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Multiplicities of a bigraded ring and intersection theory

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Introduction

The Samuel multiplicity of an $\mathfrak{m}$-primary ideal in a local ring $(A, \mathfrak{m})$ can be used to define the intersection number of an irreducible component of an intersection of two projective varieties X and Y in $\mathbb{P}^n_K$. If X and Y intersect improperly, one must also assign intersection numbers to certain embedded components of $X \cap Y$, see for example [10], [7]. Some of such components are defined over the base field K and are called K-rational, others are defined over a field extension of K. In [1] it was proved that K-rational components correspond to ideals of maximal analytic spread and in [2] a multiplicity was defined for such ideals, which generalizes Samuel's multiplicity. Here we define a multiplicity sequence $c_0(I, A), \ldots, c_d(I, A)$ for an arbitrary ideal I of a d-dimensional local ring $(A, \mathfrak{m})$ (see 2.2), which is closely related to the Stückrad–Vogel intersection cycle. Our main result, Theorem 4.1, implies that each number of the multiplicity sequence equals the (local) degree of the part of the cycle in a certain dimension (see Corollary 4.2). As applications we obtain an interpretation of the Segre classes of a subscheme as multiplicities in a bigraded ring (see Corollary 4.3) and a local version of Bezout's theorem, which improves the one of [3]. If the ideal I has maximal analytic spread, then $c_0(I, A)$ coincides with the multiplicity $\mu(I, A)$ defined in [2], and when I is $\mathfrak{m}$-primary, $c_0(I, A)$ is the Samuel multiplicity of I and it is the only element of the sequence which is different from zero. Another case, in which the sequence reduces to only one element different from zero, is when the embedded join of X and Y has minimal dimension.

The multiplicity sequence is defined by means of the bigraded ring $G_{\mathfrak{m}}(G_I(A))$, where $G_I(A)$ is the associated graded ring of A with respect to I. For this reason, in Sect. 1 we recall some known facts on Hilbert functions of bigraded rings. In

Sect. 2 we define the multiplicity sequence $c_0(I,A),\dots,c_d(I,A)$ and we prove that $\sum_{k=0}^{d} c_k(I,A) = e(G_I(A))$. In Sect. 3 , we introduce some extensions of the Stückrad–Vogel intersection algorithm to filter-regular sequences and study the deformation to the normal cone in the sense of van Gastel [11]. This allows the geometric interpretation of the multiplicity sequence, that will be given in Sect. 4.

1. Hilbert functions of bigraded rings

In this section we recall some well-known facts on Hilbert functions and Hilbert polynomials of bigraded rings, which will play a central role in the next section.

In the following, by a bigraded ring we mean a ring $R = \oplus_{i,j=0}^{\infty} R_{ij}$ such that

(i) R_{ij} are additive subgroups,
(ii) $R_{ij} \cdot R_{kl} \subseteq R_{i+k,j+l}$ for all nonnegative integers i,j,k,l,
(iii) R is as an R_{00}-algebra finitely generated by elements of R_{01} and R_{10}.

In particular, a polynomial ring $S = S_{00}[x_0,\dots,x_n,y_0,\dots,y_m]$ in two sets of variables $x_0,\dots,x_n$ and $y_0,\dots,y_m$ is a bigraded ring, where S_{ij} are the additive groups of polynomials homogeneous of degree i in the first set of variables and homogeneous of degree j in the second one. Every bigraded ring is isomorphic to a quotient of such a ring S by a bihomogeneous ideal $\mathfrak{a}$, that is, an ideal $\mathfrak{a}$ such that $\mathfrak{a} = \oplus_{i,j=0}^{\infty} \mathfrak{a} \cap S_{ij}$.

Let $R = \oplus_{i,j=0}^{\infty} R_{ij}$ be a bigraded ring of dimension d and assume that R_{00} is an Artinian ring. The *Hilbert function* of R is defined to be

$$h(i,j) = h_R(i,j) = \text{length}_{R_{00}}(R_{ij}) .$$

The *Hilbert series* of R is the formal power series

$$H_R(s,t) = \sum_{i,j=0}^{\infty} h_R(i,j) s^i t^j .$$

For i,j sufficiently large, the function $h_R(i,j)$ becomes a polynomial $p_R(i,j)$, the *Hilbert polynomial* of R, which can be written in the form

$$p_R(i,j) = \sum_{\substack{k,l \geq 0 \\ k+l \leq d-2}} a_{kl} \binom{i}{k} \binom{j}{l}$$

with $a_{kl} \in \mathbb{Z}$ and $a_{k,d-2-k} \geq 0$ (see [21], Theorem 7, p. 757 and Theorem 11, p. 759). Moreover, if $R = S/\mathfrak{a}$ is as above and $\mathfrak{a}$ is a prime ideal that is *not projectively irrelevant*, that is, it does not contain a power of $(x_0,\dots,x_n)$ or of $(y_0,\dots,y_m)$, then at least one of the coefficients $a_{k,d-2-k}$ is positive.

Let $h^{(1,0)}(i,j) = \sum_{u=0}^{i} h(u,j)$ be the so-called *sum transform* of h with respect to the first variable and let

$$h^{(1,1)}(i,j) = \sum_{v=0}^{j} h^{(1,0)}(i,v) = \sum_{v=0}^{j}\sum_{u=0}^{i} h(u,v)\,.$$

This means for the corresponding series that $H^{(1,0)}(s,t) = \frac{1}{1-s} H(s,t)$ and

$$H^{(1,1)}(s,t) = \frac{1}{1-t} H^{(1,0)}(s,t) = \frac{1}{(1-s)(1-t)} H(s,t)\,.$$

From this description it is clear that, for i, j sufficiently large, also $h^{(1,0)}$ and $h^{(1,1)}$ become polynomials with integer coefficients of degree at most $d-1$ and exactly d respectively, that can be written in the form

$$p_R^{(1,0)} = \sum_{\substack{k,l\geq 0\\ k+l\leq d-1}} a_{k,l}^{(1,0)} \binom{i}{k}\binom{j}{l}$$

with $a_{k+1,l}^{(1,0)} = a_{k,l}$ for $k,l \geq 0$, $k+l \leq d-2$ and

$$p_R^{(1,1)} = \sum_{\substack{k,l\geq 0\\ k+l\leq d}} a_{k,l}^{(1,1)} \binom{i}{k}\binom{j}{l}$$

with $a_{k+1,l+1}^{(1,1)} = a_{k,l}$ for $k,l \geq 0$, $k+l \leq d-2$.

Definition 1.1. *For the coefficients of the terms of highest degree in $p_R^{(1,1)}$ we introduce the symbol*

$$c_k := c_k(R) := a_{k,d-k}^{(1,1)} \quad \text{for } k = 0,\ldots,d\,.$$

The integers c_k can be computed by using computer algebra systems as CALI [12], CoCoA [4] and Macaulay 2 [13], in which the calculation of the numerator polynomial of the Hilbert series of a multigraded ring has been implemented.

Proposition 1.2. *Let $R = \oplus_{i,j} R_{ij}$ be a d-dimensional bigraded ring such that R_{00} is a field. Then, for each $k = 0,\ldots,d$:*

(i)
$$c_k(R) = \sum_{\mathfrak{Q}} c_k(R/\mathfrak{Q})$$

where $\mathfrak{Q}$ runs through all primary ideals of highest dimension in an irredundant primary decomposition of the zero ideal of R.

(ii) *If $\mathfrak{Q}$ is a bigraded $\mathfrak{P}$-primary ideal of R, then*

$$c_k(R/\mathfrak{Q}) = \text{length}(R/\mathfrak{Q})_{\mathfrak{P}} \cdot c_k(R/\mathfrak{P})\,.$$

Proof. As at the beginning of Sect. 1, let us write $R \cong S/\mathfrak{a}$, where $\mathfrak{a}$ is a bigraded ideal in the bigraded polynomial ring $S := S_{00}[x_0, \ldots, x_n, y_0, \ldots, y_m]$. We observe that the numbers $c_k(R) = c_k(S/\mathfrak{a})$ are obtained by doing sum transforms of the Hilbert function of R with respect to both sets of variables of S, that is,

$$c_k(R) = c_k(S/\mathfrak{a}) = a_{k,d-k}(S[x,y]/\mathfrak{a}S[x,y])$$

with the new variables x and y added to the former sets $x_0, \ldots, x_n$ and $y_0, \ldots, y_m$ respectively. Hence we can apply the results of [21] to the latter coefficients $a_{k,d-k}$, the so-called *degrees* of $\mathfrak{a}S[x,y]$.

Now (i) follows by [21], Theorem 8, p. 758 and the fact that the extension ideal of $\mathfrak{a}$ in the ring $S[x,y]$ has no irrelevant components.

The result (ii) is a consequence of [21], Sect. 32, p. 767, since

$$\operatorname{length}(R/\mathfrak{Q})_{\mathfrak{P}} = \operatorname{length}(R[x,y]/\mathfrak{Q}R[x,y])_{\mathfrak{P}R[x,y]}\,.$$

□

Proposition 1.3. ([21], Sect. 33, p. 768). *Let $S = K[x_0, \ldots, x_n, y_0, \ldots, y_m]$ be a bigraded polynomial ring over a field K with algebraic closure $\overline{K}$, let $\mathfrak{P}$ be a bihomogeneous prime ideal in S and let $R = S/\mathfrak{P}$. Put $\mathbb{P}^{n+1} = \operatorname{Proj}(\overline{K}[x_0, \ldots, x_n, x])$ and $\mathbb{P}^{m+1} = \operatorname{Proj}(\overline{K}[y_0, \ldots, y_m, y])$. Then, for $k = 0, \ldots, \dim R =: d$, the coefficients $c_k(R)$ are the numbers of points in which the subvariety of $\mathbb{P}^{n+1} \times \mathbb{P}^{m+1}$ defined by $\mathfrak{P}$ meets a subvariety given by k general linear equations in $x_0, \ldots, x_n, x$ and $d-k$ general linear equations in $y_0, \ldots, y_m, y$.*

Proof. The proposition follows immediately from [21], Sect. 33, p. 768 applied to the prime ideal $\mathfrak{P}S[x,y]$ in the bigraded ring $S[x,y]$. □

If R_{00} is a local ring with maximal ideal $\mathfrak{n}$, then $R = \oplus_u(\oplus_{i+j=u} R_{ij})$ is a simply graded ring with the unique homogeneous maximal ideal $\mathfrak{M} = \mathfrak{n} \oplus (\oplus_{i+j>0} R_{ij})$. We denote by $e(R)$ the Samuel multiplicity of R with respect to $\mathfrak{M}$.

Proposition 1.4. (cf. [5], Proposition 13.3, or [22]). *Let $R = \oplus R_{ij}$ be a bigraded ring such that R_{00} is an Artinian local ring and let $R_+ := (R_{01} + R_{10})R$. Then*

$$e(R) = e(R_+, R) = \sum_{k=0}^{d} c_k(R)\,.$$

Proof. Making use of the well-known binomial identity

$$\sum_{i+j=n} \binom{i+k}{k}\binom{j+l}{l} = \binom{n+k+l+1}{k+l+1}$$

one can compare the leading coefficients of the corresponding Hilbert polynomials in order to get the second equality. The remaining equality $e(R) = e(R_+, R)$ follows since R_+ is a reduction of the homogeneous maximal ideal of R with respect to the total grading. □

Proposition 1.5. *Let $R = \oplus_{i,j} R_{ij}$ be a bigraded ring such that R_{00} is a field, and let $\mathfrak{Q} \subset R$ be a bigraded $\mathfrak{P}$-primary ideal of (Krull-)dimension d. Then:*

(i) $c_0(R/\mathfrak{Q}) \neq 0$ *if and only if* $R_{10} \subseteq \mathfrak{P}$*, and in this case* $c_0(R/\mathfrak{Q}) = e(R/\mathfrak{Q})$.
(ii) $c_d(R/\mathfrak{Q}) \neq 0$ *if and only if* $R_{01} \subseteq \mathfrak{P}$*, and in this case* $c_d(R/\mathfrak{Q}) = e(R/\mathfrak{Q})$.
(iii) $c_k(R/\mathfrak{Q}) = 0$ *for all* $1 \leq k \leq d-1$ *if and only if* $R_{01} \subseteq \mathfrak{P}$ *or* $R_{10} \subseteq \mathfrak{P}$*, that is, if and only if* $\mathfrak{Q}$ *is projectively irrelevant.*

Proof. Since the case $d = 0$ is trivial, we will assume $d > 0$.

We will prove (iii) at first. If $R_{01} \subseteq \mathfrak{P}$ or $R_{10} \subseteq \mathfrak{P}$, then the Hilbert polynomial of $R/\mathfrak{Q}$ is zero, hence $0 = a_{k,d-2-k}(R/\mathfrak{Q}) = c_{k+1}(R/\mathfrak{Q})$ for $0 \leq k \leq d-2$. Vice versa, if $c_k(R/\mathfrak{Q}) = 0$ for all $1 \leq k \leq d-1$, then $a_{k,d-2-k}(R) = 0$ for all $0 \leq k \leq d-2$, that is, the Hilbert polynomial of $R/\mathfrak{Q}$ is zero. This means that all bihomogeneous forms of bidegree (i,j) with i and j sufficiently large, say $i \geq i_0$ and $j \geq j_0$, are in $\mathfrak{Q}$, hence $R_{01}^{i_0} \cdot R_{10}^{j_0} \subseteq \mathfrak{Q} \subseteq \mathfrak{P}$. Consequently R_{01} or R_{10} must be in $\mathfrak{P}$.

In order to prove (i), we observe that $c_0(R/\mathfrak{Q}) = a_{0,d-1}((R/\mathfrak{Q})[x])$ and $c_k(R/\mathfrak{Q}) = a_{k-1,d-1-k}(R/\mathfrak{Q}) = a_{k,d-1-k}((R/\mathfrak{Q})[x])$, see the proof of 1.2. Now assume that $R_{10} \subseteq \mathfrak{P}$. Then by (iii) $a_{k,d-1-k}((R/\mathfrak{Q})[x]) = 0$ for all $1 \leq k \leq d-1$. Since the Hilbert polynomial of $(R/\mathfrak{Q})[x]$ cannot be zero ($QR[x]$ is a relevant ideal in $R[x]$), the remaining coefficient $a_{0,d-1}((R/\mathfrak{Q})[x]) = c_0(R/\mathfrak{Q})$ must be positive. Vice versa, if $c_0(R/\mathfrak{Q}) \neq 0$, then by Propositions 1.3 and 1.2 the subvariety of $\mathbb{P}^{n+1} \times \mathbb{P}^m$ given by $d-1$ general linear equations in the variables $y_0, \ldots, y_m$ meets the subvariety of $\mathbb{P}^{n+1} \times \mathbb{P}^m$ defined by $\mathfrak{P}R[x]$ in $c_0(R/\mathfrak{P}) = c_0(R/\mathfrak{Q})/\operatorname{length}(R/\mathfrak{Q})_{\mathfrak{P}}$ points. This means in particular that the radical $\mathfrak{P}$ of the ideal $\mathfrak{Q}$ must contain $x_0, \ldots, x_n$, that is, R_{10}.

The remaining assertion $c_0(R/\mathfrak{Q}) = e(R/\mathfrak{Q})$ follows from (iii), the first part of (ii) (which one proves in the same way as (i)) and Proposition 1.4. □

Proposition 1.6. *Let $R = \oplus_{i,j} R_{ij}$ be a d-dimensional bigraded ring such that R_{00} is a field. If $\ell_{01} \in R_{01}$ and ℓ_{01} is a nonzero-divisor of R, then*

$$c_k(R) = c_k(R/\ell_{01}R) \quad \textit{for } k = 0, \ldots, d-1.$$

In the same way, if $\ell_{10} \in R_{10}$ and ℓ_{10} is a nonzero-divisor of R, then

$$c_k(R) = c_{k-1}(R/\ell_{10}R) \quad \textit{for } k = 1, \ldots, d.$$

Proof. The proposition follows from [21], Theorem 5, p. 756. □

2. A generalized Samuel multiplicity

In this section, for an arbitrary proper ideal I in a local ring $(A, \mathfrak{m})$ we define a sequence of nonnegative integers $c_k = c_k(I, A)$, $k = 0, \ldots, d$ which generalizes Samuel's multiplicity of an $\mathfrak{m}$-primary ideal.

Notation 2.1. Let $(A, \mathfrak{m})$ be a local ring of dimension d, let $I \subseteq A$ be a proper ideal, and let $G := G_I(A)$ be the associated graded ring of A with respect to I. It is well-known that $\dim G = \dim A =: d$. Put $R = G_{\mathfrak{m}}(G_I(A))$. Then $R = \oplus_{i,j=0}^{\infty} R_{ij}$ with

$$R_{ij} = G_{\mathfrak{m}}^i(G_I^j(A)) = (\mathfrak{m}^i I^j + I^{j+1})/(\mathfrak{m}^{i+1} I^j + I^{j+1})$$

is a bigraded ring of (Krull-)dimension d, and $R_{00} = A/\mathfrak{m}$ is a field.

Definition 2.2. *We call the sequence of nonnegative integers*

$$c_k := c_k(I, A) := c_k(G_{\mathfrak{m}}(G_I(A)) \qquad (0 \leq k \leq d)$$

the **multiplicity sequence** *of the ideal $I \subset A$.*

The following result is a slight extension of an unpublished result of P. Schenzel. Our proof is different since it contains also the case of height zero.

Proposition 2.3. *Let $(A, \mathfrak{m})$ be a local ring of dimension d, and let I be a proper ideal of A. Let $s = s(I)$ be the analytic spread and $q = \dim(A/I)$. Then*

(i) $c_k = 0$ *for $k < d - s$ and $k > q$;*
(ii) $c_{d-s} = \sum_{\mathfrak{P}} e(\mathfrak{m}G_{\mathfrak{P}}) \cdot e(G/\mathfrak{P})$,
where $\mathfrak{P}$ runs through all highest dimensional associated prime ideals of $G/\mathfrak{m}G$ such that $\dim G/\mathfrak{P} + \dim G_{\mathfrak{P}} = \dim G$;
(iii) $c_q = \sum_{\mathfrak{p}} e(IA_{\mathfrak{p}}) \cdot e(A/\mathfrak{p})$,
where $\mathfrak{p}$ runs through all highest dimensional associated prime ideals of A/I such that $\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} = \dim A$.

Proof. The coefficients $c_k = a_{k,d-k}^{(1,1)}$ $(0 \leq k \leq d)$ stem from the Hilbert polynomial of the sum transform $h^{(1,1)}$ of the Hilbert function

$$h(i, j) = \text{length}(G_{\mathfrak{m}}^i(G_I^j(A))\,.$$

We observe that $h^{(1,0)}(i, j) = \text{length}(G^j/\mathfrak{m}^{i+1}G^j) = \text{length}(I^j/(\mathfrak{m}^{i+1}I^j + I^{j+1}))$ and put

$$M_{ij} := (I^j \cap \mathfrak{m}^{i+1} + I^{j+1})/(\mathfrak{m}^{i+1}I^j + I^{j+1})\,.$$

Using the exact sequences

$$0 \to M_{ij} \to I^j/(\mathfrak{m}^{i+1}I^j + I^{j+1}) \to A/(\mathfrak{m}^{i+1} + I^{j+1}) \to A/(\mathfrak{m}^{i+1} + I^j) \to 0$$

and the additivity of the length function we get

$$h^{(1,1)}(i, j) = \text{length}(A/\mathfrak{m}^{i+1} + I^{j+1}) + \text{length}(\oplus_{k=0}^{j} M_{ik})\,. \tag{1}$$

To prove (ii) and the first part of (i), we fix $i \gg 0$. Then for $j \gg 0$ the function $h^{(1,1)}(i, j)$ becomes a polynomial in j. Let us first consider the case in which the analytic spread $s(I) = \dim G/\mathfrak{m}G = 0$, that is, the ideal I is nilpotent, say $I^m = 0$. Then for fixed $i \gg 0$ and all $j \gg 0$ we have that

$$h^{(1,1)}(i, j) = \text{length}(A/\mathfrak{m}^{i+1}) + \text{length}(\oplus_{k=0}^{m-1} M_{ik})\,, \tag{2}$$

which does not depend on j. Hence c_d is the only coefficient of the c_k's different from zero. We observe that, by Artin–Rees,

$$M_{ik} \subseteq (\mathfrak{m}^{i+1-c}I^k + I^{k+1})/(\mathfrak{m}^{i+1}I^k + I^{k+1}),$$

hence

$$\begin{aligned} \operatorname{length}(M_{ik}) &\leq \operatorname{length}((\mathfrak{m}^{i+1-c}I^k + I^{k+1})/(\mathfrak{m}^{i+1}I^k + I^{k+1})) \\ &= \sum_{t=0}^{c-1} \operatorname{length}((\mathfrak{m}^{i+1-c+t}I^k + I^{k+1})/(\mathfrak{m}^{i+2-c+t}I^k + I^{k+1})) \quad (3) \end{aligned}$$

and the lengths in the sum become polynomials in i of degree $\dim(I^k/I^{k+1}) - 1 \leq \dim(A/I) - 1$. In the case $s = h = d = 0$ it is easy to check that $c_d = \operatorname{length}(A) = e(A) = \operatorname{length}(G) = e(G)$ can be written as in (ii) or (iii). If $d > 0$, the leading coefficient of the polynomial in i on the right-hand side of (2) comes from $\operatorname{length}(A/\mathfrak{m}^{i+1}) = \operatorname{length}(G/G_I(\mathfrak{m}^{i+1}, A))$, which implies that $c_d = e(A) = e(G)$. For the latter equality note that $G_I(\mathfrak{m}^{i+1}, A)$ contains $\mathfrak{N}^{i+m}$, where $\mathfrak{N}$ is the homogeneous maximal ideal of G. On the other hand, by Artin–Rees, $G_I(\mathfrak{m}^{i+1}, A)$ is contained in $\mathfrak{N}^{i-c}$. Taking into account that I is nilpotent and also $\mathfrak{m}G$ is a reduction of $\mathfrak{N}$, it is now easy to check that c_d can be written as in (ii) or (iii).

If $s(I) > 0$, then for fixed $i \gg 0$ the leading coefficient of the polynomial in $j (\gg 0)$ of the right-hand side of (1) comes from the last term, that is,

$$h^{(1,1)}(i,j) = e(\mathfrak{N}, G/\mathfrak{m}^{i+1}G)\binom{s+j}{s} + \text{terms of lower degree}$$

with $s = \dim(G/\mathfrak{m}^{i+1}G) = \dim(G/\mathfrak{m}G) = s(I)$. To see this, note that $M_{ik} = [G/\mathfrak{m}^{i+1}G]_k$ if $k > i$ and that the ideal of all elements of positive degree of $G/\mathfrak{m}^{i+1}G$ is a reduction of $\mathfrak{N} \cdot (G/\mathfrak{m}^{i+1}G)$. By Nagata's additivity and reduction formula we obtain

$$e(\mathfrak{N}, G/\mathfrak{m}^{i+1}G) = \sum_{\mathfrak{P} \in \operatorname{Assh} G/\mathfrak{m}G} \operatorname{length}(G_{\mathfrak{P}}/\mathfrak{m}^{i+1}G_{\mathfrak{P}}) \cdot e(\mathfrak{N}, G/\mathfrak{P}).$$

But

$$\operatorname{length}(G_{\mathfrak{P}}/\mathfrak{m}^{i+1}G_{\mathfrak{P}}) = e(\mathfrak{m}G_{\mathfrak{P}}, G_{\mathfrak{P}})\binom{p+i}{p} + \text{terms of lower degree}$$

with $p := \dim G_{\mathfrak{P}}$ and the assertions (ii) and the first part of (i) follow.

In order to prove (iii) and the second part of (i), we fix $j \gg 0$ in (1). Then we get for $i \gg 0$ a polynomial in i whose leading coefficient comes from $\operatorname{length}(A/\mathfrak{m}^{i+1} + I^{j+1})$ since by (3) the length of M_{ik} does not contribute to the term of highest degree. Hence

$$h^{(1,1)}(i,j) = e(A/I^{j+1})\binom{q+i}{q} + \text{terms of lower degree},$$

where $q := \dim A/I$. Nagata's additivity and reduction formula implies

$$e(A/I^{j+1}) = \sum_{\mathfrak{p}\in \operatorname{Assh} A/I} \operatorname{length}(A_{\mathfrak{p}}/I^{j+1}A_{\mathfrak{p}}) \cdot e(A/\mathfrak{p})\,.$$

Putting $r := \dim A_{\mathfrak{p}}$ and taking into account that

$$\operatorname{length}(A_{\mathfrak{p}}/I^{j+1}A_{\mathfrak{p}}) = e(IA_{\mathfrak{p}}, A_{\mathfrak{p}})\binom{r+j}{r} + \text{terms of lower degree}\,,$$

the assertions (iii) and the second part of (i) follow. □

Corollary 2.4. *Let the notation be as in Proposition* 2.3*.*

(i) *If I is $\mathfrak{m}$-primary, that is,* $\operatorname{ht}(I) = s(I) = d$*, then* $c_0(I,A) = e(I,A)$ *and* $c_1(I,A) = \cdots = c_d(I,A) = 0$.
(ii) *If $s(I) = d$, then $c_0(I,A) = \mu(I,A)$, the multiplicity defined in* [2], (1.2)*.*
(iii) *If $s(I) = \operatorname{ht}(I) =: h$, then $c_k(I,A) = 0$ for all $k \neq d-h$ and*

$$c_{d-h}(I,A) = \sum_{\mathfrak{p}\in \operatorname{Assh}(A/I)} e(IA_{\mathfrak{p}}) \cdot e(A/\mathfrak{p}) = \sum_{\mathfrak{P}} e(\mathfrak{m}G_{\mathfrak{P}}) \cdot e(G/\mathfrak{P})\,,$$

where $\mathfrak{P}$ runs through all highest dimensional associated prime ideals of $G/\mathfrak{m}G$ such that $\dim G/\mathfrak{P} + \dim G_{\mathfrak{P}} = \dim G$*.*

Proof. The corollary is an immediate consequence of Proposition 2.3. For (ii) note that if $s(I) = \dim A$ and $\mathfrak{P} \in \operatorname{Assh}(G/\mathfrak{m}G)$, then the local ring $G_{\mathfrak{P}}$ is Artinian. For (iii) observe that $s(I) = \operatorname{ht}(I)$ implies $\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} = \dim A$ for all $\mathfrak{p} \in \operatorname{Assh}(A/I)$. □

Proposition 2.5. ([5], Sect. 14) *Let $(A,\mathfrak{m})$ be a local ring of dimension d and let I be a proper ideal of A. Then*

$$e(G_I(A)) = \sum_{k=0}^{d} c_k(I,A)\,.$$

Proof. Let $R := G_{\mathfrak{m}}(G_I(A))$ and let $\mathfrak{M}$ denote the unique homogeneous maximal ideal of R. By 1.4 it remains to be shown that $e(R) = e(G)$. If $\mathfrak{N}$ denotes unique homogeneous maximal ideal of $G = G_I(A)$ and $G^+ := \oplus_{k>0} G^k$, then

$$\mathfrak{N}^u = (\mathfrak{m}G^0 \oplus G^+)^u = \mathfrak{m}^u G^0 \oplus \mathfrak{m}^{u-1}G^1 \oplus \cdots \oplus \mathfrak{m}G^{u-1} \oplus (\oplus_{k\geq u} G^k)$$

and

$$\begin{aligned} \mathfrak{N}^u/\mathfrak{N}^{u+1} &\cong (\mathfrak{m}^u G^0/\mathfrak{m}^{u+1}G^0) \oplus (\mathfrak{m}^{u-1}G^1/\mathfrak{m}^u G^1) \oplus \cdots \oplus (G^u/\mathfrak{m}G^u) \\ &\cong \oplus_{i+j=u} R_{ij}\,. \end{aligned}$$

On the other hand, since $\mathfrak{M}$ is generated by R_{01} and R_{10}, we have that $\mathfrak{M}^u$ is generated by all R_{ij} with $i+j=u$, and the assertion follows. □

From the preceding proposition and Corollary 2.4 we have at once the following corollary, which generalizes and improves a result of N. V. Trung ([18], Theorem 2.2) for ideals of the principal class, that is, ideals I that can be generated by $\mathrm{ht}(I)$ elements. Note that for ideals of the principal class $s(I) = \mathrm{ht}(I)$.

Corollary 2.6. *Let $(A, \mathfrak{m})$ be a local ring of dimension d, and let I be a proper ideal of A. If $s(I) = \mathrm{ht}(I)$, then*

$$e(G_I(A)) = e(G_{\mathfrak{m}}(G_I(A)) = \sum_{\mathfrak{p}\in \mathrm{Assh}(A/I)} e(IA_{\mathfrak{p}}) \cdot e(A/\mathfrak{p}) = \sum_{\mathfrak{P}} e(\mathfrak{m}G_{\mathfrak{P}}) \cdot e(G/\mathfrak{P})\,,$$

where $\mathfrak{P}$ runs through all highest dimensional associated prime ideals of $G/\mathfrak{m}G$ such that $\dim G/\mathfrak{P} + \dim G_{\mathfrak{P}} = \dim G$. □

Remark 2.7. For an ideal I in a local ring $(A, \mathfrak{m})$ of maximal analytic spread, let $\mu(I, A)$ be the multiplicity of I defined in [2]. By Corollary 2.4 and Proposition 2.5 we have

$$\mu(I, A) = e(G) - \sum_{k=1}^{d} c_k(I, A) = e(G/\mathfrak{m}^t G) \geq e(G/\mathfrak{m}G)$$

for t sufficiently large.

3. Intersection algorithms

In this section we recall the construction of the Stückrad–Vogel intersection cycle in the projective space. L. J. van Gastel has shown that the Stückrad–Vogel cycle is invariant under the deformation to the normal cone. We will give an analogous construction in a local ring and show that also this local intersection algorithm can be deformed to the normal cone.

3.A. The intersection algorithm of Stückrad and Vogel

Let X, Y be equidimensional closed subschemes of $\mathbb{P}^n_K = \mathrm{Proj}(K[x_0, \ldots, x_n])$, where K is an arbitrary field. For indeterminates u_{ij} $(0 \leq i, j \leq n)$ let L be the pure transcendental field extension $K(u_{ij})_{0\leq i,j\leq n}$ and $X_L := X \otimes_K L$ etc. Proving a Bezout theorem for improper intersections, Stückrad and Vogel (see [7]) introduced a cycle $v(X, Y) = v_0 + \cdots + v_n$ on $X_L \cap Y_L$, which is obtained by an intersection algorithm on the ruled join variety

$$J := J(X_L, Y_L) \subset \mathbb{P}^{2n+1}_L = \mathrm{Proj}(L[x_0, \ldots, x_n, y_0, \ldots, y_n])$$

as follows:

Let Δ be the "diagonal" subspace of $\mathbb{P}^{2n+1}_L$ given by the equations

$$x_0 - y_0 = \cdots = x_n - y_n = 0\,,$$

let $H_i \subseteq J$ be the divisor given by the equation

$$\ell_i := \sum_{j=0}^{n} u_{ij}(x_j - y_j) = 0$$

and put $\underline{\ell} := (\ell_0, \ldots, \ell_n)$. Then one defines inductively cycles β_k and v_k by setting $\beta_0 := [J]$. If β_k is already defined, decompose the intersection

$$\beta_k \cap H_k = v_{k+1} + \beta_{k+1} \qquad (0 \leq k \leq \dim J),$$

where the support of v_{k+1} lies in Δ and where no component of β_{k+1} is contained in Δ. It follows that v_k is a $(\dim J - k)$-cycle on $X_L \cap Y_L \cong J \cap \Delta$. In general, $v(X, Y) := v(\underline{\ell}, J) := \sum v_k$ is a cycle defined over L. By a result of van Gastel ([11], Proposition 3.9), a K-rational subvariety C of $X_L \cap Y_L$ occurs in $v(X, Y)$ if and only if C is a distinguished variety of the intersection of X and Y in the sense of Fulton ([10], p. 95), and this is equivalent to the maximality of the analytic spread (see [2]) or the maximality of the dimension of the so-called limit of join variety (see [8]).

All these facts follow more or less from the invariance of the Stückrad–Vogel cycle under the deformation to the normal cone. In order to state the precise result, let $C_{J\cap\Delta}J$ denote the normal cone of $J \cap \Delta$ in J, that is,

$$C_{J\cap\Delta}J = \mathrm{Spec}(\oplus_{i=0}^{\infty} \mathscr{I}^i/\mathscr{I}^{i+1}),$$

where $\mathscr{I} \subset \mathscr{O}_J$ denotes the ideal sheaf of $J \cap \Delta \subset J$. Let $\underline{\ell}^* = (\ell_0^*, \ldots, \ell_n^*)$ be the sequence of initial forms of $\underline{\ell} := (\ell_0, \ldots, \ell_n)$ in $G_{\mathscr{I}}(\mathscr{O}_J) := \oplus_{i=0}^{\infty} \mathscr{I}^i/\mathscr{I}^{i+1}$. Then we have the following result of van Gastel [11], Theorem 3.2, see also [7], Theorem 2.4.7:

Proposition 3.1 (Deformation to the normal cone).

$$v(\underline{\ell}, J) = v(\underline{\ell}^*, C_{J\cap\Delta}J)$$

as cycles on $J \cap \Delta \cong X_L \cap Y_L \subseteq \mathbb{P}_L^n$.

3.B. Intersection algorithms for filter-regular sequences

In [2] we introduced two intersection algorithms in a local ring, which are counterparts of the construction of the Stückrad–Vogel cycle, and compared them with analogous algorithms in the associated graded ring. In order to extend some of the results of [2] from ideals of maximal analytic spread to arbitrary ideals in a local ring, we introduce the following notation:

Let now A be an arbitrary Noetherian ring, let I be an ideal in the Jacobson radical of A and let $G := G_I(A)$ be the associated graded ring. Consider a sequence $\underline{a} = (a_1, \ldots, a_t)$ of elements of I such that $\sqrt{\underline{a}A} = \sqrt{I}$ and the sequence $\underline{a}^* = (a_1^*, \ldots, a_t^*)$ of the initial forms of $a_1, \ldots, a_t$ in G is contained in G^1 and is a filter-regular sequence with respect to the ideal $G^+ = \oplus_{j\geq 1} I^j/I^{j+1}$, that is

$$(a_1^*, \ldots, a_{k-1}^*)G :_G a_k^* \subseteq (a_1^*, \ldots, a_{k-1}^*)G :_G \langle G^+ \rangle$$
$$:= \{g \in G \mid g \cdot (G^+)^m \subseteq (a_1^*, \ldots, a_{k-1}^*)G \text{ for some } m \in \mathbb{N}\}$$

for $k = 1, \ldots, t$, or equivalently, $a_k^* \notin \mathfrak{P}$ for all relevant associated prime ideals $\mathfrak{P} \in \mathrm{Ass}_G(G/(a_1^*, \ldots, a_{k-1}^*)G$ for $k = 1, \ldots, t$ (see, for example, [17], Definition 1, p. 252). In particular this implies that $\underline{a} = (a_1, \ldots, a_t)$ is a filter-regular sequence in A with respect to I, see, for example, [2], (2.2).

We define a cycle $v(\underline{a}, A)$ of A supported on $V(I)$ by the following *intersection algorithm in A*:

Set $\mathfrak{a}_{-1} := (0)$, $a_0 := 0$, and inductively

$$\mathfrak{a}_k := (\mathfrak{a}_{k-1} + a_k A) :_A \langle I \rangle \qquad (0 \le k \le t)\,.$$

Observe that $\mathfrak{a}_t = A$. Then

$$v_k(\underline{a}, A) := \sum_{\mathfrak{p}} \mathrm{length}(A/\mathfrak{a}_{k-1} + a_k A)_{\mathfrak{p}}\,[\mathfrak{p}]\,,$$

where the sum is taken over all highest dimensional associated prime ideals $\mathfrak{p}$ of $A/\mathfrak{a}_{k-1}+a_kA$ that contain I and $[\mathfrak{p}]$ denotes the cycle associated with $\mathfrak{p}$. We define $v(\underline{a}, A) := \sum_{k=0}^t v_k(\underline{a}, A)$, and, if $(A, \mathfrak{m})$ is a local ring, the *degree* of $v_k(\underline{a}, A)$ by

$$\deg v_k(\underline{a}, A) := \sum_{\mathfrak{p}} \mathrm{length}(A/\mathfrak{a}_{k-1} + a_k A)_{\mathfrak{p}} \cdot e(A/\mathfrak{p})\,.$$

In the same way, replacing $\underline{a}$ by $\underline{a}^*$ and I by G^+, we define a cycle $v(\underline{a}^*, G)$ by an *intersection algorithm in* $G = G_I(A)$ with $\tilde{\mathfrak{a}}_{-1} := 0 \cdot G$, $a_0^* := 0$, and

$$\tilde{\mathfrak{a}}_k := (\tilde{\mathfrak{a}}_{k-1} + a_k^* G) :_G \langle G^+ \rangle \qquad (0 \le k \le t)\,.$$

We put

$$v_k(\underline{a}^*, G) := \sum_{\mathfrak{P}} \mathrm{length}(G/(\tilde{\mathfrak{a}}_{k-1} + a_k^* G))_{\mathfrak{P}}\,[\mathfrak{P}]\,,$$

where the sum is over all highest dimensional associated prime ideals $\mathfrak{P}$ of $G/(\tilde{\mathfrak{a}}_{k-1}+a_k^*G)$ that contain G^+. Observe that the prime ideals of $v(\underline{a}; A)$ contain I and hence correspond to prime ideals in the ring A/I. On the other hand, the prime ideals of $v(\underline{a}^*, G)$ contain G^+ and correspond to their contraction ideals in $G_I^0(A) = A/I$. So both cycles $v(\underline{a}; A)$ and $v(\underline{a}^*, G)$ can be considered as cycles of A/I. In order to compare them, we need the following lemma:

Lemma 3.2. *Let A be a Noetherian ring, let I be an ideal in the Jacobson radical of A and let $a_1, \ldots, a_t \in I$ such that $\sqrt{(a_1, \ldots, a_t)A} = \sqrt{I}$ and their initial forms $a_1^*, \ldots, a_t^*$ in $G = G_I(A)$ are of degree one and a filter-regular sequence with respect to $G^+ = \oplus_{j \ge 1} I^j/I^{j+1}$. Then, for $k = 1, \ldots, t$, it holds*

$$G_I(\mathfrak{a}_{k-1}, A) :_G \langle G^+ \rangle \;=\; \tilde{\mathfrak{a}}_{k-1} \;=$$
$$=\; G_I((a_1, \ldots, a_{k-1}), A) :_G \langle G^+ \rangle \;=\; (a_1^*, \ldots, a_{k-1}^*)G :_G \langle G^+ \rangle\,.$$

Proof. Analyzing the proof of [2], Lemma (3.4), one gets the result. □

Theorem 3.3 (Deformation to the normal cone).

$$v(\underline{a}, A) = v(\underline{a}^*, G)$$

as cycles of A/I.

Proof. At first we will show that there is a 1–1 correspondence of the prime ideals $\mathfrak{p}$ of $v(\underline{a}, A)$ and $\mathfrak{P}$ of $v(\underline{a}^*, G)$ given by $\mathfrak{p}/I = \mathfrak{P} \cap G_I^0(A)$. Let $\mathfrak{P}$ be a prime ideal of the cycle $v(\underline{a}^*, G)$ and let $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\mathfrak{p}/I = \mathfrak{P} \cap G_I^0(A)$. Since $G^+ \subseteq \mathfrak{P}$, it holds $\dim(G/\mathfrak{P}) = \dim(A/\mathfrak{p}) = \dim A - k$, and Lemma 3.2 implies $(\mathfrak{a}_{k-1} + I)/I \subseteq [\tilde{\mathfrak{a}}_{k-1}]_0 \subseteq \mathfrak{p}/I$, hence $\mathfrak{a}_{k-1} + a_k A \subseteq \mathfrak{p}$. This shows that $\mathfrak{p} \in \operatorname{Assh}(A/\mathfrak{a}_{k-1} + a_k A)$, because $\dim(A/\mathfrak{a}_{k-1} + a_k A) = \dim(A/\mathfrak{p})$. (Note that, also by Lemma 3.2, $\mathfrak{a}_{k-1} \neq A$ if $\tilde{\mathfrak{a}}_{k-1} \neq G$.) Moreover $\mathfrak{p}$ contains I, hence $[\mathfrak{p}]$ appears in $v_k(\underline{a}, A)$.

Now let $\mathfrak{p}$ be a prime ideal of $v_k(\underline{a}, A)$. This means that $\mathfrak{p}$ contains I and is a highest dimensional associated prime ideal of $A/\mathfrak{a}_{k-1} + a_k A$. We will show that $\mathfrak{P} := \mathfrak{p}/I + G^+$ appears in $v_k(\underline{a}^*, G)$, that is, $\mathfrak{P} \in \operatorname{Assh}(G/\tilde{\mathfrak{a}}_{k-1} + a_k^* G)$. Because of the dimensions it is sufficient to show that $\tilde{\mathfrak{a}}_{k-1} + a_k^* G \subseteq \mathfrak{P}$, and since $G^+ \subseteq \mathfrak{P}$, it is even enough to show $\tilde{\mathfrak{a}}_{k-1} \subseteq \mathfrak{P}$. The latter statement is equivalent with $\tilde{\mathfrak{a}}_{k-1} G_{\mathfrak{P}} \neq G_{\mathfrak{P}}$, and by Lemma 3.2 with $G_I(\mathfrak{a}_{k-1}, A) G_{\mathfrak{P}} :_{G_{\mathfrak{P}}} \langle G_{\mathfrak{P}}^+ \rangle \neq G_{\mathfrak{P}}$. We observe that $G_{\mathfrak{P}} \cong G \otimes_A A_{\mathfrak{p}}$ and $G_I(\mathfrak{a}_{k-1}, A) G_{\mathfrak{P}} \cong G_{IA_{\mathfrak{p}}}(\mathfrak{a}_{k-1} A_{\mathfrak{p}}, A_{\mathfrak{p}})$. If it was $\tilde{\mathfrak{a}}_{k-1} G_{\mathfrak{P}} = G_{\mathfrak{P}}$ the ideal $G_I(\mathfrak{a}_{k-1}, A) G_{\mathfrak{P}}$ would contain a power of G^+, hence $\mathfrak{a}_{k-1} A_{\mathfrak{p}}$ would contain a power of $IA_{\mathfrak{p}}$, and this contradicts the definition of $\mathfrak{a}_{k-1}$.

It remains to prove the equality of the coefficients of a prime ideal $\mathfrak{p}$ of $v_k(\underline{a}, A)$ and the corresponding prime ideal $\mathfrak{P}$ of $v_k(\underline{a}^*, G)$, that is,

$$\operatorname{length}(A/\mathfrak{a}_{k-1} + a_k A)_{\mathfrak{p}} = \operatorname{length}(G/\tilde{\mathfrak{a}}_{k-1} + a_k^* G)_{\mathfrak{P}} .$$

But this follows with the same arguments as in the proof of [2], Proposition (3.6) by rewriting the lengths as Samuel multiplicities. □

In order to state and to prove our Main Theorem 4.1 we need a third intersection algorithm for a filter-regular sequence in a bigraded ring, which allows to calculate the generalized Samuel multiplicities $c_k(I, A)$ of Sect. 2. Assuming the notation of 2.1, consider the bigraded ring $R = G_{\mathfrak{m}}(G_I(A))$ with $R_{ij} = G_{\mathfrak{m}}^i(G_I^j(A))$ and take elements $a_1, \ldots, a_s$ in I such that $\sqrt{\underline{a}A} = \sqrt{I}$ and their images $a_1^o, \ldots, a_s^o$ in $R_{01} = I/\mathfrak{m}I$ are a filter-regular sequence with respect to R_{01}. Note that these conditions imply that $\underline{a}^o = (a_1^o, \ldots, a_s^o)$ is a system of parameters for $G/\mathfrak{m}G = \oplus_{j=0}^{\infty} R_{0j}$ and s is the analytic spread of I. Moreover, since the ideal (R_{01}) is the initial ideal $G_{\mathfrak{m}}(G^+, G)$ of $G^+ = \oplus_{j \geq 1} I^j/I^{j+1}$ in $R = G_{\mathfrak{m}}(G)$, it follows by a standard argument (see, e. g. [2], (2.2)) that the initial forms $a_1^*, \ldots, a_s^*$ of $a_1, \ldots, a_s$ in G are a filter-regular sequence of degree one with respect to G^+.

Imitating the intersection algorithms in A and G introduced at the beginning of 3.B, we will use now $\underline{a}^o = (a_1^o, \ldots, a_s^o)$ and (R_{01}) to define an *intersection algorithm in R* that produces a cycle $v(\underline{a}^o, R)$ of R as follows:

Set $\bar{\mathfrak{a}}_{-1} := (0)$, $a_0^o := 0$, and inductively

$$\bar{\mathfrak{a}}_k := (\bar{\mathfrak{a}}_{k-1} + a_k^o R) :_R \langle R_{01} \rangle \qquad (0 \le k \le s)\,.$$

Then

$$v_k(\underline{a}^o, R) := \sum_{\mathfrak{p}} \operatorname{length}(R/\bar{\mathfrak{a}}_{k-1} + a_k^o R)_{\mathfrak{p}}\,[\mathfrak{p}]\,,$$

where the sum is taken over all highest dimensional associated prime ideals of $R/\bar{\mathfrak{a}}_{k-1} + a_k^o R$ which contain R_{01}. We define $v(\underline{a}^o, R) := \sum_{k=0}^{t} v_k(\underline{a}^o, R)$.

Proposition 3.4. *With the preceding notation,*

$$c_k(R) = \deg v_{d-k}(\underline{a}^o, R) \quad \textit{for } 0 \le k \le d = \dim R\,.$$

Proof. For $k = d$ we have to prove that

$$c_d(R) = \sum_{\mathfrak{p}} \operatorname{length}(R_{\mathfrak{p}}) \cdot e(R/\mathfrak{p})\,,$$

where $\mathfrak{p}$ runs through all highest dimensional associated prime ideals of R that contain R_{01}. But this follows from Proposition 1.2, since for such primes $\mathfrak{p}$ one has $e(R/\mathfrak{p}) = c_d(R/\mathfrak{p})$, see Proposition 1.5, (ii).

Again by 1.2 and 1.5, for $k = 0, \ldots, d-1$ we have that $c_k(R) = c_k(R/\bar{\mathfrak{a}}_0)$. Observe that the filter-regular element a_1^o avoids all the associated prime ideals of R which do not contain R_{01}, hence all associated prime ideals of $R/\bar{\mathfrak{a}}_0$. Thus by 1.6 we obtain

$$c_k(R) = c_k(R/\bar{\mathfrak{a}}_0) = c_k(R/\bar{\mathfrak{a}}_0 + a_1^o R)$$

for $k = 0, \ldots, d-1$. Now the same argument used above in the case $k = d$ can be applied to the $(d-1)$-dimensional bigraded ring $R/\bar{\mathfrak{a}}_0 + a_1^\circ R$ and gives $c_{d-1}(R) = \deg v_1(\underline{a}^o, R)$. Repeating the reasoning one gets the proposition. □

In order to prove our main result 4.1 we need to compare the intersection algorithm in the associated graded ring $G = G_I(A)$ with the intersection algorithm in the bigraded ring $R = G_{\mathfrak{m}}(G_I(A))$. This will be done with the help of the following analogue of Lemma 3.2:

Lemma 3.5. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, I a proper ideal in A, $G = G_I(A)$ and $R = G_{\mathfrak{m}}(G_I(A))$. Let $a_1, \ldots, a_t \in I$ such that their images $a_1^o, \ldots, a_s^o$ in $R_{01} = I/\mathfrak{m}I$ are a filter-regular sequence with respect to R_{01} and a system of parameters for $G/\mathfrak{m}G$. Denote by $a_1^*, \ldots, a_k^*$ the initial forms of $a_1, \ldots, a_t \in I$ in G. Then, for the ideals $\tilde{\mathfrak{a}}_k$ and $\bar{\mathfrak{a}}_k$ $(k = 0, \ldots, d)$ produced by the intersection algorithms in G and R respectively, it holds*

$$\begin{aligned} G_{\mathfrak{m}}(\tilde{\mathfrak{a}}_k, G) :_R \langle R_{01} \rangle \;&=\; \bar{\mathfrak{a}}_k \;= \\ = \; G_{\mathfrak{m}}((a_1^*, \ldots, a_k^*)G, G) :_R \langle R_{01} \rangle \;&=\; (a_1^o, \ldots, a_k^o)R :_R \langle R_{01} \rangle\,. \end{aligned}$$

Proof. Analyzing the proof of [2], Lemma (3.4), one gets the result. □

4. The main theorem and applications

Theorem 4.1. *Let $(A, \mathfrak{m})$ be a d-dimensional local Noetherian ring, I be a proper ideal in A, $G := G_I(A)$ and $R := G_{\mathfrak{m}}(G_I(A))$. Let $a_1, \ldots, a_s$ be elements of I such that the images $a_1^o, \ldots, a_s^o$ in $R_{01} = I/\mathfrak{m}I$ are a filter-regular sequence with respect to R_{01} and a system of parameters for $G/\mathfrak{m}G$. Then*

$$c_k(R) = \deg v_{d-k}(\underline{a}^o, R) = \deg v_{d-k}(\underline{a}^*, G) = \deg v_{d-k}(\underline{a}, A)$$

for $k = 0, \ldots, d = \dim A$.

Proof. By 3.3 and 3.4 it remains to prove the second equality

$$\deg v_{d-k}(\underline{a}^o, R) = \deg v_{d-k}(\underline{a}^*, G)\,.$$

From the intersection algorithms in R and G it follows that

$$\deg v_{d-k}(\underline{a}^o, R) = e(R/\bar{\mathfrak{a}}_{k-1}) - e(R/\bar{\mathfrak{a}}_k)$$

and

$$\deg v_{d-k}(\underline{a}^*, G) = e(G/\tilde{\mathfrak{a}}_{k-1}) - e(G/\tilde{\mathfrak{a}}_k)\,,$$

respectively. So we have to prove

$$e(R/\bar{\mathfrak{a}}_k) = e(G/\tilde{\mathfrak{a}}_k) \qquad \text{for } k = -1, \ldots, d\,. \tag{4}$$

For $k = -1$ this means that $e(R) = e(G)$, which has already been shown (cf. the proof of 2.5). In the general case we obtain by Nagata's additivity and reduction formula (see [16], (23.5))

$$e(R/\bar{\mathfrak{a}}_k) = \sum_{\mathfrak{P}} e(R/\mathfrak{P}) \cdot \text{length}(R_{\mathfrak{P}}/\bar{\mathfrak{a}}_k R_{\mathfrak{P}})\,, \tag{5}$$

where $\mathfrak{P}$ runs through all highest dimensional associated prime ideals of $R/\bar{\mathfrak{a}}_k$. By the definition of $\bar{\mathfrak{a}}_k$, these prime ideals $\mathfrak{P}$ do not contain R_{01}, hence 3.5 implies $\bar{\mathfrak{a}}_k R_{\mathfrak{P}} = G_{\mathfrak{m}}(\tilde{\mathfrak{a}}_k, G) R_{\mathfrak{P}}$. Consequently

$$e(R/\bar{\mathfrak{a}}_k) = \sum_{\mathfrak{P}} e(R/\mathfrak{P}) \cdot \text{length}(G_{\mathfrak{m}}(G)/G_{\mathfrak{m}}(\tilde{\mathfrak{a}}_k, G))_{\mathfrak{P}}\,. \tag{6}$$

Let $\mathfrak{N}$ denote the unique homogeneous maximal ideal of G. Note that $R = G_{\mathfrak{m}}(G) = \oplus_{i,j} G^i_{\mathfrak{m}}(G^j_I(A))$ is the same ring as $G_{\mathfrak{N}}(G) = \oplus_{n=0}^{\infty}(\oplus_{i+j=n} R_{ij}) = \oplus_{n,j} G^{n-j}_{\mathfrak{m}}(G^j_I(A))$, but taken with another grading (see the proof of 2.5). Since $\tilde{\mathfrak{a}}_k$ is a graded ideal of $G_I(A)$, the initial ideals $G_{\mathfrak{m}}(\tilde{\mathfrak{a}}_k, G)$ and $G_{\mathfrak{N}}(\tilde{\mathfrak{a}}_k, G)$ coincide. It follows that

$$\begin{aligned} e(R/\bar{\mathfrak{a}}_k) &= \sum_{\mathfrak{P}} e(G_{\mathfrak{N}}(G)/\mathfrak{P}) \cdot \text{length}(G_{\mathfrak{N}}(G)/G_{\mathfrak{m}}(\tilde{\mathfrak{a}}_k, G))_{\mathfrak{P}} \qquad (7) \\ &= \sum_{\mathfrak{P}} e(G_{\mathfrak{N}}(G)/\mathfrak{P}) \cdot \text{length}(G_{\mathfrak{N}}(G/\tilde{\mathfrak{a}}_k))_{\mathfrak{P}}\,, \end{aligned}$$

where the sum is over the highest dimensional associated primes of $G_{\mathfrak{N}}(G/\tilde{\mathfrak{a}}_k)$ that do not contain R_{01}. On the other hand,

$$\begin{aligned} e(G/\tilde{\mathfrak{a}}_k) &= e(G_{\mathfrak{N}}(G/\tilde{\mathfrak{a}}_k)) = e(G_{\mathfrak{N}}(G)/G_{\mathfrak{N}}(\tilde{\mathfrak{a}}_k, G)) \\ &= \sum_{\mathfrak{P}} e(G_{\mathfrak{N}}(G)/\mathfrak{P}) \cdot \text{length}(G_{\mathfrak{N}}(G/\tilde{\mathfrak{a}}_k))_{\mathfrak{P}} , \end{aligned} \tag{8}$$

where the sum is over the highest dimensional associated primes of $G_{\mathfrak{N}}(G/\tilde{\mathfrak{a}}_k)$. To prove (4), in view of (7) and (8) it remains to prove that the highest dimensional associated prime ideals $\mathfrak{P}$ of $G_{\mathfrak{N}}(G/\tilde{\mathfrak{a}}_k)$ do not contain R_{01}. By construction of $\tilde{\mathfrak{a}}_k$ one has $\dim R/\mathfrak{P} = d - k$. Put $R_{*0} := \oplus_{i=0}^{\infty} R_{i,0} = G_{\mathfrak{m}}(A/I)$ and $\mathfrak{p} := \mathfrak{P} \cap R_{*0}$. If $\mathfrak{P}$ contained R_{01}, then $\dim G_{\mathfrak{m}}(A/I)/\mathfrak{p} = \dim R/\mathfrak{P} = d - k$. On the other hand, by construction $\mathfrak{p}$ must contain $G_{\mathfrak{N}}(\tilde{\mathfrak{a}}_k, G) \cap R_{*0}$, hence

$$\dim G_{\mathfrak{m}}(G_I^0(A))/\mathfrak{p} \leq \dim G_I^0(A)/(\tilde{\mathfrak{a}}_k \cap G_I^0(A)) = \dim G/\tilde{\mathfrak{a}}_k + G^+ .$$

But $\dim G/\tilde{\mathfrak{a}}_k + G^+ < d - k$ since by definition no associated prime ideal of $G/\tilde{\mathfrak{a}}_k$ contains G^+. This gives the contradiction $\dim G_{\mathfrak{m}}(G_I^0(A))/\mathfrak{p} < d-k$, which finishes the proof. □

Now we want to apply Theorem 4.1 to the intersection algorithm of Stückrad and Vogel for two equidimensional subschemes X and Y of $\mathbb{P}^n$, see Sect. 3.A. We will use the notation introduced in Sect. 3.A. For an arbitrary subvariety Z of $\mathbb{P}_L^n$ we put $Z_\Delta := J(Z, Z) \cap \Delta$. By $\hat{J}$ and $\hat{Z}_\Delta$ we denote the affine cones of the ruled join $J := J(X_L, Y_L) \subset \mathbb{P}_L^{2n+1}$ and Z_Δ in the affine space $\mathbb{A}_L^{2n+2}$. Then we apply Theorem 4.1 to the local ring $A := \mathcal{O}_{\hat{J}, \hat{Z}_\Delta}$ with maximal ideal $\mathfrak{m}$, the ideal $I \subset A$ of the diagonal subspace Δ, whose analytic spread we denote by s, the images of the "generic" elements $\ell_0, \dots, \ell_{s-1}$ in I and the bigraded ring $R := G_{\mathfrak{m}}(G_I(A))$. In particular we allow Z to be the empty subvariety of $\mathbb{P}^n$. Then A becomes the homogeneous ring of coordinates of the ruled join $J \subset \mathbb{P}_L^{2n+1}$ localized at the irrelevant maximal ideal, that is, we obtain a global picture of the intersection algorithm. For a subvariety $C \subseteq \mathbb{P}_L^n$ that occurs in the cycle $v = v(X, Y)$ we denote by $j(X, Y; C)$ the intersection number of Stückrad and Vogel, and we put $j(X, Y; C) = 0$ if C is not in $v(X, Y)$.

Corollary 4.2. *With the preceding notation, if $Z = \emptyset$, then $d = \dim A = \dim J + 1$,*

$$c_k(R) = \deg v_{d-k} \quad (1 \leq k \leq d) \qquad \textit{and} \quad c_0(R) = j(X, Y; \emptyset) .$$

If $k > \dim(X \cap Y) + 1$, then $c_k(R) = 0$.

If $Z \neq \emptyset$ and K-rational, then $d = \dim A = \dim J - \dim Z$ and

$$c_k(R) = \sum_C j(X, Y; C) \cdot e(\mathcal{O}_{C,Z}) \qquad (0 \leq k \leq d) ,$$

where C runs over all subvarieties of $\mathbb{P}_L^n$ with $C \supseteq Z$ and $\dim C = \dim Z + k$. If $k > \dim(X \cap Y) - \dim Z$, then $c_k(R) = 0$.

Proof. We apply Theorem 4.1 to the local ring $A = \mathcal{O}_{\hat{J},\hat{Z}_\Delta}$, the ideal $I \subset A$ of the diagonal subspace Δ and the images of the "generic" elements $\ell_0, \dots, \ell_{s-1}$ in A, where s is the analytic spread of I. Note that these "generic" elements satisfy the filter-regularity condition of Theorem 4.1 since, by an argument as in [20], proof of (2.7), p. 66 (here one needs that Z is K-rational), they avoid the required prime ideals. In the same way one can see that the images of $\ell_0, \dots, \ell_{s-1}$ in $I/\mathfrak{m}I$ are a system of parameters for $G/\mathfrak{m}G$. The statement of the corollary follows by the fact that localization commutes with doing the intersection algorithm. At the same time the intersection numbers $j(X, Y; C)$ do not change since they are by definition lengths of primary ideals contained in the prime ideal at which we localize. □

By a result of van Gastel [11], Corollary 3.7, see also [7], Corollary 2.4.9, the Segre class of the normal cone $s(C_{J\cap\Delta}J)$ is related to the Stückrad–Vogel cycle $v(\underline{\ell}, J) = \sum v_k$ by

$$v_k = \sum_{i=0}^{k} \binom{k-1}{i-1} c_1(\mathcal{O}(1))^{k-i} \cap s^i(C_{J\cap\Delta}J)$$

and

$$s^k(C_{J\cap\Delta}J) = \sum_{i=0}^{k} \binom{k-1}{i-1} (-1)^{k-i} c_1(\mathcal{O}(1))^{k-i} \cap v_i \,.$$

Here $s^i(C_{J\cap\Delta}J)$ denotes the part of $s(C_{J\cap\Delta}J)$ of codimension i in $C_{J\cap\Delta}J$, and we use the convention that $\binom{m}{-1} := 0$ for $m \le 0$ and $\binom{-1}{-1} := 1$.

Corollary 4.3. *With the preceding notation, if* $Z = \emptyset$ *then* $d = \dim A = \dim J + 1$ *and, for* $k = 0, \dots, d-1$,

$$c_{d-k}(R) = \sum_{i=0}^{k} \binom{k-1}{i-1} \deg s^i(C_{J\cap\Delta}J)$$

and

$$\deg s^k(C_{J\cap\Delta}J) = \sum_{i=0}^{k} \binom{k-1}{i-1} (-1)^{k-i} c_{d-i}(R)\,.$$

□

From Corollary 4.2 we obtain immediately a local version of Bezout's theorem which improves [3], Theorem 2:

Corollary 4.4 (Local version of Bezout's theorem). *Under the assumptions of Corollary* 4.2 *it holds*

$$e(\mathcal{O}_{X,Z}) \cdot e(\mathcal{O}_{Y,Z}) = e(A) \le e(G_I(A)) = \sum_C j(X, Y; C) \cdot e(\mathcal{O}_{C,Z})\,,$$

where C *runs over all subvarieties of* $\mathbb{P}^n_L$ *with* $C \supseteq Z$.

Proof. For the first equality see for example [1], (1.1), the inequality $e(A) \leq e(G_I(A))$ is known by [5], (6.10), and the second equality follows by 2.5 and 4.2. □

Remark 4.5 (Embedded join). Let X and Y be equidimensional subschemes of $\mathbb{P}^n$ and let A be the homogeneous ring of coordinates of the ruled join $J = J(X_L, Y_L) \subset \mathbb{P}_L^{2n+1}$ localized at the irrelevant maximal ideal and let $I \subset A$ be the ideal of the diagonal subspace. Denoting by XY the embedded join of X and Y in $\mathbb{P}^n$ one has

$$\dim XY = s(I) - 1\,,$$

see for example [9], 3.9 or [7], 2.5.9. It is well-known that $\mathrm{ht}(I) \leq s(I) \leq \dim A$, hence

$$\mathrm{ht}(I) - 1 \leq \dim XY \leq \dim A - 1\,.$$

This means that XY *has minimal dimension* if and only if $\mathrm{ht}(I) = s(I)$ and XY *has maximal dimension* if and only if $s(I) = \dim A$. From 2.5 and 4.2 it follows that XY has minimal dimension if and only if $e(G_I(A)) = c_{d-s}(G_\mathfrak{m}(G_I(A))$, that is,

$$\deg X \cdot \deg Y = \sum_C j(X, Y; C) \cdot \deg C\,,$$

where C runs through the irreducible components of $X \cap Y$ with $\dim C = \dim(X \cap Y)$. This improves one implication of [6], Corollary (3.8).

We conclude with two problems.

Problem 4.6 (Buchsbaum–Rim multiplicities). In the situation of Corollary 4.2 we have a setup in which the generalized Buchsbaum–Rim multiplicities of Kleiman and Thorup [14], [15] are defined.

With the notation of Kleiman and Thorup one can take

$$G = L[x_0, \ldots, x_n, y_0, \ldots, y_n]$$

and consider the following situation: H is the L-vector subspace of G_1 generated by $x_0 - y_0, \ldots, x_n - y_n$, Z is the closed subscheme of $\mathrm{Proj}(G)$ defined by the homogeneous ideal generated by H, M is the r-dimensional homogeneous ring of coordinates of the join variety and $\mathscr{M}$ its associated sheaf.

In this situation, what is the relation between the Buchsbaum–Rim multiplicities $e^{i,k}([\mathscr{M}]_r)$ and our multiplicity sequence $c_0(R), \ldots, c_d(R)$ of 4.2?

Problem 4.7 (Analytic case). In the recent paper [19], P. Tworzewski has constructed an intersection cycle of complex analytic subsets X and Y of a manifold M, which do not intersect necessarily properly. His construction is based on a pointwise defined intersection multiplicity $g(x) = g(X \times Y, \Delta_M, x)$ for a point $x \in \Delta_M$, where Δ_M is the diagonal of $M \times M$ and $g(x)$ is the sum of the coordinates of the so-called extended index of intersection $\tilde{g}(x)$, see [19], Definition (4.2), p. 185.

Let $A = \mathcal{O}_{X\times Y,x}$, let $I = \mathcal{I}_{\Delta_M} \cdot \mathcal{O}_{X\times Y,x}$, and put $R = G_{\mathfrak{m}}(G_I(A))$. It is a natural question to ask whether

$$g(x) = e(G_I(A))$$

and whether $\tilde{g}(X)$ is composed of our numbers $c_0(R), \ldots, c_{\dim(X\cap Y)}(R)$ and of zeros.

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