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Sharp estimates and the Dirac operator

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Abstract. This paper establishes and extends a conjecture posed by M. Gromov which states that every riemannian metric g on S^n that strictly dominates the standard metric g_0 must have somewhere scalar curvature strictly less than that of g_0. More generally, if M is any compact spin manifold of dimension n which admits a distance decreasing map $f : M \to S^n$ of non-zero degree, then either there is a point $x \in M$ with normalized scalar curvature $\tilde{\kappa}(x) < 1$, or M is isometric to S^n. The distance decreasing hypothesis can be replaced by the weaker assumption f is contracting on 2-forms. In both cases, the results are sharp.

An explicit counterexample is given to show that the result is no longer valid if one replaces 2-forms by k-forms with $k \geq 3$.

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1 Introduction

The main result in this paper could be characterized as establishing a global conservation phenomenon for the scalar curvature function on a Riemannian manifold. A classical example of this phenomenon is given by the Gauss-Bonnet Theorem in dimension two, which states that the average of the scalar curvature depends only on the topology of the surface. In high dimensions any conservation phenomenon for the scalar curvature must be far more delicate since the scalar curvature is a rather weak measure of the Riemannian structure. Nevertheless, using Dirac operator methods, certain sharp results are established. Let κ_g denote the scalar curvature function associated with the metric g on a Riemannian manifold M^n of dimension n. In [2], M. Gromov conjectured the following.

Conjecture. *Let g be a Riemannian metric on S^n such that $g \geq g_0$ where g_0 is the standard metric of constant curvature. Then the scalar curvature κ_g must become* *small somewhere, more precisely,* $inf\,\kappa_g \le c(n)\kappa_{g_0}$ *, where* $c(n) \le 1$ *is a constant that depends on the dimension* n*, with best constant when* $c(n) = 1$.

Let $\tilde{\kappa}_g$ denote the normalized scalar curvature, i.e., $\frac{\kappa_g}{n(n-1)}$. Then the following proves the above conjecture by M. Gromov.

Theorem A. *Let* g *be any Riemannian metric on* S^n *with the property that* $g \ge g_0$*. Then, either there exists some* $x \in M$ *with* $\tilde{\kappa}_g(x) < 1$*, or* $g \equiv g_0$*.*

This result can be extended in the following way. A map $f : M \longrightarrow N$ between Riemannian manifolds is said to be ϵ-contracting if $\| f_* v \| \le \epsilon \| v \|$ for tangent vectors v to M.

Theorem B. *Let* M *be a compact Riemannian spin manifold of dimension* n*. Suppose there exists a* 1*-contracting map* $f : (M, g) \longrightarrow (S^n, g_0)$ *of non-zero degree. Then, either there exists* $x \in M$ *with* $\tilde{\kappa}_g(x) < 1$*, or* $M \equiv S^n$ *and* f *is an isometry.*

Note that the result is sharp since the identity $Id : (S^n, g_0) \longrightarrow (S^n, g_0)$ is 1-contracting and $\tilde{\kappa}_g \equiv 1$.

The idea of the proof is to pull back a suitable spinor bundle on S^n to M, and use it to twist a similar spinor bundle on M. If the assertion on the scalar curvature were wrong, then a Bochner argument would imply the vanishing of the index. On the other hand, the Atiyah-Singer index theorem shows that the index is not equal to zero. This is a well-known method that has been used many times in vanishing type arguments [see [1, 3, 4, 6]], but the key point of these results is using the method for the appropriate twisting bundle. The right choice of the twisting bundle factor makes possible the sharp results.

By weakening the contraction hypothesis on f, Theorem B can be generalized into

Theorem C. *The statement of Theorem B continues to hold if the condition that* f *be* 1*-contracting is replaced by the condition that* f *be* $(1, \Lambda^2)$*-contracting.*

The hypothesis on f can not be weakened further in that sense, that is, an explicit counterexample shows that Theorem B is false for $(1, \Lambda^k)$-contracting maps with $k \ge 3$.

2 Definitions and background results

Let (M, g) be a spin compact Riemannian manifold with metric g. Let (S^n, g_0) be the unit sphere in $\mathbf{R}^n$ with the standard metric g_0. Given a map f between two compact manifolds, the *degree of f* is defined as

$$deg(f) = \sum_{p \in f^{-1}(q)} sign(det\ f_*)_p$$

where q is a regular value. A map $f : M \longrightarrow N$ is said to be ϵ*-contracting* if

$$\| f_* v \| \leq \epsilon \| v \|,$$

for all tangent vectors v to M. A map $f : M \longrightarrow N$ is said to be (ϵ, Λ^k)-*contracting* if

$$\| f^* \varphi \| \leq \epsilon \| \varphi \|,$$

for all k-forms $\varphi \in \Lambda^k(N)$.

The *normalized scalar curvature* of a manifold M of dimension n is defined as

$$\tilde{\kappa} = \frac{\kappa}{n(n-1)},$$

where κ is the usual scalar curvature.

Spin structure

A *spin manifold* is an oriented manifold with a spin structure on its tangent bundle. Let E be an oriented vector bundle, a *spin structure* on E is a 2-sheeted covering

$$\xi : P_{Spin_n}(E) \longrightarrow P_{SO_n}(E)$$

such that $\xi(p.g) = \xi(p).\xi_0(g)$ for all $p \in P_{Spin_n}(E)$ and $g \in Spin_n$, where

$$\xi_0 : Spin_n \longrightarrow SO_n$$

is the universal covering homomorphism with kernel $\mathbf{Z}_2$, and $P_{Spin_n}(E)$ and $P_{SO_n}(E)$ are principal $Spin_n$- and SO_n-bundle, respectively.

Note that a manifold M is spin if and only if the first and second Whitney classes of M, ω_1 and ω_2, are both zero.

A *spinor bundle* on E is a bundle of the form

$$S(E) = P_{Spin_n}(E) \times_\lambda V$$

where V is a left module over $\mathbf{K}$ (= $\mathbf{R}$ or $\mathbf{C}$) for the Clifford algebra $Cl(\mathbf{K}^n) = Cl_n$ and $\lambda : Spin_n \longrightarrow SO(V)$ is a representation by left multiplication by elements of $Spin_n \subseteq Cl_n^0(\mathbf{K}^n) = Cl_n^0$ The Clifford algebra $Cl(V)$ is generated by V subject to the relations $v.v = -\| v \|^2$ for all $v \in V$. The automorphism $\alpha : Cl_n \longrightarrow Cl_n$ that extends the map $\alpha(v) = -v$ gives rise to a decomposition

$$Cl_n = Cl_n^0 \oplus Cl_n^1,$$

where $Cl_n^i = \{\varphi \in Cl_n : \alpha(v) = (-1)^i v\}$ are the eigenspaces of α. $Spin_n = Pin_n \cap Cl_n^0$, where Pin_n is defined as the subgroup of $Cl_n - \{0\}$ generated by the elements v, with $\| v \| \neq 0$. Given a manifold M, $Cl(M)$ will be the Clifford bundle of M, which is the bundle over M whose fiber at a point $p \in M$ is the Clifford algebra $Cl(T_pM)$ of the tangent space at p. Notice that $T(M) \subseteq Cl(M)$. We extend the metric and the connection of M to $Cl(M)$ with the connection ∇ preserving the metric and such that

$$\nabla(\varphi \cdot \psi) = (\nabla\varphi) \cdot \psi + \varphi \cdot (\nabla\psi)$$

for all sections φ and $\psi \in \Gamma(Cl(M))$. Let us consider the following complex bundle over (M, g), where M is spin compact $2n$-dimensional Riemannian manifold

$$S = P_{Spin_{2n}}(M) \times_\lambda \mathbf{C}l_{2n}$$

with the induced connection, where λ is the representation by left multiplication and $\mathbf{C}l_{2n}$ denotes the complexification of Cl_{2n}. Notice that the bundle S is obtained by taking 2^n copies of the fundamental spinor bundle. S has a $\mathbf{Z}_2$-grading. Fix $p \in M$ and choose local point wise orthonormal tangent vector fields around p, $\{e_1, e_2, \ldots, e_{2n}\}$ such that $(\nabla e_k)_p = 0$. Let ω be the oriented "volume element" $\omega = i^n e_1 \cdot e_2 \ldots \cdot e_{2n}$, where $\cdot$ denotes Clifford multiplication. This is a globally defined section of $\mathbf{C}l(M)$ with the following properties:

$$\begin{array}{rrcl} i) & \nabla\omega & = & 0 \\ ii) & \omega^2 & = & 1 \\ iii) & \omega \cdot e & = & -e \cdot \omega \text{ for any } e \in TM. \end{array}$$

Then S has the decomposition

$$S = S^+ \oplus S^-$$

into the +1 and -1 eigenvalues of Clifford multiplication by ω. For any $e \in TM$,

$$e \cdot S^+ \subseteq S^- \text{ and } e \cdot S^- \subseteq S^+.$$

Over (S^{2n}, g_0) we can carry out the same construction to get the bundle

$$E_0 = P_{Spin_{2n}}(S^{2n}) \times_\lambda \mathbf{C}l_{2n}$$

with the induced metric and connection from (S^{2n}, g_0). Fix $x \in S^{2n}$ and choose local point wise orthonormal tangent vector fields around x, $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{2n}\}$ such that $(\nabla e_k)_x = 0$. Let ω_0 be the "volume element"

$$\omega_0 = i^n \epsilon_1 \cdot \epsilon_2 \cdot \ldots \cdot \epsilon_{2n}$$

As before, ω_0 gives the splitting

$$E_0 = E_0^+ \oplus E_0^-$$

into the +1 and -1 eigenspaces of ω_0 .

Suppose that $f : (M^{2n}, g) \longrightarrow (S^{2n}, g)$ is a non-zero degree map. We can pull-back the vector bundle E_0 to the vector bundle $E = f^*E_0$ over (M, g), which as a bundle over M has also the splitting

$$E = E^+ \oplus E^- = f^*E_0^+ \oplus f^*E_0^-.$$

We consider the tensor product bundle $S \otimes E$ over M with the tensor product metric and connection, and

$$S \otimes E = (S^+ \otimes E) \oplus (S^- \otimes E).$$

The *Dirac operator* of S

$$D : \Gamma(S) \longrightarrow \Gamma(S),$$

in terms of an orthonormal basis of tangent vectors at p, is given by

$$D = \sum_{k=1}^{2n} e_k \cdot \nabla_{e_k}$$

Moreover, we can consider the *twisted Dirac operator* D_E on $S \otimes E$; which, on simple elements $\varphi \otimes v \in \Gamma(S \otimes E)$, is defined by

$$D_E(\varphi \otimes v) = \sum_{k=1}^{2n}(e_k \cdot \nabla_{e_k}\varphi) \otimes v + \sum_{k=1}^{2n}(e_k \cdot \varphi) \otimes (\nabla_{e_k} v)$$

This first order operator D_E preserves E^+, i.e.

$$S \otimes E = S \otimes E^+ \oplus S \otimes E^-$$

and

$$D_E(\Gamma(S \otimes E^+)) \subseteq \Gamma(S \otimes E^+).$$

In fact, $E = f^*(E_0^+) = f^*(\{v \in E_0 : \omega_0 \cdot v = v\}) = \{v \in E : (f^*\omega_0) \cdot v = v\}$, so if $\varphi \otimes v \in \Gamma(S \otimes E^+)$, then

$$\begin{aligned} D_E(\varphi \otimes v) &= \textstyle\sum_{i=1}^{2n} e_i \cdot \nabla_{e_i}(\varphi \otimes v) \\ &= \textstyle\sum_{i=1}^{2n} e_i \cdot (\nabla_i \varphi \otimes v + \varphi \otimes \nabla_i v) \\ &= \textstyle\sum_{i=1}^{2n}(e_i \cdot \nabla\varphi) \otimes v + \sum_{i=1}^{2n}(e_i \cdot \varphi \otimes \nabla_i v) \end{aligned}$$

where $\nabla_i = \nabla_{e_i}$.

If $v \in E^+$, then $\nabla_i v \in E^+$ because $\nabla_i v = \nabla_i(\omega \cdot v) = (\nabla_i \omega) \cdot v + \omega \cdot (\nabla_i v)$ and $(\nabla_i \omega) = 0$. Therefore, $D_E(\sigma \otimes v) \in \Gamma(S \otimes E^+)$. Since any element of $S \otimes E^+$ is the sum of simple elements of the form $\sigma \otimes v$, we can write

$$D_{E^+} = D_{E|_{S \otimes E^+}}.$$

Furthermore, since $e \cdot S^\pm \subseteq S^\mp$, then

$$D_{E^+} = D_{E^+}^+ \oplus D_{E^+}^-$$

and

$$D_{E^+}^\pm : \Gamma(S^\pm \otimes E^+) \longrightarrow \Gamma(S^\mp \otimes E^+)$$

Bochner-Lichnerowicz-Weitzenböck formula

We now recall the fundamental B-L-W formula for the twisted Dirac operator D_E of the bundle $S \otimes E$ over M,(see [6])

$$D_E^2 = \nabla^* \nabla + \frac{1}{4}\kappa + R^E,$$

where the operator $\nabla^*\nabla : \Gamma(S \otimes E) \longrightarrow \Gamma(S \otimes E)$ is defined in terms of a local basis of point wise orthonormal tangent vector fields by

$$\nabla^*\nabla = -\sum_{k=1}^{2n} \nabla_{e_k}\nabla_{e_k} + \nabla_{\nabla_{e_k}} e_k,$$

$\kappa = \sum_{i,j} g(\mathrm{R}_{e_i e_j} e_j, e_i)$ is the scalar curvature of M, g is the Riemannian metric and R the curvature tensor of M. R^E is defined on simple elements $\sigma \otimes v \in \Gamma(S \otimes E)$ by $R^E(\sigma \otimes v) = \frac{1}{2}\sum_{k=1}^{2n}(e_i \cdot e_j \cdot \sigma) \otimes (\mathrm{R}^E_{e_i e_j} v)$, where R^E denotes the curvature tensor of E.

Notice that R^E depends linearly on the curvature tensor R^E of E. For a more detailed description see [6].

3 Results for S^n

In this section we examine the problem, originally posed by M. Gromov [2], of studying perturbations of the canonical metric g on the n-sphere with normalized curvature 1.

Theorem 3.1. *Let g be any Riemannian metric on S^n with the property that $g \geq g_0$. Then, either there exists some $x \in S^n$ with $\tilde{\kappa}(x) < 1$, or $g \equiv g_0$.*

Example 3.2.

$$(S^n, g) \xrightarrow{Id} (S^n, g_0),$$

where $g = (1+\epsilon)g$, $\epsilon > 0$, then $g \geq g$. In this case $\tilde{\kappa}_g = \frac{1}{1+\epsilon} < 1$ for all x. This result is also true when the map between the spheres is of non-zero degree and not necessarily the identity.

Theorem 3.3. *Let $f : (S^n, g) \longrightarrow (S^n, g_0)$ be a map of non-zero degree. Suppose that f is 1-contracting. Then, either there exists some $x \in S^n$ with $\tilde{\kappa}_g(x) < 1$, or f is an isometry.*

Remark 3.4. This result is sharp since the identity $Id : (S^n, g) \longrightarrow (S^n, g_0)$ is 1-contracting and $\tilde{\kappa}_{g_0} \equiv 1$.

The proof of this result given in this paper, depends only on the existence of a spin structure on (S^n, g), and therefore, (S^n, g) can be replaced by any compact spin manifold M.

4 Results for a compact spin manifold

Theorem 3.2 can be extended in the following way.

Theorem 4.1. *Let M be a compact Riemannian spin manifold of dimension n. Suppose there exists a 1-contracting map $f : (M, g) \to (S^n, g)$ of non-zero degree. Then, either there exists $x \in M$ with $\tilde{\kappa}_g(x) < 1$, or $M \equiv S^n$ and f is an isometry.*

Idea of the proof

Theorem 4.1 will be proved separately for the cases when M is an even and an odd dimensional manifold. In both cases, the proofs are done by contradiction assuming that $\tilde{\kappa}_g(x) \geq 1$ everywhere. We consider the twisted spinor bundle $S \otimes E^+$ over M together with its Dirac operator D_{E^+}; and we calculate the index of this Dirac operator in two ways. First, assuming that $\tilde{\kappa}_g(x) > 1$ all over M and considering the B-L-W formula for D_{E^+} we see that $Index(D_{E^+}) = 0$. Then, using the Atiyah-Singer Index Theorem we conclude that $Index(D_{E^+}) \neq 0$. Finally, assuming that $\tilde{\kappa}_g(x) \equiv 1$, we show that f must be an isometry.

Proof of Theorem 4.1

Even Dimensional Case. Let M be a compact spin $2n$-dimensional Riemannian manifold with metric g. Let S^{2n} be the unit $2n$-sphere with standard metric g_0. Let $f : M \to S^{2n}$ be a 1-contracting map of non-zero degree. By contradiction, assume that $\tilde{\kappa}_g > 1$ all over M. We consider the twisted vector bundle $S \otimes E^+$ over M and its Dirac operator D_{E^+} as we did in Sect. 2. Recall that $D_{E^+} = D_E|_{S\otimes E^+}$. Fix $p \in M$. Let $\{e_1, ..., e_{2n}\}$ be a g-orthonormal tangent frame near $p \in M$ such that $(\nabla e_k)_P = 0$ for each k. Let $\{\epsilon_1, ..., \epsilon_{2n}\}$ be a g_0-orthonormal tangent frame near $f(p) \in S^{2n}$ such that $(\nabla \epsilon_k)_{f(p)} = 0$ for each k. Moreover, the bases $\{e_1, ..., e_{2n}\}$ and $\{\epsilon_1, ..., \epsilon_{2n}\}$ can be chosen so that $\epsilon_j = \lambda_j f_* e_j$ for appropriate $\{\lambda_j\}_{j=1}^{2n}$. This is possible since f_* is symmetric.

Notice also that $\lambda_j \geq 1$ since f is 1-contracting,

$$1 = g_0(\epsilon_j, \epsilon_j) = g_0(\lambda_j f_* e_j, \lambda_j f_* e_j) = \lambda_j^2 g_0(f_* e_j, f_* e_j)$$

therefore,

$$1 = \lambda_j^2 g_0(f_* e_j, f_* e_j) \leq \lambda_j^2 g(e_j, e_j) = \lambda_j^2 \qquad (4.2)$$

Considering the inner product $<,>$ on the space $\Gamma(S \otimes E)$ of cross-sections defined by

$$< \phi, \psi >= \int_M g_x(\phi, \psi) \qquad \forall \phi, \psi \in \Gamma(S \otimes E),$$

we can write the B-L-W formula as

$$\begin{aligned} < D_E^2\phi, \phi > &= < \nabla^*\nabla\phi, \phi > + \tfrac{1}{4}\kappa < \phi, \phi > + < R^E\phi, \phi > \\ &= < \nabla\phi, \nabla\phi > + \tfrac{1}{4}\kappa \parallel \phi \parallel^2 + < R^E\phi, \phi > \end{aligned}$$

therefore,

$$< D_E^2\phi,\phi > \geq \frac{1}{4}\kappa \parallel \phi \parallel^2 + < R^E\phi,\phi > .$$

In order to establish the result, we must look at the term $< R^E\phi,\phi >$ in more detail. On simple elements $\sigma \otimes v \in \Gamma(S \otimes E)$, R^E is defined by

$$R^E(\sigma \otimes v) = \frac{1}{2}\sum_{i,j=1}^{2n}(e_i \cdot e_j \cdot \sigma) \otimes (\mathrm{R}^E_{e_i e_j} v).$$

More explicitly, (see [6])

$$\mathrm{R}^E_{e_i e_j} = \mathrm{R}^{E_0}_{f_* e_i f_* e_j} = \frac{1}{4}\sum_{k,l} g_0(\mathrm{R}^{E_0}_{f_* e_i f_* e_j}\epsilon_k, \epsilon_l)\epsilon_k \cdot \epsilon_l$$

where R^{E_0} is the curvature tensor on S^{2n} and it is considered as an endomorphism of $E_p \equiv (E_0)_{f(p)}$.

Lemma 4.3.

$$\mathrm{R}^E_{e_i e_j} \equiv \frac{1}{2}\frac{1}{\lambda_i\lambda_j}\epsilon_j \cdot \epsilon_i \qquad i \neq j$$

Proof.

$$\begin{aligned}
\mathrm{R}^E_{e_i e_j} &= \tfrac{1}{4}\sum_{k,l=1}^{2n}[g_0(f_* e_i, \epsilon_l)g_0(f_* e_j, \epsilon_k) - g_0(f_* e_j, \epsilon_l)g_0(f_* e_i, \epsilon_k)]\epsilon_k \cdot \epsilon_l \\
&= \tfrac{1}{4}\sum_{k,l=1}^{2n}[g_0(\tfrac{\epsilon_i}{\lambda_i}, \epsilon_l)g_0(\tfrac{\epsilon_j}{\lambda_j}, \epsilon_k) - g_0(\tfrac{\epsilon_j}{\lambda_j}, \epsilon_l)g_0(\tfrac{\epsilon_i}{\lambda_i}, \epsilon_k)]\epsilon_k \cdot \epsilon_l \\
&= \tfrac{1}{4}\sum_{k,l=1}^{2n}[\tfrac{1}{\lambda_i\lambda_j}\delta_{il}\delta_{jk} - \tfrac{1}{\lambda_i\lambda_j}\delta_{jl}\delta_{ik}]\epsilon_k \cdot \epsilon_l \\
&= \tfrac{1}{4}[\tfrac{1}{\lambda_i\lambda_j}\epsilon_j \cdot \epsilon_i - \tfrac{1}{\lambda_i\lambda_j}\epsilon_i \cdot \epsilon_j] \\
&= \tfrac{1}{4}\tfrac{1}{\lambda_i\lambda_j}2\epsilon_j \cdot \epsilon_i \qquad i \neq j
\end{aligned}$$

Thus,

$$\mathrm{R}^E_{e_i e_j} = \frac{1}{2}\frac{1}{\lambda_i\lambda_j}\epsilon_j \cdot \epsilon_i \qquad i \neq j$$

Let $\{\sigma_\alpha\}_{\alpha=1}^{2^{2n}}$ be a basis for S and $\{v_\beta\}_{\beta=1}^{2^{2n}}$ be a basis for E_0. Then for any $\phi \in \Gamma(S \otimes E)$,

$$\phi = \sum_{\alpha,\beta} a_{\alpha\beta}\sigma_\alpha \otimes v_\beta$$

and

$$< R^E\phi, \phi > = < R^E(\textstyle\sum_{\alpha,\beta} a_{\alpha\beta}\sigma_\alpha \otimes v_\beta), \sum_{k,l} a_{kl}\sigma_k \otimes v_l > \quad (4.4)$$

$$= < \tfrac{1}{2}\textstyle\sum_{i\neq j}\sum_{\alpha,\beta} a_{\alpha\beta} e_i \cdot e_j \cdot \sigma_\alpha$$

$$\otimes \tfrac{1}{2}\tfrac{1}{\lambda_i\lambda_j}\epsilon_j \cdot \epsilon_i \cdot v_\beta, \textstyle\sum_{k,l} a_{kl}\sigma_k \otimes v_l >$$

$$= \tfrac{1}{4}\textstyle\sum_{i\neq j}\sum_{k,l}\sum_{\alpha,\beta} \frac{a_{\alpha\beta}a_{kl}}{\lambda_i\lambda_j} < e_i \cdot e_j \cdot \sigma_\alpha, \sigma_k >< \epsilon_j \cdot \epsilon_i \cdot v_\beta, v_l >$$

This suggests choosing the bases $\{\sigma_\alpha\}_\alpha$ and $\{v_\beta\}_\beta$ "invariant" by $e_i \cdot e_j$ and $\epsilon_j \cdot \epsilon_i$, respectively.

For each fixed pair (i, j), consider the following orthonormal bases. Let $\sigma_1 \in \Gamma(S)$ with $\| \sigma_1 \| = 1$, since $e_i \cdot e_j : S \to S$ is such that $(e_i \cdot e_j)^2 = -1$, the subspace $\{\sigma_1, e_i \cdot e_j \cdot \sigma_1\}$ is invariant under $e_i \cdot e_j$.

Moreover, σ_1 and $e_i \cdot e_j \cdot \sigma_1$ are orthonormal. In fact, $< \sigma_1, e_i \cdot e_j \cdot \sigma_1 >= - < e_i \cdot \sigma_1, e_j \cdot \sigma_1 >=< e_j \cdot e_i \cdot \sigma_1, \sigma_1 >= - < e_i \cdot e_j \cdot \sigma_1, \sigma_1 >$; therefore, $< \sigma_1, e_i \cdot e_j \cdot \sigma_1 >= 0$; and $< e_i \cdot e_j \cdot \sigma_1, e_i \cdot e_j \cdot \sigma_1 >=< e_j \cdot \sigma_1, e_j \cdot \sigma_1 >=< \sigma_1, \sigma_1 > =\| \sigma_1 \|^2 = 1$.

Let $\sigma_2 \in \Gamma(S)$ with $\| \sigma_2 \| = 1$ and $\sigma_2 \perp \{\sigma_1, e_i \cdot e_j \cdot \sigma_1\}$, i.e., $< \sigma_2, \sigma_1 >= 0$ and $< \sigma_2, e_i \cdot e_j \cdot \sigma_1 >= 0$.

Then, $\{\sigma_1, e_i \cdot e_j \cdot \sigma_1, \sigma_2, e_i \cdot e_j \cdot \sigma_2\}$ is an orthonormal set. It remains to verify that $\sigma_1 \perp e_i \cdot e_j \cdot \sigma_2$ and that $e_i \cdot e_j \cdot \sigma_1 \perp e_i \cdot e_j \cdot \sigma_2$. From the Clifford structure, we have that $< \sigma_1, e_i \cdot e_j \cdot \sigma_2 >= - < e_i \cdot \sigma_1, e_j \cdot \sigma_2 >=< e_j \cdot e_i \cdot \sigma_1, \sigma_2 >= - < e_i \cdot e_j \cdot \sigma_1, \sigma_2 >= 0$ by construction. Also, $< e_i \cdot e_j \cdot \sigma_1, e_i \cdot e_j \cdot \sigma_2 >=< e_j \cdot \sigma_1, e_j \cdot \sigma_2 >=< \sigma_1, \sigma_2 >= 0$ by construction. Continuing with this process, we obtain an orthonormal basis, $\{\sigma_\alpha, e_i \cdot e_j \cdot \sigma_\alpha\}_{\alpha=1}^{2^{2n-1}}$ for S. The operator $e_i \cdot e_j$ leaves invariant each of these two-dimensional subspaces and permutes the basis up to a sign. Analogous considerations are true for E_0 and we can obtain an orthonormal basis $\{v_\beta, \epsilon_j \cdot \epsilon_i \cdot v_\beta\}_{\beta=1}^{2^{2n-1}}$. Each of the pairs $\{\sigma_\alpha, e_i \cdot e_j \cdot \sigma_\alpha\}$ and $\{v_\beta, \epsilon_j \cdot \epsilon_i \cdot v_\beta\}$ will give the following four orthonormal basis elements for $S \otimes E$.

$$\sigma_\alpha \otimes v_\beta \qquad \sigma_\alpha \otimes \epsilon_j \cdot \epsilon_i \cdot v_\beta \qquad e_i \cdot e_j \cdot \sigma_\alpha \otimes v_\beta \qquad e_i \cdot e_j \cdot \sigma_\alpha \otimes \epsilon_j \cdot \epsilon_i \cdot v_\beta$$

A generic element $\phi \in \Gamma(S \otimes E)$, in terms of the tensor product basis can be written as

$$\phi = \sum_{\alpha,\beta}(a_{\alpha\beta}\sigma_\alpha \otimes v_\beta + b_{\alpha\beta}\sigma_\alpha \otimes \epsilon_j \cdot \epsilon_i \cdot v_\beta + c_{\alpha\beta} e_i \cdot e_j \cdot \sigma_\alpha \otimes v_\beta + d_{\alpha\beta} e_i \cdot e_j \cdot \sigma_\alpha \otimes \epsilon_j \cdot \epsilon_i \cdot v_\beta)$$

Therefore, we have the following

Lemma 4.5. *For each fixed pair* (i, j),

$$< \sum_{\alpha,\beta} a_{\alpha\beta} e_i \cdot e_j \cdot \sigma_\alpha \otimes \epsilon_j \cdot \epsilon_i \cdot v_\beta, \sum_{k,l} a_{kl}\sigma_k \otimes v_l > \geq - \| \phi \|^2,$$

where $\{\sigma_\alpha\}_\alpha$ *and* $\{v_\beta\}_\beta$ *denote generic bases as in (4.4)*

Proof. Using the basis for the tensor product constructed above, the sums in each factor of $<,>$, can be rewritten in four-term sums. In fact,

$$< (a_{\alpha\beta}e_i \cdot e_j \cdot \sigma_\alpha \otimes \epsilon_j \cdot \epsilon_i \cdot v_\beta + b_{\alpha\beta}e_i \cdot e_j \cdot \sigma_\alpha \otimes (\epsilon_j \cdot \epsilon_i)^2 \cdot v_\beta +$$
$$c_{\alpha\beta}(e_i \cdot e_j)^2 \cdot \sigma_\alpha \otimes \epsilon_j \cdot \epsilon_i \cdot v_\beta + d_{\alpha\beta}(e_i \cdot e_j)^2 \cdot \sigma_\alpha \otimes (\epsilon_j \cdot \epsilon_i)^2 \cdot v_\beta),$$
$$(a_{kl}\sigma_k \otimes v_l + b_{kl}\sigma_k \otimes \epsilon_j \cdot \epsilon_i \cdot v_l + c_{kl}e_i \cdot e_j \cdot \sigma_k \otimes v_l + d_{kl}e_i \cdot e_j \cdot \sigma_k \otimes \epsilon_j \cdot \epsilon_i \cdot v_l) >=$$
$$a_{\alpha\beta}d_{\alpha\beta}\delta_{\alpha k}\delta_{\beta l} - b_{\alpha\beta}c_{\alpha\beta}\delta_{\alpha k}\delta_{l\beta} - b_{\alpha\beta}c_{\alpha\beta}\delta_{k\alpha}\delta_{\beta l} + a_{\alpha\beta}d_{\alpha\beta}\delta_{k\alpha}\delta_{l\beta}$$

and the only terms of each four-term sum that remain are

$$a_{\alpha\beta}d_{\alpha\beta} - b_{\alpha\beta}c_{\alpha\beta} - b_{\alpha\beta}c_{\alpha\beta} + a_{\alpha\beta}d_{\alpha\beta} = 2a_{\alpha\beta}d_{\alpha\beta} - 2b_{\alpha\beta}c_{\alpha\beta}.$$

Since

$$2a_{\alpha\beta}d_{\alpha\beta} \geq -(a^2_{\alpha\beta} + d^2_{\alpha\beta})$$

and

$$-2b_{\alpha\beta}c_{\alpha\beta} \geq -(b^2_{\alpha\beta} + c^2_{\alpha\beta}),$$

we get that each four-term sum for fixed (α, β) is bounded below by

$$-(a^2_{\alpha\beta} + b^2_{\alpha\beta} + c^2_{\alpha\beta} + d^2_{\alpha\beta}).$$

Summing over α and β, we obtain $\| \phi \|^2$.

Now we can find a lower bound for $< R^E\phi, \phi >$.

$$\begin{aligned} < R^E\phi, \phi > &= \tfrac{1}{4}\textstyle\sum_{i\neq j}\sum_{k,l}\sum_{\alpha,\beta} \frac{a_{\alpha\beta}a_{kl}}{\lambda_i\lambda_j} < e_i \cdot e_j \cdot \sigma_\alpha, \sigma_k > \\ & \quad \times < \epsilon_j \cdot \epsilon_i \cdot v_\beta, v_l > \\ &\geq -\tfrac{1}{4}\textstyle\sum_{i\neq j} \frac{1}{\lambda_i\lambda_j} \| \phi \|^2 \qquad \text{(by Lemma 4.5)} \end{aligned}$$

$$< R^E\phi, \phi > \geq -\frac{1}{4}\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j} \| \phi \|^2 \qquad (4.6)$$

Recall that $\lambda_i \geq 1$ by (4.2) and therefore,

$$\begin{aligned} < R^E\phi, \phi > &\geq -\tfrac{1}{4}\textstyle\sum_{i\neq j} \| \phi \|^2 \\ &\geq -\tfrac{1}{4}2n(2n-1) \| \phi \|^2 \end{aligned}$$

Consequently,

$$< D_E^2\phi,\phi > \quad \geq \quad \tfrac{1}{4}\kappa \parallel \phi \parallel^2 - \tfrac{1}{4}2n(2n-1) \parallel \phi \parallel^2$$

$$< D_E^2\phi,\phi > \quad \geq \quad \tfrac{1}{4}2n(2n-1)(\tilde{\kappa}-1) \parallel \phi \parallel^2$$

where $k = \frac{\kappa}{2n(2n-1)}$ is the normalized scalar curvature of M. Therefore; if $\tilde{\kappa} > 1$, then $ker(D_E^2) = 0$. But $ker(D_E^2) = ker(D_E)$, see [4]. Since the operator $D_E : \Gamma((S \otimes E^+) \oplus (S \otimes E^-)) \to \Gamma((S \otimes E^+) \oplus (S \otimes E^-))$ preserves the direct sum,

$$0 = ker(D_{E_{|_{S\otimes E^+}}}) = ker(D_E).$$

We also have that $D_{E^+} = D^+_{E^+} \oplus D^-_{E^+}$, where $D^{\pm}_{E^+} : S^{\pm} \otimes E^+ \to S^{\mp} \otimes E^+$. Since $0 = ker(D_{E^+}) = ker(D^+_{E^+}) \oplus ker(D^-_{E^+})$, we get $ker(D^+_{E^+}) = ker(D^-_{E^+}) = 0$. The index of D_{E^+} is given by

$$Index(D_{E^+}) = dim(ker(D^+_{E^+})) - dim(ker(D^-_{E^+})) = 0.$$

However, this index is not zero. In fact, using the Atiyah-Singer Index Theorem, we get

$$Index(D_{E^+}) = \{ch(E^+)\hat{A}(M)\}[M],$$

where $ch(E^+)$ is the Chern character of E^+ and $\hat{A}$ is the total $\hat{A}$-class of M.

Recall that $ch(E^+) = dim(E^+) + ch^1(E^+) + \ldots + ch^n(E^+)$, where ch^i is the i^{th} symmetric polynomial in the Chern class c_i, with $ch^i \in H^{2i}(M)$, [see [5]]; and that E^+ is the pull-back bundle of E_0^+ through f^*. On S^{2n} the Chern character of the vector bundle E_0^+ is given by

$$ch(E_0^+) = dim(E_0^+) + \frac{1}{(n-1)!}c_n(E_0^+).$$

On the pull-back,

$$ch(E^+) \quad = \quad dim(E_0^+) + \tfrac{1}{(n-1)!} f^* c_n(E_0^+)$$

$$ch(E^+) \quad = \quad 2^{2n-1} + \tfrac{1}{(n-1)!} f^* c_n(E_0^+).$$

Applying the B-L-W type formula, Atiyah, Hitchin, Lichnerowicz and Singer showed that a compact spin manifold M with $\kappa > 0$ must have $\hat{A}(M) = 1$. In our case we are under the assumption that $\tilde{\kappa} > 1$, therefore $\kappa > 0$ and $\hat{A}(M) = 1$. Consequently,

$$Index(D_{E^+}) \quad = \quad \{(2^{2n-1} + \tfrac{1}{(n-1)!} f^* c_n(E_0^+))\hat{A}[M]\}[M]$$

$$Index(D_{E^+}) \quad = \quad \tfrac{1}{(n-1)!} f^* c_n(E_0^+)[M]$$

$$= \quad \tfrac{1}{(n-1)!} \int_M f^* c_n(E_0^+)$$

$$= \quad \tfrac{1}{(n-1)!} deg(f) \int_{S^{2n}} c_n(E_0^+) \neq 0 \qquad (4.7)$$

since $deg(f) \neq 0$ and $c_n(E_0^+) \neq 0$ on S^{2n} because it is non-zero multiple of the Euler number of S^{2n}.

If $\tilde{\kappa} \equiv 1$, then f is an isometry. In fact, $\tilde{\kappa} \equiv 1$ is equivalent to $\kappa = 2n(2n-1)$, then inequality (4.6) gives

$$< D_E^2\phi, \phi > \quad \geq \quad \tfrac{1}{4}\kappa \parallel \phi \parallel^2 - \tfrac{1}{4}\sum_{i\neq j}^{2n} \tfrac{1}{\lambda_i\lambda_j} \parallel \phi \parallel^2$$

$$< D_E^2\phi, \phi > \quad \geq \quad \tfrac{1}{4} \parallel \phi \parallel^2 [2n(2n-1) - \sum_{i\neq j}^{2n} \tfrac{1}{\lambda_i\lambda_j}]$$

$$\geq \quad \tfrac{1}{4} \parallel \phi \parallel^2 [\sum_{i\neq j}^{2n}(1 - \tfrac{1}{\lambda_i\lambda_j})].$$

Since $Index(D_{E^+}) \neq 0$, $ker(D_E) \neq 0$, there exists $0 \neq \varphi \in \Gamma(S \otimes E)$ such that $D\varphi = 0$, so

$$0 \geq \frac{1}{4} \parallel \phi \parallel^2 [\sum_{i\neq j}^{2n}(1 - \frac{1}{\lambda_i\lambda_j})] \qquad (4.8)$$

Recall that each $\lambda_i \geq 1$, so $1 - \frac{1}{\lambda_i\lambda_j} \geq 0$; therefore,

$$0 \leq 1 - \frac{1}{\lambda_i\lambda_j} \leq 0 \qquad \forall i \neq j$$

or equivalently,

$$\lambda_i\lambda_j \equiv 1.$$

Therefore, $\lambda_i \equiv 1$ for all $1 \leq i \leq 2n$ and f is an isometry.

Odd dimensional case

Let M be a compact spin manifold of dimension $2n-1$, with Riemannian metric g. Let S_r^{2n-1} be $(2n-1)$-sphere of radius r with the standard metric g_0. Let $f : M \to S^{2n-1}$ be a 1-contracting map of non-zero degree. We want to show that there exists $x \in M$ where $\kappa(x) < 1$. Consider

$$M \times S_r^1 \overset{f\times\frac{1}{r}id}{\longrightarrow} S^{2n-1} \times S^1 \overset{h}{\longrightarrow} S^{2n-1} \wedge S^1 \cong S^{2n}$$

where S_r^1 is the one dimensional sphere of radius r, $f \times \frac{1}{r}id$ is defined as $(f \times \frac{1}{r}id)(p,t) = (f(p), \frac{t}{r})$ $\forall (p,t) \in M \times S^1$, and where h is a 1-contracting map into the smash product of non-zero degree.

Consider now the following metrics. On $M \times S_r^1$, $g + ds^2$ where ds is the standard metric on S_r^1; on $S^{2n-1} \times S^1$, $g_0 + ds^2$ where ds is the standard metric on S^1; and on S^{2n}, $\tilde{g}$ is the standard metric of the unit sphere S^{2n}.

The compose map $\tilde{f} = h \circ (f \times \frac{1}{r}id)$ is of non-zero degree from $M^{2n-1} \times S^1 \to S^{2n}$. It is also 1-contracting,

$$\begin{aligned}\| \tilde{f}_*(v,t) \| = \| h_*(f_*v, \tfrac{t}{r}) \| &\leq \| f_*v \| + \| \tfrac{t}{r} \| \\ &\leq \| v \| + \tfrac{1}{r} \| t \| \\ &\leq \| v \| + \| t \| .\end{aligned}$$

We assume $r > 1$.

We can now apply the same method we used for the even-dimensional case. Construct complex spinor bundles S over $M^{2n-1} \times S^1_r$ and E_0 over S^{2n}, respectively; and consider the bundle $S \otimes E$ over $M^{2n-1} \times S^1_r$, where $E = \tilde{f}^* E_0^+$.

Choose a basis $\{e_1, \ldots, e_{2n-1}, e_{2n}\}$ of $(g + ds^2)$-orthonormal adapted tangent vectors around $x \in M^{2n-1} \times S^1_r$ such that $(\nabla e_k)_x = 0$ for each k and such that $e_1, \ldots, e_{2n-1}$ are tangent to M^{2n-1} and e_{2n} is tangent to S^1_r. As before choose $\tilde{g}$-orthonormal basis $\{\epsilon_1, \ldots, \epsilon_{2n}\}$ around $\tilde{f}(x)$ in S^{2n}.

Therefore, we can find positive scalars $\{\lambda_i\}_{i=1}^{2n}$ such that $\epsilon_i = \lambda_i \tilde{f}_* e_i$. Then we have that

$$1 = \tilde{g}(\epsilon_i, \epsilon_i) = \tilde{g}(\lambda_i \tilde{f}_* e_i, \lambda_i \tilde{f}_* e_i) = \lambda_i^2 \tilde{g}(\tilde{f}_* e_i, \tilde{f}_* e_i),$$

thus for $1 \leq i \leq 2n-1$

$$1 = \lambda_i^2 \tilde{g}(\tilde{f}_* e_i, \tilde{f}_* e_i) \leq \lambda_i^2 g_0(f_* e_i, f_* e_i) \leq \lambda_i^2 g(e_i, e_i) = \lambda_i^2$$

and $1 \leq \lambda_i^2$.

For $j = 2n$

$$\begin{aligned} 1 &= \lambda_{2n}^2 \tilde{g}(\tilde{f}_* e_{2n}, \tilde{f}_* e_{2n}) \leq \lambda_{2n}^2 ds^2(\tfrac{e_{2n}}{r}, \tfrac{e_{2n}}{r}) \\ 1 &\leq \tfrac{\lambda_{2n}}{r^2} \\ r^2 &\leq \lambda_{2n}^2 . \end{aligned}$$

In the B-L-W formula for the twisted bundle $S \otimes E$ and its Dirac operator D_E,

$$D_E^2 = \nabla^* \nabla + \frac{1}{4}\kappa + R^E, \qquad (4.9)$$

the curvature term (4.3) can be bounded by separating the terms arising from the S^1 factor, as follows (cf. Lemma 4.5),

$$
\begin{aligned}
< R^E\phi, \phi > \quad = \quad & \tfrac{1}{4}\textstyle\sum_{i\neq j}^{2n-1}\sum_{k,l}\sum_{\alpha,\beta}\frac{a_{\alpha\beta}a_{kl}}{\lambda_i\lambda_j} < e_i\cdot e_j\cdot\sigma_\alpha, \sigma_k > \\
& \times < \epsilon_j\cdot\epsilon_i\cdot v_\beta, v_l > \\
+ \quad & \tfrac{1}{4}\textstyle\sum_{i=1}^{2n-1}\sum_{k,l}\sum_{\alpha,\beta}\frac{a_{\alpha\beta}a_{kl}}{\lambda_i\lambda_{2n}} < e_i\cdot e_{2n}\cdot\sigma_\alpha, \sigma_k > \\
& \times < \epsilon_{2n}\cdot\epsilon_i\cdot v_\beta, v_l > \\
+ \quad & \tfrac{1}{4}\textstyle\sum_{j=1}^{2n-1}\sum_{k,l}\sum_{\alpha,\beta}\frac{a_{\alpha\beta}a_{kl}}{\lambda_{2n}\lambda_j} < e_{2n}\cdot e_j\cdot\sigma_\alpha, \sigma_k > \\
& \times < \epsilon_j\cdot\epsilon_{2n}\cdot v_\beta, v_l >
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
< R^E\phi, \phi > \quad \geq \quad & \tfrac{1}{4}\textstyle\sum_{i\neq j}^{2n-1}\sum_{k,l}\sum_{\alpha,\beta} a_{\alpha\beta a_{kl}} < \sigma_\alpha, \sigma_k >< v_\beta, v_l > \\
- \quad & 2[\tfrac{1}{4}\textstyle\sum_{i=1}^{2n-1}\sum_{k,l}\sum_{\alpha,\beta} a_{\alpha\beta}a_{kl} < \sigma_\alpha, \sigma_k >< v_\beta, v_l >] \\
< R^E\phi, \phi > \quad \geq \quad & -\tfrac{1}{4}\textstyle\sum_{i\neq j}^{2n-1} \parallel \phi \parallel^2 - 2\frac{1}{4r}\sum_{i=1}^{2n-1} \parallel \phi \parallel^2 \\
< R^E\phi, \phi > \quad \geq \quad & -\tfrac{1}{4}(2n-1)(2n-2) \parallel \phi \parallel^2 - \tfrac{1}{2r}(2n-1) \parallel \phi \parallel^2
\end{aligned}
$$

Note that in (4.9) k is the unnormalized scalar curvature of $M^{2n-1}\times S_r^1$ which is equal to the unnormalized scalar curvature of M^{2n-1}. The normalized scalar curvature of M^{2n-1} is $\tilde{\kappa} = \frac{\kappa}{(2n-1)(2n-2)}$, consequently,

$$
\begin{aligned}
< D_E^2\phi, \phi > \quad \geq \quad & [\tfrac{1}{4}\kappa - \tfrac{1}{4}(2n-1)(2n-2) - \tfrac{2n-1}{2r}] \parallel \phi \parallel^2 \\
< D_E^2\phi, \phi > \quad \geq \quad & \tfrac{1}{4}(2n-1)(2n-2)[\tilde{\kappa} - 1 - \tfrac{2}{(2n-2)r}] \parallel \phi \parallel^2 .
\end{aligned}
$$

As before, if $\tilde{\kappa} \equiv 1$, since f is a 1-contracting map, f is an isometry [see (4.8)].

If $\tilde{\kappa} > 1$, since the last inequality is valid for all $r > 1$, then

$$ker(D_E^2) = ker(D_E) = 0.$$

And $ker(D_{E^+}^+) = 0$, hence $Index(D_{E^+}) = 0$. But the Atiyah-Singer Index Theorem gives (see (4.7))

$$Index(D_{E^+}) \neq 0$$

Remark 4.10. Recall that a map $f : M \to N$ between Riemannian manifolds is (ϵ, Λ^k)-contracting if

$$\| f^*\alpha \| \leq \epsilon \| f^*\alpha \| \qquad \forall \alpha \in \Lambda^k(N).$$

Note that "1-contracting" means $(1, \Lambda^1)$-contracting.

We have the following immediate consequence.

Theorem 4.11. *Let M be a compact Riemannian spin manifold of dimension n. Suppose there exists a $(1, \Lambda^2)$-contracting map $f : (M, g) \to (S^n, g_0)$ of non-zero degree. Then, either there exists $x \in M$ with $\tilde{\kappa}(x) < 1$, or $M \equiv S^n$ and f is an isometry.*

Proof. It follows from the proof of Theorem 4.1. We only need to point out that $\{\lambda_i\}_{i=1}^n$ satisfy

$$\begin{aligned} 1 &= \| \epsilon_i \wedge \epsilon_j \|_{g_0} = \| \lambda_i f_* e_i \wedge \lambda_j f_* e_j \|_{g_0} = \lambda_i \lambda_j \| f_*(e_i \wedge e_j) \|_{g_0} \\ &\leq \lambda_i \lambda_j \| e_i \wedge e_j \|_g = \lambda_i \lambda_j \end{aligned}$$

Thus, $\lambda_i \lambda_j \geq 1$. But this was the condition needed in the proof of Theorem 4.1, rather than requiring that each $\lambda_i \geq 1$.

5 Weaker hypothesis

Are these same results true when the map f is $(1, \Lambda^k)$-contracting for $3 \leq k \leq n$? The answer is no. The hypothesis on f cannot be further weakened in that sense. The following construction provides a counterexample for $k = 3$.

Counterexample 5.1. In $\mathbf{R}^4$ we consider the manifold that we get by rotating a curve α_m about the x_1-axis.

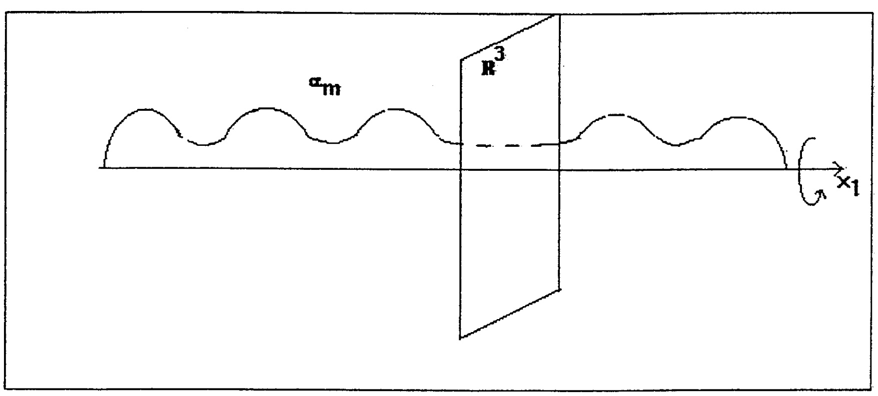

Fig. 1.

We can assume that α_m is parametrized by its arc-length. Define the map f_m onto S^3 that sends the curve α_m into the half circle that when rotated about x_1 gives the sphere S^3 and each S^2 orthogonal to x_1 in M_m^3 into S^2 orthogonal to x_1 in

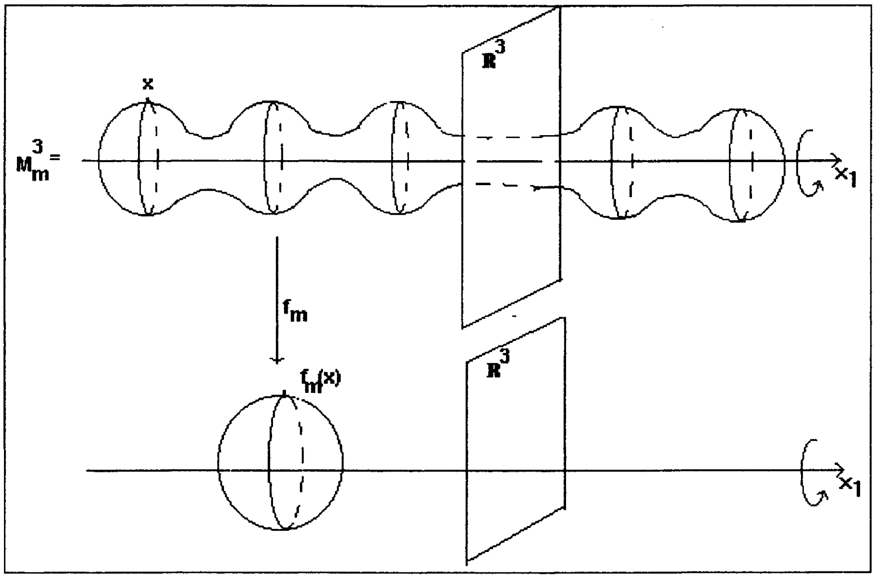

Fig. 2.

S^3. We provide M^3_m with a metric g_m such that $\tilde{\kappa}_{g_m} \geq 1$. For a point $x \in M^3_m$ we have that

$$df_m : T_x(M^3_m) \longrightarrow T_{f_m(x)}(S^3)$$

$$e_1 \longmapsto \frac{1}{l_m} e_1$$

$$e_j \longmapsto c(s)e_j \qquad j = 2, 3$$

where l_m is the function length of α_m and $c(s)$ is some bounded function of s, say $a \leq c(s) \leq b$. These constants a and b are independent of m. Thus, f_m is a $(\frac{b^2}{l_m}, \Lambda^3)$-contracting non-zero degree map and $\tilde{\kappa}_{g_m} > 1$ everywhere.

Remark 5.2 It has not been established to which extend the spin hypothesis is necessary for the results. Certainly the method used in this paper lies entirely on the spin structure of the manifold M.

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