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Kernels of surjections from $\mathscr{L}_1$-spaces with an application to Sidon sets

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0. Introduction

In 1981, J. Bourgain [2] solved a long outstanding problem in Banach space theory by showing the existence of an uncomplemented subspace of L^1 which is isomorphic to ℓ^1. In that paper, he raises the question of whether it is possible to find an uncomplemented translation invariant subspace of $L^1(G)$, where G is a compact abelian group, which is isomorphic to L^1. As a special case he mentions the question of whether the closed linear subspace of L^1 spanned by the complement of the Rademachers in the Walsh functions is (a) isomorphic to L^1, or (b) an $\mathscr{L}_1$-space. Bourgain [2, Problem 6] attributes the question to Pisier. As far as we could trace, it was also previously considered by Kisliakov and Zippin.

Suppose G is a compact abelian group and Γ its character group. For any subset A of Γ, we define $L^1_A(G)$ as the closure in $L^1(G)$ of the linear span of $\{\gamma : \gamma \in A\}$. We put $\widetilde{A} = \Gamma \setminus A$. Answering in negative questions (a) and (b), we shall show that if $S \subset \Gamma$ is an infinite Sidon set, then the canonical image of $L^1_{\tilde{S}}(G)$ is uncomplemented in its second dual and it is not an $\mathscr{L}_1$-space (Corollary 5.1).

We approach the problem from a purely Banach space point of view. Note that if S is a Sidon set, then the map $Q : L^1(G) \to c_0(S)$ defined by $Qf = \{\hat{f}(\gamma)\}_{\gamma \in S}$, where $\hat{f}$ denotes the Fourier transform, is a surjection so that $\ker Q = L^1_{\tilde{S}}(G)$ is the kernel of a quotient map from $L^1(G)$ onto a space isomorphic to $c_0(S)$.

We first show (Proposition 2.2) that if μ is a finite measure and E is a Banach space containing an isomorphic copy of c_0, then the canonical image of the kernel

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of any surjection Q from $L^1(\mu)$ onto E is uncomplemented in its second dual; consequently, $\ker Q$ is non-isomorphic to a Banach lattice. Our argument depends on an old lifting principle of Lindenstrauss [22].

We then turn to part (b) of the question. Here the idea is to study subspaces X of an $\mathscr{L}_1$-space F which are GT-spaces. (Recall, cf. [30], that a Banach space X is a GT-space if every bounded linear operator from X into ℓ^2 is absolutely summing.) Let $E = F/X$ so that X is the kernel of a quotient map onto E. Then we show (Theorem 3.1) that X is a GT-space, if and only if every short exact sequence

$$0 \to \ell^2 \to Z \to E \to 0$$

splits, i.e. in the language of [18] every twisted sum of ℓ^2 and E is naturally isomorphic to the Cartesian product $\ell^2 \oplus E$. This leads us to the general question of characterizing such Banach spaces E. We show that if every twisted sum of ℓ^2 and E splits, then E (i) fails to have any type $p > 1$ (Corollary 4.1, cf. also [8]), and (ii) has cotype $q < \infty$ (Corollary 4.2). In particular, if E contains a subspace isomorphic to c_0, then X is not a GT-space and a fortiori, by the Grothendieck Theorem fails to be an $\mathscr{L}_1$-space.

Coming back to Sidon sets, we would like to mention that our techniques do not establish whether, if S is an infinite Sidon set, the space $L^1_{\hat{S}}(G)$ can have local unconditional structure. (However, see the remark at the end of the section.) Furthermore, we do not know whether the spaces $L^1_{\hat{S}}(G)$ depend essentially on the choice of Sidon set (i.e. if S_1 and S_2 are infinite countable Sidon sets in Γ_1 and Γ_2 respectively, are the spaces $L^1_{\hat{S}_1}(G_1)$ and $L^1_{\hat{S}_2}(G_2)$ isomorphic?)

In order to keep the paper self contained we include proofs of several facts on twisted sums which have been known for about 20 years but which seem to be not available in the literature. Many of these facts are contained in the preprint of Domański [8]. We are indebted to Paweł Domański, who read the preliminary version of the paper, for supplying us with additional references and for many valuable comments.

Remark. After the initial preparation of the paper, W.B. Johnson showed, using related techniques, that the kernel of the quotient map of L_1 onto c_0 fails to have (Gordon-Lewis) local unconditional structure.

1. Auxiliary lemmas

In this section we state three essentially known lemmas on short exact sequences of Banach spaces (cf. [24, Chapt. III] in the language of homology; [28], [18], [20], [7], [32], [33] in the setting of Banach spaces and topological vector spaces).

If $u : X \to Y$ is a (bounded) linear operator acting between normed spaces X and Y, then we put

$$\rho(u) = \inf\{\eta > 0 : \forall y \in u(X)\ \exists\, x \in u^{-1}(y) \text{ with } \|x\| \le \eta\|y\|\}.$$

Lemma 1.1. *Let E, X, F be Banach spaces, $u : E \to X$ an isomorphic embedding, $v : X \to F$ a surjection, $u(E) = \ker v$. Then there exists an equivalent norm* *on X and $\beta > 0$ such that if X_1 denotes X equipped with the new norm, then $u : E \to X_1$ is a linear isometric embedding and $\beta v : X_1 \to F$ is a quotient map, hence E is isometrically isomorphic to a subspace of X_1 and F is isometrically isomorphic to the quotient of X_1 by this subspace.*

Proof. Let B_Y denote the unit ball of a normed space Y and let $cv(W)$ denote the absolute — for real spaces (resp. circled — for complex spaces) closed convex hull of a set $W \subset Y$.

We define the new norm on X to be the gauge functional of the set

$$cv(u(B_E) \cup \alpha B_X) \cap \beta v^{-1}(B_F)$$

where the positive numbers α and β are chosen so that

$$cv(u(B_E) \cup \alpha B_X) \cap u(E) = u(B_E) \text{ and } v(\alpha B_X) \supset \beta B_F.$$

The existence of $\alpha > 0$ and $\beta > 0$ in question follows from the assumptions that u is an isomorphic embedding and that v, being a surjection, is open. □

Our next lemma in the setting of Banach spaces is often called "Kisliakov's Lemma".

Lemma 1.2. *Let X, Y, X_1 be Banach spaces, let X be a subspace of X_1 and let $u : X \to Y$ be a bounded linear operator. Then there exist a Banach space Y_1 and a linear operator $u_1 : X_1 \to Y_1$ such that Y is a subspace of Y_1, u_1 is a norm preserving extension of u and the quotient spaces X_1/X and Y_1/Y are isometrically isomorphic. Precisely the following diagram commutes*

$$\begin{array}{ccccc} X & \xrightarrow{\ j\ } & X_1 & \xrightarrow{\ q\ } & X_1/X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle I} \\ Y & \xrightarrow[\ J\ ]{} & Y_1 & \xrightarrow[\ Q\ ]{} & Y_1/Y \end{array}$$

where j and J are natural inclusions, q and Q quotient maps, and I is an isometric isomorphism. Moreover, if u is an isometric isomorphism or an isometric embedding, then so is u_1; in general $\rho(u) = \rho(u_1)$.

For a proof except the "moreover part" see [5, pp. 316–317].

Proof of the "moreover part". Without loss of generality assume that $\|u\| = 1$. Our assumption says that there is c, with $0 < c \leq 1$ such that $\|u(x)\|_Y \geq c\|x\|_X$ for $x \in X$. Recall that Y_1 is defined to be the quotient space of the ℓ^1-sum $X_1 \oplus_1 Y$ by the subspace

$$W = \{(x, -u(x)) \in X_1 \oplus_1 Y : x \in X\},$$

and u_1 is defined to be the restriction of the quotient map $X_1 \oplus_1 Y \to Y_1$ to the subspace $X_1 \oplus_1 \{0\}$ naturally identified with X_1. For fixed $x_1 \in X_1$ we have

$$\begin{aligned} \|u_1(x_1)\|_{Y_1} &= \inf_{x\in X}(\|x_1 - x\|_{X_1} + \|u(x)\|_Y) \\ &\geq \inf_{x\in X}(\|c(x_1 - x)\|_{X_1} + c\|x\|_X) \\ &\geq c\|x_1\|_{X_1}. \end{aligned}$$

□

We also need a dual version of Kisliakov's Lemma.

Lemma 1.3. *Let Y, Z, Z_1 be Banach spaces. Let Z be a quotient of Y via the quotient map q and let $u : Z_1 \to Z$ be a bounded linear operator. Then there exist a Banach space Y_1 such that Z_1 is a quotient of Y_1 via the quotient map q_1 and a linear operator $u_1 : Y \to Y_1$ with $\|u\| = \|u_1\|$ such that $qu_1 = uq_1$ and the spaces $X = \ker q$ and $X_1 = \ker q_1$ are isometrically isomorphic. Precisely the following diagram commutes*

$$\begin{array}{ccccc} X & \xrightarrow{\ j\ } & Y & \xrightarrow{\ q\ } & Z \\ \uparrow{\scriptstyle I} & & \uparrow{\scriptstyle u_1} & & \uparrow{\scriptstyle u} \\ X_1 & \xrightarrow[J_1]{} & Y_1 & \xrightarrow[q_1]{} & Z_1 \end{array}$$

where I is an isometric isomorphism and j and j_1 are natural inclusions.

Moreover, if u is a quotient map onto a subspace of Z, then so is u_1, precisely $\rho(u) = \rho(u_1)$.

Proof. Put
$$Y_1 = \{(y, z_1) \in Y \oplus_\infty Z_1 : q(y) = u(z_1)\}$$
where the norm in $Y \oplus_\infty Z_1$ is defined by $\|(y, z_1)\| = \max(\|y\|, \|z_1\|)$.

Define $q_1 : Y_1 \to Z_1$ and $u_1 : Y_1 \to Y$ by

$$q((y, z_1)) = z_1 \quad \text{and } u_1((y, z_1)) = y.$$

We omit the routine verification. □

2. Quotient maps from an L^1-space whose kernels are uncomplemented in their second duals

We begin with a result which was known to several experts in the field. It generalizes an old theorem of Lindenstrauss [22]. To make the paper self-contained we include the proof which is essentially the same as Lindenstrauss' original argument. For related references cf. [18] and [19, Chapt. VI] where the results refer to p-homogeneous spaces ($0 < p < 1$); [8] and [9] in the setting of operator ideals.

Here and in the sequel we identify a Banach space with its canonical image in its second dual.

Proposition 2.1. *(Lindenstrauss Lifting Principle) Let Y and E be Banach spaces and let $Q : Y \to E$ be a surjection. Assume*

$$\ker Q \quad \textit{is complemented in its second dual.} \tag{2.1}$$

Then for every $\mathscr{L}_1$-space F every bounded linear operator $T : F \to E$ admits a lifting, i.e. there exists a bounded linear operator $\widetilde{T} : F \to Y$ such that $Q\widetilde{T} = T$

Proof. The assumption that F is an $\mathscr{L}_1$-space means that there exist a $\sigma \in [1, \infty)$ and a subnet $(F_\alpha)_{\alpha \in \Omega}$ of the net of all finite dimensional subspaces of F directed by inclusion such that each F_α is at most σ isomorphic to $\ell^1_{\dim F_\alpha}$. Let $T_\alpha = T_{|F_\alpha}$, where $T_{|F_\alpha}$ denotes the restriction of T to F_α. The lifting property of $\ell^1_{\dim F_\alpha}$ yields the existence of a linear operator $\widetilde{T}_\alpha : F_\alpha \to Y$ with $Q\widetilde{T}_\alpha = T_\alpha$ and $\|\widetilde{T}_\alpha\| \leq \sigma$. Since Q is a surjection, the open mapping theorem yields $\rho(Q) > 0$. Thus, given $\eta > \rho(Q)$, there exists a function $\varphi : E \to Y$ (in general neither linear nor continuous) such that $Q\varphi(e) = e$ and $\|\varphi(e)\| \leq \eta\|e\|$ for $e \in E$. Now, put $Y_E = \ker Q$ and, for $0 \leq r < \infty$ let

$$B(r) = \{y^{**} \in (Y_E)^{**} : \|y^{**}\| \leq r\} \text{ equipped with the } (Y_E)^* \text{ topology of } (Y_E)^{**}.$$

Then $B(r)$ is a compact topological space. Hence, by the Tychonoff theorem, the product

$$\prod = \prod_{f \in F} B((\sigma + \eta)\|T\|\|f\|)$$

is also compact. For every $\alpha \in \Omega$ define $\pi_\alpha \in \Pi$ by

$$\pi_\alpha(f) = \begin{cases} \widetilde{T}_\alpha(f) - \varphi T(f) & \text{for } f \in F_\alpha \\ 0 & \text{for } f \notin F_\alpha \end{cases}$$

Let π be a limit point of the set $(\pi_\alpha)_{\alpha \in \Omega}$; the existence of π is a consequence of the compactness of Π.

Let us put

$$\widetilde{T} = P\pi + \varphi T,$$

where $P : (Y_E)^{**} \to Y_E$ is the projection granted by (2.1).

To verify that $\widetilde{T}$ is the desired operator, first note that $\widetilde{T}_\alpha(f) - \varphi T(f) \in Y_E$ for every $f \in F_\alpha$ and for $\alpha \in \Omega$; hence $\pi_\alpha(f) \in B((\sigma + \eta)\|T\|\|f\|) \cap Y_E$ for every $f \in F$ and for every $\alpha \in \Omega$. Thus $\pi(f) \in B((\sigma + \eta)\|T\|\|f\|)$ and $P\pi(f) \in B((\sigma + \eta)\|P\|\|T\|\|f\|) \cap Y_E$. Since $\|P\| \geq 1$, we get

$$\|\widetilde{T}(f)\| \leq \|P\|(\sigma + 2\eta)\|T\|\|f\| \quad \text{for } f \in F. \tag{2.2}$$

To complete the proof one has to verify the linearity of $\widetilde{T}$. In view of (2.1) and the fact that $\bigcup_{\alpha \in \Omega} F_\alpha = F$, it is enough to show

$$\widetilde{T}(f' + f'') = \widetilde{T}(f') + \widetilde{T}(f'') \quad \text{for } f', f'' \in \textstyle\bigcup_{\alpha \in \Omega} F_\alpha. \tag{2.3}$$

To this end pick α_0 so that $f', f'' \in F_{\alpha_0}$ and let $F_\alpha \supset F_{\alpha_0}$. Taking into account that $\widetilde{T}_\alpha(f'+f'') - \widetilde{T}_\alpha(f') - \widetilde{T}_\alpha(f'') = 0$ and $\varphi T(f'+f'') - \varphi T(f') - \varphi T(f'') \in Y_E$ we get

$$\pi_\alpha(f'+f'') - \pi_\alpha(f') - \pi_\alpha(f'') \quad = \quad -[\varphi T(f'+f'') - \varphi T(f') - \varphi T(f'')] \quad \text{for } \alpha \in \Omega \text{ such that } F_\alpha \supset F_{\alpha_0}.$$

Thus, remembering that π is a limiting point of the net $(\pi_\alpha)_{\alpha\in\Omega}$, we get

$$\pi(f'+f'') - \pi(f') - \pi(f'') = -[\varphi T(f'+f'') - \varphi T(f') + \varphi T(f'')] \in Y_E\,.$$

Applying to both sides of the latter identity P, we get

$$P\pi(f'+f'') - P\pi(f') - P\pi(f'') = -[\varphi T(f'+f'') - \varphi T(f') - \varphi T(f'')].$$

which yields (2.3). □

Next we discuss relationships of Lindenstrauss' Lifting Principle with the Radon Nikodym Property (=RNP). We follow the terminology and notation of [6].

Recall ([6, Chap. III]) that a linear operator $T : L^1(\mu) \to E$ (E a Banach space; μ a finite measure on a measure space (Ω, Σ, μ)) is representable if there exists a Bochner integrable function $e(\cdot) \in L^\infty(\mu; E)$ such that $\|e(\cdot)\|_\infty = \|T\|$ and

$$Tf = \int_\Omega f(s)e(s)\mu(ds) \quad \text{for } f \in L^1(\mu).$$

A Banach space E has RNP provided for every finite measure μ (equivalently for some non purely atomic μ) every bounded linear operator from $L^1(\mu)$ into E is representable.

It is interesting to compare Proposition 2.1 with the well known

Fact. *The assertion of Proposition 2.1 remains valid if the assumption (2.1) is replaced by*

$$E \text{ has RNP and } F \text{ is an abstract } L\text{-space.} \tag{2.4}$$

Proof. Assume first that $F = L^1(\mu)$ with μ finite. Then every $e(\cdot) \in L^\infty(\mu; E)$ can be represented as a sum of an absolutely convergent series in $L^\infty(\mu; E)$ of countably valued functions. Hence, by the open mapping principle there exist $\delta > 0$ and a function $y(\cdot) \in L^\infty(\mu; Y)$ such that $Q(y(s)) = e(s)$ for $s \in \Omega$ μ a.e. and $\|y(\cdot)\|_\infty \le \delta \|e(\cdot)\|_\infty$. Thus if $e(\cdot)$ represents $T : L^1(\mu) \to E$, then we define $\widetilde{T} : L^1(\mu) \to Y$ by

$$\widetilde{T}f = \int f(s)y(s)\mu(ds) \quad \text{for } f \in L^1(\mu).$$

The general case follows from the observation that by Kakutani's representation theorem every abstract L-space is the ℓ^1-sum of a family of spaces $(L^1(\mu_\alpha))_{\alpha\in A}$ with μ_α finite for all $\alpha \in A$. □

A simple consequence of Proposition 2.1 is

Corollary 2.1. *If E fails RNP and $Q : \ell^1(A) \to E$ is a surjection, then* $\ker Q$ *is not complemented in* $(\ker Q)^{**}$.

Proof. If E fails RNP, then there exists a finite measure μ and a bounded linear operator $T : L^1(\mu) \to E$ which is not representable. Since every bounded linear operator $u : L^1(\mu) \to \ell^1(A)$ is representable ([6, p.83, Corollary 8]), so is Qu. If $\ker Q$ were complemented in $(\ker Q)^{**}$ then, by Proposition 2.1, T would be of the form Qu, a contradiction. □

Our next result is more sophisticated.

Proposition 2.2. *Let $Q : L^1(\mu) \to E$ be a surjection. Assume that the measure μ is finite and E contains a subspace, say E_1, isomorphic to c_0. Then* $\ker Q$ *is uncomplemented in* $(\ker Q)^{**}$.

In particular the kernel of a surjection of $L^1(\mu)$ onto $c_0(A)$ is uncomplemented in its second dual whenever A is infinite and μ is a finite measure.

Proof. Assume first that E is separable. Then, by Sobczyk's Theorem ([23, I, Theorem 2.f.5]), there exists a projection $P : E \xrightarrow{\text{onto}} E_1$. Let $(e_n, e_n^*)_{n=1}^{\infty}$ be the biorthogonal system in (E_1, E_1^*) induced by the unit vector basis of c_0. Put $\varphi_n = (PQ)^*(e_n^*)$ for $n = 1, 2, \ldots$. Then $(\varphi_n) \subset L^\infty(\mu) = [L^1(\mu)]^*$. Regarding (φ_n) as a sequence in $L^2(\mu)$ we infer that $\varphi_n \to 0$ weakly in $L^2(\mu)$ as $n \to \infty$ (because $e_n^* \to 0$ in the c_0 topology of $\ell^1 = (c_0)^*$ as $n \to 0$). By Mazur's Theorem some convex combinations of the φ_n's tend to 0 strongly in $L^2(\mu)$. Hence, by a result of F. Riesz, a subsequence of these convex combinations tends to 0 μ-almost everywhere. Thus there is an increasing sequence of the indices $0 = k_0 < k_1 < \cdots$, a sequence $(\psi_n) \subset L^\infty(\mu)$ such that $\psi_n = \sum_{j=k_{n-1}+1}^{k_n} a_j \varphi_j$ with $a_j \geq 0$ and $\sum_{j=k_{n-1}+1}^{k_n} a_j = 1$ $(n = 1, 2, \ldots)$, and

$$\lim_n \psi_n(s) = 0 \quad \text{for } s \in \Omega \ \mu\text{-a.e.} \tag{2.5}$$

We put

$$R\left(\sum_{j=1}^{\infty} t_j e_j\right) = \sum_{n=1}^{\infty}\left(\sum_{j=k_{n-1}+1}^{k_n} t_j a_j\right)\left(\sum_{j=k_{n-1}+1}^{k_n} e_j\right) \text{ for } (t_j) \in c_0.$$

Then R is a projection from E_1 onto its subspace E_0 isomorphic to c_0 and spanned by the sequence of "characteristic functions", $\left(\sum_{j=k_{n-1}+1}^{k_n} e_j\right)_{n=1}^{\infty}$. Thus the natural embedding $J : E_0 \to E$ satisfies $RPJ = \mathrm{id}_{E_0}$. Clearly E_0 being isomorphic to c_0 fails RNP. Thus there is a bounded linear operator $T : L^1 \to E_0$ which is not representable (L^1 denotes the space of absolutely Lebesgue integrable scalar valued functions on $[0, 1]$). Now, if $\ker Q$ were complemented in $(\ker Q)^{**}$ then, by Proposition 2.1, there would exist a bounded linear operator $\widetilde{S} : L^1 \to L^1(\mu)$ such that $Q\widetilde{S} = JT$. Thus $RPQ\widetilde{S} = RPJT = T$. Now, observe that

$$RPQ(f) = \sum_{n=1}^{\infty} \int_{\Omega} f(s)\psi_n(s)\mu(ds) \left(\sum_{j=k_{n-1}+1}^{k_n} e_j \right) \quad \text{for } f \in L^1(\mu).$$

Note that the sequence $(\sum_{j=k_{n-1}+1}^{k_n} e_j)_{n=1}^{\infty}$ is equivalent to the unit vector basis of c_0. Thus the condition (2.5) yields that the operator RPQ is representable (cf. [6, p.75, Remark after the proof of Lemma 2.11]). Hence $T = RPQ\widetilde{S}$ would be representable because an operator from $L^1(\mu)$ to a Banach space is representable iff it factors through $\ell^1(A)$ (cf. [6, Chapt. III, Sect. 1, proof of Theorem 8]), a contradiction. □

The argument for non separable E is almost the same. Instead of Sobczyk's Theorem we use the following generalization.

Lemma 2.1. *Let E be a quotient of $L^1(\mu)$ with μ-finite measure. Assume that E contains a subspace E_1 isomorphic to c_0. Then E_1 is complemented in E.*

Proof. If μ is finite then the natural injection of $L^2(\mu)$ into $L^1(\mu)$ is bounded and has a dense range, hence $L^1(\mu)$ is a WCG space. Thus E is a WCG space. Therefore every separable subspace of E is contained in a separable subspace which is a range of a contractive projection from E (cf. [4, pp. 237–240]). Combining this fact with Sobczyk's Theorem we get the desired conclusion. □

Remarks

1. The assertion of Proposition 2.2 remains valid if the assumption "μ-finite" is replaced by "μ σ- finite" because every $L^1(\nu)$ with ν σ-finite is isomorphic as a Banach space with $L^1(\mu)$ for some finite measure μ.
2. After reading a preliminary version of this paper, S. Kwapien has shown us an alternative proof of Proposition 2.2 which does not use the Lindenstrauss Lifting Principle. We present his argument with his permission.

Let X_0 be a subspace of a Banach space X. Then $(X/X_0)^*$ can be identified with $X_0^{\perp}$ and X_0^{**} with $X_0^{\perp\perp}$ where

$$\begin{aligned} X_0^{\perp} &= \{x^* \in X^* : x^*(x) = 0 \text{ for } x \in X_0\}, \\ X_0^{\perp\perp} &= \{x^{**} \in X^{**} : x^{**}(x^*) = 0 \text{ for } x^* \in X_0^{\perp}\}. \end{aligned}$$

The subspace $X + X_0^{\perp\perp}$ of X^{**} is norm closed. The condition X_0 is complemented in X_0^{**} is equivalent to the existence of a bounded linear projection $p : X_0^{\perp\perp} \underset{\text{onto}}{\longrightarrow} X_0$.

For $j = 0, 1, 2$ denote by $(\delta_n^{(j)})_{n=1}^{\infty}$ the unit vector basis of c_0, ℓ^1, ℓ^2 respectively.

Proposition A. *If X_0 is complemented in X_0^{**} then for every bounded operator $T : c_0 \to X/X_0$ there exists a weakly null sequence $(x_n) \subset X$ such that $T(\delta_n^{(0)}) = Q(x_n)$ for $n = 1, 2, \ldots$ where $Q : X \to X/X_0$ is the quotient map.*

Proof. First note that the formula

$$\tilde{p}(x + x^{**}) = x + p(x^{**}) \quad (x \in X, x^{**} \in X_0^{\perp\perp})$$

well defines a projection from $X \oplus X_0^{\perp\perp}$ onto X with $\|\tilde{p}\| \le \|p\| + 2$.

Let $S = IT^* : X_o^{\perp} \xrightarrow{T^*} \ell^1 \xrightarrow{I} \ell^2$ where $I : \ell^1 \to \ell^2$ is the natural embedding since I is 2-summing, so is S. Thus S extends to a bounded operator $\tilde{S} : X^* \to \ell^2$. Put $x_n^{**} = (\tilde{S})^*(\delta_n^{(2)})$ for $n = 1, 2, \ldots$. Then $x_n^{**} \in X \in X_0^{\perp\perp}$. Indeed, pick $y_n \in X$ so that $Q(y_n) = T(\delta_n^{(0)})$. Then $x_n^{**} = y_n + (x_n^{**} - y_n)$ and $x_n^{**} - y_n \in X_0^{\perp\perp}$ because for every $x^* \in X_0^{\perp}$ one has

$$x_n^{**}(x^*) = \delta_n^{(2)}(\tilde{S}x^*) = \delta_n^{(2)}(Sx^*) = \delta_n^{(1)}(T^*x^*) = x^*(T(\delta_n^{(0)})).$$

Now put $x_n = \tilde{p}(x_n^{**})$ for $n = 1, 2, \ldots$. Since $x_n = \tilde{p}(\tilde{S})^*(\delta_n^{(2)})$ and $(\delta_n^{(2)})$ is a weakly null sequence in ℓ^2, so is (x_n) in X. Finally $Q(x_n) = Q(y_n)$ because $x_n = \tilde{p}(y_n + (x_n^* - y_n)) = y_n + p(x_n^* - y_n)$ and $p(x_n^{**} - y_n) \in X_0$. □

Definition. *A Banach space X has property (K) if for an arbitrary weak* null sequence $(\varphi_n^*) \subset X^*$ there exists a CCC sequence (ψ_n^*) such that*

$$\lim_k \psi_k^*(x_k) = 0 \text{ for every weakly null sequence } (x_k) \subset X. \tag{2.6}$$

“CCC” stands for “consecutive convex combinations”; (ψ_k^*) is a CCC sequence for (φ_n^*) if there exist an increasing sequence of the indices $0 = n_0 < n_1 < \ldots$ and a sequence (λ_n) of non-negative scalars such that

$$\psi_k^* = \sum_{j=n_{k-1}+1}^{n_k} \lambda_j \varphi_j^* \text{ with } \sum_{j=n_{k-1}+1}^{n_k} \lambda_j = 1 \quad (k = 1, 2, \ldots).$$

Proposition B. *$L^1(\mu)$ has (K) for every finite measure μ.*

Proof. If $(\varphi_n^*) \subset L^\infty(\mu) = (L^1(\mu))^*$ is a weak* star null sequence then some subsequence of (φ_n^*) is a weakly null sequence in $L^2(\mu)$. Hence, by Mazur’s theorem, some CCC sequence (ψ_k^*) for (φ_n^*) strongly converges to zero in $L^2(\mu)$. Since weakly null sequences in L^1 are equi-integrable one easily checks that (ψ_k^*) satisfies (2.6) (cf. the argument at the end of the proof of Lemma 5.2) □

The space c_0 fails (K) in the following strong sense:

Proposition C. *If (y_k^*) is a CCC sequence for $(\delta_n^{(1)}) \subset \ell^1 = (c_0)^*$ then there exists a bounded linear projection $V : c_0 \to c_0$ such that*

$$\inf_k y_k^*(V(\delta_k^{(0)})) > 0 \quad \square \tag{2.7}$$

Proof of Proposition 2.2. Let $U : c_0 \to E$ be an isomorphic embedding. Since Sobczyk’s theorem generalizes to quotients of $L^1(\mu)$ for μ finite (Lemma 2.1), there exists a projection, $P : E \to E$ with $P(E) = U(c_0)$. Put $\varphi_n^* = Q^*P^*(U^{-1})^*(\delta_n^{(1)}) \in (L^1(\mu))^*$ for $n = 1, 2, \ldots$. Since $(\delta_n^{(1)})$ is a weak* null sequence in $\ell^1 = (c_0)^*$, so is (φ_n^*) in $(L^1(\mu))^*$. By Proposition B there exists a

CCC sequence (ψ_k^*) for (φ_n^*) which satisfies (2.6). Let (y_k^*) be the CCC sequence for $(\delta_n^{(1)})$ of the same convex combinations as the sequence (ψ_k^*) for (φ_n^*). Let $V : c_0 \to c_0$ be that of Proposition C. Put $T : UV : c_0 \to E$. Assume that $\ker Q = X_0 \subset L^1(\mu)$ were complemented in X_0^{**}. Then, by Proposition A (applied to $X = L^1(\mu)$) there would exist a weakly null sequence sequence $(x_k) \subset L^1(\mu)$ such that $T(\delta_k^{(0)}) = Q(x_k)$ for $k = 1, 2, \ldots$. On the other hand we have

$$\psi_k^*(x_k) = y_k^*(V(\delta_k^{(0)})) \quad \text{for } k = 1, 2, \ldots$$

This would lead to a contradiction with (2.7), because $\lim_k \psi_k^*(x_k) = 0$ by (2.6).□

Finally note that Proposition B and accordingly Proposition 2.2 can be generalized to preduals of C^*-algebras with finite faithful trace.

Our last result in this section is a partial converse to Corollary 2.1.

Proposition 2.3. *Suppose E has RNP and is complemented in E^{**}. Then for every quotient map $Q : \ell^1(A) \to E$, $\ker Q$ is complemented in $(\ker Q)^{**}$.*

Proof. Denote by P a bounded projection from E^{**} onto E and by $\Pi : \ell^1(A)^{**} \to \ell^1(A)$ the canonical projection.

We start by observing that if Z is an abstract L-space, then every operator $T : Z \to E$ factors through a space $\ell^1(B)$. This follows from the fact that Z can be decomposed as an ℓ^1-sum of $L^1(\mu_\alpha)$ spaces where each μ_α is a finite measure and from the Lewis-Stegall theorem (cf. [23], [6, Chapt. III, Sect. 1, Theorem 8]). Since $\ell^1(B)$ is projective, this implies that T also factors through Q. We apply these remarks to $Z = (\ell^1(A))^{**}$ and to $T = PQ^{**}$ to deduce the existence of a bounded linear operator $S : [\ell^1(A)]^{**} \to \ell^1(A)$ such that $PQ^{**} = QS$.

Now let $V = S + (I - S)\Pi : [\ell^1(A)]^{**} \to \ell^1(A)$. Clearly V is a projection onto $\ell^1(A)$. Furthermore $V(\ker Q^{**}) = \ker Q$ (indeed let $x^{**} \in \ker Q^{**}$. Put $x = \pi x^{**} \in \ell^1(A)$. Then $QSx^{**} = PQ^{**}x^{**} = 0$ and $PQ^{**}x = PQx = Qx$. Therefore $QVx^{**} = Q(I - S)x = Qx - PQ^{**}x = Q$). However $\ker Q^{**}$ is the weak*-closure of $\ker Q$ in $[\ell^1(A)]^{**}$ (since Q is a quotient map) and so is naturally isomorphic to $(\ker Q)^{**}$. □

3. Subspaces of an $\mathscr{L}_1$-space which are GT spaces

Recall that a Banach space Y is a twisted sum of Banach spaces X and Z, in symbols $Y = X \oplus\!\!\!\!\smile Z$ provided

$$0 \to X \xrightarrow{j} Y \xrightarrow{q} Z \to 0$$

is a short exact sequence, i.e. $j(X) = \ker q$, with j being an isometrically isomorphic embedding and q being a quotient map. We say that a twisted sum $Y = X \oplus\!\!\!\!\smile Z$ splits provided it is naturally isomorphic to the Cartesian product $X \oplus Z$, i.e. there exists a bounded linear operator $v : Z \to Y$ such that

$$qv = \mathrm{id}_Z \tag{3.1}$$

where id_Z denotes the identity operator on Z. Note that if v satisfies (3.1) then $\mathrm{id}_Y - vq$ is a projection from Y onto X. Conversely, if $p : Y \to X$ is a bounded linear projection onto X then the formula $v(z) = y - p(y)$ for any $y \in Y$ such that $q(y) = z$ well defines $v : Z \to Y$ which satisfies (3.1), hence $X \tilde{\oplus} Z$ splits.

Next recall that every Banach space is a quotient space of $\ell^1(A)$ for an appropriate set A.

We begin with a "purely formal" but useful fact.

Proposition 3.1. *Suppose we are given Banach spaces E and H and a quotient map $q_E : \ell^1(A) \to E$ and let $X_E = \ker q_E$.*

Then the following conditions are equivalent

(i) every bounded linear operator from X_E into H extends to a bounded linear operator from $\ell^1(A)$ into H;
(ii) every twisted sum $H \tilde{\oplus} E$ splits.

Proof. (ii) $\Rightarrow$ (i). By Lemma 1.2 a bounded linear operator $u : X_E \to H$ extends to a linear operator $u_1 : \ell^1(A) \to H \tilde{\oplus} E$. By (ii) there exists $v : E \to H \tilde{\oplus} E$ with $qv = \mathrm{id}_E$ where $q : H \tilde{\oplus} E \to E$ is the quotient map with $\ker q = H$. Put $p = \mathrm{id}_{H \tilde{\oplus} E} - vq$. Then p is a projection from $H \tilde{\oplus} E$ onto H and pu_1 is the desired extension of u.
(i) $\Rightarrow$ (ii). Fix a twisted sum $Y = H \tilde{\oplus} E$ and let $q : Y \to E$ be the quotient map with $\ker q = H$. Fix $c > 1$. The lifting property of $\ell^1(A)$ yields the existence of a bounded linear operator $\varphi : \ell^1(A) \to Y$ such that $q\varphi = q_E$ and $\|\varphi\| < c$. Obviously $\varphi(X_E) \subset H = \ker q$. Thus, by (i), the restriction of φ to X_E extends to a bounded linear operator, say $v : \ell^1(A) \to H$. Let us consider the operator $\varphi - v : L^1(A) \to Y$. Clearly $\ker(\varphi - v) \supset X_E$. Hence $\varphi - v$ factors through q_E, precisely the formula $u(e) = (\varphi - v)(\xi)$ for $e = q_E(\xi) \in E$ well defines a bounded linear operator $u : E \to Y$ such that $uq_E = \varphi - v$. Since $qv = 0$, we have

$$quq_E = q\varphi = q_E.$$

Thus $qu = \mathrm{id}_E$ because $q_E(\ell^1(A)) = E$. Hence $H \tilde{\oplus} E$ splits. □

Remark. For an analogue of Proposition 3.1 in Frechet spaces cf. [33], [34].

Under the additional assumption that H is complemented in its second dual, Proposition 3.1 generalizes to $\mathscr{L}_1$-spaces.

Proposition 3.2. *Let E and H be Banach spaces. Assume that H is complemented in its second dual. Then the following conditions are equivalent*

(ii) every twisted sum $H \tilde{\oplus} E$ splits;
(iii) for every $\mathscr{L}_1$-space F and every quotient map $q_E : F \to E$ every bounded linear operator from $X_E = \ker q_E$ into H extends to a bounded linear operator from F into H;

(iv) *there exists an* $\mathscr{L}_1$*-space* F *and a quotient map* $q_E : F \to E$ *such that* $X_E = \ker q_E$ *has the above extension property.*

Proof. (ii) $\Rightarrow$ (iii). The proof is the same as that of the implication (ii) $\Rightarrow$ (i) of Proposition 3.1.
(iii) $\Rightarrow$ (iv). Trivial.
(iv) $\Rightarrow$ (ii). The proof differs from the proof of the implication (i) $\Rightarrow$ (ii) of Proposition 3.1 only how the bounded linear operator φ which lifts q_E is constructed. Instead of using the lifting property of $\ell^1(A)$ we apply Proposition 2.1. At this place the assumption that H is complemented in H^{**} is used. □

Remark. Condition (ii) has isomorphic character in the following sense. If (H, E) is a pair of Banach spaces such that every twisted sum $H \oplus_t E$ splits and if (H_1, E_1) is another pair such that H is isomorphic to H_1 and E is isomorphic to E_1 then every twisted sum $H_1 \oplus_t E_1$ splits. This is an immediate consequence of Lemma 1.1. Thus Propositions 3.1 and 3.2 remain valid if one replaces E by a Banach space isomorphic to E or equivalently if q_E is an arbitrary surjection onto E.

Next we discuss a qualitative and a local version of condition (ii).

For a twisted sum $Y = X \oplus_t Z$ put

$$spl(Y) = \begin{cases} +\infty & \text{if } Y \text{ does not split} \\ \inf(\|v\|^2 + \|\mathrm{id}_Y - vq\|^2)^{\frac{1}{2}} \end{cases} \tag{3.2}$$

where the infimum extends over all $v : Z \to Y$ satisfying (3.1).

Our next proposition can be deduced from some results in Domański's Ph.D. Thesis (Poznań 1986) which are stated in terms of operator ideals (cf. [8, Theorem II.1.1 and Theorem II.3.1]; cf. also [9]).

Proposition 3.3. *For every pair of Banach spaces* E *and* H *the following conditions are equivalent*

(ii) *every twisted sum* $H \oplus_t E$ *splits;*
(v) $\sup spl(H \oplus_t E) = C < +\infty$, *where the supremum extends over all twisted sums* $H \oplus_t E$.

Moreover, if H *is complemented in* H^{**} *and there exists a family* $(H_\alpha)_{\alpha\in\Omega}$ *of finite dimensional subspaces of* H *directed by inclusion and such that* $\cup_{\alpha\in\Omega} H_\alpha$ *is dense in* H *and each* H_α *is the range of projection* $\pi_\alpha : H \to H_\alpha$ *with* $\sup_{\alpha\in\Omega} \|\pi_\alpha\| < +\infty$ *then the equivalent conditions (ii) and (v) are equivalent to*

(vi) $\sup spl(H_\alpha \oplus_t E) = C_1 < \infty$, *where the supremum extends over all* $\alpha \in \Omega$ *and all twisted sums* $H_\alpha \oplus_t E$.

Proof. (ii) $\Rightarrow$ (v). Let $L(X, Y)$ denote the Banach space of all bounded linear operators from X into Y. A restatement of condition (i) of Proposition 3.1 says that the restriction operator maps $L(\ell^1(A), H)$ onto $L(X_E, H)$. Hence, by the open

mapping principle, there exists an $M \in (0, \infty)$ such that every $u \in L(X_E, H)$ extends to an $u_1 \in L(\ell^1(A), H)$ with $\|u_1\| \le M\|u\|$. Now the analysis of the proof of the implication (i) $\Rightarrow$ (ii) yields (v) with $C \le \sqrt{M^2 + (M+1)^2}$.
(v) $\Rightarrow$ (ii). Trivial.
(v) $\Rightarrow$ (vi). Fix a twisted sum $H_\alpha \oplus E$ and denote by $i : H_\alpha \to H$ the natural inclusion. By Lemma 1.2 there exists a twisted sum $H \oplus E$ and an isometrically isomorphic embedding $I : H_\alpha \oplus E \to H \oplus E$ which extends i. By (v) for every $a > 1$ there exists a projection $p : H \oplus E \overset{\text{onto}}{\to} H$ with $\|p\| \le aC$. Put $P = \pi_\alpha p I$. Then $P : H_\alpha \in E \to H_\alpha$ is a projection with $P(H_\alpha \oplus E) = H_\alpha$ and $\|P\| \le aC \sup_\alpha \|\pi_\alpha\| = C_2$. Thus $spl(H_\alpha \oplus E) \le C_1$ where $C_1 = \sqrt{C_2^2 + (1 + C_2)^2}$.
(vi) $\Rightarrow$ (v). Let $H \oplus E$ be a twisted sum. By Lemma 1.2 for each $\alpha \in \Omega$ there exist a twisted sum $H_\alpha \oplus E$ and a linear operator Π_α which extends π_α and satisfies $\|\Pi_\alpha\| = \|\pi_\alpha\|$. Fix $a > 1$. By (vi) there exists a projection $p_\alpha : H_\alpha \oplus E \to H_\alpha$ with $p_\alpha(H_\alpha \oplus E) = H_\alpha$ and $\|p_\alpha\| \le aC_1$. Consider the family of operators $(i_\alpha p_\alpha \Pi_\alpha)_{\alpha \in \Omega}$. Clearly

$$i_\alpha p_\alpha \Pi_\alpha : H \oplus E \to H_\alpha \subset H \subset H^{**} \text{ and } \|i_\alpha p_\alpha \Pi_\alpha\| \le aC_1 \sup_\alpha \|\pi_\alpha\|.$$

Note that

$$\text{if } h \in H_\alpha \text{ and } H_{\alpha'} \supset H_\alpha \text{ then } i_\alpha p_\alpha \Pi_\alpha(h) = i_{\alpha'} p_{\alpha'} \Pi_{\alpha'}(h) = h. \tag{3.3}$$

Now, in the Stone-Čech compactification $\beta(\Omega)$ of the discrete set Ω, consider the family $(cl\mathscr{O}_\alpha)_{\alpha \in \Omega}$ where

$$\mathscr{O}_\alpha = \{\alpha' \in \Omega : H_{\alpha'} \supset H_\alpha\} \quad \text{for } \alpha \in \Omega$$

and clW denotes the closure of a set W. Clearly the family $(\mathscr{O}_\alpha)_{\alpha \in \Omega}$ is centered. Hence the intersection $\bigcap_{\alpha \in \Omega} cl\mathscr{O}_\alpha$ is non-empty. Pick a $\phi \in \bigcap_{\alpha \in \Omega} cl\mathscr{O}_\alpha$. Denote by $\lim_\phi(f)$ the evaluation at ϕ of the unique continuous extension of a bounded scalar-valued function f on Ω to a continuous function on $\beta(\Omega)$. For each $y \in H \oplus E$ and each $h^* \in H^*$ let f_{y,h^*} be the scalar valued function on Ω defined by $f_{y,h^*}(\alpha) = [i_\alpha p_\alpha \prod_\alpha(y)](h^*)$ (we identify H with its canonical image in H^{**}). For each $y \in H \oplus E$ define the function $T(y)$ on H^* by $T(y)(h^*) = \lim_\phi f_{y,h^*}$. It can be easily verified that $T(y) \in H^{**}$ and $T : H \oplus E \to H^{**}$ defined by $y \to T(y)$ for $y \in H \oplus E$ is a linear operator with $\|T\| \le aC_1 \sup_\alpha \|\pi_\alpha\|$. It follows from (3.3) and the density of $\bigcap_{\alpha \in \Omega} H_\alpha$ in H that $T(h) = h$ for $h \in H$. Now if S is a bounded linear projection from H^{**} onto H then $P = ST$ is the desired projection from $H \oplus E$ onto H with $\|P\| \le aC_1\|S\| \sup_\alpha \|\pi_\alpha\| = uC_0$. This yields (v) with $C \le \sqrt{C_0^2 + (C_0 + 1)^2}$. □

We get an important application of Proposition 3.1–3.3 by specifying H to be an infinite dimensional Hilbert space, say $H = \ell^2$. The theory of absolutely summing operators enters. First we recall some results on absolutely summing operators essentially due to Grothendieck [16] with Maurey's [25] improvement of (jjj) for $p < 1$ (cf. [30, pp. 60]).

(G) *Let X be a closed linear subspace of an $\mathscr{L}_1$-space F. Then for every bounded linear operator $u : X \to \ell^2$ the following conditions are equivalent*

- *(j) u extends to a bounded linear operator from F into ℓ^2;*
- *(jj) u is 2-absolutely summing;*
- *(jjj) u is p-absolutely summing for $p \in [0;2]$;*
- *(jjjj) for every bounded linear operator $v : \ell^2 \to X$ the composition uv is in the Hilbert-Schmidt class.*

Recall that a Banach space X is called a GT-space (cf. [23]) if every bounded linear operator from X into ℓ^2 is 1-absolutely summing.

Combining (G) with Propositions 3.1–3.3 and with the Remark after Proposition 3.2 we get

Theorem 3.1. *Let E be a Banach space and let Q be a linear surjection from an $\mathscr{L}_1$-space F onto E. Let $X_E = \ker Q$. Then the following conditions are equivalent*

- *(+) X_E is a GT-space;*
- *(++) every twisted sum $\ell^2 \oplus E$ splits;*
- *(+++) $\sup_n spl(\ell^2_n \oplus E) < +\infty$, where the supremum extends over all twisted sums $\ell^2_n \oplus E$ and over positive integers n.*

4. Banach spaces E with a non-trivial twisted sum of ℓ^2 and E

We begin with two known lemmas (cf. e.g. [8, Chapt. I Sect. 5 and Chapt. II Sect. 5]).

Lemma 4.1. *Let E, F, H be Banach spaces. Assume that there exist linear operators $\varphi : E \to F$ and $\psi : F \to E$ such that $\varphi\psi = \mathrm{id}_F$, $\|\psi\| \le C$ and $\|\varphi\| \le C$, so that F is C^2-equivalent to a C-complemented subspace of E. Then there exists a twisted sum $H \oplus E$ such that*

$$spl(H \oplus E) \ge C^{-2}\left(\frac{1}{8}\sup spl\ Y - \frac{1}{4}\right) \tag{4.1}$$

where the supremum extends over all twisted sums $Y = H \oplus F$.

In particular, if there exists a twisted sum $H \oplus F$ which does not split, then there is a twisted sum $H \oplus E$ which does not split.

Proof. Pick a twisted sum $H \oplus F$ so that $2spl(H \oplus F) \ge \sup spl(Y)$. By Lemma 1.3 there exist a twisted sum $H \oplus E$ and a linear operator $\Phi : H \oplus E \to H \oplus F$ such that $\|\Phi\| = \|\varphi\|$ and $q\Phi = \varphi q_1$ where $q : H \oplus F \to F$ and $q_1 : H \oplus E \to E$ are quotient maps with $\ker q = \ker q_1 = H$. If $H \oplus E$ does not split, we are done. Otherwise there exists a bounded linear operator $v_1 : E \to H \oplus E$ such that $q_1 v_1 = \mathrm{id}_E$ and $\|v_1\| \le 2spl(H \oplus E)$. Put $v = \Phi v_1 \psi : F \to H \oplus F$. Clearly,

$$qv = q\Phi v_1 \psi = \varphi q_1 v_1 \psi = \varphi\psi = \mathrm{id}_F$$

and

$$\|v\| \leq C^2\|v_1\| \leq 2C^2 spl(H \oplus E).$$

On the other hand,

$$2\|v\| + 1 \geq spl(H \oplus F) \geq 2^{-1} \sup spl(Y).$$

The last two chains of inequalities obviously yield (4.1). □

Remembering that the adjoint of a linear isometric embedding is a quotient map and vice versa, and applying directly formula (3.2) we get

Lemma 4.2. *Let $H \oplus E$ be a twisted sum of Banach spaces, precisely*

$$0 \to H \xrightarrow{j} H \oplus E \xrightarrow{q} E \to 0.$$

Then

$$0 \to E^* \xrightarrow{j^*} (H \oplus E)^* \xrightarrow{q^*} H^* \to 0$$

is also a twisted sum, say $E^* \oplus H^*$; we have

$$spl(H \oplus E) \geq spl(E^* \oplus H^*).$$

Moreover, if $E \oplus H$ is reflexive, equivalently if E and H are reflexive, then

$$spl(H \oplus E) = spl(E^* \oplus H^*).$$

Next we recall the profound result due to Enflo, Lindenstrauss and Pisier [12] (cf. [17], [18] for further examples).

(ELP) *There exists a twisted sum $\ell^2 \oplus \ell^2$ which does not split.*

A simple and known consequence of (ELP) are the next two Corollaries (cf. [8, Theorem IV.6.1].

Corollary 4.1. *If a Banach space E contains ℓ_n^2 uniformly isomorphic and uniformly complemented* $(n = 1, 2, \ldots)$ *then there exists a twisted sum $\ell^2 \oplus E$ which does not split.*

Proof. By (ELP) and Theorem 3.1 for $n = 1, 2, \ldots$ there exists a twisted sum $Y_n = \ell_n^2 \oplus \ell^2$ such that $\sup_n spl(Y_n) = +\infty$. Thus, by Lemma 4.1 we have $\sup_n spl(Y_n^*) = +\infty$ because $Y_n^* = \ell^2 \oplus \ell_2^n$ is reflexive $(n = 1, 2, \ldots)$. By our assumption on E there is a $C \in [1, \infty)$ such that for $n = 1, 2, \ldots$ there are linear operators $\varphi_n : E \to \ell_n^2$ and $\psi_n : \ell_n^2 \to E$ satisfying (4.1) with $F = \ell_n^2$, $\varphi = \varphi_n$ and $\psi = \psi_n$. Thus, by Lemma 4.1 there exists a sequence (Z_n) of twisted sums $\ell^2 \oplus E$ such that $\sup_n spl(Z_n) = +\infty$. Thus, by Proposition 3.3 there exists a twisted sum $\ell^2 \oplus E$ which does not split. □

Corollary 4.2. *If a Banach space E is K-convex and infinite dimensional, then there exists a twisted sum $\ell^2 \oplus E$ which does not split.*

Proof. By [5, Theorem 19.3] every infinite dimensional K-convex Banach space satisfies the assumption of Corollary 4.1. (For the definition of K-convex Banach spaces see [5, Chapt. 13]. □

A less obvious although also "formal" consequence of (ELP) is the following

Theorem 4.1. *If E is an infinite dimensional Banach space such that E^* is of cotype 2, then there exists a twisted sum $\ell^2 \oplus_{\tau} E$ which does not split.*

Proof. Similarly, as in the proof of Corollary 4.2, pick for $n = 1, 2, \ldots$ twisted sums $Y_n = \ell^2_n \oplus_{\tau} \ell^2$ so that $\sup_n spl(Y_n) = \sup_n spl(Y_n^*) = +\infty$. By the dual version of Dvoretzky's Theorem ([31, Theorem 7.1]) there exists a bounded linear surjection $u_n : E \to \ell^2_n$ with $\|u_n\|[\rho(u_n)]^{-1} \leq 2$. Thus, by Lemma 1.3 there exist a twisted sum $W_n = \ell^2 \oplus_{\tau} E$ and a surjection $U_n : W_n \to Y_n^*$ such that $q_n U_n = u_n Q_n$ where $q_n : Y_n^* \to \ell^2_n$ and $Q_n : W_n \to E$ are quotient maps with the kernels isometrically isomorphic to ℓ^2 and $\|U_n\|\rho(U_n) \leq 2$.

By Proposition 3.3 it is enough to show that $\sup_n spl\, W_n = +\infty$. Assume on the contrary that $\sup_n spl\, W_n = C < +\infty$. Then, by Lemma 4.2 $\sup_n spl(W_n^*) \leq C$. Thus W_n^* would be C-isomorphic to the cartesian product $E^* \oplus \ell^2$ hence the cotype 2 constant of W_n^* would be bounded by a constant C_2 independent of n. Since Y_n is reflexive and U_n is a surjection, U_n^* is an isomorphic embedding of Y_n into W_n^*, moreover the Banach Mazur distance $d(Y_n, U_n^*(Y_n)) \leq 2$. Thus the cotype 2 constants of the Y_n's would be uniformly bounded by $2C_2$. Since the Y_n's are twisted sums of Hilbert spaces, they have for every $\varepsilon > 0$ uniformly bounded type $(2 - \varepsilon)$ constants (cf. [12]).

Thus, by a result of Pisier [29], the K-convexity constants of the Y_n's are uniformly bounded, say by C_3. Thus, by the Maurey-Pisier duality theorem (cf. [27], [31, Proposition 12.8]) the Y_n^*'s would have uniformly bounded type 2 constants, say by $C_4 = C_4(C_1, C_2)$. Now let $q_n : Y_n^* \to \ell^2_n$ be the quotient map with $X_n = \ker q_n$ isometrically isomorphic to ℓ^2. Then the map id_{X_n} could be regarded as an operator from a subspace X_n of a space Y_n^* of type 2 into a space ℓ^2 of cotype 2. Thus, by Maurey's extension theorem (cf. [26]; [31, Theorem 13.13]), id_{X_n} would extend to a projection $p_n : Y_n^* \to X_n$. Moreover, the norms $\|p_n\|$ would be uniformly bounded by a constant depending on C_4 only. Hence $\sup_n spl(Y_n^*) < +\infty$, a contradiction. □

The analysis of the proofs of Corollary 4.2 and Theorem 4.1 shows that the "local version" of these assertions also holds. Precisely one has

Corollary 4.3. *For each $C \in [1, \infty)$ there exists a sequence (a_n^c) of positive numbers with $\lim_n a_n^c = +\infty$ such that if E is an n-dimensional Banach space such that either the K-convexity constant of E or the cotype 2 constant of E^* does not exceed C, then there exists a twisted sum $\ell^2 \oplus_{\tau} E$ with $spl(\ell^2 \oplus_{\tau} E) \geq a_n^c$ ($n = 1, 2, \ldots$).*

Corollary 4.4. *If a Banach space* E *either contains* ℓ_n^∞ *uniformly isomorphic or for some* $p \in (1,\infty)$ *contains* ℓ_n^p *uniformly isomorphic and uniformly complemented* ($n = 1, 2, \ldots$)*, then there exists a twisted sum* $\ell^2 \oplus E$ *which does not split.*

Proof. Recall the following well known facts. For each fixed $p \in (1,\infty)$ the spaces ℓ_n^p have uniformly bounded K-convexity constants and the spaces $(\ell_n^\infty)^* = \ell_n^1$ have uniformly bounded cotype 2 constants ($n = 1, 2, \ldots$); the space ℓ_n^∞ is norm one complemented in every larger Banach space in which it is isometrically embedded ($n = 1, 2, \ldots$). Now, combine Corollary 4.3 with Lemma 4.2 and Proposition 3.3. $\square$

Corollary 4.5. *For every infinite set A there exists a twisted sum* $\ell^2 \oplus c_0(A)$ *which does not split.*

Remarks:

1. Corollaries 4.4 and 4.5 can be deduced from some results of the forthcoming paper [3] where a different argument is used.
2. The results of this section indicates that there are "few" Banach spaces E for which every twisted sum $\ell^2 \oplus E$ splits. Since the $\{0\}$ space is a GT space, Theorem 3.1 yields that if E is an $\mathscr{L}^1$ space, then every twisted sum $\ell^2 \oplus E$ splits. A more sophisticated example is the following. Let X be a subspace of ℓ^1 which is isomorphic to ℓ^1 but uncomplemented in ℓ^1 (The existence of X follows from a result of Bourgain [2]). Put $E = \ell^1/X$. Since X being isomorphic to ℓ^1 is a GT-space, Theorem 3.1 yields that every twisted sum $\ell^2 \oplus E$ splits. On the other hand, E is not an $\mathscr{L}_1$-space because otherwise it would follow from Proposition 2.1 that id_E lifts to ℓ^1, i.e. there exists $\widetilde{T} : E \to \ell^1$ such that $Q\widetilde{T} = \mathrm{id}_E$ where $Q : \ell^1 \to \ell^1/X$ is the quotient map. This would yield that X is complemented in ℓ^1, a contradiction. It seems to be an interesting problem to characterize all Banach spaces E such that every twisted sum $\ell^2 \oplus E$ splits.

5. An application to Sidon sets

Let G be a compact abelian group, Γ its dual. $L^p(G)$ denotes the L^p space with respect to the normalized Haar measure of G denoted either by dx or by λ ($1 \le p \le \infty$). $M(G)$ stands for the space of all complex Borel measures on G with finite variation.

Given a set $A \subset \Gamma$, put $\tilde{A} = \Gamma \setminus A$,

$$L^p_{\tilde{A}} = \left\{ f \in L^p(G) : \hat{f}(\gamma) = 0 \quad \text{for } \gamma \in A \right\} \quad (1 \le p \le \infty),$$

where

$$\hat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}dx \quad (\gamma \in \Gamma,\ f \in L^1(G)).$$

Note that for $1 \le p < \infty$,

$$L^p_{\tilde{A}} = \text{ the closed linear subspace of } L^p \text{ generated by } A.$$

Recall that an $S \subset \Gamma$ is a *Sidon set* if S regarded as a subspace of $L^\infty(G)$ is equivalent to the unit vector basis of $\ell^1(S)$. A classical characterization of a Sidon set says

(*) *S is a Sidon set iff the map $Q : L^1(G) \to c_0(S)$ defined by $Qf = (\hat{f}(\gamma))_{\gamma\in S}$ is a surjection. Clearly* $\ker Q = L^1_{\tilde{S}}(G)$.

Thus,

Corollary 5.1. *Let S be an infinite Sidon set. Then*

(i) The canonical image of $L^1_{\tilde{S}}(G)$ is uncomplemented in $(L^1_{\tilde{S}}(G))^{**}$.
(ii) $L^1_{\tilde{S}}(G)$ not isomorphic to a complemented subspace of a Banach lattice.
(iii) There exists a bounded linear operator from $L^1_{\tilde{S}}(G)$ into a Hilbert space which is not 2-absolutely summing.
(iv) $L^1_{\tilde{S}}(G)$ is not an $\mathscr{L}_1$-space.

Proof. (i). Combine (*) with Proposition 2.2.
(ii). First note that $L^1_{\tilde{S}}(G)$ is a weakly sequentially complete Banach space being a closed subspace of a weakly sequentially complete Banach space $L^1(G)$. Next, observe that if a Banach space X is complemented in its second dual, then every complemented subspace of X has the same property. Furthermore, a weakly sequentially complete complemented subspace of a Banach lattice is isomorphic to a complemented subspace of a weakly sequentially complete Banach lattice ([23, II, Proposition 1.c.6]; [13]). Finally, a weakly sequentially complete Banach lattice is complemented in its second dual ([23, II, Theorem 1.c.4]). Combining these facts with (i), we get (ii). Note that this argument also applies to the kernel of any surjection onto c_0 in place of $L^1_{\tilde{S}}(G)$.
(iii). Combine (*) with Theorem 3.1 and Corollary 4.5.
(iv). Combine (iii) with (G) in Sect. 3. □

It is interesting to compare Corollary 5.1 (iii) with the following known fact (cf. [21, Theorem 2.1 and Proposition 3.2]).

Corollary 5.2. *Let $S \subset \Gamma$ be a Sidon set. Let $u : L^1_{\tilde{S}}(G) \to H$ be a bounded linear operator into a Hilbert space H. Let $I_{\tilde{S}} : L^2_{\tilde{S}}(G) \to L^1_{\tilde{S}}(G)$ be the natural embedding. Then a) $uI_{\tilde{S}} : L^1_{\tilde{S}}(G) \to H$ belongs to the Hilbert-Schmidt class; b) if $v : L^1_{\tilde{S}}(G) \to L^2_{\tilde{S}}(G)$ is a translation invariant bounded linear operator, then v is p-absolutely summing for all $p \in [0, 2]$.*

Proof. Since S is a Sidon set, there exists a $\mu \in M(G)$ such that

$$|\hat{\mu}(\gamma)| \geq 1 \quad \text{for } \gamma \in \tilde{S} \text{ and } \hat{\mu}(\gamma) = 0 \text{ for } \gamma \in S. \tag{5.1}$$

The existence of a $\mu \in M(G)$ satisfying (5.1) is a consequence of Drury's Lemma [10] (cf. also [15] and [14, Chapt. 2]).

For a) consider the factorization $I_{\tilde{S}} = M_\mu I i \hat{\mu}_{\tilde{S}}^{-1}$, i.e.

$$I_{\tilde{S}} : L^2_{\tilde{S}}(G) \xrightarrow{\hat{\mu}_{\tilde{S}}^{-1}} L^2_{\tilde{S}}(G) \xrightarrow{i} L^2(G) \xrightarrow{I} L^1(G) \xrightarrow{M_\mu} L^1_{\tilde{S}}(G)$$

where $\hat{\mu}_{\tilde{S}}^{-1}$ is the diagonal operator in the character basis $(\gamma)_{\gamma\in\tilde{S}}$ taking γ into $\hat{\mu}(\gamma)^{-1} \cdot \gamma$ for $\gamma \in \tilde{S}$; i the natural isometrically isomorphic inclusion; I the natural embedding; M_μ the operator of convolution with μ. Thus $uI_{\tilde{S}}$ factors through $L^1(G)$ and the desired conclusion follows from (G) of Sect. 3.

For b), note that the assumption on v implies that $v(\gamma) = \alpha_\gamma \gamma$ and for $\gamma \in \tilde{S}$ for some scalar function $(\alpha_\gamma)_{\gamma\in\tilde{S}}$. By a), $vI_{\tilde{S}}$ is in the Hilbert Schmidt class, hence $\sum_{\gamma\in\tilde{S}} |\alpha_\gamma|^2 = \sum_{\gamma\in\tilde{S}} \|v(\gamma)\|_2^2 < +\infty$. Thus $\tilde{v} : L^1(G) \to L^2_{\tilde{S}}(G)$ defined by $\tilde{v}(\gamma) = \alpha_\gamma\gamma$ for $\gamma \in \tilde{S}$ and $\tilde{v}(\gamma) = 0$ for $\gamma \in S$ is a bounded linear operator and we have the factorization $v = \tilde{v}i$ where $i : L^1_{\tilde{S}}(G) \to L^1(G)$ denotes the natural isometrically isomorphic inclusion. The desired conclusion again follows from (G) of Sect. 3. □

Our last result shows that if S is a Sidon set then the space $L^1_{\tilde{S}}(G)$ shares an important property of L^1 spaces.

Recall that a linear operator $T : X \to Y$ is called Dunford-Pettis or completely continuous if it takes weakly compact sets in X into norm compact sets. A Banach space X is said to have the Dunford-Pettis property provided every weakly compact operator from X is Dunford-Pettis.

Theorem 5.1. *Let S be a Sidon set. Then $L^1_{\tilde{S}}(G)$ has the Dunford-Pettis property.*

Remark. For the special case where S consists of the Rademachers, this is stated without proof by Bourgain [2].

The proof is based upon some properties of the Lions-Peetre K-functional interpolating between the ℓ^1 and ℓ^2 norms on Γ.

Let $\Psi \in L^2(G)$. Let (γ_m) be an enumeration of *supp* $\widehat{\Psi} = \{\gamma \in \Gamma : \widehat{\Psi}(\gamma) \neq 0\}$ into a sequence such that the sequence $(|\widehat{\Psi}(\gamma_m)|)$ is non-increasing. Put

$$K_{1,2}(\widehat{\Psi}, t) = \sum_{1\leq m\leq t^2} |\widehat{\Psi}(\gamma_m)| + t \left(\sum_{m>t^2} |\widehat{\Psi}(\gamma_m)|^2 \right)^{\frac{1}{2}} \quad \text{for } t > 0. \tag{5.2}$$

If the set *supp* $\widehat{\Psi}$ is finite, say it has m_0 elements, then the right hand side of (5.2) is understood to be equal $\sum_{m=1}^{m_0} |\widehat{\Psi}(\gamma_m)|$ whenever $t^2 \geq m_0$.

The qualitative version of the following deep result of Asmar and Montgomery-Smith ([1, Theorem 3.9]) plays the crucial role in our proof

(AM) *Let $S \subset \Gamma$ be a Sidon set. Then there is a constant $c > 0$ depending only on S so that if $\Psi \in L^2_S(G)$, then $\lambda\{|\Psi| \geq c^{-1}K_{1,2}(\widehat{\Psi}, t)\} \geq c^{-1}e^{-ct^2}$ for all $t > 0$.*

We apply the qualitative analogue of (AM) via a "correction lemma" which is stated next.

Lemma 5.1. *Let $S \subset \Gamma$. Assume that there exist $c > 0$ and a strictly decreasing function $a : (0, +\infty) \to (0, +\infty)$ with $\lim_{t\to\infty} a(t) = 0$ such that*

$$\lambda\{|\Psi| \geq c^{-1}K_{1,2}(\widehat{\Psi}, t)\} \geq a(t) \qquad (t > 0, \Psi \in L^2_S(G)).$$

Let $\varepsilon_0 = \lim_{t\to 0} \sqrt{a(t)}$. Then there exists a function $\delta : (0, \varepsilon_0) \to (0, +\infty)$ with $\lim_{\eta\to 0} \delta(\eta) = 0$ such that if $\phi \in L^\infty(G)$ and $\varepsilon \in (0, \varepsilon_0)$ satisfy

$$\sum_{\gamma\in\tilde{S}} |\hat{\phi}(\gamma)|^2 \leq \varepsilon^2 \|\phi\|^2_\infty, \tag{5.3}$$

then there exists $\varphi \in L^\infty(G)$ such that

$$\begin{array}{llll} (i) & \hat{\varphi}(\gamma) & = & \hat{\phi}(\gamma) \quad \textit{for } \gamma \in \widetilde{S}; \\ (ii) & \|\varphi\|_\infty & \leq & (2c+1)\|\phi\|_\infty; \\ (iii) & \|\varphi\|_2 & \leq & \|\phi\|_\infty \delta(\varepsilon). \end{array} \tag{5.4}$$

Proof. If $\phi \in L^\infty_S(G)$ put $\varphi = 0$. Let $\phi \notin L^\infty_S(G)$. Put $\Psi = \sum_{\gamma\in S} \hat{\phi}(\gamma)\gamma$. Clearly $\Psi \in L^2_S(G)$, $\phi \neq \Psi$ and (by (5.3))

$$\|\Psi - \phi\|^2_2 = \sum_{\gamma\in\widetilde{S}} |\hat{\phi}(\gamma)|^2 \leq \varepsilon^2 \|\phi\|^2_\infty. \tag{5.5}$$

If $\varepsilon \in (0, \varepsilon_0)$ then $\varepsilon^2 = a(t)$ for some $t \in (0, +\infty)$. Hence

$$\lambda\{|\Psi| \geq c^{-1}K_{1,2}(\hat{\Psi}, t)\} \geq \varepsilon^2. \tag{5.6}$$

On the other hand, combining (5.5) with the inclusion

$$\{|\Psi| \geq 2\|\phi\|_\infty\} \subset \{|\Psi - \phi| \geq \|\phi\|_\infty\}$$

and with a weak type estimate

$$\lambda\{|\Psi - \phi| \geq \|\phi\|_\infty\} \leq \|\Psi - \phi\|^2_2 \cdot \|\phi\|^{-2}_\infty$$

we obtain

$$\lambda\{|\Psi| \geq 2\|\phi\|_\infty\} \leq \varepsilon^2. \tag{5.7}$$

Moreover, the equality in (5.7) implies $|\Psi| = 2\|\phi\|_\infty 1_{\{|\Psi|\geq 2\|\phi\|_\infty\}}$ because the equality in the weak type estimate implies $|\Psi - \phi| = 1_{\{|\Psi-\phi|\geq\|\phi\|_\infty\}}$.

The inequalities (5.6), (5.7) and the "moreover" remark imply

$$K_{1,2}(\widehat{\Psi}, t) \leq 2c\|\phi\|_\infty. \tag{5.8}$$

We put $\varphi = \phi - \sum_{1\leq m\leq t^2} \widehat{\Psi}(\gamma_m)\gamma_m$. Then (5.4) (i) is obvious. Since

$$\|\varphi\|_\infty \leq \|\phi\|_\infty + \sum_{1\leq m\leq t^2} |\widehat{\Psi}(\gamma_m)| \leq \|\phi\|_\infty + K_{1,2}(\widehat{\Psi}, t),$$

the inequality (5.8) implies (5.4) (ii).

Let $b : (0, \varepsilon_0) \to (0, +\infty)$ be the inverse function of the function $\sqrt{a}$. In particular $b(\varepsilon) = t$. Define $\delta : (0, \varepsilon_0) \to (0, +\infty)$ by

$$\delta(\eta)^2 = \eta^2 + 4c^2[b(\eta)]^{-2} \quad \text{for } \eta \in (0, \varepsilon_0).$$

Clearly $\lim_{\eta\to 0} \delta(\eta) = 0$ because $\lim_{\eta\to 0} b(\eta) = +\infty$. We have

$$\begin{aligned}
\|\varphi\|_2^2 &= \|\phi - \Psi\|_2^2 + \sum_{m>t^2} |\hat{\phi}(\gamma_m)|^2 \\
&\leq \varepsilon^2\|\phi\|_\infty^2 + t^{-2}[K_{1,2}(\widehat{\Psi}, t)]^2 \quad \text{(by (5.2) and (5.5))} \\
&\leq (\varepsilon^2 + 4c^2[b(\varepsilon)]^{-2})\|\phi\|_\infty^2 \quad \text{(by (5.8))} \\
&= (\delta(\varepsilon)\|\phi\|_\infty)^2
\end{aligned}$$

which verifies (5.4) (iii). □

Let $B_{\tilde{S},2}$ denote the closed unit ball of $L^2_{\tilde{S}}(G)$. Since $B_{\tilde{S},2}$ is a weakly compact subset of $L^1_{\tilde{S}}(G)$, every Dunford-Pettis operator from $L^1_{\tilde{S}}(G)$ maps $B_{\tilde{S},2}$ into a norm compact set. Conversely one has

Lemma 5.2. *Let $S \subset \Gamma$ satisfy the assertion of Lemma 5.1. Let $T : L^1_{\tilde{S}}(G) \to Y$ (Y an arbitrary normed space) be a bounded linear operator such that $T(B_{\tilde{S},2})$ is norm compact. Then T is Dunford-Pettis.*

Proof. Let $K \subset L^1_{\tilde{S}}(G)$ be a weakly compact set. Then

$$\sup\{\|h\|_1 : h \in K\} = A < +\infty,$$

and K is uniformly absolutely integrable, in particular, there is a function $\omega : (0, +\infty) \to (0, +\infty)$ such that

$$\begin{aligned}
\lim_{\alpha\to\infty} \omega(\alpha) &= 0, \\
\|h - h^\alpha\|_1 &\leq \omega(\alpha) \quad \text{for } \alpha > 0 \text{ and } h \in K,
\end{aligned} \tag{5.9}$$

where $h^\alpha = h \cdot 1_{\{|h|\leq\alpha\}}$.

We shall show that $T(K)$ is a norm totally bounded set, hence it is norm compact. To this end it suffices to verify

$$\begin{aligned}
&\text{for every } \sigma > 0 \text{ there exists } \varepsilon > 0 \text{ such that} \\
&T(K) \subset \sigma B_Y + T\left(\tfrac{2A}{\varepsilon} B_{\tilde{S},2}\right)
\end{aligned} \tag{5.10}$$

(B_Y denotes the unit ball of Y).

Fix $f \in K$. Let $\varepsilon \in (0, \varepsilon_0)$ be given (ε_0 is that of Lemma 5.1). We shall choose ε for σ later on. Put

$$\rho = \inf\{\|T(f) - T\left(\frac{2A}{\varepsilon} g\right)\| : g \in B_{\tilde{S},2}\}.$$

By the Separation Theorem ([11, Theorem V.2.12]) there exists a linear functional $x^* \in Y^*$ such that

$$\rho = \sup\{\operatorname{Re} x^*(y) : \|y\| \le \rho\} = \inf\{\operatorname{Re} x^*(T(f) - T\left(\frac{2A}{\varepsilon} g\right)) : g \in B_{\tilde{S},2}\}.$$

Clearly $\|x^*\| = 1$. Let $\phi^* = T^*(x^*) \in (L^1_{\tilde{S}}(G))^*$ and let $\phi \in L^\infty(G) = (L^1(G))^*$ be a norm preserving extension of ϕ^*. Then

$$\begin{aligned} \int_G \phi h\,dx &= x^*(T(h)) \quad \text{for } h \in L^1_{\tilde{S}}(G), \\ \|\phi\|_\infty &\le \|T\|. \end{aligned}$$

Since $0 \in B_{\tilde{S},2}$, we have

$$\rho \le \left|\int_G \phi f\,dx\right| \le A\|\phi\|_\infty. \tag{5.11}$$

Thus, if $g \in B_{\tilde{S},2}$ then

$$\operatorname{Re}\int \phi \frac{2A}{\varepsilon} g\,dx \ge -\rho + \operatorname{Re}\int \phi f\,dx \ge -2A\|\phi\|_\infty.$$

Consequently, taking into account that $B_{\tilde{S},2}$ is a circled set

$$\left|\int \phi g\,dx\right| \le \varepsilon\|\phi\|_\infty \quad \text{for } g \in B_{\tilde{S},2}.$$

The latter inequality implies (5.3).

Now, given $\sigma > 0$ invoking (5.9), one picks $\alpha > 0$ so that

$$\|f - f^a\|_1 \le \sigma(2(2c+1)(\|T\|+1))^{-1} \quad \text{for } f \in K.$$

Next we choose $\varepsilon \in (0, \varepsilon_0)$ so that $\delta(\varepsilon) \le \sigma(2\sqrt{\alpha}(\|T\|+1))^{-1}$ where $\delta(\cdot)$ is the function of Lemma 5.1. Finally, for ϕ which has been constructed for fixed $f \in K$ and ε just chosen (ϕ satisfies (5.3) and $\|\phi\|_\infty \le \|T\|$) we apply Lemma 5.1 to construct φ satisfying (5.4). Then

$$\begin{aligned} \left|\int \phi f\,dx\right| &= \left|\int \varphi f\,dx\right| \qquad \text{(by (i) because } f \in K \subset L^1_{\tilde{S}}(G)) \\ &\le \left|\int_G \varphi(f - f^\alpha)dx\right| + \left|\int_G \varphi f^\alpha\,dx\right| \\ &\le \|\varphi\|_\infty \|f - f^\alpha\|_1 + \|\varphi\|_2\sqrt{\alpha} \\ &\le \sigma \qquad \text{(by (ii) and (iii)).} \end{aligned}$$

Thus, by (5.11), $\rho \le \sigma$. Hence $T(f) \in \sigma B_Y + T(\frac{2A}{\varepsilon} B_{\tilde{S},2})$ which yields (5.9). □

Lemma 5.3. *Assume that for some $S \subset \Gamma$ there exists a $\mu \in M(G)$ satisfying (5.1). Let T be a weakly compact operator from $L^1_{\tilde{S}}(G)$. Then $T(B_{\tilde{S},2})$ is a norm compact set.*

Remark. Note that the condition "$T(B_{\tilde{S},2})$ norm compact" is equivalent to "the operator $TI_{|L^2_{\tilde{S}}(G)}$ is compact" where $I : L^2(G) \to L^1(G)$ is the natural embedding.

Proof. Let $M_\mu : L^1(G) \to L^1_{\tilde{S}}(G)$ be the operator of convolution with μ (by (5.1), $M_\mu(L^1(G)) \subset L^1_{\tilde{S}}(G)$). Then the set $TM_\mu(B_{\tilde{S},2})$ is norm compact because $L^1(G)$ has the Dunford-Pettis property ([11, Theorem VI.8.12]). Then $T(B_{\tilde{S},2})$ is norm compact because if M_μ is regarded as an operator from $L^2(G)$ into $L^2_{\tilde{S}}(G)$ then the restriction $M_{\mu|L^2_{\tilde{S}}(G)}$ is, by (5.1), invertible. □

Proof of Theorem 5.1. If $S \subset \Gamma$ is a Sidon set, then, by [1], S satisfies the assumption of Lemma 5.1, while by the result of Drury [10] mentioned above, S satisfies the assumption of Lemma 5.3. Now Theorem 5.1 follows directly from Lemmas 5.1–5.3. □

Remark. We do not know if a subspace X of L_1 has the Dunford-Pettis property whenever L_1/X is isomorphic to c_0.

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