}{\partial y}$ is an odd function with respect to y. For small y we have the Taylor expansion

$$\begin{aligned} 2\frac{\partial u}{\partial y}(x,y) &= \frac{\partial u}{\partial y}(x,y) - \frac{\partial u}{\partial y}(x,-y) \\ &= \sum_{n=1}^{\infty} \frac{\frac{\partial^{n+1}u}{\partial y^{n+1}}(x,-y)}{n!}(2y)^n. \end{aligned}$$

Using the estimates in Lemma 6 to the partial derivatives in the series and the relation $|\underline{q}| \leq \frac{\delta(\nu)}{4}|1-q|$, we have

$$\begin{aligned} |I_1| &\leq \frac{4C_\nu}{\delta^2(\nu)|1-q|^3}\sum_{n=1}^{\infty}(n+1)\left(\frac{2|\underline{q}|}{|1-q|\delta(\nu)}\right)^{n-1} \\ &\leq \frac{4C_\nu}{\delta^2(\nu)|1-q|^3}\sum_{n=1}^{\infty}\frac{n+1}{2^{n-1}} \\ &\leq \frac{C_\nu}{|1-q|^3}. \end{aligned}$$

The proof of the estimate for I_2 is similar and left to the interested reader (see [Q5] for a complete proof in a more general case).

Now consider the case "+". Assume $b \in H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,+})$ and $\psi(q) = \sum_{i=1}^{\infty} b(i) P^{(i)}(q)$. The Kelvin inversion then gives $I(\psi)(q) = \sum_{i=-1}^{-\infty} b'(i)P^{(i-1)}(q)$, where $b'(z) = b(-z) \in H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,-})$. Since $I(\psi) = \tau^0(\psi^0)$, where $\psi^0(z) = \sum_{i=-1}^{-\infty} b'(i) z^{i-1} = \frac{1}{z}\sum_{i=-1}^{-\infty} b'(i)z^i \in \mathbf{H}^{\mathbf{c}}_{\omega,-}$, the argument for dealing with ϕ in the above considered case, and hence the conclusions there, all apply to $I(\psi)$. Using the relation $\psi = I^2(\psi) = E(q)I(\psi)(q^{-1})$ and the relation $q \in \mathbf{H}_{\nu,+}$ if and only if $q^{-1} \in \mathbf{H}_{\nu,-}$, we have

$$|\psi(q)| = |E(q)I(\psi)(q^{-1})| \leq \frac{1}{|q|^3}C_\nu\frac{1}{|1-q^{-1}|^3} = C_\nu\frac{1}{|1-q|^3}, \quad q \in \mathbf{H}_{\nu,+}.$$

The proof for the case $b \in H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$ is thus complete.

Thanks to the following observation which enables us to extend the result for $b \in H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$ to functions $b \in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega})$.

Observation. *If $b \in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$, then $\overline{b}(\overline{z})$ is in the same class with the same bounds. We observe that $g(z) = \frac{1}{2}(b(z) + \overline{b}(\overline{z}))$ and $h(z) = \frac{1}{2i}(b(z) - \overline{b}(\overline{z}))$ both belong to $H^{\infty,\mathbf{r}}(\mathbf{S}^{\mathbf{c}}_{\omega,\pm})$ with the same bounds, and $b = g + ih$.*

The proof of Theorem 1 is complete.

A consequence of Theorem 1 is

Corollary 1. *Let* $b \in H^\infty(\mathbf{S}^{\mathbf{c}}_\omega)$ *and* $\phi(q) = {\sum_{i=-\infty}^{\infty}}' b(i)P^{(i)}(q)$. *Then* $\phi \in K(\mathbf{H}_\omega)$.

The converse of Theorem 1 holds. Before we state the result we need to extend the index k in the functions $P^{(k)}$ to complex numbers. First, the domain of the mapping τ^0 can be extended to holomorphic functions, not necessarily intrinsic, defined in intrinsic sets in $\mathbf{C}$, using the decomposition given in the above observation. In light of Lemma 2 and 3, we may define, for any $z \in \mathbf{S}^{\mathbf{c}}_\omega$,

$$P^{(z)} = \tau^0((\cdot)^z), \qquad z \in \mathbf{S}^{\mathbf{c}}_{\omega,-}$$

and

$$P^{(z)} = \tau^0((\cdot)^{z+2}), \qquad z \in \mathbf{S}^{\mathbf{c}}_{\omega,+},$$

where $(\cdot)^z = \exp(z \ln(\cdot))$, where in the first case the ln function is defined by cutting the positive x-axis, and in the second case defined by cutting the negative x-axis.

Theorem 2. *If* $\phi(q) = \sum_{i=\pm 1}^{\pm\infty} b_i P^{(i)}(q) \in K(\mathbf{H}_{\omega,\pm})$, *then for every* $\nu \in (0,\omega)$ *there exists a function* $b^\nu \in H^\infty(\mathbf{S}^{\mathbf{c}}_{\nu,\pm})$ *such that* $b_i = b^\nu(i), i = \pm 1, \pm 2, \ldots$ *Moreover,*

$$b^\nu(z) = \lim_{r\to 1-} \frac{1}{2\pi^2} \int_{L^\pm(\nu)} P^{(z)}(p^{-1})E(p)n(p)\phi(r^{\pm 1}p)d\sigma(p),$$

where $L^\pm(\nu) = \overrightarrow{\exp(\mathbf{i}l^\pm(\nu))}$ *and* $l^\pm(\nu)$ *is defined in Theorem B.*

The proof of Theorem 2 is similar to that of Theorem B (see [Q1] and [Q2]). The integral formula for b^ν in Theorem 2 is related to the formula given in Proposition 4 below.

Remark 4. *From the proofs of Theorems 1 and 2 we can obtain the bounds of* $\phi \in K(\mathbf{H}_{\omega,\pm})$ *and the bounds of* $b^\nu \in H^\infty(\mathbf{S}^{\mathbf{c}}_{\nu,\pm})$, *respectively. For instance, out of the proof of Theorem 1 we can further conclude that for any* $0 < \nu < \nu' < \omega$, *we have*

$$|\phi(q)| \leq C_\nu \|b\|_{L^\infty(\mathbf{S}^{\mathbf{c}}_{\nu',\pm})} \frac{1}{|1-q|^3}, \quad q \in \mathbf{S}^{\mathbf{c}}_{\nu,\pm}.$$

Remark 5. *As in the complex variable case proved in [Q2]. Theorem 1 and Theorem 2 can be extended to the cases where* b *is holomorphic, bounded near the origin and satisfies* $|b(z)| \leq C_\nu |z|^s$ *for* $|z| > 1$ *in smaller sectors* $\mathbf{S}^{\mathbf{c}}_{\nu,\pm}$, *where* s *is any real number. Details will not be included here, but the following result, with a proof similar to that of Theorem 1 (also see [Q2]), will be used in the proof of part (ii) of Theorem 4 (see Lemma 8) and Remark 9.*

Theorem 3. *Let $-\infty < s < \infty, s \neq -3, -4, \ldots$ and b a holomorphic function in $\mathbf{S}^{\mathbf{c}}_{\omega,\pm}$ satisfying the estimates*

$$|b(z)| \leq C_\nu |z \pm 1|^s, \quad \text{in every } \ \mathbf{S}^{\mathbf{c}}_{\nu,\pm}, 0 < \nu < \omega.$$

Then $\phi(q) = \sum_{i=\pm 1}^{\pm\infty} b(i) P^{(i)}(q)$ can be regularly extended to $\mathbf{H}_{\omega,\pm}$ satisfying

$$|\phi^{\pm}(q)| \leq C_\nu \| \frac{b(\cdot)}{|(\cdot) \pm 1|^s} \|_{L^\infty(\mathbf{S}^{\mathbf{c}}_{\nu'})} \frac{1}{|1-q|^{s+3}}, \quad q \in \mathbf{H}_{\nu,\pm}, 0 < \nu < \nu' < \omega.$$

Remark 6. *Theorems 1 and 3 can be interpreted as regular continuations of Taylor and principal series. For instance, for $(b_i)_{i=1}^{\infty} \in l^\infty$ the series $\phi(q) = \sum_{i=1}^{\infty} b_i P^{(i)}(q)$ is naturally defined and regular in the unit ball in $\mathbf{Q}$. Theorem 1 asserts that if there exists $b \in H^\infty(\mathbf{S}^{\mathbf{c}}_{\omega,+})$ such that $b_i = b(i)$, then ϕ can be regularly extended to $\mathbf{H}_{\omega,+}$, and actually belongs to $K(\mathbf{H}_{\omega,+})$. A similar interpretation applies to principal series.*

2 Singular integrals and Fourier multipliers

A surface Σ is said to be a star-shaped Lipschitz surface, if it is star-shaped about the origin and there exists a constant $M < \infty$ such that $q, q' \in \Sigma$ implies that

$$\frac{|\ln |q^{-1}q'||}{\arg(q, q')} \leq M.$$

The minimum value of M is called the Lipschitz constant of Σ, denoted by $N = \text{Lip}(\Sigma)$.

Since locally $\ln |q^{-1}q'| = \ln(1+(|q^{-1}q'|-1))| \approx (|q^{-1}q'|-1) \approx |q^{-1}|(|q'|-|q|) \approx (|q'| - |q|)$, the above defined sense of Lipschitz is the same as that of the usual one. According to the definition, we have that if $q, q' \in \Sigma$, then $q^{-1}q' \in \mathbf{H}_\omega$ for any $\omega \in (\arctan(N), \frac{\pi}{2})$.

From now on we will be working on a fixed star-shaped Lipschitz surface Σ and we assume that $\omega \in (\arctan(N), \frac{\pi}{2})$.

Let

$$\rho = \min\{|q| : q \in \Sigma\} \text{ and } \tau = \max\{|q| : q \in \Sigma\}.$$

We will be working on $L^2(\Sigma) = L^2(\Sigma, d\sigma)$. The norm of $f \in L^2(\Sigma)$ is denoted by $\|f\|$.

As in [CM] and [GQW], we consider the following subclass of $L^2(\Sigma)$:

$$\mathscr{A} = \{f(q) : f(q) \text{ is left} - \text{regular in } \rho - s < |q| < \tau + s \text{ for some } s > 0\}.$$

Proposition 3. *The subclass $\mathscr{A}$ is dense in $L^2(\Sigma)$.*

Proof. For $f, g \in L^2(\Sigma)$, denote by (f, g) the bilinear form

$$\int_\Sigma f \overline{g} d\sigma.$$

It is easy to verify that, for $q \in \mathbf{Q}$,

$$(f,f) = \|f\|^2, \ \overline{(f,g)} = (g,f), \ (qf,g) = q(f,g).$$

If $(f,g) = 0$, then we say that f is orthogonal to g. Assume that $\mathscr{A}$ was not dense. Since the bilinear form $(\cdot,\cdot)$ satisfies the requirements for inner products over quaternions, the basic Hilbert space methods are adaptable to this case. In particular, there would exist a function $0 \neq g \in L^2(\Sigma)$ orthogonal to all the functions in $\mathscr{A}$, and so in particular to $E(\cdot - q')$, q' being outside an annuls $\rho - s < |q| < \tau + s$.

We would have, therefore,

$$\int_\Sigma E(q - q')n(q)h(q)d\sigma(q) = 0, \tag{13}$$

where

$$h(q) = \overline{n(q)g(q)}$$

was a function in $L^2(\Sigma)$. Since the integral in (13) is absolutely convergent, it would remain valid for all $q' \notin \Sigma$, by regular continuation.

Let p be a point on Σ and $q' = rp, q^* = r^{-1}p$. We would have, as a consequence of the CMcM theorem on Lipschitz surfaces ([CMcM]) and (13),

$$0 = h(p) = \lim_{r\to 1-} \frac{1}{2\pi^2}\int_\Sigma (E(q-q') - E(q-q^*))n(q)h(q)d\sigma(q)$$

for almost all $p \in \Sigma$, and so $g(p) = 0$ for almost all $p \in \Sigma$. This is a contradiction and the proof is complete.

From [Su], we have, for $f \in \mathscr{A}$,

$$f(q) = \sum_{n=0}^{\infty}\sum_{\alpha\in\sigma_n}\{P_\alpha(q)a_\alpha + E_\alpha(q)b_\alpha\}, \quad \rho - s < |q| < \tau + s, \tag{14}$$

where

$$a_\alpha = \frac{1}{2\pi^2}\int_\Sigma E_\alpha(p)n(p)f(p)d\sigma(p)$$

and

$$b_\alpha = \frac{1}{2\pi^2}\int_\Sigma P_\alpha(p)n(p)f(p)d\sigma(p).$$

The following result is expected.

Proposition 4. *If $f \in \mathscr{A}$, then*

$$f(q) = \sum_{k=-\infty}^{\infty}\frac{1}{2\pi^2}\int_\Sigma P^{(k)}(p^{-1}q)E(p)n(p)f(p)d\sigma(p), \quad \rho - s < |q| < \tau + s. \tag{15}$$

In general, if the annuls is centered at q^0, then for a left-regular function f in the annuls, we have

$$f(q) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi^2} \int_{\Sigma_{q^0}} P^{(k)}(p^{-1}(q - q^0))E(p)n(p)f(p)d\sigma(p),$$

$$\rho - s < |q - q^0| < \tau + s,$$

where Σ_{q^0} *is a star-shaped Lipschitz surface about* q^0.

Corollary 2. *If* f *is left-regular in* $U \subset \mathbf{Q}$ *and* $q^0 \in U$, *then*

$$f(q) = \sum_{k=0}^{\infty} \frac{1}{2\pi^2} \int_{B(q^0,r)} P^{(k)}(p^{-1}(q - q^0))E(p)n(p)f(p)d\sigma(p),$$

where $B(q^0, r)$ *is the solid ball centered at* q^0 *with radius* r *such that* $B(q^0, r) \subset U$. *Moreover, the Taylor series is absolutely convergent in any ball* $B(q^0, r) \subset U$.

We will only prove the equality (15). The other conclusions will then become obvious.

Proof. We first deduce some relations between $P^{(k)}$ and P_α, E_α.

Let $|q| < |p|$. The equality (6) implies

$$E(1 - p^{-1}q)E(p) = \sum_{n=0}^{\infty} P^{(n)}(p^{-1}q)E(p).$$

On the other hand, from (15),

$$\begin{aligned} E(1 - p^{-1}q)E(p) &= E(p - q) \\ &= \sum_{n=0}^{\infty} \sum_{\alpha \in \sigma_n} P_\alpha(q)E_\alpha(p) \\ &= \sum_{n=0}^{\infty} \sum_{\alpha \in \sigma_n} E_\alpha(p)P_\alpha(q). \end{aligned}$$

Comparing the two expressions, we have

$$P^{(n)}(p^{-1}q)E(p) = \sum_{\alpha \in \sigma_n} P_\alpha(q)E_\alpha(p) = \sum_{\alpha \in \sigma_n} E_\alpha(p)P_\alpha(q). \tag{16}$$

The obtained relation (16) can be extended to any p, q such that $p^{-1}q \neq 1$ using regular continuation.

Similarly, for $|q| > |p|$, from formula (5) we have

$$E(1 - p^{-1}q)E(p) = \sum_{n=1}^{\infty} -P^{(-n)}(p^{-1}q)E(p).$$

On the other hand,

$$\begin{aligned} E(1-p^{-1}q)E(p) &= -E(q-p) \\ &= \sum_{n=0}^{\infty}\sum_{\alpha\in\sigma_n} -P_\alpha(p)E_\alpha(q) \\ &= \sum_{n=0}^{\infty}\sum_{\alpha\in\sigma_n} -E_\alpha(q)P_\alpha(p). \end{aligned}$$

Comparing the two expressions, we have, for $n = 1, 2, \ldots$,

$$P^{(-n)}(p^{-1}q)E(p) = \sum_{\alpha\in\sigma_{n-1}} P_\alpha(p)E_\alpha(q) = \sum_{\alpha\in\sigma_{n-1}} E_\alpha(q)P_\alpha(p). \tag{17}$$

The relation (17) can be extended to any p, q such that $p^{-1}q \neq 1$ by using regular continuation.

The relations (16), (17) imply that the projections of f onto the spaces $U_k, k = \ldots - 5, -4, -3, 0, 1, 2, 3, \ldots$ are given by convolution integrals using $P^{(n)}$ as kernels. In particular, from (14), we have

$$\begin{aligned} f(q) &= \sum_{n=0}^{\infty}\frac{1}{2\pi^2}\Big(\int_\Sigma \sum_{\alpha\in\sigma_n} P_\alpha(q)E_\alpha(p)n(p)f(p)d\sigma(p) + \\ &\quad + \int_\Sigma \sum_{\alpha\in\sigma_n} E_\alpha(q)P_\alpha(p)n(p)f(p)d\sigma(p)\Big) \\ &= \sum_{n=0}^{\infty}\frac{1}{2\pi^2}\int_\Sigma P^{(n)}(p^{-1}q)E(p)n(p)f(p)d\sigma(p) + \\ &\quad + \sum_{n=-1}^{-\infty}\frac{1}{2\pi^2}\int_\Sigma P^{(n)}(p^{-1}q)E(p)n(p)f(p)d\sigma(p) \end{aligned}$$

as desired. The proof is complete.

As a bi-product of the argument in deducing (16) and (17), we have, for any integer n and $p \neq q$,

$$E(p)P^{(n)}(qp^{-1}) = P^{(n)}(p^{-1}q)E(p) \tag{18}$$

and therefore, for any $\phi \in K(\mathbf{H}_\omega)$,

$$E(p)\phi(qp^{-1}) = \phi(p^{-1}q)E(p). \tag{19}$$

Owing to (18) and (19) all the integral expressions that involve $P^{(n)}(p^{-1}q)E(p)$ and $\phi(p^{-1}q)E(p)$ in their integrands, e.g. in Proposition 4, Corollary 2 and Theorem 4 have alternative expressions using $E(p)P^{(n)}(qp^{-1})$ and $E(p)\phi(qp^{-1})$ in their integrands instead.

Remark 7. *If f^0 is a holomorphic function in the annuls $\rho - s < |\eta| < \tau + s$ in* $\mathbf{C}$, *σ a star-shaped Lipschitz curve in the annuls, then the Laurent series of f^0has the expression*

$$f^0(z) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_\sigma (\eta^{-1}z)^k f^0(\eta) \frac{d\eta}{\eta}, \quad \rho - s < |\eta| < \tau + s.$$

Comparing this with Proposition 4 and its Corollary 2, we can see that the functions $P^{(k)}$ play the same role as the functions $(\cdot)^k$ in the one complex variable case. For right-regular functions we have symmetric results. For instance, the counterpart of Proposition 4 in that case is

$$f(q) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi^2} \int_\Sigma f(p)n(p)E(p)P^{(k)}(qp^{-1})d\sigma(p), \quad \rho - s < |q| < \tau + s.$$

Now for a function $b \in \mathbf{S}^{\mathbf{c}}_\omega$ we introduce the following multiplier operator $M_b : \mathscr{A} \to \mathscr{A}$:

$$M_b f(q) = \sum_{n=1}^{\infty} b(n) \sum_{\alpha\in\sigma_n} P_\alpha(q)a_\alpha + \sum_{n=1}^{\infty} b(-n) \sum_{\alpha\in\sigma_n} E_\alpha(q)b_\alpha, \quad \rho - s < |q| < \tau + s.$$

Now we show that M_b has a singular integral expression whose kernel, apart from a constant multiple, is the function in $K(\mathbf{H}_\omega)$ associated with b as specified in Theorem 1

Now for $q \in \Sigma, r \approx 1$ but $r < 1$ consider the function

$$\begin{aligned} M_b^r f(q) &= \sum_{n=1}^{\infty} b(n) \sum_{\alpha\in\sigma_n} P_\alpha(rq)a_\alpha + \sum_{n=1}^{\infty} b(-n) \sum_{\alpha\in\sigma_n} E_\alpha(r^{-1}q)b_\alpha \\ &= P^r(q) + Q^r(q), \\ &\qquad \rho - s < |q| < \tau + s. \end{aligned}$$

Using the convolution expressions of the projections, we have

$$\begin{aligned} P^r(q) &= \sum_{n=1}^{\infty} b(n) \frac{1}{2\pi^2} \int_\Sigma P^{(n)}(p^{-1}rq)E(p)n(p)f(p)d\sigma(p) \\ &= \frac{1}{2\pi^2} \int_\Sigma (\sum_{n=1}^{\infty} b(n)P^{(n)}(p^{-1}rq))E(p)n(p)f(p)d\sigma(p) \\ &= \frac{1}{2\pi^2} \int_\Sigma \phi^+(p^{-1}rq)E(p)n(p)f(p)d\sigma(p), \end{aligned}$$

where $\phi^+ = \sum_{n=1}^{\infty} b(n)P^{(n)} \in K(\mathbf{H}_{\omega,+})$ as proved in Theorem 1. Similarly, we have

$$Q^r(q) = \frac{1}{2\pi^2} \int_\Sigma \phi^-(p^{-1}r^{-1}q)E(p)n(p)f(p)d\sigma(p),$$

where $\phi^- \in K(\mathbf{H}_{\omega,-})$.

Now letting $r \to 1-$, since the series defining $M_b^r f$ is uniformly convergent as $r \to 1-$, we can exchange the order of taking limit and summation to obtain

$$M_b f(q) = \lim_{r\to 1-} \frac{1}{2\pi^2} \int_\Sigma (\phi^+(p^{-1}rq) + \phi^-(p^{-1}r^{-1}q))E(p)n(p)f(p)d\sigma(p).$$

Theorem 4. *(i) If $b \in H^\infty(\mathbf{S}^c_\omega)$, then for any $f \in \mathscr{A}$ and $q \in \Sigma$, we have*

$$\begin{aligned} M_b f(q) &= \lim_{r\to 1-} \frac{1}{2\pi^2} \int_\Sigma (\phi^+(p^{-1}rq) + \phi^-(p^{-1}r^{-1}q))E(p)n(p)f(p)d\sigma(p) \\ &= \lim_{\epsilon\to 0} \frac{1}{2\pi^2} \{ \int_{|1-p^{-1}q|>\epsilon, p\in\Sigma} \phi(p^{-1}q)E(p)n(p)f(p)d\sigma(p) \\ &\quad + \phi^1(\epsilon, q)f(q)\}, \end{aligned}$$

where $\phi = \phi^+ + \phi^-$ is the function associated with b as specified in Corollary 1 and ϕ^1 the bounded continuous function: $\phi^1 = \phi^{+,1} + \phi^{-,1}$, where

$$\phi^{\pm,1}(\epsilon, q) = \int_{S(\epsilon,q,\pm)} \phi^\pm(p)E(p)n(p)d\sigma(p),$$

where $S(\epsilon, q, \pm)$ is the part of the surface $|1 - p^{-1}q| = \epsilon$ inside or outside Σ, depending on $\pm$ taking + or −.

(ii) For any $b \in H^\infty(\mathbf{S}^c_\omega), M_b$ can be extended to a bounded operator from $L^2(\Sigma)$ to $L^2(\Sigma)$. Moreover,

$$\|M_b\|_{L^2(\Sigma)\to L^2(\Sigma)} \le C_\nu \|b\|_{L^\infty(\mathbf{S}^c_\nu)}, \quad \arctan(N) < \nu < \omega.$$

Proof of part (i) Only the second equality requires a proof. Let us only consider the part related to ϕ^+; the other part can be dealt with similarly. We will adapt the proof of Theorem 6.1 of [McQ1]. For each $\epsilon > 0$, the integral can be decomposed into

$$\begin{aligned} \lim_{r\to 1-} \{ &\int_{|1-p^{-1}q|>\epsilon, p\in\Sigma} \phi^+(p^{-1}rq)E(p)n(p)f(p)d\sigma(p) + \\ &+ \int_{|1-p^{-1}q|\le\epsilon, p\in\Sigma} \phi^+(p^{-1}rq)E(p)n(p)f(p)d\sigma(p)\}. \end{aligned}$$

As $r \to 1-$, the first part tends to

$$\int_{|1-p^{-1}q|>\epsilon, p\in\Sigma} \phi^+(p^{-1}q)E(p)n(p)f(p)d\sigma(p).$$

The second part can be decomposed into

$$\begin{aligned} &\int_{|1-p^{-1}q|\le\epsilon, p\in\Sigma} \phi^+(p^{-1}rq)E(p)n(p)(f(p) - f(q))d\sigma(p) + \\ &+ \int_{|1-p^{-1}q|\le\epsilon, p\in\Sigma} \phi^+(p^{-1}rq)E(p)n(p)d\sigma(p)f(q). \end{aligned}$$

As $\epsilon \to 0$, the first integral tends to zero uniformly with respect to $r \to 1-$; while if we invoke Cauchy's theorem, for a fixed ϵ, the second integral tends to $\phi^{+,1}(\epsilon, q)f(q)$ as $r \to 1-$. The proof is complete.

The proof in [GQW] for the analogous result for a complex variable can be closely followed to give a proof of part 2 of Theorem 4. Due to the consideration that the particulars in relation to the spherical Dirac operator in the present case may be worthwhile pointing out, we choose to incorporate the proof.

We need some preparation on Hardy spaces of regular functions on the surface Σ (see [Mi] for the theory of Clifford monogenic Hardy spaces on higher-dimensional Lipschitz graphs).

Let Δ and Δ^c be the bounded and unbounded connected components of $\mathbf{Q} \setminus \Sigma$. For $\alpha > 0$, define the *non-tangential approach regions* $\Lambda_\alpha(q)$ and $\Lambda^c_\alpha(q)$ to a point $q \in \Sigma$ to be

$$\Lambda_\alpha(q) = \Lambda_\alpha(q, \Delta) = \{p \in \Delta : |p - q| < (1 + \alpha)\mathrm{dist}(p, \Sigma)\},$$

and

$$\Lambda^c_\alpha(q) = \Lambda_\alpha(q, \Delta^c) = \{p \in \Delta^c : |p - q| < (1 + \alpha)\mathrm{dist}(p, \Sigma)\}.$$

It is easy to show, similarly to the complex variable case considered in [K1] and [JK], that there exists a positive constant α_0, depending on the Lipschitz constant of Σ only, such that $\Lambda_\alpha(q) \subset \Delta$ and $\Lambda^c_\alpha(q) \subset \Delta^c$ for $0 < \alpha < \alpha_0$ and all $q \in \Sigma$. The following argument is independent of specially chosen $\alpha \in (0, \alpha_0)$. We choose and fix α from now on.

The *interior non-tangential maximal function* $N_\alpha(f)$ is defined by

$$N_\alpha(f)(q) = \sup\{|f(p)| : p \in \Lambda_\alpha(q)\}, \qquad q \in \Sigma.$$

The *exterior non-tangential maximal function* $N^c_\alpha(f)$ is similarly defined.

For $0 < p_0 < \infty$, the (left-) Hardy space $H^{p_0}(\Delta)$ is defined by

$$H^{p_0}(\Delta) = \{f : f \text{ is left} - \text{regular in } \Delta, \text{ and } N_\alpha(f) \in L^{p_0}(\Sigma)\}.$$

If $f \in H^{p_0}(\Delta)$, then $\|f\|_{H^{p_0}(\Delta)}$ is defined as the L^{p_0} norm of $N_\alpha(f)$ on Σ.

The space $H^{p_0}(\Delta^c)$ is defined similarly, except that the functions in $H^{p_0}(\Delta^c)$ are assumed to vanish at infinity. Similarly to the monogenic Hardy space case studied in [Mi], one can prove

Proposition 5. *If $f \in H^{p_0}(\Delta), p_0 > 1$, then the non-tangential limit of f,*

$$\lim_{p \to q, p \in \Lambda_\alpha(q)} f(p)$$

exists almost everywhere with respect to the surface measure on Σ. If the limit is still denoted by f, then

$$C_{N,p_0}\|f\|_{H^{p_0}(\Delta)} \leq \|f\|_{L^{p_0}(\Sigma)} \leq C'_{N,p_0}\|f\|_{H^{p_0}(\Delta)},$$

where C_{N,p_0}, C'_{N,p_0} depend on the Lipschitz constant N and p_0.

In other words, for $p_0 > 1$, the $H^{p_0}(\Delta)$ norm of a function is equivalent to the L^{p_0} norm of its non-tangential limit on the boundary. A similar result holds for the functions in the Hardy spaces associated with Δ^c.

In polar coordinate system the Dirac operator D can be decomposed into

$$D = \zeta\partial_r - \frac{1}{r}\partial_\zeta = \zeta(\partial_r - \frac{1}{r}\Gamma_\zeta),$$

where Γ_ζ is a first order differential operator depending only on the angular coordinates (see [DSS] and note that our Γ_ζ is their $-\Gamma_\zeta$).

Recall that $U_k, k \in \mathbf{Z} \setminus \{-1, -2\}$, denotes the subspace of k-homogeneous regular functions. It is known that

$$\Gamma_\zeta f(\zeta) = kf(\zeta), f \in U_k \tag{20}$$

(see, e.g. page 162-163, [DSS]).

For $f \in \mathscr{A}$, by defining $\Gamma_\zeta f$ to be the regular extension of $\Gamma_\zeta(f|_{\mathbf{S}})$, the definition of Γ_ζ can be extended to $\Gamma_\zeta : \mathscr{A} \to \mathscr{A}$.

Proposition 6. *Suppose that $f \in H^2(\Delta)$. Then the norm $\|f\|_{H^2(\Delta)}$ is equivalent to the norm*

$$\int_0^1 \int_\Sigma |(\Gamma_\zeta^j f)(rq)|^2(1-r)^{2j-1}d\sigma(q)\frac{dr}{r}, \quad j = 1, 2, \ldots$$

The proof is similar to that of the corresponding result for Lipschitz graphs studied in [Mi] (also see [JK]). A similar result holds for $f \in H^2(\Delta^c)$.

The following is equivalent to the CMcM theorem on Σ ([CMcM]).

Proposition 7. *Suppose that $f \in L^2(\Sigma)$. Then there exist $f^+ \in H^2(\Delta)$ and $f^- \in H^2(\Delta^c)$ such that their non-tangential boundary limits, still denoted by f^+, f^-, respectively, lie in $L^2(\Sigma)$, and $f = f^+ + f^-$. The mappings $f \to f^\pm$ are bounded on $L^2(\Sigma)$.*

It is easy to see that if $f \in \mathscr{A}$, then the natural decomposition of f into its power series and principal series parts induces the decomposition given in Proposition 7.

Denote by $\Sigma_r, 0 < r < 1$, the surface $\{rq : q \in \Sigma\}$.

Lemma 7. *Suppose $q_0 \in \Sigma$. Let $0 < r < 1$, and $q = rq_0$. Then there exists a constant C_N such that*

$$|1 - p^{-1}q| \geq C_N\{(1-\sqrt{r})^2 + \theta^2\}^{\frac{1}{2}}, \quad p \in \Sigma_{\sqrt{r}},$$

where $\theta = \arg(q, p)$.

Proof. Owing to the relation $\arg(q, p) = \arg(p^{-1}q, 1) = \arg((p^{-1}q)_0 + \mathbf{i}|\underline{(p^{-1}q)}|)$, where the last arg stands for the angle of the complex number, the argument in the proof of Lemma 3.4 of [GQW] then can be followed to conclude the lemma.

Proof of part (ii) Let $f \in \mathscr{A}$. Using the decomposition of f defined in Proposition 7, we have $f = f^+ + f^-$, where $f^+ \in H^2(\Delta)$, $f^- \in H^2(\Delta^c)$, $\|f^\pm\|_{L^2(\Sigma)} \leq C_N \|f\|_{L^2(\Sigma)}$. We also have $M_b f = M_{b^+} f^+ + M_{b^-} f^-$, where

$$M_{b^\pm} f^\pm(q) = \lim_{r \to 1-} \int_\Sigma \phi^\pm(r^{\pm 1} p^{-1} q)) E(p) n(p) f(p) d\sigma(p), \quad q \in \Sigma.$$

$M_{b^\pm} f^\pm(q)$ can be left-regularly extended to Δ and Δ^c using

$$M_{b^\pm} f^\pm(q) = \int_\Sigma \phi^\pm(p^{-1} q)) E(p) n(p) f(p) d\sigma(p),$$

for $q \in \Delta$ and $q \in \Delta^c$, respectively.

Owing to Proposition 5, we need to show

$$\|M_{b^\pm} f^\pm\|_{H^2} \leq C_N \|f^\pm\|_{H^2}.$$

We will now only prove the inequality for the case "+"; the case "-" can be dealt with similarly. We will suppress the superscript "+" in below for simplicity. Using the Taylor series expansion of f, and accordingly that of $M_b f$, we easily have, for $q \in \Delta$,

$$\Gamma_\zeta M_b f(q) = \frac{1}{2\pi^2} \int_\Sigma \phi(p^{-1} q) E(p) n(p) \Gamma_\zeta f(p) d\sigma(p),$$

where exchange of the order of taking differentiation Γ_ζ and the infinite summation is justified by first assuming $|q| < \rho$, and then performing regular continuation. We also easily have, by exchanging the order of taking integration and differentiation,

$$\Gamma_\zeta^2 M_b f(q) = \frac{1}{2\pi^2} \int_\Sigma \Gamma_\zeta(\phi(p^{-1} q)) E(p) n(p) \Gamma_\zeta f(p) d\sigma(p)$$

which is justified by the following

Lemma 8. *If* $\nu \in (\arctan(N), \omega)$, *then*

$$|\Gamma_\zeta(\phi(p^{-1} q)|) \leq C_\nu \frac{1}{|1 - p^{-1} q|^4}, \qquad p \in \Sigma, q \in \Delta.$$

Proof. Applying Γ_ζ with respect to q to the series

$$\phi(p^{-1} q) E(p) = \sum_{n=1}^{\infty} b(n) P^{(n)}(p^{-1} q) E(p) = \sum_{n=1}^{\infty} b(n) \sum_{\alpha \in \sigma_n} P_\alpha(q) E_\alpha(p)$$

term by term, justified by first assumimg $|q| < |p|$ and then performing regular continuation, owing to (26), we obtain

$$\Gamma_\zeta(\phi(p^{-1} q)) = \sum_{n=1}^{\infty} n b(n) P^{(n)}(p^{-1} q).$$

Applying Theorem 3 with $s = 1$ to the multiplier $b'(z) = zb(z)$, we conclude the lemma.

Now we return to the proof of part (ii). Using Lemma 8 and 7, we have, for $q \in \Sigma_r$.

$$
\begin{aligned}
|\Gamma_\zeta^2 M_b f(q)| &\leq C \left(\int_{\Sigma_{\sqrt{r}}} |\Gamma_\zeta(\phi(p^{-1}q))| \frac{d\sigma(p)}{|p|^3} \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\Sigma_{\sqrt{r}}} |\Gamma_\zeta(\phi(p^{-1}q))| |\Gamma_\zeta f(p)|^2 \frac{d\sigma(p)}{|p|^3} \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{\Sigma_{\sqrt{r}}} \frac{1}{|1-p^{-1}q|^4} \frac{d\sigma(p)}{|p|^3} \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\Sigma_{\sqrt{r}}} \frac{1}{|1-p^{-1}q|^4} |\Gamma_\zeta f(p)|^2 \frac{d\sigma(p)}{|p|^3} \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{\Sigma} \frac{1}{((1-\sqrt{r})^2+\theta_0^2)^2} d\sigma(p) \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\Sigma} \frac{1}{((1-\sqrt{r})^2+\theta_0^2)^2} |\Gamma_\zeta f(\sqrt{r}p)|^2 d\sigma(p) \right)^{\frac{1}{2}},
\end{aligned}
$$

where θ_0 is the angle between q and p.

Since

$$
\begin{aligned}
\int_\Sigma \frac{1}{((1-\sqrt{r})^2+\theta_0^2)^2} d\sigma(p) &\leq C \int_0^\pi \frac{\sin^2\theta_0}{((1-\sqrt{r})^2+\theta_0^2)^2} d\theta_0 \\
&\leq C \int_0^\pi \frac{\theta_0^2}{((1-\sqrt{r})^2+\theta_0^2)^2} d\theta_0 \\
&= C \frac{1}{1-\sqrt{r}} \\
&\leq C \frac{1}{1-r},
\end{aligned}
$$

we have

$$
\begin{aligned}
\|M_b f\|^2_{H^2(\Delta)} &\approx \int_0^1 \int_\Sigma |\Gamma_\zeta^2(M_b f)(rq)|^2 (1-r)^3 d\sigma(q) \frac{dr}{r} \\
&\leq C \int_0^1 \int_\Sigma \frac{1}{1-r} \Big(\int_\Sigma \frac{1}{((1-\sqrt{r})^2+\theta_0^2)^2} |\Gamma_\zeta f(\sqrt{r}p)|^2 d\sigma(p) \Big) \\
&\quad \times (1-r)^3 \frac{dr}{r} d\sigma(q) \\
&\leq C \int_0^1 \int_\Sigma |\Gamma_\zeta f(\sqrt{r}p)|^2 \Big(\int_\Sigma \frac{(1-r)}{((1-\sqrt{r})^2+\theta_0^2)^2} d\sigma(q) \Big)
\end{aligned}
$$

$$
\begin{aligned}
& \times(1-r)d\sigma(p)\frac{dr}{r} \\
\leq \quad & C\int_0^1\int_\Sigma |\Gamma_\zeta f(\sqrt{r}p)|^2(1-r)d\sigma(p)\frac{dr}{r} \\
\leq \quad & C\int_0^1\int_\Sigma |\Gamma_\zeta f(rp)|^2(1-r)d\sigma(p)\frac{dr}{r} \\
\approx \quad & C\|f\|^2_{H^2(\Delta)}.
\end{aligned}
$$

The bounds of the operator norm $\|M_b\|$ can be derived from the proof of Lemma 8 and the estimates obtained in Theorem 3. The proof is complete.

Remark 8. *Since the surface measure on Σ satisfies the so called doubling condition, the standard Calderón-Zygmund method ([St]) can be applied to the operators M_b, based on the L^2-boundedness supplied by Theorem 4, so to conclude the L^p- and the weak-L^1-boundedness of the operators with the same bounds for the operator norms in terms of $\|b\|_{L^\infty(\mathbf{S}^{\mathbf{c}}_\nu)}$ as given in Theorem 4.*

Remark 9. *As in the standard cases the Hilbert transform on the unit sphere and on star-shaped Lipschitz surfaces should be defined using the Fourier multiplier $b(z) = -i\,\mathrm{sgn}(z)$, where $\mathrm{sgn}(z)$ is the signum function which takes value $+1$ for $\mathrm{Re}(z) > 0$ and -1 for $\mathrm{Re}(z) < 0$, whose singular integral expression is given by the kernel $-\frac{i}{\pi^2}E(1-q)$, as derived in Sect. 1. The associated singular integral kernels*

$$
\frac{1}{2\pi^2}{\sum_{k=-\infty}^{\infty}}{}' b(k)P^{(k)}(q)
$$

of a general function $b \in H^\infty(\mathbf{S}^{\mathbf{c}}_\omega)$ can be alternatively obtained from the formula

$$
\frac{1}{2\pi^2}\tau^0(\phi^0),
$$

where

$$
\phi^0 = \phi^{0,+} + \phi^{0,-}, \phi^{0,+}(z) = \sum_{k=1}^{\infty} b(k)z^{k+2}, \phi^{0,-}(z) = \sum_{k=-\infty}^{-1} b(k)z^k.
$$

If, instead, we use $\phi^0(z) = {\sum_{k=-\infty}^{\infty}}{}' b(k)z^k$, then the difference between the two image functions under τ^0 is

$$
\frac{1}{2\pi^2}[b(2) + \sum_{k=1}^{\infty}(b(k+2) - b(k))P^{(k)}]
$$

which, owing to Theorem 3 for the case $s = -1$, is an integrable function on the surface, and so gives rise to a bounded operator.

We now close the section by giving Parseval's identity on the sphere between the kernel functions and the bounded holomorphic multipliers.

Corollary 3. *Let f be a smooth function on the sphere, $b \in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega})$, and ϕ the function in $K(\mathbf{H}_{\omega})$ associated with b as specified in Theorem 1. Then we have*

$$\sum_{n=-\infty}^{\infty} b(n) \int_{\mathbf{S}} P^{(-n-1)}(p)n(p)f(p)d\sigma(p) =$$

$$\lim_{\epsilon\to 0+} \{ \int_{|1-p|>\epsilon, p\in\mathbf{S}} I(\phi)(p)n(p)f(p)d\sigma(p) + \phi^1(\epsilon,1)f(1)\},$$

where $b(0) = \frac{1}{2\pi^2}\int_{\mathbf{S}} I(\phi)(p)n(p)d\sigma(p)$ and $\phi^1(\epsilon, q)$ is defined in Theorem 4.

Proof. Subsequently performing the following steps: decomposing f into $f = f^+ + f^-$, where f^+ and f^- are the power and principal series part of f, respectively; adopting the argument in the proof of part (i) of Theorem 4 to each of f^+ and f^- for $q = 1$; replacing, in virtue of (18) and (19), $P^{(n)}(p^{-1})E(p)$ and $\phi(p^{-1})E(p)$ by $P^{(-n-1)}(p)$ and $I(\phi)(p)$ respectively, we obtain the identity.

3 Holomorphic functional calculus of the spherical Dirac operator

We wish to point out that the class of the bounded operator M_b studied in Sect. 2 constitutes a functional calculus of Γ_ζ, and in fact is identical to the Cauchy-Dunford's bounded holomorphic functional calculus of Γ_ζ. For a discussion in relation to the domains of Γ_ζ in various spaces we refer the reader to [LMcQ] and [Mc3] where examples in relation to Dirac operators on Lipschitz graphs are given.

The operators M_b enjoy the following properties, according to which the class $M_b, b \in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega})$ is said to constitute a bounded holomorphic functional calculus.

Let $0 < \omega < \frac{\pi}{2}, 0 < p_0 < 1,\ \tan\omega > N = \mathrm{Lip}(\Sigma),\ b, b_1, b_2 \in H^{\infty}(\mathbf{S}^{\mathbf{c}}_{\omega})$, and $\alpha_1, \alpha_2 \in \mathbf{C}$. Then

$$\|M_b\|_{L^{p_0}(\Sigma)\to L^{p_0}(\Sigma)} \le C_{p_0,\nu}\|b\|_{L^{\infty}(\mathbf{S}^{\mathbf{c}}_{\nu})}, \quad \arctan(N) < \nu < \omega;$$

$$M_{b_1 b_2} = M_{b_1} \circ M_{b_2};$$

$$M_{\alpha_1 b_1 + \alpha_2 b_2} = \alpha_1 M_{b_1} + \alpha_2 M_{b_2}.$$

The first assertion is concluded in Remark 8 . The second and the third can be easily derived by using Laurent series expansions of test functions.

Denote by

$$R(\lambda, \Gamma_\zeta) = (\lambda I - \Gamma_\zeta)^{-1}$$

the resolvent operator of Γ_ζ at $\lambda \in \mathbf{C}$. We show that for non-real $\lambda, R(\lambda, \Gamma_\zeta) = M_{\frac{1}{\lambda-(\cdot)}}$. In fact, a direct computation using the property (20) shows that the Fourier multiplier $\lambda - k$ is associated with the operator $\lambda I - \Gamma_\zeta$, and therefore the Fourier multiplier $(\lambda - k)^{-1}$ is associated with $R(\lambda, \Gamma_\zeta)$. The property of the functional calculus regarding the boundedness then asserts that for $1 < p_0 < \infty$

$$\|R(\lambda, \Gamma_\zeta)\|_{L^{p_0}(\Sigma)\to L^{p_0}(\Sigma)} \leq \frac{C_\nu}{|\lambda|}, \qquad \lambda \notin \mathbf{S}^{\mathbf{c}}_\nu. \tag{21}$$

Owing to the estimate (21), for $b \in \mathbf{S}^{\mathbf{c}}_\omega$ with good decays at zero and infinity, the Cauchy-Dunford integral

$$b(\Gamma_\zeta) = \frac{1}{2\pi i}\int_\Pi b(\lambda)R(\lambda, \Gamma_\zeta)d\lambda \tag{22}$$

defines a bounded operator, where Π is a path consisting of four rays: $\{s\exp(-i\theta) : s \text{ is from } \infty \text{ to } 0\} \cup \{s\exp(i\theta) : s \text{ is from } 0 \text{ to } \infty\} \cup \{s\exp(i(\pi-\theta)) : s \text{ is from } \infty \text{ to } 0\} \cup \{s\exp(i(\pi+\theta)) : s \text{ is from } 0 \text{ to } \infty\}$, where $\arctan(N) < \theta < \omega$. The functions of this sort form a dense subclass of $H^\infty(\mathbf{S}^{\mathbf{c}}_\omega)$ in the sense specified in the convergence lemma of McIntosh in [Mc2]. Using the lemma, we can define a bounded functional calculus $b(\Gamma_\zeta)$ on general functions $b \in H^\infty(\mathbf{S}^{\mathbf{c}}_\omega)$ extending the definition given by (22) for functions with good decays.

Now we show $b(\Gamma_\zeta) = M_b$. Assume again that b has good decays at zero and ∞, and $f \in \mathscr{A}$. Then change of order of the integration and the summation in the following chain of equalities can be easily justified, and we have

$$\begin{aligned}
b(\Gamma_\zeta)f(q) &= \frac{1}{2\pi i}\int_\Pi b(\lambda)R(\lambda, \Gamma_\zeta)d\lambda f(q) \\
&= \frac{1}{2\pi i}\int_\Pi b(\lambda)\sum_{k=-\infty}^{\infty}{}'(\lambda-k)^{-1}\frac{1}{2\pi^2} \\
&\quad \times \int_\Sigma P^{(k)}(p^{-1}q)E(p)n(p)f(p)d\sigma(p)d\lambda \\
&= \sum_k{}'(\frac{1}{2\pi i}\int_\Pi b(\lambda)(\lambda-k)^{-1}d\lambda)\frac{1}{2\pi^2} \\
&\quad \times \int_\Sigma P^{(k)}(p^{-1}q)E(p)n(p)f(p)d\sigma(p) \\
&= \sum_k{}' b(k)\frac{1}{2\pi^2}\int_\Sigma P^{(k)}(p^{-1}q)E(p)n(p)f(p)d\sigma(p) \\
&= M_b f(q).
\end{aligned}$$

Denote by $P^\pm$ the projection operators such that $P^\pm f = f^\pm$ as defined in Proposition 7. A consequence of estimate (21) is that both $\Gamma_\zeta P^\pm$ are *type-ω* operators (see [Mc2]).

$\Gamma_\zeta P^\pm$, as well as Γ_ζ, are identical to their *dual operators* on $L^2(\Sigma)$, respectively, in the *dual pair* $(L^2(\Sigma), L^2(\Sigma))$ under the bilinear pairing

$$<< f, f' >> = \frac{1}{2\pi^2}\int_\Sigma f(q)n(q)f'(q)d\sigma(q). \tag{23}$$

That is

$$<< \Gamma_\zeta P^\pm f, f' >> = << f, \Gamma_\zeta P^\pm f' >>$$

and

$$<< \Gamma_\zeta f, f' >>=<< f, \Gamma_\zeta f' >> .$$

These can be easily derived from Parseval's identity

$$\sum_{n=0}^{\infty} \sum_{\alpha \in \sigma_n} a_\alpha a'_\alpha + b_\alpha b'_\alpha = \frac{1}{2\pi^2} \int_{\mathbf{S}} f(q) n(q) f'(q) d\sigma(q),$$

in the notation of (14), and the relation (20).

Similar conclusions hold for the Banach space dual pairs $(L^{p_0}(\Sigma), L^{p'_0}(\Sigma))$, $1 < p_0 < \infty$, $\frac{1}{p_0} + \frac{1}{p'_0} = 1$, under the same bilinear pairing (23).

Hilbert and Banach space properties of general type-ω operators are well studied respectively in [Mc2] and [CDMcY]. The results of [Mc2] and [CDMcY] therefore are all applicable to the operators $\Gamma_\zeta P^{\pm}$, and so to Γ_ζ as well.

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