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On the characterization of Grauert tubes covered by the ball

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1 Introduction

Let X^n be a real analytic manifold, a natural way to complexify it is by thicking each coordinate patch $U \subset \mathbb{R}^n$ to $\mathbb{C}U \subset \mathbb{C}^n$. This process makes $\mathbb{C}X$ a complex manifold since the coordinate changes of X are real analytic maps, they can be complexified as holomorphic transition functions. X is then a maximal totally real submanifold of $\mathbb{C}X$. Suppose now that X is equipped with a Riemannian metric, one interesting problem is to know how to associate $\mathbb{C}X$ with a Kähler metric such that the corresponding Kähler structure is canonically defined. Since all Kähler forms come from taking derivatives with respect to strictly plurisubharmonic functions, it is kind of natural to think about the strictly plurisubharmonic functions that Grauert had discovered on this complexification process: given compact real analytic manifold X, there exists a neighborhood M of X in $\mathbb{C}X$ and a smooth strictly plurisubharmonic function $\rho : M \to [0, 1)$ such that X is the zero set of ρ. This ρ is clearly not uniquely defined, since $c\rho$ and $e^\rho - 1$ keep the positivity and the strictly plurisubharmonicity for any positive c. Recently, Lempert and Szöke (and independently Guillemin and Stenzel) put two extra datum to assert the uniqueness of such a ρ. They first asked that the Kähler metric induced by the Kähler form $\frac{i}{2}\partial\bar\partial\rho$ coincides with the original Riemannian metric g when restricted to X, and $\sqrt{\rho}$ satisfies the complex homogeneous Monge-Ampère equation $(\partial\bar\partial\sqrt{\rho})^n = 0$ on $M - X$. Such a ρ is uniquely defined for any given real analytic compact Riemannian manifold (cf. [L-S]), we view this as the canonical way to complexify a Riemannian manifolds and call the set $X^r_{\mathbb{C}} = \{\rho < r^2\}$ as the Grauert tube of radius r over center $X = \{\rho = 0\}$. The

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author has shown in [K], by computing a global CR invariant on boundaries of Grauert tubes, that $X_{\mathbb{C}}^{r_1}$ is not biholomorphic to $X_{\mathbb{C}}^{r_2}$ for different r_1 and r_2.

From the construction, it is easy to see that three elements: the center X, the Riemannian metric of the center and the radius of the tube determine the Grauert tube uniquely. Naturally, giving two isometric Riemannian manifolds (X_1, g_1) and (X_2, g_2), we expect that the two Grauert tubes of the same radius constructed over these two centers respectively are biholomorphically equivalent, which was proved by Lempert and Szöke in [L-S]. Burns [B] proved the other direction of the theorem, namely, if two Grauert tubes of the same radius are biholomorphically equivalent then their corresponding centers are isometrically the same. These two theorems provide some kind of uniqueness of Grauert tubes. However, due to the strong symmetry of the tube, we expect that for a Grauert tube of fixed radius, any two centers are not only isometrically equivalent but also identically the same as point sets. More precisely, the uniqueness problem we are interested in is the following. Let $\Omega = X_{\mathbb{C}}^r$ be a Grauert tube of some real analytic compact Riemannian manifold (X, g) of radius $r < \infty$. Could Ω be the Grauert tube $Y_{\mathbb{C}}^r$ of another compact Riemannian manifold (Y, h) of radius r? (The reason we keep the same radius is because rescaling the original metric will give us a trivial example of nonuniqueness.) If not, this uniqueness property shall tell us more about the behavior of the isometry group of X and the automorphism group of $X_{\mathbb{C}}^r$, namely, the center is unique if and only if Isom(X) is isomorphic to Aut(Ω), and therefore the rigidity of Grauert tubes. The answer to this problem is in general unknown except for the homogeneous cases proved by Burns in [B]. Burns proved that if $X_{\mathbb{C}}^r$ is the Grauert tube constructed over a homogeneous Riemannian manifold X of finite radius r, then this X is the only possible center we could find inside this Grauert tube $\Omega = X_{\mathbb{C}}^r$.

In this paper, we would like to show that the uniqueness statement holds for some special Grauert tubes–Grauert tubes covered by the unit ball. We show further that there is only one way to obtain such Grauert tubes: all Grauert tubes covered by B^n come from the complexification of real compact hyperbolic space. We state the main result as following:

Theorem. *Let $\Omega = X_{\mathbb{C}}^r$ be a Grauert tube covered by the unit ball B^n, then $X = H^n/\Gamma$ for some discrete subgroup Γ of $O(n, 1)$ and $\Omega = B^n/\Gamma$.*

Remark. Grauert tubes covered by the ball are Stein manifolds with compact spherical boundary. On the other hand, D. Burns informed me he has a proof that Stein manifolds with compact spherical boundary are in fact covered by the unit ball. In this case, our result would imply that: Let $\Omega = X_{\mathbb{C}}^r$ be a spherical Grauert tube, then $X = H^n/\Gamma$ for some discrete subgroup Γ of $O(n, 1)$ and $\Omega = B^n/\Gamma$.

At the end, we give a couple of examples to assert that the compactness of the center and the finiteness of the radius are essential to the above theorem.

2 Some properties of B^n and Grauert tubes covered by the ball

A standard way to realize the complex unit ball in $\mathbb{C}^n$ is by writing

$$B^n = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^2 < 1\}.$$

The Bergman metric g_B of B^n has constant sectional curvature -1 at all points. We first examine some interesting properties, important to us later, of antiholomorphic involutions of the unit ball.

Proposition 2.1. *Let τ be an antiholomorphic involution of the unit ball B^n. There exists $f \in Aut(B^n)$ such that $f\tau f^{-1} = \sigma$, the standard antiholomorphic involution of B^n fixing $B^n \cap \mathbb{R}^n$. The fixed point set X of τ is an n-dimensional totally real submanifold, such that $f(X) = B^n \cap \mathbb{R}^n$.*

Proof. We first show that τ has a fixed point inside B^n. Pick $p \in B^n$, if p is not a fixed point of τ, then $p \neq \tau(p)$. Since (B^n, g_B) has negative sectional curvature, there exist a unique geodesic $\gamma(t)$ joining p and $\tau(p)$.

$$\gamma(0) = p, \qquad \gamma(1) = \tau(p).$$

As every antiholomorphic map of B^n is an isometry of (B^n, g_B), the curve $\tau(\gamma(t))$ is a geodesic as well.

$$\tau(\gamma(0)) = \tau(p), \qquad \tau(\gamma(1)) = \tau(\tau(p)) = p.$$

By the uniqueness of geodesic through two points in a manifold of nonpositive sectional curvature, $\tau(\gamma(t))$ is simply a reparametrization of $\gamma(t)$.

$$\tau(\gamma(t)) = \gamma(-t + 1).$$

The point $\gamma(\frac{1}{2}) \equiv q \in B^n$ is then fixed by τ.

For any given $q \in B^n$, there exists a $\phi \in Aut(B^n)$ exchanging q and the origin. $\phi \cdot \tau \cdot \phi^{-1}$ is an antiholomorphic involution of B^n.

$$\phi \cdot \tau \cdot \phi^{-1}(0) = \phi \cdot \tau(q) = \phi(q) = 0.$$

$\overline{\phi \cdot \tau \cdot \phi^{-1}}$ is a biholomorphic map of B^n, fixing the origin. By a classical result of Cartan, $\overline{\phi \cdot \tau \cdot \phi^{-1}}$ is a linear transformation, *i.e.*, there exist a matrix $U \in U(n)$, such that

$$\phi \cdot \tau \cdot \phi^{-1}(z) = \overline{U(z)} = \overline{U}(\overline{z}), \quad \forall z \in B^n. \tag{2.1}$$

$\phi \cdot \tau \cdot \phi^{-1}$ is an involution implies $\overline{U}U(z) = z, \ \forall z \in B^n$, *i.e.*,

$$\overline{U}U = I_{n\times n}. \tag{2.2}$$

On the other hand, every unitary matrix could be diagonalized.

That is to say that there exists $B \in U(n)$ such that

$$BUB^{-1} = \begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{pmatrix} = D, \tag{2.3}$$

a diagonal matrix, where $\theta_j \in [0, 2\pi], j = 0, \cdots, n$.

Denoting $G = \overline{B}B^{-1} = (g_{kl}) \in U(n)$, the condition $\overline{U}U = I$ holds if and only if

$$\overline{D}GD = G. \tag{2.4}$$

It is not hard to examine the matrix G more closely, since D is a relatively simple matrix, a diagonal one. In terms of the notation (g_{kl}), there are two possible cases by solving (2.4).

(1) $\theta_k = \theta_l,\ \forall k, l$. Then $D = e^{i\theta} I_{n\times n} = U$, for some $\theta \in [0, 2\pi]$.
(2) $g_{kl} = e^{i\eta_k}\delta_{kl}$ for some $\eta_k \in [0, 2\pi]$,*i.e.*, G is a diagonal matrix.

Notice that, the standard antiholomorphic involution of the unit ball is $\sigma(z) = \overline{z}$. In the first case,

$$\begin{aligned} \phi \cdot \tau \cdot \phi^{-1}(z) &= \overline{U}(\overline{z}) = \overline{U}(\sigma(z)) \\ &= e^{-i\theta} I_{n\times n}\sigma(z) \\ &= e^{\frac{-i\theta}{2}} I_{n\times n}\sigma(e^{\frac{i\theta}{2}} z) \end{aligned}$$

Let Z $= e^{\frac{i\theta}{2}} z$, then

$$\begin{aligned} e^{\frac{i\theta}{2}}\phi \cdot \tau \cdot \phi^{-1}(e^{\frac{-i\theta}{2}})Z &= \sigma(Z) \\ e^{\frac{i\theta}{2}}\phi \cdot \tau \cdot (e^{\frac{i\theta}{2}}\phi)^{-1}Z &= \sigma(Z) \end{aligned}$$

$f = e^{\frac{i\theta}{2}}\phi$ does the job.

In the second case, let

$$H = \begin{pmatrix} e^{\frac{-i(\eta_1-\theta_1)}{2}} & & & \\ & e^{\frac{-i(\eta_2-\theta_2)}{2}} & & \\ & & \ddots & \\ & & & e^{\frac{-i(\eta_n-\theta_n)}{2}} \end{pmatrix},$$

$$\overline{H}^{-1} = H.$$

Let $f = H\overline{B}\phi$, then

$$
\begin{aligned}
f\cdot\tau\cdot f^{-1}(z) &= H\overline{B}\phi\cdot\tau\cdot\phi^{-1}\overline{B}^{-1}H^{-1}(z)\\
&= H\overline{BU}\,\overline{(\overline{B}^{-1}H^{-1}(z))}\\
&= H\overline{BU}B^{-1}\overline{H}^{-1}\overline{z}\\
&= H\overline{BB}^{-1}\overline{DB}B^{-1}H\overline{z}\\
&= H\overline{D}GH\overline{z}\\
&= \overline{z} = \sigma(z).
\end{aligned}
$$

Since

$$
\begin{aligned}
f\cdot\tau\cdot f^{-1}(B^n\cap\mathbb{R}^n) &= \sigma(B^n\cap\mathbb{R}^n) = B^n\cap\mathbb{R}^n,\\
\tau\cdot f^{-1}(B^n\cap\mathbb{R}^n) &= f^{-1}(B^n\cap\mathbb{R}^n).
\end{aligned}
$$

$f^{-1}(B^n\cap\mathbb{R}^n)$ is then part of the fixed point set of τ. On the other hand, if y is not in $B^n\cap\mathbb{R}^n, f^{-1}(y)$ is fixed by τ. As

$$\sigma(y) = f\cdot\tau\cdot f^{-1}(y) = f\cdot f^{-1}(y) = y.$$

y is fixed by τ and hence is in $B^n\cap\mathbb{R}^n$, a contradiction. The fixed point set X of τ is exactly $f^{-1}(B^n\cap\mathbb{R}^n)$. Therefore, $f(X) = B^n\cap\mathbb{R}^{\ltimes}$. ☐

Remark. Antiholomorphism as well as involution is crucial in the above proposition. We give two examples here.

(1) For $0\neq|a|<1, \phi_a(z) = \frac{a-z}{1-\bar{a}z}$ is a holomorphic involution of B which fixes a single point $\frac{1-\sqrt{1+|a|^2}}{\bar{a}}$.

(2) $\psi_a(z) = \overline{\frac{a-z}{1-\bar{a}z}}$ is an antiholomorphic map of B which has no fixed point inside the unit ball. The fixed point set contains two boundary points $Re\ a \pm \sqrt{-1}\sqrt{1-(Re\ a)^2}$ only.

It is interesting that not all of the fixed point sets of antiholomorphic involutions of B^n have good symmetries, i.e. not all of them come from rotations of $B^n\cap\mathbb{R}^n$. A simple example is the following: $\phi_{\frac{1}{2}}\sigma\phi_{\frac{1}{2}}$ has $\phi_{\frac{1}{2}}(B^n\cap\mathbb{R}^n)$ as its fixed point set which stays away from the origin.

In order to describe the automorphism group of the unit ball, we need another way of viewing the ball. Let $Z = (Z_0,\cdots,Z_n)$ be the homogeneous coordinates for $\mathbb{C}P^n$, $\langle\ ,\ \rangle$ be the standard positive Hermitian inner product of $\mathbb{C}^{n+1}$. An $(n+1)\times(n+1)$ matrix A is in $U(n,1)$ if A satisfies $ACA^* = C$ where

$$C = \begin{pmatrix} -1 & 0\\ 0 & I_{n\times n}\end{pmatrix}.$$

Define an inner product $\langle\langle\ ,\ \rangle\rangle$ on $\mathbb{C}P^n$:

$$
\begin{aligned}
\langle\langle Z,Z\rangle\rangle &= \langle CZ,Z\rangle\\
&= -Z_0\overline{Z}_0 + Z_1\overline{Z}_1 + \cdots + Z_n\overline{Z}_n.
\end{aligned}
$$

We shall identify the unit ball B^n as

$$\begin{aligned} B^n &= \{Z \in \mathbb{C}P^n : -Z_0\overline{Z}_0 + Z_1\overline{Z}_1 + \cdots + Z_n\overline{Z}_n = 0\} \\ &= \{Z \in \mathbb{C}P^n : \langle\langle Z, Z\rangle\rangle = 0\}. \end{aligned}$$

For any $A \in U(n, 1)$,

$$\langle\langle AZ, AZ\rangle\rangle = \langle CAZ, AZ\rangle = \langle A^*CAZ, Z\rangle = \langle CZ, Z\rangle = \langle\langle Z, Z\rangle\rangle.$$

A is a unitary matrix with respect to this inner product $\langle\langle\ ,\ \rangle\rangle$ and keeps B^n invariant. Actually $PU(n, 1) \equiv U(n, 1)/center$, a non-compact Lie group, is the full group of biholomorphism of B^n.

Still, there is the third way to look at the ball. We consider the real hyperbolic space H^n as the unit ball in $\mathbb{R}^n$.

$$H^n = \{x \in \mathbb{R}^n : |x|^2 < 1\}$$

with the complete Riemannian metric

$$g = (n+1)\frac{(1-|x|^2)(\sum dx_i^2) + (\sum x_i dx_i)^2}{(1-|x|^2)^2}, \tag{2.5}$$

which has constant sectional curvature $\frac{-4}{(n+1)}$ when $n > 1$. Geodesics are straight line segments. The natural complexification of H^n is the unit ball $B^n = \{z \in \mathbb{C}^n : |z|^2 < 1\}$ in which H^n is contained as $B^n \cap \mathbb{R}^n$. The complexified metric turns out to be the Bergman metric of the ball

$$g_B = (n+1)\frac{(1-|z|^2)(\sum dz_i d\overline{z_i}) + (\sum \overline{z_i} dz_i)(\sum z_i d\overline{z_i})}{(1-|z|^2)^2}. \tag{2.6}$$

We usually call (B^n, g_B) the complex hyperbolic space. This metric has very good properties:

(1) The sectional curvature as well as holomorphic sectional curvature is $\frac{-4}{(n+1)}$.
(2) Biholomorphic maps and antiholomorphic maps of the ball act as isometries of (B^n, g_B)
(3) Giving $\eta_1, \eta_2 \in \partial B^n$, there exist a unique geodesic $\eta(t)$ with endpoints η_1 and η_2,

$$\begin{aligned} \eta(t) &= \frac{1+iw}{1+e^t+iw}\eta_1 + \frac{e^t}{1+e^t+iw}\eta_2, \quad t \in \mathbb{R}, \\ w &= \frac{Im < \eta_1, \eta_2 >}{1 - Re < \eta_1, \eta_2 >} \in \mathbb{R}. \end{aligned}$$

We also review some necessary background about Grauert tubes in this section. A Grauert tube $X_{\mathbb{C}}^r$ is a Stein manifold since we could take $-\log(r^2 - \rho)$ as an exhaustion function. Its automorphism group $\mathrm{Aut}(X_{\mathbb{C}}^r)$ is a compact Lie group (cf.[Sz],[M]). This shows us that the unit ball can't be a Grauert tube since its automorphism group is not compact.

More generally, we would like to consider those Grauert tubes everywhere locally like the ball. A connected real hypersurface M in a complex manifold

X is spherical if, at every point $p \in M$, there is a local holomorphic coordinate system $(z_1, \cdots, z_n)$ of X such that M is defined by

$$|z_1|^2 + \cdots + |z_n|^2 = 1.$$

Call a Grauert tube $\Omega = X^r_{\mathbb{C}}$ a spherical Grauert tube if its boundary $\partial\Omega = \{\rho = r\}$ is a spherical hypersurface.

In [K] and [L], the authors constructed Grauert tubes by taking quotient to the unit ball and the hyperbolic center. They are the only spherical Grauert tubes we could find so far. It is interesting to know whether this is the only possible case. We first notice that if a Grauert tube is covered by the unit ball then it must be a spherical Grauert tube since its boundary is just part of the sphere quotient. They are part of the family of Stein manifolds with compact spherical boundary.

The characterization of Stein manifolds with compact spherical boundaries has been well-known since 1976 by Burns and Shnider (cf.[B-S]). They proved that a Stein manifold with compact spherical boundary M is either the complex ball B^n or M has infinite fundamental group.

As Grauert tubes have connected boundaries, this theorem will imply that the fundamental group of Grauert tubes covered by the unit ball is infinite.

3 Proof of the theorem

Let $\Omega = X^1_{\mathbb{C}} = \{\rho < 1\}$ be a Grauert tube covered by the ball, τ_X be the corresponding antiholomorphic involution of Ω which has X as its fixed point set. We denote this tube as $(\Omega, X, \rho, \tau_X)$. Let $\partial\Omega = \{\rho = 1\}$ be the spherical hypersurface. The fundamental group Π, acting freely and properly discontinuously on B^n as a covering transformation, of Ω lifts X to a totally real n-dimensional submanifold $\tilde{X}$ of B^n; ρ to a non-negative strictly plurisubharmonic function $\tilde{\rho}$; τ to an antiholomorphic involution $\tilde{\tau}$ of B^n

$$\tilde{\rho}(z) = \rho([z]), \quad \forall z \in B^n.$$

The fixed point set of $\tilde{\tau}$ is exactly $\tilde{X}$, $\tilde{X} = \{z \in B^n : \tilde{\rho}(z) = 0\}$. The fundamental group Π of Ω lifts the Grauert tube to a Monge-Ampère model $\{B^n, \tilde{X}, \tilde{\rho}, \tilde{\tau}\}$, which is, roughly speaking, a Grauert tube of complete center.

Proposition 3.1. *Let $(\Omega, X, \rho, \tau_X)$ be a Grauert tube covered by the ball, then the fundamental group Π of Ω lifts the compact center X to a non-compact set $\tilde{X}$; the spherical boundary $\partial\Omega$ to $S^{2n-1} - S^{n-1}_X$, where $S^{n-1}_X = \partial\tilde{X}$ is a totally real $(n-1)$–sphere.*

Proof. The fundamental group Π of Ω lifts the Grauert tube $(\Omega, X, \rho, \tau_X)$ to $\{B^n, \tilde{X}, \tilde{\rho}, \tilde{\tau}\}$, by the discussion above, where $\tilde{\tau}$ is an antiholomorphic involution of B^n. Proposition 2.1 tells us, without loss of generality, we may assume

$$\tilde{X} = B^n \cap \mathbb{R}^n, \quad \tilde{\tau}(z) = \overline{z}$$
$$\partial\tilde{X} = \partial(B^n \cap \mathbb{R}^n) = S^{n-1}, \quad \text{the real } (n-1)\text{– sphere.}$$

Let $\Omega_r = \{\rho < r\}$. The fundamental group Π lifts Ω_r to a strictly pseudoconvex domain $\tilde{\Omega}_r$ in B^n.

$$\tilde{\Omega}_r = \{0 \le \tilde{\rho} < r\} \subset B^n, \quad \forall 0 < r < 1.$$

As $B^n = \{\tilde{\rho} < 1\}$,

$$\tilde{\Omega}_r \subset \tilde{\Omega}_s \subset B^n, \quad 0 < r < s < 1.$$

$\{\tilde{\Omega}_r\}$ is actually a family of strictly pseudoconvex domains exhausting B^n; each $\tilde{\Omega}_r$ contains $B^n \cap \mathbb{R}^n$ as a subset. As $\tilde{\Omega}_r = \{0 \le \tilde{\rho} < r\}$ and $\tilde{X} = \{\tilde{\rho} = 0\}$,

$$\begin{aligned} \partial\tilde{\Omega}_r &= \{\tilde{\rho} = r\} \cup \partial\{\tilde{\rho} = 0\} \\ &= \{\tilde{\rho} = r\} \cup S^{n-1}. \end{aligned}$$

where $\{\tilde{\rho} = r\} = \Pi(\{\rho = r\})$ is a $(2n-1)-$dimensional hypersurface approaching the boundary of the unit ball when r goes to 1. Therefore,

$$\{\tilde{\rho} = 1\} = \partial B^n - S^{n-1} = S^{2n-1} - S^{n-1}.$$

So,

$$\Pi(\partial\Omega) = \Pi(\{\rho = 1\}) = \{\tilde{\rho} = 1\} = S^{2n-1} - S^{n-1}. \qquad \square$$

Let $(\Omega, X, \rho, \tau_X)$ be a Grauert tube covered by the ball, suppose there exists another center Y such that $(\Omega, Y, \varphi, \tau_Y)$ is a Grauert tube, and

$$\begin{aligned} \Omega &= \{\rho < 1\} = \{\varphi < 1\}, \\ \partial\Omega &= \{\rho = 1\} = \{\varphi = 1\}. \end{aligned}$$

At one hand, the fundamental group Π of Ω lifts $\partial\Omega$ to $S^{2n-1} - \partial\tilde{X}$. On the other hand, Π lifts $\partial\Omega$ to $S^{2n-1} - \partial\tilde{Y}$. We conclude that: above on the universal covering B^n of Ω, both $\tilde{X}$ and $\tilde{Y}$ share the same boundary,

$$\partial\tilde{X} = \partial\tilde{Y} = \text{ an } (n-1)\text{-circle}. \tag{3.1}$$

We would like to show that $\tilde{X}$ and $\tilde{Y}$ are actually the same point set. For this purpose, we need the help of some nice metric on the ball. Equip the ball with the Bergman metric g_B on (2.6).

$\tilde{X}$ is the fixed point sets of the antiholomorphic maps(hence isometry). $\tilde{X}$ with the induced metric is then a totally geodesic submanifolds of (B^n, g_B), this means every geodesic of $\tilde{X}$ with respect to the induced metric is a geodesic of (B^n, g_B). Moreover, giving any two points $p, q \in \partial\tilde{X}$, there exists a unique geodesic lying on $\tilde{X}$ with endpoints p and q. Similarly, the situation hold for $\tilde{Y}$ with the induced metric.

Since $\partial\tilde{X} = \partial\tilde{Y}$, we choose $p, q \in \partial\tilde{X} \cap \partial\tilde{Y}$, a geodesic γ_X on $\tilde{X}$ and a geodesic γ_Y on $\tilde{Y}$ ending at the same points p and q. Both γ_X and γ_Y are geodesics of (B^n, g_B), simply because $\tilde{X}$ and $\tilde{Y}$ are totally geodesic submanifolds. $\gamma_X = \gamma_Y$ then follows from the uniqueness of the geodesics. Notice that all points of $\tilde{X}$ and $\tilde{Y}$ will be covered by such kind of geodesics. We conclude that $\tilde{X} = \tilde{Y}$.

By the construction, $X = \tilde{X}/\Pi = \tilde{Y}/\Pi = Y$, which proves that the center of a spherical Grauert tube is unique.

A simple application is the following corollary:

Corollary. *Let* $f \in Aut(B^n) = PU(n,1)$, *fixing* $\partial(B^n \cap \mathbb{R}^n)$. *Then* f *sends* $B^n \cap \mathbb{R}^n$ *to itself, i.e.,* $f \in O(n,1)$.

Let $(B, \tilde{X}, \tilde{\rho}, \tilde{\tau})$ be the universal lifting of a Grauert tube (Ω, X, ρ, τ), f be the biholomorphic map of B^n sending $\tilde{X}$ to $B^n \cap \mathbb{R}^n$. The Kähler metric induced by the Kähler form $\tilde{\omega} = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\tilde{\rho}$ is well-defined since $\tilde{\rho}$ is Π invariant. We could take the target space as

$$(B^n, f(\tilde{X}), \tilde{\rho}f^{-1}, f\tilde{\tau}f^{-1}) = (B^n, B^n \cap \mathbb{R}^n, \tilde{\rho}f^{-1}, \sigma).$$

Let Π' be the conjugate group $\Pi' = f\Pi f^{-1}$.

$$(B^n/\Pi', (B^n \cap \mathbb{R}^n)/\Pi', \tilde{\rho}f^{-1}, \sigma) \tag{3.1}$$

is a Grauert tube of radius 1; the Riemannian metric of the center $B^n \cap \mathbb{R}^n/\Pi'$ is $\frac{i}{2}\partial\overline{\partial}(\tilde{\rho}f^{-1})|_{B^n\cap\mathbb{R}^n/\Pi'}$.

On the other hand,

$$(B^n/\Pi', H^n/\Pi', \varphi, \sigma) \tag{3.2}$$

is a Grauert tube of radius 1, where

$$\varphi(z) = \frac{4}{\pi}\tan^{-1}\tanh\ (\text{the Kobayashi distance of } z \text{ to } H^n)$$

is $O(n,1)$ invariant and the Kähler metric induced from the Kähler form $\frac{i}{2}\partial\overline{\partial}\varphi$ is the real hyperbolic metric when restricted to the center H^n/Π' (cf. [K],[L]).

Comparing (3.1) and (3.2), two Grauert tube structures of the same radius 1 are given in the same set B^n/Π'. Both of the strictly plurisubharmonic functions $\tilde{\rho}f^{-1}$ and φ satisfy the complex homogeneous Monge-Ampère equation on $B^n/\Pi' - (B^n \cap \mathbb{R}^n)/\Pi'$; are continuous up to the boundary of $B^n/\Pi' - (B^n \cap \mathbb{R}^n)/\Pi'$; and share the same values on boundary points. The maximal principle of Monge-Ampère equations (cf.[B-T]) confirms $\tilde{\rho}f^{-1}$ and φ are identically the same on the whole tube B^n/Π'. We therefore conclude that the Kähler metric induced from the function $\tilde{\rho}f^{-1}$ coincides with the one induced from φ. The hyperbolic metric of the center and of the submanifold $(B^n \cap \mathbb{R}^n)/\Pi'$, as well as of the original manifold X must have negative curvature since H^n/Π' has constant curvature -1. This completes the proof of our main theorem. □

Finally, we would like to assert the essence of the compactness of centers and the finiteness of radii by examining the difference of Grauert tubes and Grauert tubes of non-compact centers.

(1) Instead of having only finitely many antiholomorphic involutions in Grauert tubes (cf.[B]), there are infinitely many anti-holomorphic involutions inside B^n. Some obvious examples are $\phi_a\sigma\phi_a$ for any a in $B^n \cap \mathbb{R}^n$.

(2) The uniqueness doesn't hold if we allow the centers of Grauert tubes to be non-compact. Taking $g \in U(n,1), g \notin O(n,1)$, this g shifts H^n to a totally real submanifold $g(H^n)$ such that $(B^n, g(H^n), \varphi g^{-1}, g\sigma g^{-1})$ is a Grauert tube of non-compact center different from $(B^n, H^n, \varphi, \sigma)$.

(3) Centers of spherical Grauert tubes are not necessarily hyperbolic if we allow the radii of centers are infinite. One example could been found in [K], where the author constructed a spherical Grauert tube of infinite radius above the unit sphere of constant curvature 1.

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References

[B-T] Bedford, E.,Taylor, B.A.: The Dirichlet problem for a complex Monge-Ampère equation. Invent. Math **37**,1-44 (1976)

[B] Burns, D.: On the uniqueness and characterization of Grauert tubes. In: Lecture Notes in Pure and Applied Math. 119-133 (1995)

[B-S] Burns, D., Shnider, S.: Spherical hypersurfaces in complex manifolds. Invent. Math. **33**, 223-246 (1976)

[G] Goldman, W.: Introduction to complex hyperbolic geometry. Lecture Note. University of Maryland (1990)

[Ja] Jacobowitz, H.: An introduction to CR structures. Providence: Am. Math. Soc. 1990

[K] Kan, S-J.: The asymptotic expansion of a CR invariant and Grauert tubes. Math. Ann **304**, 63-92 (1996)

[L] Lempert, L.: Elliptic and hyperbolic tubes. In: Proceedings of the Special Year in Complex Analysis at the Mittag-Leffler Institute. Princeton: Princeton University Press 1993

[L-S] Lempert, L., Szöke, R.: Global solutions of the homogeneous complex Monge-Ampère equations and complex structures on the tangent bundle of Riemannian manifolds. Math. Ann. **290**, 689-712 (1991)

[M] Mok, N.: Rigidity of holomorphic self-mappings and the automorphism groups of hyperbolic Stein spaces. Math. Ann. **266**, 433-447, (1984)

[Sz] Szöke, R.: Automorphisms of certain Stein manifolds. Math. Z. **219** (3), 357-385 (1995)