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Einstein metrics on S^2-bundles

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Abstract. New Einstein metrics are constructed on the associated $\mathbb{RP}^2$, S^2, and $\mathbb{R}^2$-bundles of principal circle bundles with base a product of Kähler-Einstein manifolds with positive first Chern class and with Euler class a rational linear combination of the first Chern classes. These Einstein metrics represent different generalizations of the well-known Einstein metrics found by Bérard Bergery, D. Page, C. Pope, N. Koiso, and Y. Sakane. Corresponding new Einstein-Weyl structures are also constructed.

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0 Introduction

Let (M_i, J_i, h_i), $i = 1, 2, \ldots, l$, be compact Kähler manifolds with positive first Chern class and real dimension $2n_i$. Since it is well-known that the M_i are simply connected, we can write the first Chern class $c_1(M_i)$ of M_i as $p_i \alpha_i$, where α_i is indivisible and p_i is a positive integer. Let $M = M_1 \times M_2 \times \cdots \times M_l$ and π_i denote the projection of M onto M_i. For non-zero integers $q_1, q_2, \ldots, q_l$, the integral cohomology class $q_1 \pi_1^* \alpha_1 + \cdots + q_l \pi_l^* \alpha_l$ is the Euler class of a principal circle bundle $P_{q_1,\ldots,q_l}$ over M. The circle acts by rotation on the complex plane $\mathbb{C}$ and its one-point compactification, the Riemann sphere S^2. We denote the total space of the associated complex line bundle by $V_{q_1,\ldots,q_l}$ and that of the associated S^2-bundle by $W_{q_1,\ldots,q_l}$.

In this paper we construct new Einstein metrics and Einstein-Weyl structures on these spaces under various additional conditions. See Theorems 1.2, 1.6, 1.7 in Sect. 1 for the precise statements. These new Einstein metrics are hermitian but in general are not Kähler with respect to the natural induced complex structure on

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the total spaces. However, their Riemann curvature tensor is invariant under the action of the complex structure. We also construct Einstein metrics and Einstein-Weyl structures on the quotient $\overline{W}_{q_1,\dots,q_l}$ of $W_{q_1,\dots,q_l}$ by the antipodal map of the fibres, which can be viewed as an associated $\mathbb{RP}^2$-bundle of $P_{q_1,\dots,q_l}$. See Theorems 1.4 and 1.8 for the precise statements.

All these new Einstein (Einstein-Weyl) manifolds do not in general have large isometry groups. However, when all factors of the base are homogeneous, the metrics are of cohomogeneity one, i.e., there is an isometric action of a compact Lie group with codimension one principal (generic) orbits.

Regarding the motivation for studying this class of manifolds, recall that the first example of a compact inhomogeneous Einstein manifold with positive scalar curvature was constructed by D. Page [P] from the Taub-NUT solution, essentially by replacing time t by $\sqrt{-1}\, t$. Bérard Bergery [BB] then made the important observation that the Page metric can be generalized to *non-Kähler* Einstein metrics on the associated S^2-bundles of those principal $U(1)$-bundles over Kähler-Einstein manifolds with positive first Chern class whose Euler class is of the form $q \cdot \alpha,\ 0 < q < p$, with α indivisible and $p \cdot \alpha$ equal to the first Chern class of the base. Somewhat later, Page and Pope [PP2] independently observed the same generalization, with the difference that while [BB] emphasized the framework of cohomogeneity one Einstein metrics, they stressed the Kaluza-Klein ansatz and the existence of a certain variable change which allows the components of the Einstein metrics to be expressed explicitly in terms of special functions, specifically the Gegenbauer polynomials. Yet another approach to these examples can be found in [JR], and orbifold solutions in dimension 4 were studied in [PZ].

Inspired by [BB], Y. Sakane [S] constructed the first non-homogeneous examples of Kähler-Einstein manifolds with positive first Chern class by studying the associated S^2-bundles of principal $U(1)$-bundles over a product of two compact hermitian symmetric spaces. He found that the existence of Kähler-Einstein metrics of Kaluza-Klein type on these S^2-bundles required not only similar conditions on the Euler class of the $U(1)$-bundle, but also the vanishing of a certain integral. His work was generalized by N. Koiso and himself in [KS1, KS2], where it was discovered that the integral is actually the Futaki invariant of the holomorphic vector field associated to the $U(1)$-action. If one uses the variable change in [PP2], then again the Koiso-Sakane metrics can be expressed in terms of certain linear functions and integrals of their products. In the Kähler context, this change of independent variable corresponds to conversion to the variable naturally associated to the moment map of the $U(1)$ action.

In view of the above developments, it is natural to study the Kaluza-Klein construction of *non-Kähler* Einstein metrics on the associated S^2-bundles of circle bundles over an arbitrary finite product of Kähler-Einstein manifolds, and to try to fit into a single framework the works of the above authors. Furthermore, in [PS2], Einstein-Weyl structures which are not locally conformally Einstein were constructed on the S^2-bundles considered by Bérard Bergery. It is therefore also natural to study the Einstein-Weyl equations for this larger family of S^2-bundles.

We shall see that replacing the base by product manifolds gives rise to several new phenomena. First, there is a new family of solutions of the Einstein equation (Theorem 1.2) in addition to the generalization of the Bérard Bergery metrics (Theorem 1.4). Second, (local) conformality to Kähler metrics is no longer automatic (Corollary 7.3). Perhaps the most thought-provoking phenomenon is the role played by the Futaki integral in Theorems 1.2 and 1.7.

The main results of this paper will be stated in Sect. 1, where the general geometrical set-up will also be presented. The existence of Einstein metrics will be taken up in Sects. 2–5. In Sect. 6, some of the topological properties of the S^2-bundles will be discussed, and in Sect. 7 some aspects of their Hermitian geometry will be described. Einstein-Weyl structures will be explored in the remaining sections of the paper.

Sections 2-6 of this paper are based on the third chapter of the Ph.D. thesis of the first author [W2] written under the supervision of the second author, whose role beyond that of supervision is to ensure that certain closely related themes are pursued and completed. Finally, we would like to thank A. Nicas for some useful discussions about topology and Sun Poon for discussions about [PePo].

1 Statement of results

Let (M_i, J_i, h_i) be a compact Kähler-Einstein manifold with positive first Chern class as in the Introduction, ω_i be its Kähler form, and ρ_i be its Ricci form. We will normalize h_i so that $\rho_i = p_i\omega_i$. Also, the multi-index subscripts on the bundles defined in the Introduction will be omitted whenever there is no confusion.

Next, we choose a connection form θ on P whose curvature $\Omega = d\theta = \sum_{i=1}^{l} q_i \pi_i^* \omega_i$. We caution the reader that these are real-valued forms, while the usual convention is for the Lie algebra of the circle to be identified with the imaginary complex numbers. Notice that Ω is harmonic with respect to any product metric of the form $a_1 h_1 + \cdots + a_l h_l$ on M, where a_i are positive constants, so that θ may be viewed as a Yang-Mills connection on P. θ induces connections on V and W which we use to lift metrics on the base M to the horizontal spaces of the bundles. We will examine the Einstein (resp. Einstein-Weyl) equations for the family of metrics of the form

$$h = dt^2 + f(t)^2\ \theta \otimes \theta + \sum_{i=1}^{l} g_i(t)^2\ \pi_i^* h_i, \tag{1.1}$$

where f and g_i are smooth non-negative functions of t defined on some interval I and satisfying suitable boundary conditions which guarantee that h defines a smooth metric on V, W, or $\overline{W}$. These conditions will be discussed in detail later on. Here we only point out that for each fixed t, the induced metric on P makes it into a Riemannian submersion with totally geodesic fibres onto a product metric on M, and, as t varies, we have an equi-distant hypersurface family as was discussed in Sect. 2 of [EW].

The connection θ can also be used to construct a complex structure J on V (resp. W) by lifting the product complex structure $J_1 \times \cdots \times J_l$ on the base M to the horizontal spaces and using the natural complex structure of $\mathbb{C}$ (resp. the Riemann sphere) on the vertical spaces. Thus V admits a natural complex structure, and W, with the complex structure J, may be viewed as the projectivization of $V \oplus \mathbf{1}$, where $\mathbf{1}$ denotes the trivial complex line bundle over M.

Theorem 1.2 *Notation as above, assume that*

$$0 < |q_i| < p_i, \quad i = 1, 2, \ldots, l,$$

and that there exists $(\varepsilon_1, \ldots, \varepsilon_l)$ *where* $|\varepsilon_i| = 1$ *with at least one* ε_i *positive such that*

$$\int_{-1}^{1} \left(\frac{p_1}{|q_1|} + \varepsilon_1 x\right)^{n_1} \left(\frac{p_2}{|q_2|} + \varepsilon_2 x\right)^{n_2} \cdots \left(\frac{p_l}{|q_l|} + \varepsilon_l x\right)^{n_l} x\, dx < 0.$$

Then there exists an Einstein metric with positive scalar curvature on $W_{q_1,\ldots,q_l}$ *which is Hermitian with respect to* J *but which is not Kählerian.*

Note that l cannot be equal to 1 in Theorem 1.2! This is because the positivity of the first Chern class implies that $p_i \le n_i + 1$, which in turn implies that when $l = 1$ the above integral is positive. Hence the Einstein metrics constructed in this theorem belong to a *different* family than the Bérard Bergery-Page-Pope metrics. Indeed, we shall see from the proof of the theorem that the metrics do not factor through the antipodal map of the S^2-fibres (cf Remark 3.4).

Suppose now that the bundle is fixed, and some choice of $(\varepsilon_1, \ldots, \varepsilon_l)$ containing both +1 and -1 makes the above integral positive. Then one easily checks that the choice $(-\varepsilon_1, \ldots, -\varepsilon_l)$ renders the integral negative. Of course, there could be several choices of $(\varepsilon_1, \ldots, \varepsilon_l)$ containing both +1 and -1 which make the integral negative.

At this point let us recall the existence theorem of Koiso-Sakane [KS1, Theorem 4.2], which we state in the following form for comparison purposes:

Theorem 1.3 (Koiso-Sakane) *There exists an Einstein metric with positive scalar curvature on* $W_{q_1,\ldots,q_l}$ *which is Kähler with respect to* J *if*

$$0 < |q_i| < p_i,$$

and if

$$\int_{-1}^{1} \left(\frac{p_1}{q_1} - x\right)^{n_1} \left(\frac{p_2}{q_2} - x\right)^{n_2} \cdots \left(\frac{p_l}{q_l} - x\right)^{n_l} x\, dx = 0$$

As was discovered in [KS1], the above integral is precisely Futaki's functional [Fu] evaluated on the (real) holomorphic vector field $f\partial/\partial t$. Comparing Theorems 1.2 and 1.3, one sees that the hypotheses in Theorem 1.2 above are much less restrictive. Furthermore, when a choice of $(\varepsilon_1, \ldots, \varepsilon_l)$ makes the integral in Theorem 1.2 *zero*, then Theorem 1.3 shows that on the bundle whose

Euler class is given by the integers $-\varepsilon_i|q_i|$ there is a Kähler-Einstein metric, because the integral in Theorem 1.2 is the Futaki invariant for $f\partial/\partial t$ on this bundle. In this sense Theorems 1.2 and 1.3 complement each other.

The generalization of the Einstein metrics of Bérard Bergery-Page-Pope to S^2-bundles over a product of Kähler-Einstein manifolds with positive first Chern class is given by

Theorem 1.4 *Let* $0 < |q_i| < p_i, i = 1, \ldots, l$. *Then there always exists an Einstein metric with positive scalar curvature on the* $\mathbb{RP}^2$*-bundle* $\overline{W}_{q_1,\ldots,q_l}$, *and hence an Einstein metric on the* S^2*-bundle* $W_{q_1,\ldots,q_l}$ *which is Hermitian but non-Kähler with respect to* J *and has fibre-wise* $\mathbb{Z}/2$ *symmetry.*

Notice that, unlike the Kähler case, the hypotheses in Theorems 1.2 and 1.4 depend only on $|q_i|$. In fact, the manifolds $W_{q_1,\ldots,q_l}$ for which the absolute values of the corresponding q_i are equal are diffeomorphic, but the natural complex structures on them (see Remark 2.5) are not equivalent in general.

The Hermitian (with respect to J) geometry of the Einstein metrics given by Theorems 1.2 and 1.4 is quite interesting. In Sect. 7 we will characterize when these Einstein metrics are conformal to J-Kähler metrics (Corollary 7.3). We also prove (see Corollary 7.5) that the Riemann curvature tensor is J-invariant, i.e., $R(X, Y, Z, W) = R(JX, JY, JZ, JW)$ for all tangent vectors X, Y, Z, W. So these Einstein metrics belong to the class $\mathscr{L}_3$ of Hermitian manifolds studied by A. Gray in [Gr]. See [FFS] for a more up-to-date study and more complete references.

Remark 1.5 Theorems 1.2 and 1.4 may be generalized as follows. Let $(M_i, g_i), i = 1, 2, \ldots, l$ be compact Einstein manifolds with Einstein constants $p_i > 0$, and suppose that there are harmonic 2-forms ω_i on M_i such that $(2\pi)^{-1}[\omega_i] \in H^2(M_i; \mathbb{Z})$. Suppose further that each ω_i satisfies the condition

$$\sum_k \omega_i(X, e_k)\omega_i(Y, e_k) = 2\lambda_i{}^2 g_i(X, Y)$$

for all vectors X, Y of M_i and some positive constant λ_i. ($\{e_k\}$ is an orthonormal basis for g_i.) As before, we can construct a principal circle bundle P over M with a connection θ whose curvature form is $\sum_{i=1}^{l} q_i\omega_i$, where q_i are nonzero integers. Then the proofs of Theorems 1.2 and 1.4 carry over to show the existence of Einstein metrics on the associated S^2 and $\mathbb{RP}^2$-bundles of P.

Our construction also produces complete Einstein metrics on the 2-plane bundles $V_{q_1,\ldots,q_l}$.

Theorem 1.6 (a) *There exists an* $l - 1$ *parameter family of complete Ricci-flat Kähler metrics on* $V_{q_1,\ldots,q_l}$ *provided that* $-q_i = p_i$ *for all* i.
(b) *Let* $0 < |q_i| < p_i$ *for all* i, *then there exists a complete non-Kähler Ricci-flat Einstein metric on* $V_{q_1,\ldots,q_l}$.
(c) *There exists a complete Kähler-Einstein metric with negative constant and infinite volume on an open disk-subbundle of* $V_{q_1,\ldots,q_l}$ *provided that* $-q_i > p_i$ *for all* i.

(*d*) *On each bundle $V_{q_1,\ldots,q_l}$ there exists at least a* 1*-parameter family of complete non-Kähler Einstein metrics with negative constant and infinite volume.*

This theorem generalizes the corresponding theorems in [BB]. A new feature is that when $l > 1$ we get continuous families of Einstein metrics in case (a), where one obtained only one Einstein metric (up to homothety) before. Also, in case (d) we get a 2-parameter family of Einstein metrics if in addition either $|q_i| > p_i$ for all i or $|q_i| < p_i$ for all i. The proof of this theorem will be given in Sect. 5.

Next we present our results on Einstein-Weyl structures. Recall that a Weyl structure on a manifold consists of a conformal class of metrics together with a linear connection on the bundle of length-scales determined by the conformal class. Such structures were introduced by H. Weyl [We] in order to have a conformally invariant analogue of Einstein's equation. The paper [Fo] gives a detailed study of various equivalent formulations of a Weyl structure. We will work with the definition of a Weyl structure as an equivalence class of pairs (h, η), where h is a Riemannian metric and η a 1-form, with $(h, \eta) \sim (\tilde{h}, \tilde{\eta})$ if there exists a smooth function w such that $\tilde{h} = \exp(2w)h$ and $\tilde{\eta} = \eta + 2dw$. The Levi-Civita connection of h and the 1-form η determine a unique torsion free affine connection D such that $Dh = \eta \otimes h$. (See, e.g., [PS1], Lemma 2.1.) A Weyl structure is Einstein if the symmetric part of the Ricci tensor of D is a function times h. This equation is invariant under the equivalence relation above. There is an extensive literature on (Einstein-)Weyl structures. We refer the reader to [Ga], [PS1], [PS2], [PPS], [Md], and [MPPS] for more information and a more complete guide to the literature.

We will consider in Sect. 8 solutions of the Einstein-Weyl equation on the S^2-bundles $W_{q_1,\ldots,q_l}$ and $\mathbb{RP}^2$-bundles $\overline{W}_{q_1,\ldots,q_l}$. The metrics h will again be of the form (1.1). As for the 1-form η we assume it to be of the form $Adt + Bf\theta$, where A, B are smooth functions of t satisfying appropriate boundary conditions described in Sect. 8.

Theorem 1.7 *Suppose that for all $i = 1, \ldots, l, 0 < |q_i| < p_i$. Suppose further that there exists $(\varepsilon_1, \ldots, \varepsilon_l)$ with $\varepsilon_i = \pm 1$ and at least one $\varepsilon_i = +1$ such that the integral*

$$\int_{-1}^{1} (\frac{p_1}{|q_1|} + \varepsilon_1 x)^{n_1} (\frac{p_2}{|q_2|} + \varepsilon_2 x)^{n_2} \cdots (\frac{p_l}{|q_l|} + \varepsilon_l x)^{n_l} x dx < 0.$$

Then on $W_{q_1,\ldots,q_l}$ there exists a 1*-parameter family of Einstein-Weyl metrics $[h, \eta]$ which are not locally conformal to the Einstein metrics in Theorem 1.2.*

The analog of Theorem 1.4 is

Theorem 1.8 *If $0 < |q_i| < p_i$ for all $i = 1, \ldots, l$, then there exists a* 1*-parameter family of Einstein-Weyl metrics on $\overline{W}_{q_1,\ldots,q_l}$ which are not locally conformal to the corresponding Einstein metrics in Theorem 1.4.*

None of these Einstein-Weyl structures are hermitian in the sense of [PPS].

2 The Einstein equations

We will study in this section the Einstein condition for metrics of the form (1.1) without regard to boundary conditions. It will be shown in particular that the Einstein equations are explicitly integrable in terms of polynomial functions. We will employ the notation established in Sect. 1. We will, however, omit the multi-index subscripts whenever there is no confusion. Recall that P is the principal $U(1)$ bundle over M with Euler class determined by the integers $q_1, \ldots, q_l$. This part of the analysis is common to the bundles $W, \overline{W}$, and V.

We saw in Sect. 1 that the above manifolds are all of the form $P \times I$, where I is a finite interval in the case of W and $\overline{W}$ and an infinite interval in the case of V, with appropriate identifications at the boundary of I. Metrically, we have an equi-distant family of hypersurfaces and so the Einstein condition can be read off from Proposition 2.1 of [EW].

Let $-U$ be the vector field on P generated by the $U(1)$-action. Then, since the connection form θ was chosen to be real-valued, it follows that $\theta(U) = 1$. If we choose as a local basis $\{N = \partial/\partial t, U, e_1, \ldots, e_{2n}\}$, where n is the complex dimension of M, and $\{e_1, \ldots, e_{2n}\}$ is a local basic orthonormal frame field with respect to the product metric $h_1 \times \ldots \times h_l$, then the shape operator L in [EW] has components $f'/f, g_i{}'/g_i$ and $tr(L) = f'/f + \sum_{i=1}^{l} 2n_i g_i{}'/g_i$. Together with the Kähler-Einstein condition $Ric(h_i) = p_i h_i$ for M_i, we see easily that the Einstein condition for (1.1) is given by the following system:

$$-\frac{f''}{f} - \sum_{i=1}^{l} 2n_i \frac{g_i''}{g_i} = c \tag{2.1}$$

$$-\frac{f''}{f} - \sum_{i=1}^{l} 2n_i \frac{f' g_i'}{f g_i} + \sum_{i=1}^{l} 2n_i \lambda_i{}^2 \frac{f^2}{g_i^4} = c \tag{2.2}$$

$$-\frac{g_i''}{g_i} - \frac{f' g_i'}{f g_i} - \sum_{j \neq i} 2n_j \frac{g_i' g_j'}{g_i g_j} - (2n_i - 1)(\frac{g_i'}{g_i})^2 + \frac{p_i}{g_i^2} - 2\lambda_i{}^2 \frac{f^2}{g_i^4} = c \tag{2.3}$$

where we have set $\lambda_i = q_i/2,\ i = 1, 2, \ldots, l$, and c is the Einstein constant. Notice that by the remark after Corollary 2.4 of [EW] there is no need to consider the Einstein condition for the off-diagonal components of the Ricci tensor.

Let us now equate (2.1) and (2.2). We obtain

$$\sum_{i=1}^{l} 2n_i \left(\frac{g_i''}{g_i} - \frac{f' g_i'}{f g_i} + \lambda_i^2 \frac{f^2}{g_i^4} \right) = 0.$$

Let us set

$$\mu_i = \frac{g_i''}{g_i} - \frac{f' g_i'}{f g_i} + \lambda_i^2 \frac{f^2}{g_i^4}. \tag{2.4}$$

In a generic situation, the μ_i would be linearly independent. Accordingly, we will consider only solutions for which $\mu_i \equiv 0$ for all i. When $l = 1$, this condition is of course automatically satisfied. We shall see in Sect. 7 that this is equivalent to the condition that the sectional curvatures of the metric h for all mixed 2-planes (i.e., spanned by a vertical and a horizontal vector) are equal. In Hermitian geometry terms, this is equivalent to requiring the identity $R(X, Y, Z, W) = R(JX, JY, JZ, JW)$ for all tangent vectors. It would be interesting to determine whether all solutions of (2.1)-(2.3) must satisfy this geometrical condition. See the end of Sect. 7 for further discussion about this point.

Remark 2.5 The complex structure J was defined by lifting the product complex structure of the base via the connection θ to the horizontal spaces and using the natural complex structure of the fibres. For a metric h of the form (1.1), it is hermitian with respect to the complex structure J_f which agrees with J on horizontal spaces and which is given by $J_f(U) = f(\partial/\partial t)$ on the fibres. (Caution: $-U$ is the infinitesimal generator of the S^1 action.) In the case of S^2-bundles, the uniformization theorem implies that the complex structures J and J_f are equivalent by an orientation (and fibre) preserving diffeomorphism. This is also the case for Ricci flat metrics on $\mathbb{R}^2$-bundles, i.e., cases (a) and (b) in Theorem 1.6. More details and the situation for cases (c) and (d) will be described in Remark 5.1. For the rest of the paper we will abuse notation and denote J_f by J. Actually, the vanishing of the torsion of J_f also needs verification, for which it is important that the Euler class of P is of type $(1, 1)$ with respect to the complex structure of the base.

Let ω denote the fundamental 2-form of the Hermitian metric h. Then

$$\omega = -fdt \wedge \theta + \sum_{i=1}^{l} g_i^2 {\pi_i}^* \omega_i .$$

It follows immediately from this that ω is closed iff

$$(g_i^2)' = -q_i f, \quad i = 1, \ldots, l. \tag{2.6}$$

As can easily be verified, this Kähler condition is a special solution of $\mu_i = 0$.

Suppose that h is a Kähler metric of the type in (1.1). Then the $U(1)$-action on our manifold is a symplectic action. One can verify that the negative of any anti-derivative of the function f is a moment map for this action. With this as motivation, we let r be an anti-derivative of f, i.e., $dr = f(t)dt$. Since f is non-negative, we can express t as a function of r and define functions α and β_i of r by

$$\alpha(r) = f(t), \quad \beta_i(r) = g_i(t).$$

Then (2.1)-(2.3) become

$$-\alpha\alpha'' - (\alpha')^2 - \sum_{i=1}^{l} 2n_i \left(\alpha^2 \frac{\beta_i''}{\beta_i} + \alpha\alpha' \frac{\beta_i'}{\beta_i} \right) = c,$$

$$-\alpha\alpha'' - (\alpha')^2 - \sum_{i=1}^{l} 2n_i \alpha\alpha' \frac{\beta_i'}{\beta_i} + \sum_{i=1}^{l} 2n_i \lambda_i^2 \frac{\alpha^2}{\beta_i^4} = c,$$

$$-\alpha^2 \frac{\beta_i''}{\beta_i} - 2\alpha\alpha' \frac{\beta_i'}{\beta_i} - \sum_{j=1}^{l} 2n_j \alpha^2 \frac{\beta_i'\beta_j'}{\beta_i\beta_j} + \alpha^2 \frac{(\beta_i')^2}{\beta_i^2} + \frac{p_i}{\beta_i^2} - 2\lambda_i^2 \frac{\alpha^2}{\beta_i^4} = c.$$

The condition $\mu_i = 0$ which we imposed now takes on the simple form

$$\frac{\beta_i''}{\beta_i} + \frac{\lambda_i^2}{\beta_i^4} = 0. \tag{2.7}$$

A first integral of (2.7) is

$$\frac{\beta_i^2(\beta_i')^2}{A_i\beta_i^2 + \lambda_i^2} = 1, \tag{2.8}$$

where A_i are constants of integration. The Kähler condition (2.6) simplifies to $\beta_i\beta_i' = -\lambda_i$ for all i, which corresponds to $A_i = 0$ for all i.

We will make one further transformation of the Einstein equations by letting $a(r) = \alpha(r)^2$ and $b_i(r) = \beta_i(r)^2$. The metric h then takes on the form

$$a(r)^{-1}\, dr^2 + a(r)\, \theta \otimes \theta + \sum_{i=1}^{l} b_i(r)\pi_i^* h_i,$$

and (2.1)-(2.3) become

$$\frac{1}{2}a'' + \frac{1}{2}a'(\log v)' + a \sum_{i=1}^{l} n_i \left(\frac{b_i''}{b_i} - \frac{1}{2}\left(\frac{b_i'}{b_i}\right)^2 \right) = -c, \tag{2.9}$$

$$\frac{1}{2}a'' + \frac{1}{2}a'(\log v)' - 2a \sum_{i=1}^{l} n_i \frac{\lambda_i^2}{b_i^2} = -c, \tag{2.10}$$

$$-\frac{p_i}{b_i} + 2a\frac{\lambda_i^2}{b_i^2} + \frac{a}{2}\left(\frac{b_i''}{b_i} - \left(\frac{b_i'}{b_i}\right)^2 \right) + \frac{a}{2}(\log v)' \frac{b_i'}{b_i} + \frac{a'b_i'}{2b_i} = -c, \tag{2.11}$$

where

$$v = \prod_{i=1}^{l} g_i^{2n_i} = \prod_i b_i^{n_i}.$$

Notice that, up to a constant, the integral of v with respect to the variable r is precisely the volume of the metric h. Also, a appears linearly in each of these equations, and for each i, (2.11) is a first order linear equation in a, which is readily integrable.

In terms of b_i, (2.8) becomes

$$(b_i')^2 = 4(A_i b_i + \lambda_i^2), \tag{2.12}$$

and the Kähler condition is given by $-b_i' = 2\lambda_i = q_i$. Indeed, solving (2.12), we obtain

$$b_i(r) = \begin{cases} A_i(r+R_i)^2 - \frac{\lambda_i^2}{A_i} & \text{if } A_i \neq 0 \\ \pm 2\lambda_i(r+R_i) & \text{otherwise,} \end{cases} \tag{2.13}$$

where R_i are constants.

Lemma 2.14 *Functions $a, b_i, i = 1, \ldots, l$ give a solution of (2.9)-(2.11) on $P \times int(I)$ such that $\mu_i \equiv 0$ and $A_i \neq 0$ for all i iff the following hold.*

i. There exist constants r_0 and E such that for all i

$$E = \frac{p_i}{A_i} + c\left(\frac{\lambda_i}{A_i}\right)^2, \tag{2.15}$$

$$b_i(r) = A_i(r+r_0)^2 - \frac{\lambda_i^2}{A_i}. \tag{2.16}$$

ii. On each subinterval of $int(I)$ on which $r + r_0$ does not vanish, there exists $\varepsilon_i = \pm 1$ such that $\varepsilon_i A_i^{-1}(A_i b_i + \lambda_i^2)^{1/2} = r + r_0$.
iii. a is given by

$$a(r) = \frac{r+r_0}{v}\int_0^r v(E - c(s+r_0)^2)(s+r_0)^{-2}ds. \tag{2.17}$$

Proof. Starting with a solution of (2.9)-(2.11) such that $\mu_i = 0$ and $A_i \neq 0$, it follows that b_i has the form given in the first case of (2.13). Using this form, we multiply (2.11) by $b_i A_i^{-1}$, and obtain upon simplification

$$(r+R_i)a' = a(1-(r+R_i)(\log v)') + E_i - c(r+R_i)^2, \tag{2.18}$$

where $E_i = \frac{p_i}{A_i} + c\left(\frac{\lambda_i}{A_i}\right)^2$. If we subtract equation j from equation i in (2.18), we have

$$(R_i - R_j)a' = -a(\log v)'(R_i - R_j) + (E_i - E_j) - c(R_i - R_j)(2r + R_i + R_j),$$

while if we subtract $(r + R_i)$ times equation j from $(r + R_j)$ times equation i in (2.18) we get

$$0 = (R_i - R_j)a + c(R_i - R_j)(r+R_i)(r+R_j) - E_i(r+R_j) + E_j(r+R_i).$$

Differentiating this last equation yields

$$(R_j - R_i)a' = c(R_i - R_j)(2r + R_i + R_j) - E_i + E_j.$$

Hence we must have $R_i = R_j = r_0$ and $E_i = E_j = E$ for constants r_0 and E. This proves (i). From this and (2.13) it follows that

$$\frac{1}{A_i}(A_i b_i + \lambda_i^2)^{1/2} = sgn(A_i)|r + r_0|,$$

which proves (ii). The formula for a follows by straight-forward integration. Conversely, if a and b_i are as given above and (2.15) holds, then (2.11) holds for all i and $\mu_i = 0$. It remains to check (2.9). It follows from using (2.18) twice that

$$a'' = \left(\frac{1}{r + r_0} - (\log v)'\right) a' - \left(\frac{1}{(r + r_0)^2} + (\log v)''\right) a - \frac{E}{(r + r_0)^2} - c.$$

Using (2.18) once more with (2.16), we obtain (2.9). This completes the proof of (2.14).

Remark 2.19 If the metric is required to be Kähler, then similar computations show that there are constants D_i and $\hat{E}$ such that for all i

$$b_i = -2\lambda_i r + D_i,$$

$$\hat{E} = \frac{cD_i - p_i}{\lambda_i}.$$

Furthermore,

$$a = v^{-1} \int_0^r v(\hat{E} - 2cs)ds.$$

3 Existence on S^2 bundles

In this section we will give the proof of Theorem 1.2. Recall that smooth functions $f, g_i, i = 1, \ldots, l$ on $I = [0, T]$ define via (1.1) a smooth metric on the S^2-bundle W iff

(a) (positivity) f is positive on $(0, T)$ and g_i are positive on $[0, T]$,
(b) (smoothness) in a neighborhood of 0, $f(t) = t\phi(t^2)$ for some smooth function ϕ with $\phi(0) = 1$, $g_i(t) = \psi_i(t^2)$ for smooth functions ψ_i, and analogous conditions hold in a neighborhood of T.

Discussions of these conditions can be found, for example, in [BB], [PP2], or [S]. Perhaps more systematically, they can also be deduced using the argument in the proof of Lemma 1 in [EW]. In any case, we choose the anti-derivative of f so that the interval $[0, T]$ corresponds, under our change of variable from t to $\tilde{r}$, to the interval $\tilde{I} = [r_0, r_0 + R]$, where we have set $\tilde{r} = r + r_0$, so that r ranges over the interval $[0, R]$. The positivity condition (a) translates into: a positive on $(0, R)$ and b_i positive on $[0, R]$. The condition $\phi(0) = 1$ (resp. $\phi(T) = -1$) becomes $a'(0) = 2$ (resp. $a'(R) = -2$). By Lemma 2.14, if we choose a so that $a(0) = 0 = a(R)$, then we can easily verify that the smoothness condition (b) is satisfied to second order. This means that we have an Einstein metric of class C^2. By Theorem 5.2 in [DK], the metric is smooth.

We will assume in the following that $c > 0$ and choose $E > 0$ in a manner to be specified later. If we substitute the boundary conditions for a into (2.11), we find that r_0 and $-(R + r_0)$ are roots of the quadratic equation

$$cx^2 + 2x - E = 0. \tag{3.1}$$

Hence the length R of the interval is $2c^{-1}$ and $r_0(R + r_0) = Ec^{-1}$. From this we see that the assumption that $E > 0$ is equivalent to the condition that the interval $\tilde{I}$ does not contain 0. So once we have chosen c and E, then the interval $\tilde{I}$ is determined by the choice of a root of (3.1). By replacing $\tilde{r}$ by $-\tilde{r}$, we can assume that the positive root of (3.1) is chosen.

Furthermore, (2.15) gives a quadratic equation from which we can solve for A_i. Of course, there are two solutions, but if the condition

$$0 < |q_i| < p_i, \quad i = 1, 2, \cdots, l$$

in the hypothesis of Theorem 1.2 holds, then the positivity condition for b_i holds no matter which A_i we pick. The verification of this is routine, although there are a number of cases to check. As an example, let $r_0 > 0$ and suppose that $A_i < 0$. Note, by the way, that in this case $\varepsilon_i = -1$ (cf 2.14(ii)). Then

$$A_i = \frac{p_i - (p_i^2 + cEq_i^2)^{1/2}}{2E}.$$

But by (2.16) and (2.15),

$$b_i = \frac{p_i}{c} + A_i\left((r + r_0)^2 - \frac{E}{c}\right) > 0 \tag{3.2}$$

iff

$$E - c(r + r_0)^2 > \frac{2Ep_i}{p_i - (p_i^2 + cEq_i^2)^{1/2}} \quad .$$

Using (3.1), we see that the last inequality holds on $[0, R]$ iff

$$\frac{1 + (1 + cE)^{1/2}}{c} = r_0 + R \; < \; \frac{Ep_i}{(p_i^2 + cEq_i^2)^{1/2} - p_i}$$

iff

$$1 + (1 + cE)^{1/2} \; < \; \frac{p_i^2}{q_i^2}\left(1 + \left(1 + Ec\frac{q_i^2}{p_i^2}\right)^{1/2}\right),$$

which holds iff $0 < |q_i/p_i| < 1$.

It remains to guarantee the positivity condition on a and ensure that $a(R) = 0$.

By (2.17), $a(R) = 0$ is equivalent to

$$\int_{r_0}^{r_0+R}\left[\prod_{j=1}^{l}\left(s^2 - (\frac{\lambda_j}{A_j})^2\right)^{n_j}\right](E - cs^2)s^{-2}ds = 0.$$

Let $x = cs - (1 + cE)^{1/2}$, then the above integral becomes (modulo constants)

$$\int_{-1}^{1}\left[\prod_{j=1}^{l}\left(x^2+2(1+cE)^{1/2}x+1+\frac{\varepsilon_j p_j(p_j^2+4cE\lambda_j^2)^{1/2}-p_j^2}{2\lambda_j^2}\right)^{n_j}\right]$$
$$\times\frac{x^2+2x(1+cE)^{1/2}+1}{(x+(1+cE)^{1/2})^2}dx.$$

Note that ε_j has appeared to account for the choice of r_0 and A_j: if r_0 is chosen to be positive, then ε_i is just the sign of A_i. Let $F(E)$ denote the value of the above integral as a function of E. We will prove that there is a positive value of E so that $F(E)=0$. With this choice of E in our preceeding discussion, we obtain in addition the desired boundary condition $a(R)=0$.

First, we compute $\lim_{E\to 0^+}F(E)$. Note that for $E>0$,

$$\int_{-1}^{1}\frac{x^2+2(1+cE)^{1/2}x+1}{(x+(1+cE)^{1/2})^2}dx=0,$$

so we can write

$$F(E) = F(E)-\int_{-1}^{1}\left[\prod_{j=1}^{l}\left(\frac{\varepsilon_j p_j(p_j^2+4cE\lambda_j^2)^{1/2}-p_j^2}{2\lambda_j^2}-cE\right)^{n_j}\right]$$
$$\times\frac{x^2+2(1+cE)^{1/2}x+1}{(x+(1+cE)^{1/2})^2}dx.$$

Let

$$z=(x+(1+cE)^{1/2})^2=c^2s^2,$$

and

$$\Gamma_j=\frac{\varepsilon_j p_j(p_j^2+4cE{\lambda_j}^2)^{1/2}-p_j^2}{2{\lambda_i}^2}-cE.$$

Then

$$F(E)=\int_{-1}^{1}\left[\prod_{j=1}^{l}(z+\Gamma_j)^{n_j}-\prod_{j=1}^{l}{\Gamma_j}^{n_j}\right]\left(\frac{z-cE}{z}\right)dx.$$

If we expand the term in square brackets, we see that z divides into the resulting expression. Hence, we may interchange the order of taking the limit as $E\to 0^+$ and integration to obtain $\lim_{E\to 0^+}F(E)=$

$$(-1)^{\sum_{\varepsilon_j=-1}n_j}\int_{-1}^{1}\left[\prod_{\varepsilon_j=1}(1+x)^{2n_j}\right]\left[\prod_{\varepsilon_j=-1}\left(\frac{4p_j^2}{q_j^2}-(x+1)^2\right)^{n_j}\right]dx$$
$$-2\prod_{j=1}^{l}\left(\frac{(\varepsilon_j-1)p_j^2}{2\lambda_j^2}\right)^{n_j}.$$

By assumption, at least one of the ε_j equals 1, so the last term in the above expression vanishes. Furthermore, since $0<|q_i|<p_i$, it is clear that $\lim_{E\to 0^+}F(E)>0$

if $\sum_{\varepsilon_j=-1} n_j$ is even, and $\lim_{E\to 0^+} F(E) < 0$ if $\sum_{\varepsilon_j=-1} n_j$ is odd. On the other hand,

$$\begin{aligned}\lim_{E\to+\infty}[F(E)E^{\frac{1}{2}-\sum_j \frac{n_j}{2}}] &= \int_{-1}^{1}\left[\prod_{j=1}^{l}\left(2c^{1/2}x + c^{1/2}\varepsilon_j \frac{p_j}{|\lambda_j|}\right)^{n_j}\right] 2c^{-1/2}xdx \\ &= (-1)^{\sum_{\varepsilon_j=-1} n_j} K\int_{-1}^{1}\prod_{j=1}^{l}\left(\frac{p_j}{|q_j|} + \varepsilon_j x\right)^{n_j} xdx,\end{aligned}$$

where K is a positive constant. Therefore, if

$$\int_{-1}^{1}\left[\prod_{j=1}^{l}\left(\frac{p_j}{|q_j|} + \varepsilon_j x\right)^{n_j}\right] xdx < 0,$$

then there exists some $E > 0$ such that $F(E) = 0$.

Finally, $a(r) > 0$ for $0 < r < R$ because $E - c(r + r_0)^2$ is monotone, so that the integral part of (2.17) has the same sign as $r + r_0$ before it becomes 0. The original variable t can now be recovered from $a(r)$ by

$$t = \int_0^r a(s)^{-1/2} ds. \tag{3.3}$$

This completes the proof of Theorem 1.2.

Remark 3.4 Since we have chosen $E > 0$, i.e., $\tilde{I} = [r_0, R + r_0]$ does not contain 0, it follows that $A_i b_i + \lambda_i^2$ is never zero, and so $(g_i^2)'$ is never zero on $(0, T)$. This means that the Einstein metric does not have fibrewise antipodal symmetry.

Remark 3.5 For the Kähler case, the above considerations simplify greatly. In particular, there is no longer any singularity in the integral defining a. The boundary conditions for $a(r)$ give $\hat{E} = 2$ and $R = 2c^{-1}$. The positivity of b_i is again guaranteed by the condition $0 < |q_i| < p_i$, and the condition $a(R) = 0$ is equivalent to

$$\int_0^R \left[\prod_{j=1}^{l}\left(\frac{p_j + q_j}{c} - q_j s\right)^{n_j}\right] (1 - cs) ds = 0.$$

Letting $x = cs - 1$, we obtain (modulo constants)

$$\int_{-1}^{1}\left[\prod_{j=1}^{l}\left(\frac{p_j}{q_j} - x\right)^{n_j}\right] xdx = 0,$$

which is the integral condition in Theorem 1.3.

4 Existence on RP^2 bundles

We turn now to the $\mathbb{RP}^2$-bundles $\overline{W}$. Recall that the circle acts on $\mathbb{RP}^2$ with circles as principal orbits which collapse on one end to a fixed point and on the other end to a circle via a double covering map. In order for functions f, g_i to define a smooth metric h of type (1.1) on $\overline{W}$, it is necessary and sufficient that

(a) (positivity) f is positive on $(0, T]$, g_i are positive on $[0, T]$,
(b) (smoothness at 0) in a neighborhood of 0, $f(t) = t\phi(t^2)$ for some smooth function ϕ with $\phi(0) = 1$, $g_i(t) = \psi_i(t^2)$ for smooth functions ψ_i,
(c) (smoothness at T) in a neighborhood of T, $f(T-t) = \hat{\phi}(t^2)$ and $g_i(T-t) = \hat{\psi}_i(t^2)$ for smooth functions $\hat{\phi}, \hat{\psi}_i$.

We will again consider solutions such that all $\mu_i \equiv 0$. In terms of the variable r, the function a must be positive on $(0, R]$ and b_i must be positive on $[0, R]$. Furthermore, $a(0) = 0, a'(0) = 2, a'(R) = 0$, and $b_i'(R) = 0$ must hold. If so, then as in the proof of Theorem 1.2, the conditions (a)-(c) will be satisfied.

Any metric on $\overline{W}$ lifts to a metric on W such that at the midpoint of the corresponding interval I the derivatives of f and g_i vanish simultaneously. Conversely, from (2.8) and Lemma 2.14, we see that for an Einstein metric h of the type under consideration, this can happen only *once* and then *only* at $\tilde{r} = 0 \in int(\hat{I})$. Therefore, back on $\overline{W}$, in contrast to the situation in Theorem 1.2, we must have $r_0 + R = 0$. It follows that the boundary condition $b_i'(R) = 0$ automatically holds. By (2.16) and the positivity condition, it follows that for all i, $A_i < 0$. Furthermore, if we evaluate (2.11) at $r = R$ and use (2.16) with $r_0 + R = 0$, we obtain $-a(R) = E < 0$. Applying the boundary conditions for a at 0 to (2.11), we conclude that r_0 is a (negative) root of the quadratic equation (3.1). So again, the Einstein constant c and the constant E determine, up to a choice of roots of (3.1), the interval $\hat{I}$.

We address next the positivity condition on the b_i. Let

$$r_0 = \frac{-1-(1+cE)^{1/2}}{c} \tag{4.1}$$

and

$$A_i = \frac{p_i - (p_i^2 + cEq_i^2)^{1/2}}{2E}, \tag{4.2}$$

with $-cE < 1$ and $-cE(q_i^2/p_i^2) < 1$ to guarantee that r_0 and A_i are real. As in the proof of Theorem 1.2, positivity of b_i on $[0, R]$ is equivalent to

$$c(r+r_0)^2 - E < \frac{-2Ep_i}{p_i - (p_i + cEq_i^2)^{1/2}},$$

which, upon using (3.1) and the above expression for r_0, is equivalent to

$$1 + (1+cE)^{1/2} < \frac{p_i^2}{q_i^2}\left(1 + \left(1 + \frac{cEq_i^2}{p_i^2}\right)^{1/2}\right).$$

This is satisfied iff $0 < |q_i| < p_i$ holds, as can be easily verified.

Remark 4.3 In the above we may also choose r_0 to be the other root of (3.1). However, the inequality would not hold if we choose the other value of A_i. We will see shortly that the above choices for r_0 and A_j are important for the rest of the proof. Hence there is less flexibility in the proof of Theorem 1.4. Notice also that when $l = 1$ (the Bérard Bergery-Page-Pope case), the consistency condition (2.15) is not required.

We will now show that there is a value of $E < 0$ with $-c^{-1} < E$ so that the boundary condition $a'(R) = 0$ holds. According to (2.17),

$$a(r) = \frac{r}{\prod_{j=1}^{l}(A_j^2 r^2 - \lambda_j^2)^{n_j}} \int_{r_0}^{r} \left(\prod_j (A_j^2 s^2 - \lambda_j^2)^{n_j} \right) (E - cs^2) s^{-2} ds.$$

Since one can easily check that $a(R + r_0) = a(0) = -E$, it follows that

$$a'(0) = \lim_{r \to 0^-} \left(\frac{a(r) + E}{r} \right).$$

So $a'(0) = 0$ iff

$$\text{(4.4)} \quad \lim_{r \to 0^-} \left[\int_{r_0}^{r} \left(\prod_j (A_j s^2 - \frac{\lambda_j^2}{A_j})^{n_j} \right) \left(\frac{E - cs^2}{s^2} \right) ds + \frac{E}{r} \prod_j (A_j r^2 - \frac{\lambda_j^2}{A_j})^{n_j} \right] = 0.$$

Let $G(E)$ denote the value of this limit as a function of E. First we compute $\lim_{E \to 0^-} G(E)$. Observe that as a Laurent polynomial in r, the expression in (4.4) in square brackets has cancelling $1/r$-terms and the polynomial part of the second term has no constant term. So, using (4.1) and (4.2) in taking the limit, and letting $s = r_0(1 - y)$, we get

$$\lim_{E \to 0^-} G(E) = -2 \int_0^1 \prod_j \left(\frac{p_j}{c} - \frac{4\lambda_j^2}{cp_j}(1 - y)^2 \right)^{n_j} dy,$$

which is negative since $0 < |q_i| < p_i$.

Next, to compute $\lim_{E \to -1/c} G(E)$, we again let $r = r_0(1 - y)$ in (4.4) and add and subtract the term

$$\prod_j \left(-\frac{\lambda_j^2}{A_j} \right)^{n_j} \int_0^y \left(\frac{-E + cr_0^2(1 - x)^2}{r_0(1 - x)^2} \right) dx$$

inside the square brackets. After recombining the terms, integrating, and using (3.1), we obtain

$$\int_0^{1^-} \left[\prod_{j=1}^{l} \left(A_j r_0^2 (1 - y)^2 - \frac{\lambda_j^2}{A_j} \right)^{n_j} - \prod_{j=1}^{l} \left(-\frac{\lambda_j^2}{A_j} \right)^{n_j} \right] \left(\frac{-E + cr_0^2(1 - y)^2}{r_0(1 - y)^2} \right) dy$$

$$+ 2(1 + cr_0) \prod_{j=1}^{l} \left(-\frac{\lambda_j^2}{A_j} \right)^{n_j} .$$

As E approach $-c^{-1}$, the second term tends to 0, and using (4.1) and (4.2), we see that the integral tends to

$$\int_0^1 \left[\prod_j \left(\frac{p_j + (p_j^2 - 4\lambda_j^2)^{1/2}}{2c} \right)^{n_j} - \prod_j \left(\frac{p_j + (p_j^2 - 4\lambda_j^2)^{1/2}}{2c} - \frac{2\lambda_j^2}{c(p_j + (p_j^2 - 4\lambda_j^2)^{1/2})} (1-y)^2 \right)^{n_j} \right] \cdot \left(\frac{1 + (1-y)^2}{(1-y)^2} \right) dy,$$

which is positive. Therefore there is some E lying between $-1/c$ and 0 such that $G(E) = 0$.

Finally, the positivity of a follows from that of b_i and the negativity of $E - c(r + r_0)^2$. This completes the proof of Theorem 1.4.

5 Complete Einstein metrics

We now consider complete Einstein metrics on the 2-plane bundles $V_{q_1,\ldots,q_l}$ of type (1.1) with $\mu_i \equiv 0$ for all i. By the theorem of Bonnet-Myers, the Einstein constant c must be non-positive. Smooth functions $f, g_i, i = 1, \ldots, l$ give a complete smooth metric on E iff

(a) (positivity) f is positive on $(0, +\infty)$, g_i are positive on $[0, +\infty)$,
(b) (smoothness at 0) in a neighborhood of 0, $f(t) = t\phi(t^2)$ for some smooth function ϕ with $\phi(0) = 1$, $g_i(t) = \psi_i(t^2)$ for smooth functions ψ_i.

Note that completeness requires the integral (3.3) to be infinity when $r = R$, where $[0, R)$ is the domain of the functions a, b_i.

We consider first complete Ricci-flat Kähler-Einstein metrics. Applying the boundary conditions at $r = 0$, we obtain immediately that $-q_i = p_i$ for all i. It follows that $\hat{E}$ in Remark 2.19 equals 2. By completeness, the length R of the interval of definition of the functions a, b_i must be infinite because the volume of these metrics must be infinite. The positivity conditions are satisfied as long as D_i are positive. Up to homothety we can fix the value of one of the D_i, but we may choose the rest arbitrarily, giving an $l - 1$ parameter family of solutions of the Einstein equation.

For Kähler-Einstein metrics with negative constant c, the boundary conditions at $r = 0$ force $-q_i$ to be larger than p_i. Choosing $\hat{E} \geq 2$ (see Remark (2.19)), it follows that $D_i = c^{-1}(\frac{1}{2}\hat{E} q_i + p_i)$ are positive. Changing the value of $\hat{E}$ only results in translating the variable r. So we obtain only one solution. Notice that

R is finite iff the volume is finite, in which case (3.3) would also be finite, contradicting completeness. Therefore the volume must be infinite.

We consider next the non-Kähler case. For $c = 0$, it follows from the consistency condition (2.15) that $p_i/A_i = E$. Since the volume must be infinite, $R = +\infty$, and, by replacing $\tilde{r}$ by $-\tilde{r}$ if necessary, we may assume that $r_0 = E/2 > 0$. Hence $A_i > 0$ and one checks easily that b_i satisfy the positivity condition iff $0 < |q_i| < p_i$ for all i. Changing the value of E only results in a homothetic metric.

Finally, if $c < 0$ in the non-Kähler case, one sees first that completeness again implies that $R = +\infty$. From the boundary conditions at $r = 0$, one obtains that r_0 is a root of (3.1). Suppose that $|q_i| \le p_i$. Then we choose $0 < E < -c^{-1}$, and

$$r_0 = \frac{1 + (1 + Ec)^{1/2}}{-c}, \quad A_i = \frac{p_i + (p_i^2 + Ecq_i^2)^{1/2}}{2E}.$$

It follows that b_i satisfies the positivity condition. Notice that if $|q_i| < p_i$, we may also choose $r_0 = -c^{-1}[1 - (1 + Ec)^{1/2}]$. As $E \to -c^{-1}$, the two values of r_0 become equal. Also, as $E \to 0^+$, A_i tends to $+\infty$.

When $|q_i| > p_i$, we may choose $0 < E < \min(p_1^2/q_1^2, \dots, p_l^2/q_l^2)$, $r_0 = -c^{-1}[1 + (1 + Ec)^{1/2}]$, and $A_i = (p_i - (p_i^2 + Ecq_i^2)^{1/2})/2E$. Then b_i will satisfy the positivity condition. We may also choose $E < 0$.

Therefore, we obtain a 2-parameter family of Einstein metrics if $|q_i| > p_i$ for all i, or if $|q_i| < p_i$ for all i. We obtain a 1-parameter family of Einstein metrics in all remaining cases. This completes the proof of Theorem 1.6.

Remark 5.1 We now give the discussion of complex structures promised in Remark 2.5. First, note that in order to compare the complex structures J_f and J, it suffices to do so on the fibres. On $\mathbb{R}^2$ consider the $U(1)$-equivariant injection $\mathscr{S}$ given in polar coordinates by $\mathscr{S}(t, \theta) = (\sigma(t), \theta)$ where σ is a solution of the equation $\sigma' f = \sigma$ satisfying appropriate boundary conditions so that $\mathscr{S}$ is a smooth map . Then $\mathscr{S}$ becomes a holomorphic map on $\mathbb{R}^2$ equipped with the complex structure J_f on the domain and J on the range. One then checks that $\mathscr{S}$ is surjective iff

$$\int_0^\infty \frac{d\rho}{a(\rho)} = \infty,$$

where $a(r) = f(t)^2$. Looking at our solutions, one finds that the above integral diverges exactly in the Ricci flat cases. Thus J_f and J are equivalent iff $c = 0$. In the negative case, J_f is equivalent to J restricted to an open disk in $\mathbb{R}^2$. This is exactly the discrepancy observed in [PePo, p.319, remark after (3.14)] between their example and the corresponding example of Bérard Bergery (see 9.129(c) in [Be]).

Remark 5.2 It is interesting to study other boundary conditions such as the vanishing of some of the g_i at $t = 0$. In a forthcoming paper by A. Dancer and the second author [DW] about Kähler-Einstein metrics of cohomogeneity one, the possibilities for the collapse of the hypersurfaces $P \times \{t\}$ to lower-dimensional manifolds will be examined in greater detail.

6 Examples

In this section we first discuss some topological properties of our S^2-bundles and then we indicate applications of Theorems 1.2–1.4 with explicit examples.

Let χ denote the Euler class $\sum_i q_i \pi_i{}^* \alpha_i \in H^2(M;\mathbb{Z})$ of the principal circle bundle $P_{q_1,\ldots,q_l}$. The associated complex line bundle $V_{q_1,\ldots,q_l}$ has a unique compatible holomorphic structure since its Euler class is of type $(1,1)$ and M is simply connected. Then $W_{q_1,\ldots,q_l}$ can be identified with the projectivized bundle $\mathbb{P}(V \oplus \mathbf{1})$, where $\mathbf{1}$ denotes the trivial complex line bundle. We will summarise below some well-known facts which allow us to compute topological invariants of $W_{q_1,\ldots,q_l}$.

Let $\xi = V \oplus \mathbf{1}$ and π be the projection map from $\mathbb{P}\xi$ onto M. Then there is an exact sequence

$$0 \to \gamma \to \pi^*\xi \to \nu \to 0, \tag{6.1}$$

where over each fibre of $\mathbb{P}\xi$, γ is the tautological line bundle and ν is the quotient bundle. Denote $-c_1(\gamma) \in H^2(\mathbb{P}\xi;\mathbb{Z})$ by s. The Leray-Hirsch theorem asserts that $H^*(W;\mathbb{Z})$ is a free $H^*(M;\mathbb{Z})$-module with generators $1, s$ and its ring structure is given by

$$s^2 + (\pi^* c_1(\xi))s + \pi^* c_2(\xi) = 0.$$

In the present situation, we have

$$s^2 = -\pi^* c_1(V)s = -\chi \cdot s, \tag{6.2}$$

using the module structure.

We will identify the holomorphic tangent bundle of W (resp. M) with the tangent bundle TW (resp. TM) and denote by $\mathscr{F}$ the (holomorphic) tangent bundle along fibres of $\pi : W \to M$. Then it is well-known that (see [H, p. 102])

$$\mathscr{F} \cong \gamma^{-1} \otimes \nu. \tag{6.3}$$

Proposition 6.4 *We retain the above notation.*
(a) *The complex manifolds $W_{q_1,\ldots,q_l}$ are simply connected with cohomology ring*

$$H^*(W;\mathbb{Z}) = H^*(M;\mathbb{Z})[s]/(s^2 + \chi \cdot s).$$

(b) $c_1(W) = \pi^*(c_1(M) + \chi) + 2s$, *and it is positive iff* $0 < |q_i| < p_i$ *for all* i. *Moreover,* $c_2(W) = \pi^* c_2(M) + \pi^* c_1(M) \cdot (\pi^*\chi + 2s)$.
(c) $p_1(W) = \pi^* p_1(M) + (\pi^*\chi)^2$.
(d) *The second Stiefel-Whitney class* $w_2(W) = \pi^* c_1(M) + \pi^*\chi$ *(mod 2). Hence W is spin iff for all i, p_i and q_i are of the same parity.*

Proof. Part (a) follows from the homotopy exact sequence and the remarks before the proposition. The Chern classes are obtained by routine computations using (6.1) and (6.3) and the fact that $TW = \pi^* TM \oplus \mathscr{F}$. The Pontrjagin classes are calculated from the Chern classes using (6.2). Since W is complex, the $2i$th Stiefel-Whitney class is just the reduction mod 2 of the ith Chern class.

It remains to examine the positivity of the first Chern class. We will compute the Ricci form of the induced hermitian metric on $\mathscr{F}$ for some suitably chosen hermitian metric on W of the form (1.1). Since $f\partial/\partial r$ is a (real) holomorphic vector field along $\mathscr{F}$, the Ricci form is given by $-id'd'' \log f^2$. This equals

$$-\frac{f''}{f}(fdr \wedge \theta) - f' \sum_i q_i \pi_i^* \omega_i .$$

So $2\pi c_1(W) = 2\pi(c_1(\mathscr{F}) + \pi^* c_1(M))$ is represented by

$$-\frac{f''}{f}(fdr \wedge \theta) - f' \sum_i q_i \pi_i^* \omega_i + \sum_i p_i \pi_i^* \omega_i .$$

We may choose a Kähler metric on W with $f(t) = \sin t$. Then the above Ricci form is positive iff $|q_i| < |p_i|,\ i = 1, 2, \dots, l$. This completes the proof of (6.4).

To see that our constructions give non-trivial Einstein metrics, we prove the following

Proposition 6.5 *Let* $M = \mathbb{CP}^n \times \mathbb{CP}^m$.

(*a*) *The total space* W_{q_1,q_2} *is not homotopy equivalent to the product* $S^2 \times M$.
(*b*) *For* $n \neq m$, W_{q_1,q_2} *is homeomorphic to* W_{r_1,r_2} *iff* $|q_1| = |r_1|$ *and* $|q_2| = |r_2|$.
(*c*) *For* $n = m$, W_{q_1,q_2} *is homeomorphic to* W_{r_1,r_2}, *iff either* $|q_i| = |r_i|,\ i = 1, 2$ *or else* $|q_1| = |r_2|,\ |q_2| = |r_1|$.

Proof. In this case, Proposition 6.4 implies that

$$H^*(W; \mathbb{Z}) = \mathbb{Z}[\alpha_1, \alpha_2, s]/(\alpha_1^{n+1}, \alpha_2^{m+1}, s^2 + (q_1\alpha_1 + q_2\alpha_2)s),$$

where α_1 is a generator of $H^2(\mathbb{CP}^n; \mathbb{Z})$ and α_2 is a generator of $H^2(\mathbb{CP}^m; \mathbb{Z})$. To prove (a), one only has to observe that if $2 \leq n \leq m$, then the cohomology ring of W does not contain a 2-dimensional class whose square is 0, and when $1 = n < m$, then the only 2-dimensional classes whose squares equal 0 are proportional to α_1. Finally, if $1 = n = m$, then such classes are proportional either to α_1 or to α_2.

Next, the first Pontrjagin class of M is $p_1(M) = (n + 1)\alpha_1^2 + (m + 1)\alpha_2^2$. So, by (6.4(c)), the first Pontrjagin class of W is

$$p_1(W) = (n + q_1^2 + 1)\alpha_1^2 + (m + q_2^2 + 1)\alpha_2^2 + 2q_1q_2\alpha_1\alpha_2.$$

Recall that our cohomology rings are torsion free and that rational Pontrjagin classes are homeomorphism invariants. Suppose there is a homeomorphism Ψ between W_{r_1,r_2} and W_{q_1,q_2} inducing a ring isomorphism Ψ^* between

$$H^*(W_{q_1,q_2};\mathbb{Z}) \approx \mathbb{Z}[\alpha_1,\alpha_2,s]/(\alpha_1^{n+1},\alpha_2^{m+1},s^2+(q_1\alpha_1+q_2\alpha_2)s)$$

and

$$H^*(W_{r_1,r_2};\mathbb{Z}) \approx \mathbb{Z}[\beta_1,\beta_2,t]/(\beta_1^{n+1},\beta_2^{m+1},t^2+(r_1\beta_1+r_2\beta_2)t).$$

Assume that

$$\Psi^*(\alpha_1) = u_1\beta_1+u_2\beta_2+u_3t$$

$$\Psi^*(\alpha_2) = v_1\beta_1+v_2\beta_2+v_3t$$

$$\Psi^*(s) = w_1\beta_1+w_2\beta_2+w_3t.$$

For $n,\ m>1$, the condition $\Psi^*(s^2)=(\Psi^*(s))^2$ gives rise to the equations

(6.6) $$w_1[w_1+q_1u_1+q_2v_1]=0,$$

(6.7) $$w_2[w_2+q_1u_2+q_2v_2]=0,$$

(6.8) $$2w_1w_2+w_1[q_1u_2+q_2v_2]+w_2[q_1u_1+q_2v_1]=0,$$

(6.9) $$2w_1w_3+w_1[q_1u_3+q_2v_3]+w_3[q_1u_1+q_2v_1]=r_1w_3[w_3+q_1u_3+q_2v_3],$$

(6.10) $$2w_2w_3+w_2[q_1u_3+q_2v_3]+w_3[q_1u_2+q_2v_2]=r_2w_3[w_3+q_1u_3+q_2v_3].$$

From $\Psi^*(p_1(W_{q_1,q_2}))=p_1(W_{r_1,r_2})$, we obtain

(6.11) $$n+r_1^2+1 = (n+q_1^2+1)u_1^2+(m+q_2^2+1)v_1^2+2q_1q_2u_1v_1,$$

(6.12) $$m+r_2^2+1 = (n+q_1^2+1)u_2^2+(m+q_2^2+1)v_2^2+2q_1q_2u_2v_2,$$

(6.13) $$r_1r_2 = (n+q_1^2+1)u_1u_2+(m+q_2^2+1)v_1v_2 +q_1q_2(u_1v_2+u_2v_1).$$

To analyse these equations, there are two cases to consider. First, if w_1 (respectively w_2) is non-zero, then using (6.8)-(6.10) together with the invertibility of Ψ^* and the fact that its determinant is therefore ± 1, it follows that $w_1=r_1w_3$ and $w_2=r_2w_3$ with $w_3^2=1$. From (6.11)-(6.13) we then obtain the equations

$$n+1 = (n+1)u_1^2+(m+1)v_1^2,$$

$$m+1 = (n+1)u_2^2+(m+1)v_2^2,$$

$$0 = (n+1)u_1u_2+(m+1)v_1v_2.$$

The desired conclusion then follows easily from these equations.

The second case to consider is when $w_1=w_2=0$, so that $w_3^2=1$ and $u_1v_2-v_1u_2=\pm 1$. Then, by looking at (6.9) and (6.10), we see that neither $q_1u_1+q_2v_1$ nor $q_1u_2+q_2v_2$ can equal to 0. It follows from (6.11)-(6.13) that this time we have

$$\begin{aligned} n+1 &\geq (n+1)u_1^2 + (m+1)v_1^2, \\ m+1 &\geq (n+1)u_2^2 + (m+1)v_2^2, \\ r_1 r_2 &= (n+1)u_1 u_2 + (m+1)v_1 v_2 + r_1 r_2 (w_3 + q_1 u_3 + q_2 v_3)^2, \end{aligned}$$

where equality holds in the first two inequalities iff $(w_3 + q_1 u_3 + q_2 v_3)^2 = 1$. The desired conclusions again follow easily from these facts.

If $n = 1 < m$, since $\alpha_1^2 = 0$ and $\beta_1^2 = 0$, we will not have (6.6) and (6.11). Also, the terms in (6.11)-(6.13) containing the factor $n + q_1^2 + 1$ will be absent. However, from the fact that $\Psi^*(\alpha_1)^2 = 0$, we obtain $u_2 = 0 = u_3$, so that $u_1^2 = 1$. Accordingly, similar arguments will yield the desired conclusions in this case as well.

As for the converses, recall that complex conjugation of the homogeneous coordinates of $\mathbb{CP}^k$ induces multiplication by -1 on $H^2(\mathbb{CP}^k; \mathbb{Z})$. Hence, pulling back P_{q_1,q_2} via an appropriate diffeomorphism of $\mathbb{CP}^n \times \mathbb{CP}^m$ yields the other circle bundle, and so the associated S^2-bundles are diffeomorphic.

Example 6.14 If $n_1 = n_2 = 1$, then the possible choices for (q_1, q_2) are $(1, 1)$ and $(1, -1)$. By Theorem 1.4, we have a non-Kähler Einstein metric on $W_{1,1}$ and $W_{1,-1}$. The integral in Theorem 1.2 for $W_{1,1}$ is positive if $(\varepsilon_1, \varepsilon_2) = (1, 1)$ and is 0 if $(\varepsilon_1, \varepsilon_2) = (1, -1)$. So we do not get any further Einstein metrics. However, we do get a Kähler-Einstein metric on $W_{1,-1}$, by the theorem of Koiso-Sakane. Note that $W_{1,1}$ is the Fano 3-fold which is the blow-up of the cone over a smooth quadric surface in $\mathbb{CP}^3$ and has unstable tangent bundle [St, p.638]. $W_{1,-1}$ is the Fano 3-fold which is the blow-up of $\mathbb{CP}^3$ with center two disjoint lines [MM, p. 157]. Notice that by 6.4(d), $W_{1,\pm 1}$ are non-spin, so they are not diffeomorphic to $S^2 \times S^2 \times S^2$. However, they are diffeomorphic to each other, since $P_{1,-1}$ is the pull-back of $P_{1,1}$ via the orientation reversing diffeomorphism on $S^2 \times S^2$ which is the identity on the first factor and a reflection on the second factor.

Example 6.15 Let $M_1 = M_2$ be any Kähler-Einstein manifold with positive first Chern class and set $n_1 = n_2 = n, p_1 = p_2 = p$. Assume further that $0 < |q_1| < |q_2| < p$. Then by Theorem 1.4 there is always a non-Kähler Einstein metric on W_{q_1,q_2}. On the other hand, by the calculation on p. 612 of [S], the integral

$$\int_{-1}^{1} x\left(\frac{p}{|q_1|} + x\right)^n \left(\frac{p}{|q_2|} - x\right)^n dx$$

is negative. It follows from Theorem 1.2 that there is a non-Kähler Einstein metric of type $(\varepsilon_1, \varepsilon_2) = (1, -1)$. The integral in Theorem 1.2 for $(\varepsilon_1, \varepsilon_2) = \pm(1, 1)$ is positive (resp. negative). Hence no further Einstein metrics arise. Of course, Sakane proved that there are no Kähler-Einstein metrics.

Example 6.16 Let $1 = n_1 < n_2 = n$, then $|q_1| = 1$ and $|q_2| = k = 1, \ldots, n$. As before, Theorem 1.4 gives a non-Kähler Einstein metric on each W_{q_1,q_2} with the above values of q_i. As for Theorem 1.2, corresponding to $(\varepsilon_1, \varepsilon_2) = (1, 1)$, the integral

$$\int_{-1}^{1} x(x+2)(x+\frac{n+1}{k})^n dx > 0,$$

as can be seen by expanding the third factor by the binomial theorem and integrating the resulting expression term by term. On the other hand, for $(\varepsilon_1, \varepsilon_2) = (-1, 1)$, the integral

$$\int_{-1}^{1} x(-x+2)(x+\frac{n+1}{k})^n dx$$

can be shown to be positive. So if we take $(\varepsilon_1, \varepsilon_2)$ to be $(1, -1)$, then the corresponding integral will be negative and we get another non-Kähler Einstein metric by Theorem 1.2. Note that again Theorem 1.3 does not give any Einstein metrics on these spaces.

7 Hermitian geometry

In this section we study the Hermitian geometry of the metrics h of the form (1.1) and then apply some of the results to the Einstein metrics constructed in Theorems 1.2, 1.4, and 1.6. Recall (see Remark 2.5) that h is Hermitian with respect to J, defined by $JN = -(1/f)U$ and the horizontal lift of the complex structure of the base.

Almost Hermitian metrics can be classified by the covariant derivative of their fundamental 2-form ω. More precisely, if we let $\overline{\nabla}$ denote the Levi Civita connection of h, then as a 1-form with values in the skew-Hermitian endomorphisms, $\overline{\nabla}\omega$ decomposes into 4 parts corresponding to the decomposition of the bundle with respect to the unitary structural group. This classification is due to A. Gray and L. Hervella [GH]. In the Hermitian case, the components of $\overline{\nabla}\omega$ lie in a subbundle denoted by these authors by $\mathscr{W}_3 \oplus \mathscr{W}_4$. Each summand is characterized by a tensor identity. For example, h belongs to the component $\mathscr{W}_4$ iff

$$h((\overline{\nabla}_A J)B, C) = \frac{-1}{n-1}\left\{h(A,B)\overline{\delta}\omega(C) - h(A,C)\overline{\delta}\omega(B) \right. \\ \left. -h(A,JB)\overline{\delta}\omega(JC) + h(A,JC)\overline{\delta}\omega(JB)\right\}, \tag{7.1}$$

where $n-1 = dim\ M$. (For details, see [GH, pp. 36-41].) Furthermore, all locally conformally Kähler Hermitian metrics (with respect to J) lie in $\mathscr{W}_4$. In order to apply this theory, we need the following

Lemma 7.2 *Let $\overline{\nabla}$ denote the Levi Civita connection of the metric h, L the shape operator of the hypersurface $P \times \{t\}$, and Ω the curvature form of the connection θ on the circle bundle P. For a horizontal vector X we let $\sum_i X_i$ denote its decomposition into components "along M_i". Let N denote the vector field $\partial/\partial t$ and U the vector field such that $\theta(U) = 1$. Then*

(*a*) $\overline{\nabla}_N J = 0 = \overline{\nabla}_U J$,
(*b*) $(\overline{\nabla}_X J)(N) = -\sum_i \left(\frac{\lambda_i f}{g_i^2} + \frac{g_i'}{g_i}\right) JX_i$,
(*c*) $(\overline{\nabla}_X J)(U) = fJ((\overline{\nabla}_X J)(N))$,
(*d*) $(\overline{\nabla}_X J)(Y) = \left(h(J(LX), Y) + \frac{1}{2} f\,\Omega(X, Y)\right) N - \left(\frac{1}{f} h(LX, Y) + \frac{1}{2}\Omega(X, JY)\right) U$.

Hence, the divergence of ω is given by

$$\overline{\delta}\omega(A) = \left(\sum_i 2n_i \left(\frac{g_i'}{f g_i} + \frac{\lambda_i}{g_i^2}\right)\right) h(U, A).$$

The proof of the lemma is by straight-forward computation, so we will leave it to the reader.

Corollary 7.3 *Let h be one of the non-Kähler Einstein metrics constructed on $W_{q_1,\ldots,q_l}$ by Theorem 1.2 or 1.4, or on $V_{q_1,\ldots,q_l}$ by Theorem 1.6. Then h lies in the family $\mathscr{W}_4$ iff q_i/A_i are independent of i, in which case h must be globally conformally equivalent to some J-Kähler metric. In particular, if q_i/p_i are not all equal, then h cannot be locally conformally Kähler.*

If h is an Einstein metric from Theorem 1.4 or from Theorem 1.6(b), then it belongs to $\mathscr{W}_4$ iff p_i/q_i are independent of i. The same is true for an Einstein metric from Theorem 1.6(d) provided that $|q_i| \le p_i$ for all i or $|q_i| > p_i$ for all i.

Proof. By the above lemma and the characterization of the class $\mathscr{W}_4$, we see that h lies in $\mathscr{W}_4$ iff

$$\frac{g_i'}{g_i} + \frac{\lambda_i f}{g_i^2} = \frac{1}{n-1} \sum_i 2n_i \left(\frac{g_i'}{g_i} + \frac{\lambda_i f}{g_i^2}\right).$$

This holds iff $g_i'/g_i + (\lambda_i f/g_i^2)$ is independent of i. Using Lemma 2.14, it follows that this is equivalent to q_i/A_i being independent of i. When the Einstein constant $c \neq 0$, then

$$\frac{q_i}{A_i} = \frac{2}{c}\left[-\frac{p_i}{q_i} \pm sgn(q_i)\left(\frac{p_i^2}{q_i^2} + cE\right)^{1/2}\right].$$

Note that if q_i/A_i are independent of i, then by the consistency condition (2.15), p_i/A_i would be independent of i, which in turn implies that q_i/p_i are independent of i. Finally, recall from [GH, Theorem 4.3] that if h belongs to $\mathscr{W}_4$ then it is locally (globally) conformally Kähler depending on whether the Lee form is closed (exact). It remains to observe that by Lemma 7.2, the Lee form is precisely

$$\frac{1}{n-1} \sum_i 2n_i \left(\frac{g_i'}{g_i} + \frac{\lambda_i f}{g_i^2}\right) dt,$$

and that all the manifolds W and V are simply connected.

In the case of Einstein metrics from Theorem 1.4, note that all A_i are negative. So if h lies in $\mathscr{W}_4$, then all q_i have the same sign. In this case, the minus sign occurs in the above formula for q_i/A_i, and it follows that q_i/A_i are all equal iff

p_i/q_i are all equal. Likewise, in case (b) of Theorem 1.6, A_i/p_i are all equal, so q_i/p_i are all equal iff q_i/A_i are all equal. For Einstein metrics constructed in Theorem 1.6(d), if $|q_i| \leq p_i$ for all i or if $|q_i| > p_i$ for all i, then there is a uniform choice of the $\pm$ sign in the formula for q_i/A_i, and so again p_i/q_i being all equal is equivalent to q_i/A_i being all equal. This completes the proof of Corollary 7.3.

We turn now to consider the curvature of h.

Lemma 7.4 *The Riemann curvature tensor $\overline{R}$ of h is given by*

1. $\overline{R}(X,Y,Z,W) = \overline{R}(JX,JY,JZ,JW)$, *where X,Y,Z,W are horizontal vector fields,*
2. $\overline{R}(X,Y,Z,U) = 0 = \overline{R}(X,Y,Z,N)$,
3. $\overline{R}(JN,JU,JU,JN) = \overline{R}(N,U,U,N) = -f''f$,
4. $\overline{R}(N,U,N,X) = 0 = \overline{R}(JN,JU,JN,JX)$,
5. $\overline{R}(X,Y,U,N) = -ff'\Omega(X,Y) + f^2\Omega(X,LY) = \overline{R}(JX,JY,JU,JN)$,
6. $\overline{R}(X,U,Y,N) = (f^2/2)\Omega(X,LY) - (1/2)ff'\Omega(X,Y) = \overline{R}(JX,JU,JY,JN)$,
7. $\overline{R}(X,N,N,Y) = -\sum_i (g_i''/g_i)h(X_i,Y_i)$,
8. $\overline{R}(X,U,U,Y) = f^2\sum_i (\frac{\lambda_i^2 f^2}{g_i^4} - \frac{f'g_i'}{fg_i})h(X_i,Y_i)$.

The proof of this lemma is again via straight-forward computation, using the Gauss equation and the O'Neill formulas for a Riemannian submersion. An immediate consequence of the lemma is

Corollary 7.5 *The Riemannian curvature tensor of a metric h of type (1.1) on $W_{q_1,\ldots,q_l}$ or $V_{q_1,\ldots,q_l}$ satisfies*

$$\overline{R}(A,B,C,D) = \overline{R}(JA,JB,JC,JD)$$

for all vectors A,B,C, and D iff

$$\frac{g_i''}{g_i} - \frac{f'g_i'}{fg_i} + \lambda_i^2\frac{f^2}{g_i^4} = 0$$

for all $i = 1,2,\ldots,l$.

This corollary gives a geometric interpretation of the assumption $\mu_i \equiv 0$ in the construction of the Einstein metrics on W or V. Furthermore, the question of whether an Einstein metric of type (1.1) necessarily satisfies $\mu_i \equiv 0$ can be rephrased in terms of the notation of Proposition 4.1 of [FFS]. For Hermitian manifolds, there is no difference between the classes $\mathscr{L}_2$ and $\mathscr{L}_3$. So the question becomes: what is the effect of the vanishing of the $\mathscr{C}_8$ component on the components $\mathscr{C}_6$ and $\mathscr{C}_7$ in the situation of the manifolds under consideration?

8 Einstein-Weyl structures

In this section we derive the analogs of Theorem 1.2 and 1.4 for Einstein-Weyl structures. Recall that the Einstein-Weyl equation for an n-dimensional manifold with Weyl structure $[h, \eta]$ is

$$Ric(h) + \left(\frac{n-2}{4}\right) \mathscr{D}\eta = \Lambda h,$$

where

$$\mathscr{D}\eta(X, Y) = (\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) + \eta(X)\eta(Y),$$

and Λ is the function given by

$$\Lambda = \left(\frac{n-2}{4}\right) |\eta|^2 + \frac{1}{2}\delta\eta + \frac{s^D}{n}.$$

In the above, $\delta\eta$ is the codifferential and s^D is the conformal scalar curvature of the Weyl connection.

As mentioned in Sect. 1, we choose a 1-form $\eta = Adt + Bf\theta$, where A and B are functions of t. Using the argument for the proof of Lemma 1 in [EW], for example, one sees that for the bundles W, the smoothness conditions for η are as follows. In a neighborhood of 0, $A(t) = t\phi(t^2)$ for some smooth function ϕ, and $(Bf)(t) = \psi(t^2)$ for some smooth function ψ such that $\psi(0) = 0$. Analogous conditions should hold in a neighborhood of T. For the bundles $\overline{W}$, the smoothness conditions for η are the same except that in a neighborhood of T, $(Bf)(T-t) = \hat{\psi}(t^2)$ for some smooth function $\hat{\psi}$ which need not vanish at T. Notice that by Theorem 2.2 of [PS2], it suffices to satisfy these conditions up to first order.

Using (2.1)-(2.3) and the calculations of [PS2] on pp. 107-108, we obtain the Einstein-Weyl equations below:

$$-\frac{f''}{f} - \sum_{i=1}^{l} 2n_i \frac{g_i''}{g_i} + nA' + \frac{n}{2}A^2 = \Lambda, \tag{8.1}$$

$$-\frac{f''}{f} - \sum_{i=1}^{l} 2n_i \frac{f' g_i'}{f g_i} + \sum_{i=1}^{l} 2n_i {\lambda_i}^2 \frac{f^2}{g_i^4} + nA\frac{f'}{f} + \frac{n}{2}B^2 = \Lambda, \tag{8.2}$$

$$-\frac{g_i''}{g_i} - \frac{f' g_i'}{f g_i} - \sum_{j\neq i} 2n_j \frac{g_i' g_j'}{g_i g_j} - (2n_i - 1)(\frac{g_i'}{g_i})^2 + \frac{L_i}{g_i^2} - 2{\lambda_i}^2 \frac{f^2}{g_i^4} + nA\frac{g_i'}{g_i} = \Lambda, \tag{8.3}$$

$$\frac{B'}{B} - \frac{f'}{f} + A = 0, \tag{8.4}$$

where $n = \sum n_i$, and $\lambda_i = q_i/2,\ i = 1, 2, \ldots, l$.

We may introduce a function $U(t)$ such that

$$B = fU^{-2}, \quad A = \frac{2U'}{U}.$$

The corresponding boundary conditions for U is that near 0, U should be even, so that in particular we must have $U'(0) = 0$, and analogously near T. Then the Einstein-Weyl equations become

$$-\frac{f''}{f} - \sum_{i=1}^{l} 2n_i \frac{g_i''}{g_i} + 2n\frac{U''}{U} = \Lambda,$$

$$-\frac{f''}{f} - \sum_{i=1}^{l} 2n_i \frac{f'g_i'}{fg_i} + \sum_{i=1}^{l} 2n_i \lambda_i{}^2 \frac{f^2}{g_i^4} + 2n\frac{U'f'}{Uf} + \frac{nf^2}{2U^4} = \Lambda,$$

$$-\frac{g_i''}{g_i} - \frac{f'g_i'}{fg_i} - \sum_{j\neq i} 2n_j \frac{g_i'g_j'}{g_ig_j} - (2n_i-1)(\frac{g_i'}{g_i})^2 + \frac{L_i}{g_i^2} - 2\lambda_i{}^2\frac{f^2}{g_i^4} + \frac{2nU'g_i'}{Ug_i} = \Lambda.$$

As before, if we equate the first two equations, we obtain

$$\sum_{i=1}^{l} 2n_i\mu_i - 2n\left(\frac{U''}{U} - \frac{U'f'}{Uf} - \frac{f^2}{4U^4}\right) = 0,$$

where μ_i is given by (2.4). We will seek a solution with constant Λ, $\mu_i \equiv 0$ for all i, and

$$\frac{U''}{U} - \frac{U'f'}{Uf} - \frac{f^2}{4U^4} = 0. \tag{8.5}$$

We now perform the same variable change as in Sect. 2, defining in addition $u(r) = U(t)^2$. Then the Einstein-Weyl equations become

$$\begin{aligned} &-\frac{1}{2}a'' - \frac{1}{2}a'(\log v)' - a\sum_{i=1}^{l} n_i\left(\frac{b_i''}{b_i} - \frac{1}{2}(\frac{b_i'}{b_i})^2\right) \\ &+\frac{n}{2}\left(\frac{a'u'}{u}\right) + na\left(\frac{u''}{u} - \frac{1}{2}\left(\frac{u'}{u}\right)^2\right) = \Lambda, \end{aligned} \tag{8.6}$$

$$-\frac{1}{2}a'' - \frac{1}{2}a'(\log v)' + 2a\sum_{i=1}^{l} n_i\frac{\lambda_i^2}{b_i^2} + \frac{n}{2}\left(\frac{a'u'}{u}\right) + \frac{n}{2}\frac{a}{u^2} = \Lambda, \tag{8.7}$$

$$\frac{p_i}{b_i} - 2a\frac{\lambda_i^2}{b_i^2} - \frac{a}{2}\left(\frac{b_i''}{b_i} - (\frac{b_i'}{b_i})^2\right) - \frac{a}{2}(\log v)'\frac{b_i'}{b_i} - \frac{a'b_i'}{2b_i} + \frac{na}{2}\left(\frac{b_i'u'}{b_iu}\right) = \Lambda. \tag{8.8}$$

If we solve for u, we obtain

$$u = C(r+\hat{r})^2 + \frac{1}{4C}, \tag{8.9}$$

where $\hat{r}$ is a constant and C is a *positive* constant. In particular, there are no solutions which are linear in r, and u is everywhere positive. One can derive the analog of Lemma 2.14 in the same way. The only differences are that c should be replaced by Λ and v by vu^{-n}, and part (i) should also assert that $\hat{r}$ equals r_0.

One can now go through the proofs of Theorems 1.2 and 1.4 and see that they can be modified easily to prove Theorems 1.7 and 1.8 respectively. In analysing the behavior of the analogs of $F(E)$ and $G(E)$, one needs only to observe that u^{-1} is uniformly bounded from above and below on a finite interval by (positive) constants. Consequently, the desired behavior can be deduced from that of $F(E)$ and $G(E)$. Furthermore, the boundary conditions for u are satisfied easily. Since the constant C is arbitrary, we actually obtain a 1-parameter family of solutions. Finally, if B is not identically 0, then η is not closed, hence it cannot be exact. Thus we obtain Einstein-Weyl structures which are not locally conformal to the Einstein structures constructed in Theorems 1.2 and 1.4.

Remark 8.10 The solution of the Einstein-Weyl equation for $l = 1$ given in Theorem 1.8 is not conformally equivalent to that in [PS2]. Indeed, applying a conformal factor to our solution that makes $g(t)$ constant, one can check using (8.9) and (2.16) that (4.4) in [PS2] does not hold.

9 Uniqueness of Einstein-Weyl structures on principal S^1-bundles

We take this opportunity to include an extension of the main result in [W1] to the Einstein-Weyl situation. Recall that for each principal circle bundle $P_{q_1,\dots,q_l}$ as in Sect. 1, it was shown in [WZ] that there is an Einstein metric of Kaluza-Klein type which submerses onto a product of the Kähler-Einstein metrics on the base. In [W1] it was proved that conversely if a principal circle bundle over a compact Kähler manifold admits an Einstein metric such that the bundle projection is a Riemannian submersion with totally geodesic fibres and that the curvature form of the principal connection is of type $(1, 1)$, then in fact the base splits isometrically into a product of Kähler-Einstein manifolds with positive scalar curvature and the Euler class of the principal bundle is of the form described in Sect. 1.

In [PS1], the above existence theorem of [WZ] was extended to the Einstein-Weyl case (see Theorem 4.1, p. 388 of [PS1]). It turns out that a similar converse holds.

Theorem 9.1 *Let P be a principal circle bundle over a compact Kähler manifold M admitting an Einstein-Weyl structure h, η such that the bundle projection is a Riemannian submersion with totally geodesic fibres onto the Kähler metric of M, the curvature form $\Omega \neq 0$ of the principal connection θ is of type $(1, 1)$, and the 1-form $\eta = f\theta$ for some smooth function f. Suppose also that the scalar curvature of M is constant. Then the eigenvalues of the Ricci tensor of M are constant over M and M is isometric to a product of Kähler-Einstein manifolds corresponding to the eigenspaces of the Ricci tensor of M.*

Proof. We refer to [W1] for notation. As before, let U satisfy $\theta(U) = 1$. We will view Ω as a closed 2-form on M. Then the Einstein-Weyl equations are given by

$$\frac{1}{4}\sum_{i,j}(\Omega_{ij})^2 + \left(\frac{n-2}{4}\right)(2U(f) + f^2) = \Lambda, \tag{9.2}$$

(9.3)
$$\frac{1}{2}\sum_j \Omega_{ijj} + \left(\frac{n-2}{4}\right) f_i = 0,$$

(9.4)
$$Ric^M_{ij} - \frac{1}{4}\sum_t \Omega_{it}\Omega_{jt} = \Lambda\delta_{ij},$$

where Λ is a function. By (9.3)

$$e_i(U(f)) = U(f_i) = -\frac{2}{n-2}U(\Omega_{ijj}) = 0.$$

Therefore, $U(f) = 0$ and f is a function constant along the fibres. On the other hand, again by (9.3)

$$\frac{n-2}{2}f_{ii} = -\Omega_{ijji} = \Omega_{jiji} = \Omega_{jiij} + K^M_{ijit}\Omega_{jt} + K^M_{ijjt}\Omega_{ti}.$$

It follows from the last equality that

$$-\Omega_{ijji} = K^M_{ijit}\Omega_{jt} = Ric_{jt}\Omega_{jt} = 0,$$

and hence $f_{ii} = 0$. We conclude that f is a constant, and therefore Ω is harmonic by (9.3). Since M is of constant scalar curvature, Ω has constant norm and the Ricci curvature of M has constant eigenvalues by (9.4). The rest of the proof is the same as that given in [W1].

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