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Density and completeness of subvarieties of moduli spaces of curves or Abelian varieties

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Introduction

Let $\mathscr{M}_g$ be the moduli space of smooth curves of genus $g \geq 2$ and let $\mathscr{A}_g$ be the moduli space of principally polarized abelian varieties (*ppav*) of dimension g over $\mathbb{C}$. A (Deligne-Mumford) stable curve of genus g is a reduced, connected and complete curve of arithmetic genus g with only nodes as singularities and with finite automorphism group. We say that a stable curve is of compact type if its generalized jacobian is an abelian variety. We denote by $\widetilde{\mathscr{M}_g}$ the moduli space of stable curves of compact type and genus g over $\mathbb{C}$. By "density" we always mean "analytic density" unless we specify otherwise.

Given a subvariety V of $\mathscr{M}_g$ or $\widetilde{\mathscr{M}_g}$ and an integer q between 1 and $g/2$, let $E_q(V)$ be the subset of V parametrizing curves whose jacobian contains an abelian variety of dimension q. We define $E_q(V)$ for V a subvariety of $\mathscr{A}_g$ in a similar fashion. It is well-known that $E_q(\mathscr{A}_g)$ is dense in $\mathscr{A}_g$ for all q. Colombo and Pirola pose the following question in [3]

Problem 1. *When is $E_q(V)$ dense in V?*

Colombo and Pirola give a sufficient condition for the density of $E_q(V)$ in V. They then show that $E_1(V)$ is dense in V for all subvarieties V of $\mathscr{M}_g$ of codimension at most $g-1$. They deduce from this a second proof of the noncompleteness of codimension $g-1$ subvarieties of $\widetilde{\mathscr{M}_g}$ which was originally proved by Diaz in [5], Corollary page 80 (Colombo and Pirola prove the noncompleteness of codimension $g-1$ subvarieties of $\widetilde{\mathscr{M}_g}$ which meet $\mathscr{M}_g$; however, if the subvariety is contained in $\widetilde{\mathscr{M}_g} \setminus \mathscr{M}_g$, its noncompleteness can

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be easily seen by mapping it (or, in some cases, a double cover of it) to a moduli space of curves of lower genus).

Using the condition of Colombo and Pirola, we show

Theorem 1. *Suppose that $g \geq 2$. Let V be a subvariety of codimension at most g of $\mathscr{M}_g$ or $\mathscr{A}_g$, then $E_1(V)$ is dense in V.*

This result brings out another fundamental difference between the moduli spaces in characteristic zero and in positive characteristic (see Section 2 below).

We also obtain

Corollary 1. *Suppose that $g \geq 2$. Let V be a complete subvariety of codimension at most g of $\mathscr{A}_g$. For all $i \in \{1, ..., g\}$, denote by $E_{1,i}(V)$ the subset of V parametrizing ppav's isogenous to a product of a ppav of dimension $g - i$ and i elliptic curves. Then*

1. *the variety V has codimension exactly g,*
2. *for all $i \in \{1, ..., g\}$, any irreducible component Z of $E_{1,i}(V)$ has the expected dimension $\frac{(g-i)(g-i+1)}{2} + i - g$; furthermore, the variety Z parametrizes ppav's isogenous to a product of i fixed elliptic curves (depending only on Z) and some ppav of dimension $g - i$,*
3. *for all $q, 1 \leq q \leq g/2$, any irreducible component of $E_q(V)$ has the expected dimension $\frac{(g-q)(g-q+1)}{2} + \frac{q(q+1)}{2} - g$,*
4. *the set $E_{1,g}(V)$ is dense in V. In particular, the set $E_{1,g}(V)$ is (countable) infinite and, for all $i \in \{1, ..., g\}$, for all $q \in \{1, ..., g/2\}$, the sets $E_{1,i}(V)$ and $E_q(V)$ are dense in V (since they contain $E_{1,g}(V)$).*

An immediate consequence of Corollary 1 is that complete subvarieties of $\mathscr{A}_g$ have codimension at least g. There are different proofs of this last fact in [12] (2.5.1 page 231) and [7] (Corollary 1.7). There are no known examples of complete subvarieties of codimension g of $\mathscr{A}_g$ (or $\widetilde{\mathscr{M}_g}$) except for $g = 2$, although g is the best known lower bound for the codimension of complete subvarieties of $\mathscr{A}_g$ (and $\widetilde{\mathscr{M}_g}$). It is conjectured in [12] (2.3 page 230) that for $g \geq 3$ the codimension of a complete subvariety of $\mathscr{A}_g$ is at least $g + 1$.

Let $\mathscr{H}_g$ be the locus of hyperelliptic curves in $\mathscr{M}_g$. Let $\widetilde{\mathscr{M}'_g}$ and $\mathscr{A}'_g$ be respectively the moduli space of curves of compact type with level n structure (for some fixed $n \geq 3$) and the moduli space of ppav's with level n structure. Let $s_a : \mathscr{A}'_g \longrightarrow \mathscr{A}_g$ and $s_c : \widetilde{\mathscr{M}'_g} \longrightarrow \widetilde{\mathscr{M}_g}$ be the natural morphisms. It is well-known that $\mathscr{A}'_g$ and $\mathscr{M}'_g := s_c^{-1}(\mathscr{M}_g)$ are smooth and that there is a universal family of abelian varieties with level n structure on $\mathscr{A}'_g$ and a universal family of (smooth) curves with level n structure on $\mathscr{M}'_g$. By a "universal" family of curves or abelian varieties with level n structure we mean a family which solves the moduli problem for curves or abelian varieties with level n structure. We note that the only properties of $\widetilde{\mathscr{M}'_g}$, $\mathscr{M}'_g$ and $\mathscr{A}'_g$ we need are the smoothness of $\mathscr{M}'_g$ and $\mathscr{A}'_g$ and the existence of the universal families (also note that with non-abelian level structures or Prym-level structures, one can get smooth covers of $\widetilde{\mathcal{M}'_g}$ as well (see [8] and [14])). We have the following technical consequence of our results.

Corollary 2. *Suppose that $g \geq 3$. Let V be a complete subvariety of codimension g of $\widetilde{\mathcal{M}_g}$ or $\mathcal{A}_g$. If $V \subset \mathcal{A}_g$, let V_0 be the smooth locus of $s_a^{-1}(V)$. If $V \subset \widetilde{\mathcal{M}_g}$, let V_0 be the smooth locus of $s_c^{-1}(V \cap (\mathcal{M}_g \setminus \mathcal{H}_g))$. Then the conormal bundle to V_0 is isomorphic to the tensor product of the Hodge bundle (the pushforward of the sheaf of relative one-forms on the universal abelian (or jacobian) variety) with a subline bundle of the Hodge bundle.*

Finally, we point out that, if $\mathcal{A}_{g,d}$ denotes the moduli space of abelian varieties of dimension g and polarization type $d = (d_1, ..., d_g)$, then there is a finite correspondence between $\mathcal{A}_g$ and $\mathcal{A}_{g,d}$ so that our results remain valid if we replace $\mathcal{A}_g$ with $\mathcal{A}_{g,d}$.

Notation

For any vector space or vector bundle (resp. affine cone) E, we denote by $\mathbb{P}(E)$ the projective space (resp. projective variety) of lines in E and by E^* its dual vector space or vector bundle. We let $E^{\otimes m}$, $S^m E$ and $\Lambda^m E$ respectively be the m-th tensor power, the m-th symmetric power and the m-th alternating power of E. For any linear map of vector spaces or vector bundles $l : E \longrightarrow F$, we denote by $\bar{l} : \mathbb{P}(E) \longrightarrow \mathbb{P}(F)$ its projectivization.

For any variety X and any point $x \in X$, we denote by $T_x X$ the Zariski tangent space to X at x and by $T_x^* X$ the dual of $T_x X$. We denote by X_{sm} the subvariety of smooth points of X.

For a ppav A, we let $\rho : H^0(\Omega_A^1)^{\otimes 2} \longrightarrow S^2 H^0(\Omega_A^1)$ be the natural linear map with kernel $\Lambda^2 H^0(\Omega_A^1)$. For a subvariety V of $\mathcal{A}'_g$ the restriction to V of the universal family on $\mathcal{A}'_g$ gives a family $\mathcal{A}_V$ of ppav's on V (we forget the level n structure). For a point t of V, we let A_t be the fiber of $\mathcal{A}_V$ at t. The Zariski-tangent space to $\mathcal{A}'_g$ at t can be canonically identified with $S^2 H^0(\Omega_{A_t}^1)^*$. We denote by $\pi_a : S^2 H^0(\Omega_{A_t}^1) \longrightarrow T_t^* V$ the codifferential at t of the embedding $V \hookrightarrow \mathcal{A}'_g$.

For a smooth curve C, we denote by ω_C the canonical sheaf of C and let κC be the image of C in the dual projective space $|\omega_C|^*$ of the linear system $|\omega_C|$ by the natural morphism associated to this linear system. If $A = JC$ is the jacobian of a smooth curve C, then $H^0(\Omega_A^1) \cong H^0(\omega_C)$. Let $m : S^2 H^0(\omega_C) \longrightarrow H^0(\omega_C^{\otimes 2})$ be multiplication and put $\mu := m\rho$. For $V \subset \mathcal{M}'_g$ the restriction to V of the universal family on $\mathcal{M}'_g$ gives a family $\mathcal{C}_V$ of curves on V (again we forget the level n structure). For $t \in V$ we let C_t be the fiber of $\mathcal{C}_V$ at t. The Zariski-tangent space to $\mathcal{M}'_g$ at t can be canonically identified with $H^0(\omega_{C_t}^{\otimes 2})^*$. We let $\pi : H^0(\omega_{C_t}^{\otimes 2}) \longrightarrow T_t^* V$ be the codifferential at t of the embedding $V \hookrightarrow \mathcal{M}'_g$.

For a stable curve C of compact type (resp. a ppav A), we will call the corresponding point of $\widetilde{\mathcal{M}_g}$ (resp. $\mathcal{A}_g$) the moduli point of C (resp. A).

1. The proofs

In this section we give the proof of Theorem 1 and its corollaries.

We first consider the case where V is contained in $\mathscr{M}_g$. We may and will replace V with its inverse image in $\mathscr{M}'_g$.

The relative jacobian of $\mathscr{C}_V$ gives us a family of ppav's on V. We can therefore apply Theorem (1) on page 162 of [3]: to show that $E_1(V)$ is dense in V it is enough to prove the following:

There exists a Zariski dense (Zariski-)open subset U of V, contained in the smooth locus V_{sm} of V, such that, for all $t \in U$, there is a subvector space W of $H^0(\omega_{C_t})$ which has dimension 1 and is such that the composition

$$W \otimes W^{\perp} \overset{\mu}{\hookrightarrow} H^0(\omega_{C_t}^{\otimes 2}) \overset{\pi}{\longrightarrow} T_t^* V$$

is injective. Here $W^{\perp}$ is the orthogonal complement of W with respect to the hermitian form on $H^0(\omega_{C_t})$ induced by the natural polarization of JC_t. We sketch briefly how this condition is obtained in the more general case where W has dimension q with $1 \leq q \leq g/2$ and $V \subset \mathscr{A}'_g$ has any dimension $\geq q(g-q)$.

To prove the density of $E_q(V)$ in V, it is enough to show that there is a Zariski dense open subset U of V_{sm}, such that, for all $t \in U$, there is an analytic neighborhood U' of t, $U' \subset U$, such that $E_q(V) \cap U'$ is dense in U'. An abelian variety A contains an abelian subvariety of dimension q if and only if $H^0(\Omega^1_A)$ contains a q-dimensional $\mathbb{C}$-subvector space which is the tensor product with $\mathbb{R}$ of a vector subspace of dimension $2q$ of $H^1(A,\mathbb{Q})$ (after identifying $H^0(\Omega^1_A)$ with $H^1(A,\mathbb{R}) \cong H^1(A,\mathbb{Q}) \otimes \mathbb{R}$ as real vector spaces). Let t be an element of V_{sm}. For a contractible analytically open set $U' \ni t$ contained in V_{sm}, let $F_{U'}$ be the Hodge bundle over U'. Then one can trivialize $F_{U'}$ as a real vector bundle using the Gauss-Manin connection. Therefore the grassmannian bundle of $2q$-dimensional real subvector spaces of $F_{U'}$ is isomorphic to $U' \times G_{\mathbb{R}}(2q,2g)$, where $G_{\mathbb{R}}(2q,2g)$ is the Grassmannian of $2q$-dimensional $\mathbb{R}$-subvector spaces of $H^1(A_t,\mathbb{R})$. Hence there is a well-defined map $\Phi : G(q,F_{U'}) \longrightarrow G_{\mathbb{R}}(2q,2g)$ where $G(q,F_{U'})$ is the Grassmannian of q-dimensional $\mathbb{C}$-subvector spaces of $F_{U'}$: The map Φ sends a q-dimensional complex subvector space of $H^0(\Omega^1_{A_s})$ (with $s \in U'$) to the image of its underlying real vector space under the isomorphism $H^1(A_s,\mathbb{R}) \overset{\cong}{\longrightarrow} H^1(A_t,\mathbb{R})$ obtained from the $\mathbb{R}$-trivialization of $F_{U'}$. Let $G_{\mathbb{Q}}(2q,2g) \subset G_{\mathbb{R}}(2q,2g)$ be the Grassmannian of $2q$-dimensional $\mathbb{Q}$-subvector spaces of $H^1(A_t,\mathbb{Q})$ and let $p : G(q,F_{U'}) \longrightarrow U'$ be the natural morphism. Then $E_q(V) \cap U' = p(\Phi^{-1}(G_{\mathbb{Q}}(2q,2g)))$. To prove the density of $E_q(V) \cap U'$ in U', it is enough to prove that there is a subset $\mathscr{Y}$ of $G(q,F_{U'})$ such that $p(\mathscr{Y}) = U'$ and $\Phi^{-1}(G_{\mathbb{Q}}(2q,2g)) \cap \mathscr{Y}$ is dense in $\mathscr{Y}$. Since $G_{\mathbb{Q}}(2q,2g)$ is dense in $G_{\mathbb{R}}(2q,2g)$, it is enough to find $\mathscr{Y}$ such that $p(\mathscr{Y}) = U'$ and $\Phi|_{\mathscr{Y}}$ is an open map. If Φ has maximal rank (i.e., the differential $d\Phi$ of Φ is surjective) everywhere on $\mathscr{Y}$, then $\Phi|_{\mathscr{Y}}$ is an open map. Therefore $E_q(V) \cap U'$ is dense in U' if for every $s \in U'$ there is a q-dimensional $\mathbb{C}$-subvector space W of $H^0(\Omega^1_{A_s})$ such that $d\Phi$ is surjective at $(W,s) \in G(q,F_{U'})$ (then $\mathscr{Y}$ would be the set of such (W, s)). The tangent space $T_{(W,s)}G(q, F_{U'})$ is isomorphic to $W \otimes W^{\perp} \oplus T_sU'$, the tangent space to $G_{\mathbb{R}}(2q, 2g)$ at $\Phi(W, s)$ is isomorphic to $W \otimes W^{\perp} \oplus \overline{W \otimes W^{\perp}} \cong W \otimes W^{\perp} \oplus (W \otimes W^{\perp})^*$ and the restriction of $d\Phi$ to the $W \otimes W^{\perp}$ summand of $T_{(W,s)}G(q, F_{U'})$ is an isomorphism onto the $W \otimes W^{\perp}$ summand of $T_{\Phi(W,s)}G_{\mathbb{R}}(2q, 2g)$. Therefore $d\Phi$ is surjective if and only if the map it induces $T_sU' = \frac{T_{(W,s)}G(q,F_{U'})}{W \otimes W^{\perp}} \longrightarrow (W \otimes W^{\perp})^* = \frac{T_{\Phi(W,s)}G_{\mathbb{R}}(2q,2g)}{W \otimes W^{\perp}}$ is surjective, i.e., if and only if the dualized map $W \otimes W^{\perp} \longrightarrow T_s^*U'$ is injective. Let F be the Hodge bundle over the Siegel upper half space $\mathscr{U}_g$. The inclusion $U' \hookrightarrow \mathscr{A}'_g$ lifts to an inclusion $U' \hookrightarrow \mathscr{U}_g$ because U' is contractible and there is a family of ppav's on U' (the restriction of $\mathscr{A}_V$). Factoring Φ through the Grassmannian of q-planes in F over $\mathscr{U}_g$, the map $W \otimes W^{\perp} \longrightarrow T_s^*U'$ can be seen to be the composition

$$W \otimes W^{\perp} \xrightarrow{\rho} S^2H^0(\Omega^1_{A_s}) \xrightarrow{\pi_a} T_s^*U' = T_s^*V \ .$$

For V contained in $\mathscr{M}'_g$, we have $\pi_a = \pi m$.

Clearly, if $\pi\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T_t^*V$ is injective, then so is $\pi\mu : W \otimes W^{\perp} \longrightarrow T_t^*V$. In view of this (and also for use in the proof of Corollary 2) we show:

Proposition 1.1. *Suppose that $g \geq 3$. Let V be a subvariety of codimension at most g of $\mathscr{M}'_g$. Let t be a point of V_{sm} and let N be the kernel of $\pi : H^0(\omega_{C_t}^{\otimes 2}) \longrightarrow T_t^*V$.*

1. *Suppose that C_t is non-hyperelliptic. Suppose that, for any one-dimensional subvector space W of $H^0(\omega_{C_t})$, the map $\pi\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T_t^*V$ is* not *injective. Then V has codimension exactly g and there is a one-dimensional subvector space W_N of $H^0(\omega_{C_t})$ such that $N = \mu(W_N \otimes H^0(\omega_{C_t}))$.*
2. *Suppose that C_t is hyperelliptic and that V is* not *transverse to $\mathscr{H}'_g := s_c^{-1}(\mathscr{H}_g)$ at t (i.e., the sum $T_tV + T_t\mathscr{H}'_g \subset T_t\mathscr{M}'_g$ is* not *equal to $T_t\mathscr{M}'_g$). Then there exists a one-dimensional subvector space W of $H^0(\omega_{C_t})$ such that the map $\pi\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T_t^*V$ is injective.*

Proof: Consider the composition

$$\mathbb{P}(H^0(\omega_{C_t})^{\otimes 2}) \xrightarrow{\overline{\rho}} \mathbb{P}(S^2H^0(\omega_{C_t})) \xrightarrow{\overline{m}} \mathbb{P}(H^0(\omega_{C_t}^{\otimes 2})) \ .$$

The kernel of m is the space $I_2(C_t)$ of quadratic forms vanishing on κC_t. Hence the *rational* map $\overline{m}$ is the projection with center $\mathbb{P}(I_2(C_t))$. Let $\overline{\mathscr{S}}$ be the image by $\overline{\rho}$ of the Segre embedding $\mathscr{S}$ of $\mathbb{P}(H^0(\omega_{C_t})) \times \mathbb{P}(H^0(\omega_{C_t}))$ in $\mathbb{P}(H^0(\omega_{C_t})^{\otimes 2})$. Let N' be the set of rank 2 symmetric tensors in $S^2H^0(\omega_{C_t})$ which lie in $m^{-1}(N)$ (then $\mathbb{P}(N')$ is the *reduced* intersection of $\overline{\mathscr{S}}$ and $\overline{m}^{-1}(\mathbb{P}(N))$).

Suppose that for all $W \subset H^0(\omega_{C_t})$ of dimension 1, the map $\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T_t^*V$ is not injective, i.e., for all $w \in H^0(\omega_{C_t})$, there is $w' \in H^0(\omega_{C_t})$ such that $\mu(w \otimes w') \in N$. This implies that the dimension of $\mathbb{P}(N')$ is at least $g - 1$. We will show below that this does not happen if C_t is hyperelliptic and V is not transverse to $\mathscr{H}'_g$ at t. If C_t is non-hyperelliptic, we will show that this implies that $\mathbb{P}(N')$ is a linear subspace of $\overline{\mathscr{S}}$ and that its inverse image in $\mathscr{S}$ is the union of two linear subspaces of $\mathscr{S}$ which are two fibers of the two projections of $\mathscr{S}$ onto $\mathbb{P}^{g-1}$ and are exchanged under the involution of $\mathscr{S}$ which interchanges the two $\mathbb{P}^{g-1}$-factors of $\mathscr{S}$. The proposition will then easily follow from this.

Suppose first that C_t is *non*-hyperelliptic. Then $m : S^2H^0(\omega_{C_t}) \longrightarrow H^0(\omega_{C_t}^{\otimes 2})$ is onto (see [2] page 117). We have

Lemma 1.2. *Suppose $g = 2$ or $g \geq 3$ and C_t is non-hyperelliptic. Suppose that for all $W \subset H^0(\omega_{C_t})$ of dimension 1, the map $\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T_t^*V$ is not injective. Then the map $\mathbb{P}(N') \longrightarrow \mathbb{P}(N)$ is generically one-to-one.*

Proof: If not, then, for all $w \in H^0(\omega_{C_t})$, there exists $w', w_1, w_1' \in H^0(\omega_{C_t})$ such that $ww' := \rho(w \otimes w')$ and $w_1w_1' := \rho(w_1 \otimes w_1')$ are not proportional but $m(ww')$ and $m(w_1w_1')$ are proportional elements of N. Therefore, supposing w general, there exits $\lambda \in \mathbb{C}, \lambda \neq 0$, such that the element $\lambda ww' - w_1w_1'$ of $S^2H^0(\omega_{C_t})$ lies in $I_2(C_t)$, i.e., defines a quadric $q(w)$ of rank 3 or 4 (in the canonical space $|\omega_{C_t}|^*$) which contains κC_t (the canonical curve κC_t is not contained in any quadric of rank ≤ 2 since it is nondegenerate). If $g \leq 3$, this is impossible because in that case $I_2(C_t) = 0$. If $g \geq 4$, the intersection L of the two hyperplanes in $|\omega_{C_t}|^*$ with equations w and w_1 is an element of a ruling of the quadric $q(w)$. Therefore L cuts a divisor of a g^1_d (a g^1_d is a pencil of divisors of degree d) on C_t with $d \leq g - 1$ (see [1], Lemmas 2 and 3 page 192). Therefore the divisor of zeros of w on C_t contains a divisor of a g^1_d. By the uniform position Theorem (see [2] Chapter 3, Sect. 1) this does not happen for w in some nonempty Zariski-open subset of $H^0(\omega_{C_t}) \setminus \{0\}$. □

Therefore, since the dimension of $\mathbb{P}(N')$ is at least $g - 1$ and the dimension of $\mathbb{P}(N)$ is at most $g - 1$, the map $\mathbb{P}(N') \longrightarrow \mathbb{P}(N)$ is birational and $\mathbb{P}(N')$ and $\mathbb{P}(N)$ have both dimension $g - 1$.

This proves, in particular, that V has codimension exactly g.

Since no quadrics of rank ≤ 2 contain κC_t, the center $\mathbb{P}I_2(C_t)$ of the projection $\overline{m}$ does not intersect $\overline{\mathscr{S}}$. In particular, the space $\mathbb{P}I_2(C_t)$ does not intersect $\mathbb{P}(N')$. Therefore $\overline{m}$ restricts to a birational *morphism* $\mathbb{P}(N') \longrightarrow \mathbb{P}(N)$ and, since $\mathbb{P}(N)$ is a linear subspace of $\mathbb{P}(H^0(\omega_{C_t}^{\otimes 2}))$, the degree of $\mathbb{P}(N')$ (in the projective space $\mathbb{P}(S^2H^0(\omega_{C_t})))$ is equal to the (generic) degree of the map $\mathbb{P}(N') \longrightarrow \mathbb{P}(N)$. Hence $\mathbb{P}(N')$ is a linear subspace of $\mathbb{P}(S^2H^0(\omega_{C_t}))$ and $\overline{m}$ restricts to an *isomorphism* $\mathbb{P}(N') \xrightarrow{\cong} \mathbb{P}(N)$.

Let N'' be the cone of decomposable tensors in $H^0(\omega_{C_t})^{\otimes 2}$ which lie in $\mu^{-1}(N)$ (then $\mathbb{P}(N'')$ is the *reduced* inverse image of $\mathbb{P}(N')$ in $\mathscr{S} \subset \mathbb{P}(H^0(\omega_{C_t})^{\otimes 2})$). The map $\mathscr{S} \longrightarrow \overline{\mathscr{S}}$ is a finite *morphism* of degree 2 ramified on the diagonal. Therefore the map $\mathbb{P}(N'') \longrightarrow \mathbb{P}(N')$ is a morphism of degree ≤ 2. Since the diagonal of $\overline{\mathscr{S}} \cong S^2\mathbb{P}(H^0(\omega_{C_t}))$ is irreducible of dimension $g - 1$ and spans $\mathbb{P}(S^2H^0(\omega_{C_t}))$, the space $\mathbb{P}(N')$ intersects this diagonal in a subvariety of dimension at most $g - 2$. Therefore the morphism $\mathbb{P}(N'') \longrightarrow \mathbb{P}(N')$ has degree 2 and $\mathbb{P}(N'')$ has degree 2 in $\mathbb{P}(H^0(\omega_{C_t})^{\otimes 2})$.

If $\mathbb{P}(N'')$ is irreducible, it spans a linear subspace $\widetilde{\mathbb{P}}$ of $\mathbb{P}(H^0(\omega_{C_t})^{\otimes 2})$ of dimension g. This implies that $\mathbb{P}(\Lambda^2 H^0(\omega_{C_t}))$ intersects $\widetilde{\mathbb{P}}$ in exactly one point. For $w_1, w_2 \in H^0(\omega_{C_t})$, let w_1', w_2' be such that $\mu(w_1 \otimes w_1'), \mu(w_2 \otimes w_2') \in N$. For w_i general, w_i' is not proportional to w_i since $\mathbb{P}(N'')$ intersects the diagonal of $\mathscr{S}$ in a subvariety of dimension at most $g-2$. Therefore the lines spanned by $w_1 \otimes w_1' - w_1' \otimes w_1$ and $w_2 \otimes w_2' - w_2' \otimes w_2$ give us elements of $\widetilde{\mathbb{P}} \cap \mathbb{P}(\Lambda^2 H^0(\omega_{C_t}))$ which is a point. Therefore, for all $w_1, w_2 \in H^0(\omega_{C_t})$ general there exists $\lambda \in \mathbb{C}, \lambda \neq 0$, such that

$$w_1 \otimes w_1' - w_1' \otimes w_1 = \lambda(w_2 \otimes w_2' - w_2' \otimes w_2)$$

Complete $\{w_1, w_2\}$ to a general basis $\{w_1, w_2, w_3, \ldots, w_g\}$ of $H^0(\omega_{C_t})$ and write $w_i' = \sum_{1 \leq j \leq g} a_{ij} w_j$ for $i = 1$ or 2. Then from the equation above we deduce $a_{1j} = a_{2j} = 0$ for $j > 2$. Therefore, w_1' belongs to the span of w_1 and w_2. Repeating this argument with w_1 and w_3 instead of w_1 and w_2, we see that w_1' also belongs to the span of w_1 and w_3. Hence w_1' is proportional to w_1 (this is the only part in the proof of Proposition 1.1 where we need $g \geq 3$). Contradiction.

Therefore $\mathbb{P}(N'')$ is reducible, i.e., it is the union of two linear subspaces of dimension $g-1$. We have

Lemma 1.3. *Suppose $g \geq 2$. All linear subspaces of dimension $g-1$ of $\mathscr{S} \subset \mathbb{P}(H^0(\omega_{C_t})^{\otimes 2}) \cong \mathbb{P}^{g^2-1}$ are elements of one of the two rulings of $\mathscr{S} \cong \mathbb{P}(H^0(\omega_{C_t})) \times \mathbb{P}(H^0(\omega_{C_t})) \cong \mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$.*

Proof: Let T be a linear subspace of dimension $g-1$ of $\mathscr{S}$. Let p_1 and p_2 be the two projections of $\mathscr{S} \cong \mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$ onto its two factors. Let H_i be a general element of $p_i^*|\mathscr{O}_{\mathbb{P}^{g-1}}(1)|$ for $i = 1$ or 2. Then $H_1 \cap T \neq H_2 \cap T$ and H_i does not contain T. In particular, the intersection $H_i \cap T$ is either empty or of dimension $g-2$. The divisor $H_1 \cup H_2$ is the intersection of a hyperplane H in $\mathbb{P}^{g^2-1}$ with $\mathscr{S}$. Since T is not contained in H_1 nor H_2, the hyperplane H does not contain T and hence $T \cap H$ is a linear space of dimension $g-2$. Since the two intersections $T \cap H_1 \neq T \cap H_2$ are both contained in the $(g-2)$-dimensional linear space $T \cap H$ and are either empty or have dimension $g-2$, we have either $H_1 \cap T = \emptyset$ or $H_2 \cap T = \emptyset$. Suppose, for instance, that $H_1 \cap T = \emptyset$. It is easily seen that $p_1^{-1}(p_1(H_1)) = H_1$ implies $p_1(H_1) \cap p_1(T) = p_1(H_1 \cap T)$. Therefore $p_1(T)$ does not intersect $p_1(H_1)$ which is a hyperplane in $\mathbb{P}^{g-1}$. Hence $p_1(T)$ is a point and T is a fiber of p_1. □

We deduce from the above Lemma that $\mathbb{P}(N'') = \mathbb{P}(N_1) \cup \mathbb{P}(N_2)$ where $\mathbb{P}(N_1)$ and $\mathbb{P}(N_2)$ are elements of the two rulings of $\mathscr{S} \cong \mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$. The spaces $\mathbb{P}(N_1)$ and $\mathbb{P}(N_2)$ are exchanged by the involution which exchanges the two factors of $\mathscr{S}$ because $\mathbb{P}(N'')$ is the inverse image of a linear subspace in $\overline{\mathscr{S}} \cong S^2\mathbb{P}^{g-1}$. Therefore there exists a one-dimensional subvector space W_N of $H^0(\omega_{C_t})$ such that, for instance, $N_1 = W_N \otimes H^0(\omega_{C_t})$ and $N_2 = H^0(\omega_{C_t}) \otimes W_N$. So $N = \mu(N_1) = \mu(W_N \otimes H^0(\omega_{C_t}))$. This proves the Proposition in the non-hyperelliptic case.

Now suppose that C_t is hyperelliptic and that V is not transverse to $\mathscr{H}_g'$ at t, i.e., the subspaces $T_t V$ and $T_t \mathscr{H}_g'$ do *not* span $T_t \mathscr{M}_g'$. Let ι be the hyperelliptic involution of C_t. Let $H^0(\omega_{C_t}^{\otimes 2})^+$ and $H^0(\omega_{C_t}^{\otimes 2})^-$ be the subvector spaces of $H^0(\omega_{C_t}^{\otimes 2})$ of ι-invariant and ι-anti-invariant quadratic differentials respectively. Then $H^0(\omega_{C_t}^{\otimes 2})^+$ is the image of $S^2H^0(\omega_{C_t})$ by m and the conormal space to $\mathscr{H}'_g$ at t can be canonically identified with $H^0(\omega_{C_t}^{\otimes 2})^-$. The non-transversality of V and $\mathscr{H}'_g$ means that $N \cap H^0(\omega_{C_t}^{\otimes 2})^- \neq \{0\}$. This implies that N is not contained in $H^0(\omega_{C_t}^{\otimes 2})^+$. Since N has dimension at most g, the dimension of $N \cap H^0(\omega_{C_t}^{\otimes 2})^+$ is at most $g-1$. Hence the dimension of $\mathbb{P}(N \cap H^0(\omega_{C_t}^{\otimes 2})^+) = \mathbb{P}(N) \cap \mathbb{P}(H^0(\omega_{C_t}^{\otimes 2})^+) = \mathbb{P}(N) \cap \overline{m}(\mathbb{P}(S^2H^0(\omega_{C_t})))$ is at most $g-2$. We have

Lemma 1.4. *Suppose $g \geq 2$ and C_t hyperelliptic. The map $\overline{m} : \overline{\mathscr{S}} \longrightarrow \overline{\overline{\mathscr{S}}} := \overline{m}(\overline{\mathscr{S}})$ is a finite morphism of degree $\frac{1}{2}\begin{pmatrix} 2g-2 \\ g-1 \end{pmatrix}$.*

Note that the lemma finishes the proof of Proposition 1.1: we saw above that the dimension of $\mathbb{P}(N) \cap \overline{m}(\mathbb{P}(S^2H^0(\omega_{C_t})))$ is at most $g-2$. A fortiori, since $\overline{m}(\mathbb{P}(S^2H^0(\omega_{C_t}))) \supset \overline{\overline{\mathscr{S}}}$, the dimension of $\mathbb{P}(N) \cap \overline{\overline{\mathscr{S}}}$ is at most $g-2$ and the dimension of $\mathbb{P}(N')$ is at most $g-2$ which is what we needed to show (see the paragraphs preceding Lemma 1.2).

Proof of Lemma 1.4: The map $\overline{m} : \overline{\mathscr{S}} \longrightarrow \overline{\overline{\mathscr{S}}}$ is a morphism if and only if the center $\mathbb{P}(I_2(C_t))$ of the projection $\overline{m}$ does not intersect $\overline{\mathscr{S}}$. This is the case because the canonical curve κC_t is nondegenerate and hence not contained in any quadrics of rank ≤ 2.

Fix a nonzero element $ww' = \rho(w \otimes w')$ of $S^2H^0(\omega_{C_t})$ and suppose that $w_1w'_1 \in S^2H^0(\omega_{C_t})$ is not proportional to ww' and $m(w_1w'_1) = \lambda.m(ww')$ for some $\lambda \in \mathbb{C}, \lambda \neq 0$. This is equivalent to $Z(w) + Z(w') = Z(w_1) + Z(w'_1)$ where $Z(w)$, for instance, is the divisor of zeros of w on the rational normal curve κC_t. So there are only a finite number of possibilities for $Z(w_1)$ and $Z(w'_1)$. This proves that $\overline{m} : \overline{\mathscr{S}} \longrightarrow \overline{\overline{\mathscr{S}}}$ is quasi-finite and hence finite since it is proper. Any divisor of degree $g-1$ on $\kappa C_t \cong \mathbb{P}^1$ is the divisor of zeros of some element of $H^0(\omega_{C_t}) = H^0(\mathscr{O}_{\mathbb{P}^1}(g-1))$, hence, since there are $\frac{1}{2}\begin{pmatrix} 2g-2 \\ g-1 \end{pmatrix}$ ways to write a fixed reduced divisor of degree $2g-2$ as a sum of two divisors of degree $g-1$, the degree of $\overline{m} : \overline{\mathscr{S}} \longrightarrow \overline{\overline{\mathscr{S}}}$ is $\frac{1}{2}\begin{pmatrix} 2g-2 \\ g-1 \end{pmatrix}$. □

Proof of Theorem 1 in the case of curves: As explained in the beginning of this section, we need to find a Zariski-dense open subset U of V_{sm}, such that, for all $t \in U$, there exists $W \subset H^0(\omega_{C_t})$ (W of dimension 1) such that $\mu(W \otimes W^{\perp}) \cap N = \{0\}$.

First suppose $g \geq 3$. We may assume that V is irreducible. If V is contained in $\mathscr{H}'_g$, then V is not transverse anywhere to $\mathscr{H}'_g$ and hence, by Proposition 1.1, we may take U to be all of V_{sm}. If $V \not\subset \mathscr{H}'_g$, take $U = V_{sm} \setminus \mathscr{H}'_g$. Suppose that there exists $t \in U$ such that, for all $W \subset H^0(\omega_{C_t})$ of dimension 1, we have $\mu(W \otimes W^{\perp}) \cap N \neq \{0\}$. Then, a fortiori, the hypotheses of part 1 of Proposition 1.1

are met and $N = \mu(W_N \otimes H^0(\omega_{C_t}))$. Then every element of $H^0(\omega_{C_t})$ is orthogonal to W_N. This is impossible given that the hermitian form on $H^0(\omega_{C_t})$ is positive definite.

Now suppose $g = 2$. Then N has dimension ≤ 2 and $\mathbb{P}(N)$ has dimension ≤ 1. For each $W \subset H^0(\omega_{C_t})$ of dimension 1, the space $W^\perp$ also has dimension 1 and hence $W \otimes W^\perp$ has dimension 1. The lines $W \otimes W^\perp$ form a real analytic subset of $\mathbb{P}(H^0(\omega_{C_t})^{\otimes 2})$ of real dimension 2. Since $\overline{\rho} : \mathscr{S} \longrightarrow \overline{\mathscr{S}}$ is finite, we deduce that the lines $\rho(W \otimes W^\perp) = \mu(W \otimes W^\perp)$ form a real analytic subset of $\mathbb{P}(S^2 H^0(\omega_{C_t})) = \mathbb{P}(H^0(\omega_{C_t}^{\otimes 2})) \cong \mathbb{P}^2$ of real dimension 2. An easy computation (with coordinates) will show that this subset is not contained in any projective line in $\mathbb{P}(H^0(\omega_{C_t}^{\otimes 2}))$ and hence is not contained in $\mathbb{P}(N)$. Hence there exists W such that the line $\mu(W \otimes W^\perp)$ is not contained in N, in other words $\mu(W \otimes W^\perp) \cap N = \{0\}$. □

We now consider the case $V \subset \mathscr{A}'_g$. As before, we first prove

Proposition 1.5. *Suppose that $g \geq 3$. Let V be a subvariety of codimension at most g of $\mathscr{A}'_g$. Let t be a point of V_{sm} and let N be the kernel of $\pi_a : S^2 H^0(\Omega^1_{A_t}) \longrightarrow T^*_t V$. Suppose that, for any one-dimensional subvector space W of $H^0(\Omega^1_{A_t})$, the map $\pi_a \rho : W \otimes H^0(\Omega^1_{A_t}) \longrightarrow T^*_t V$ is* not *injective. Then V has codimension exactly g and there is a one-dimensional subvector space W_N of $H^0(\Omega^1_{A_t})$ such that $N = \rho(W_N \otimes H^0(\Omega^1_{A_t}))$.*

Proof: If the map $\pi_a \rho : W \otimes H^0(\Omega^1_{A_t}) \longrightarrow T^*_t V$ is *not* injective, then $\rho(W \otimes H^0(\Omega^1_{A_t})) \cap N \neq \{0\}$. If this holds for every $W \subset H^0(\Omega^1_{A_t})$ of dimension 1, then $\mathbb{P}(N)$ has dimension $g - 1$ and is contained in $\overline{\mathscr{S}} \cong S^2 \mathbb{P}(H^0(\Omega^1_A))$. It follows that V has codimension g. The rest of the argument is now analogous to the proof of part 1 of Proposition 1.1 with $N' = N$. □

Proof of Theorem 1 in the case of abelian varieties: This proof is now as in the case of curves. □

Proof of Corollary 1: Let V be a complete subvariety of codimension $g - d$ ($d \geq 0$) of $\mathscr{A}_g$. By Theorem 1, the set $E_1(V)$ is dense in V. In particular, it is nonempty. Let Y be an irreducible component of $E_1(V)$. Let r and s be integers such that for every ppav A with moduli point in Y there is an elliptic curve E, a ppav B and an isogeny $\nu : E \times B \longrightarrow A$ of degree at most r such that the inverse image of the principal polarization of A by ν is a polarization of degree at most s. Let Y' be an irreducible component of the variety parametrizing such quadruples (E, B, A, ν). Then Y' is a finite cover of Y. The morphism $Y' \longrightarrow \mathscr{A}_1$ which to (E, B, A, ν) associates the isomorphism class of E is constant since Y' is complete (and irreducible) and $\mathscr{A}_1$ is affine.

For any irreducible component Z of $E_1(\mathscr{A}_g)$, there is a finite correspondance between Z and $\mathscr{A}_{g-1} \times \mathscr{A}_1$. In particular, the codimension of Z in $\mathscr{A}_g$ is $\frac{g(g+1)}{2} - (\frac{g(g-1)}{2} + 1) = g - 1$. The variety Y is an irreducible component of the intersection of V with such a Z, hence there is a nonnegative integer e_0 such that the codimension of Y in V is $g - 1 - e_0$. So the codimension of Y in

$\mathscr{A}_g$ is $g-d+g-1-e_0 = 2g-d-1-e_0$. Since Y' maps to a point in $\mathscr{A}_1$, its image V_1 in $\mathscr{A}_{g-1}$ by the second projection has dimension equal to the dimension of Y. Therefore V_1 has dimension $g(g+1)/2-(2g-d-1-e_0) = (g-1)g/2-(g-1-d-e_0)$, i.e., codimension $g-1-d-e_0 \leq g-1$ in $\mathscr{A}_{g-1}$. By Theorem 1, the set $E_1(V_1)$ is dense in V_1. In particular, the set $E_1(V_1)$ is nonempty. Let Y_1 be an irreducible component of $E_1(V_1)$ and let Y_1' be the analogue of Y' for Y_1. Then, as before, the variety Y_1 has codimension $g-1-d-e_0+g-2-e_1$ in $\mathscr{A}_{g-1}$ (for some nonnegative integer e_1), the variety Y_1' maps to a point in $\mathscr{A}_1$ and its image V_2 in $\mathscr{A}_{g-2}$ has codimension $g-2-d-e_0-e_1$. Repeating the argument, we obtain V_i in $\mathscr{A}_{g-i}$ of codimension $g-i-d-e_0-\ldots-e_{i-1}$ containing Y_i of codimension $g-i-d-e_0-\ldots-e_{i-1}+g-i-1-e_i$ in $\mathscr{A}_{g-i}$. For $i=g-2$, we can repeat the argument one last time for $V_{g-2}\subset\mathscr{A}_2$ to obtain Y_{g-2}' with image V_{g-1} in $\mathscr{A}_1$ with codimension $1-d-e_0-\ldots-e_{g-2}$. Since $\mathscr{A}_1$ is affine, the variety V_{g-1} is a point and $d=e_0=\ldots=e_{g-2}=0$. Therefore Y has codimension $2g-1$ in $\mathscr{A}_g$, all the varieties Y_i have codimension $g-i+g-i-1 = 2g-2i-1$ in $\mathscr{A}_{g-i}$, V has codimension g in $\mathscr{A}_g$ and V_i has codimension $g-i$ in $\mathscr{A}_{g-i}$. In particular, the first part of Corollary 1 is proved.

For each i, there is an irreducible subvariety Z_i of V which parametrizes ppav's isogenous to the product of an element of V_i and i fixed elliptic curves ($Z_1 = Y$) because all the maps $Y_i' \longrightarrow \mathscr{A}_1$ (and also $Y'\longrightarrow\mathscr{A}_1$) are constant. It follows from the above that Z_i has the expected dimension $\frac{(g-i)(g-i+1)}{2}+i-g$. Since our choices of the Y_i's (and Y) and hence our choices of the Z_i's were arbitrary, we have proved the second part of the Corollary as well.

To prove the third part, first observe that a dimension count (similar to the case of Y) shows that the dimension of any irreducible component X of $E_q(V)$ is at least $\frac{q(q+1)}{2}+\frac{(g-q)(g-q+1)}{2}-g$. Let X' be the analogue of Y' for X. Then the images X_q and X_{g-q} of X' by the two projections to $\mathscr{A}_q$ and $\mathscr{A}_{g-q}$ are complete subvarieties of $\mathscr{A}_q$ and $\mathscr{A}_{g-q}$ whose codimensions are at least q and $g-q$ respectively by part 1 of the Corollary. So we have

$$\frac{q(q+1)}{2}+\frac{(g-q)(g-q+1)}{2}-g \leq \dim(X) = \dim(X') = \dim(X_q)+\dim(X_{g-q}) \leq$$
$$\leq \frac{q(q+1)}{2}-q+\frac{(g-q)(g-q+1)}{2}-(g-q) = \frac{q(q+1)}{2}+\frac{(g-q)(g-q+1)}{2}-g\,.$$

Therefore we have equality everywhere and part 3 is proved.

Now let V' be the analytic closure of $E_{1,g}(V)$ in V. Since, by Theorem 1, the set $E_1(V_{g-2})$ is dense in V_{g-2} (which is a curve), we see that V' contains Z_{g-2}. Since all of our choices for the Y_i and Y (and hence for the Z_i) were arbitrary, we see that V' contains $E_{1,g-2}(V)$. Repeating this reasoning, we see that V' contains $E_{1,i}(V)$ for all i, hence V' contains $E_1(V)$ and $V'=V$ by Theorem 1. $\square$

Proof of Corollary 2: Let V be a complete codimension g subvariety of $\widetilde{\mathscr{M}_g'}$ or $\mathscr{A}_g'$. Again, by Theorem 1, the set $E_1(V)$ is nonempty. Let $Y\subset V$ be an irreducible component of $E_1(V)$ and define Y' as in the proof of Corollary 1. As in loc. cit. the variety Y is a complete subvariety of V, of codimension at most $g-1$ in V (codimension exactly $g-1$ by Corollary 1 if $V\subset\mathscr{A}_g'$).

Suppose that $V \subset \widetilde{\mathcal{M}'_g}$. Again, since Y' is irreducible and complete and $\mathcal{A}_1$ affine, the map $Y' \longrightarrow \mathcal{A}_1$ is constant, hence its differential has rank 0 everywhere. It follows from [3] pages 172-173 that, for all $t \in Y \cap V_0$ and every one-dimensional subvector space W of $H^0(\omega_{C_t})$, the map $\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T^*_t V$ is *not* injective. Since this noninjectivity is a closed condition and $E_1(V_0)$ is dense in V_0, it follows that it holds for all $t \in V_0$.

Therefore, by Proposition 1.1 and with the notation there, for all $t \in V_0$, there is a one-dimensional subvector space W_N of $H^0(\omega_{C_t})$ such that $N = \mu(W_N \otimes H^0(\omega_{C_t}))$.

Let us globalize the constructions in the proof of Proposition 1.1. Let F_0 be the Hodge bundle on V_0 and let $S^2\mathbb{P}(F_0)$ be the quotient of the fiber product $\mathbb{P}(F_0) \times_{V_0} \mathbb{P}(F_0)$ by the involution σ exchanging the two factors of the fiber product. Let $T^*\mathcal{M}'_g$ be the cotangent bundle of $\mathcal{M}'_g$ and let $\mathcal{N}_0 \subset T^*\mathcal{M}'_g|_{V_0}$ be the conormal bundle to V_0. Denote by $\mathcal{N}''$ (resp. $\mathcal{N}'$) the subcone of decomposable tensors (resp. rank 2 symmetric tensors) in $F_0 \otimes F_0$ (resp. S^2F_0) lying in the inverse image of $\mathcal{N}_0$ by the multiplication map $S^2F_0 \longrightarrow T^*\mathcal{M}'_g|_{V_0}$. Then, by Proposition 1.1 and with the notation there, the fibers of $\mathcal{N}''$, $\mathcal{N}'$, and $\mathcal{N}_0$ at t are respectively $W_N \otimes H^0(\omega_{C_t}) \cup H^0(\omega_{C_t}) \otimes W_N$, $\rho(W_N \otimes H^0(\omega_{C_t}))$ and $\mu(W_N \otimes H^0(\omega_{C_t}))$. Hence the morphism $m : \mathcal{N}' \longrightarrow \mathcal{N}_0$ is an isomorphism because it is an isomorphism on each fiber and the map $\mathbb{P}(\mathcal{N}'') \longrightarrow \mathbb{P}(\mathcal{N}_0)$ is a double cover which splits on each fiber. Since the double cover of V_0 parametrizing the rulings of the fibers of $\mathbb{P}(F_0) \times_{V_0} \mathbb{P}(F_0)$ over V_0 is split, the double cover $\mathbb{P}(\mathcal{N}'') \longrightarrow \mathbb{P}(\mathcal{N}') \cong \mathbb{P}(\mathcal{N}_0)$ is globally split and hence the variety $\mathbb{P}(\mathcal{N}'')$ is the union of two subvarieties of $\mathbb{P}(F_0) \times_{V_0} \mathbb{P}(F_0)$ exchanged by σ and both isomorphic to $\mathbb{P}(\mathcal{N}')$ (by the quotient morphism $\mathbb{P}(F_0) \times_{V_0} \mathbb{P}(F_0) \longrightarrow S^2\mathbb{P}(F_0)$) and to $\mathbb{P}(F_0)$ by either of the two projections $\mathbb{P}(F_0) \times_{V_0} \mathbb{P}(F_0) \longrightarrow \mathbb{P}(F_0)$. In particular, the two components of $\mathbb{P}(\mathcal{N}'')$ are projective bundles on V_0 and $\mathcal{N}''$ is the union of two vector bundles $\mathcal{N}''_1$ and $\mathcal{N}''_2$ with respective fibers $W_N \otimes H^0(\omega_{C_t})$ and $H^0(\omega_{C_t}) \otimes W_N$ at t. Furthermore, we have $\mathcal{N}''_1 \xrightarrow{\cong} \mathcal{N}_0 \xleftarrow{\cong} \mathcal{N}''_2$ (checked on fibers again). Since $\mathbb{P}(\mathcal{N}''_1)$ is isomorphic to $\mathbb{P}(F_0)$, there is a line bundle $\mathcal{W}$ such that $\mathcal{N}''_1 \cong \mathcal{W} \otimes F_0$. So $\mathcal{N}_0 \cong \mathcal{W} \otimes F_0$.

From the injection $\mathcal{N}''_1 \hookrightarrow F_0 \otimes F_0$ we deduce the injection $\mathcal{W} \hookrightarrow F_0$ which is the composition of the morphism $\mathcal{W} \hookrightarrow F_0 \otimes F_0 \otimes F_0^*$ (obtained from $\mathcal{W} \otimes F_0 \cong \mathcal{N}''_1 \hookrightarrow F_0 \otimes F_0$) with the morphism $F_0 \otimes (F_0 \otimes F_0^*) \xrightarrow{id \otimes tr} F_0$ which is the product of the identity $F_0 \xrightarrow{id} F_0$ and the trace morphism $F_0 \otimes F_0^* \cong End(F_0) \xrightarrow{tr} \mathcal{O}_{V_0}$.

For $V \subset \mathcal{A}'_g$ the proof is similar to (and simpler than) the above and uses Proposition 1.5 instead of Proposition 1.1. $\square$

2. Appendix: A remark on density in positive characteristic

In this section we use the notation of the introduction to denote moduli spaces of curves and abelian varieties over an algebraically closed field k of characteristic

$p > 0$. The subvariety V_0 of $\mathscr{A}_g$ parametrizing ppav's of p-rank 0 is a complete (connected if $g > 1$ by [11] (2.6)(c)) subvariety of codimension g of $\mathscr{A}_g$ (see [13], (2) in the introduction and [10], the proof of Theorem 1.1a pages 98–99). We explain below how to deduce from the results of [4, 6, 9] and [11] that the moduli points of non-simple abelian varieties in V_0 are contained in a proper closed subset of V_0 when $g \geq 3$.

The formal group of an abelian variety A of p-rank 0 is isogenous to a sum

$$\sum_{1 \leq i \leq r} G_{m_i, n_i}$$

where m_i and n_i are relatively prime positive integers for each i, the sum $m_i + n_i$ is less than or equal to g for all i, the formal group G_{m_i,n_i} has dimension m_i and its dual is G_{n_i,m_i} (see [9] chapter IV, Sect. 2). The decomposition is symmetric, i.e., the group G_{m_i,n_i} appears as many times as G_{n_i,m_i}. We call the unordered r-tuple $((m_i, n_i))_{1 \leq i \leq r}$ the formal isogeny type of the abelian variety. As in [11], we define the Symmetric Newton Polygon of A to be the lower convex polygon in the plane $\mathbb{R}^2$ which starts at $(0, 0)$ and ends at $(2g, g)$, whose break-points have integer coordinates and whose slopes (arranged in increasing order because of lower convexity) are $\lambda_i = \frac{m_i}{m_i+n_i}$ with multiplicity $m_i + n_i$ (i.e., on the polygon, the x-coordinate grows by $m_i + n_i$ and the y-coordinate grows by m_i). The polygon is symmetric in the sense that if the slope λ appears, then the slope $1 - \lambda$ appears with the same multiplicity. Following [11], we shall say that the Newton Polygon β is above the Newton Polygon α if for all real numbers $x \in [0, 2g], y, z \in [0, g]$ such that $(x, z) \in \beta$, $(x, y) \in \alpha$, we have $z \geq y$. We shall say that β is strictly above α if β is above α and $\beta \neq \alpha$. Again as in [11], for a Symmetric Newton Polygon α, we denote by W_α the set of points in $\mathscr{A}_g$ corresponding to abelian varieties whose Newton Polygon is above α. By [4] page 91, Newton polygons go up under specialization. By [6] page 143 Theorem 2.3.1 and Corollary 2.3.2 (see also [11], 2.4), for any Newton polygon α, the set W_α is closed in V_0. By [11] Theorem (2.6)(a) and Remark (3.3), the abelian variety A_0 with moduli point the generic point of V_0 has formal isogeny type $((1, g-1), (g-1, 1))$. Therefore, since $g \geq 3$, the abelian variety A_0 is simple. Let α_0 denote the Symmetric Newton Polygon of A_0. The moduli point of a non-simple ppav of p-rank 0 is in W_β for some Symmetric Newton Polygon β strictly above α_0. Therefore the set of non-simple ppav's in V_0 is contained in $\cup_{\beta \text{ strictly above } \alpha_0} W_\beta$. Since there are only a finite number of Symmetric Newton Polygons (below the line $x = 2y$ and) above α_0, we deduce that all points of V_0 corresponding to nonsimple abelian varieties are in a proper closed subset of V_0 (which is $\cup_{\beta \text{ strictly above } \alpha_0} W_\beta$).

Therefore V_0 is an example of a subvariety V of codimension g of $\mathscr{A}_g$ (for all $g \geq 3$) or of $\widetilde{\mathscr{M}_3}$ such that $E_q(V)$ is not Zariski-dense in V for any q.

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