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An infinite collection of Heegaard splittings that are equivalent after one stabilization

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1. Preliminaries

Throughout this paper all three-manifolds and surfaces will be assumed to be compact, orientable and connected.

A *complete collection of compressing disks* for a three-manifold M is a collection of properly embedded disks Δ such that M cut along Δ is a collection of 3-balls.

A *handlebody* H is a homeomorph of a regular neighborhood of a connected graph in S^3. The image of the graph, Σ, is called the *spine of the handlebody* H. Equivalently, a handlebody is a three-manifold which has a complete collection of compressing disks.

Let F be a surface with a single boundary component which is embedded in a three-manifold M. Let Σ be a graph embedded in the surface F such that F cut along Σ is an annulus. Then we call Σ a *spine of the surface* F . Note that as a regular neighborhood of F, $N(F)$, is a handlebody which retracts onto F which in turn retracts onto Σ, Σ is also a spine for the handlebody $N(F)$.

A *Heegaard splitting* of a three-manifold M is a decomposition, $M = H_1 \cup_F H_2$, where H_1 and H_2 are handlebodies such that $F = \partial H_1 = \partial H_2 = H_1 \cap H_2$. The *genus of the splitting* is the genus of the surface F.

Note that the spine Σ of the handlebody H_1 determines a Heegaard splitting $M = N(\Sigma) \cup N(\Sigma)^C$ which is isotopic to the original splitting, $M = H_1 \cup_F H_2$. As we are only interested in Heegaard splittings up to isotopy and the spine Σ represents this splitting, we will call Σ a *spine of the Heegaard splitting* $M = H_1 \cup_F H_2$. Also note that an ambient isotopy of Σ or manipulating Σ by

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edgeslides (see [ST]) does not change the isotopy class of the Heegaard splitting. Thus if Σ can be manipulated to Σ' through isotopies and edgeslides we will write $\Sigma \sim \Sigma'$, and consider the two to be equivalent. As Heegaard splittings determined by the handlebody, the spine of the handlebody, and the neighborhood of the spine are equivalent we will use these designations interchangeably.

Call $H_1 \cup N(\alpha)$ a *stabilization of the Heegaard splitting* $M = H_1 \cup_F H_2$ if α is an arc properly embedded in H_2 that is parallel to an arc on F. This stabilized splitting is represented by a *stabilization of the spine* $\Sigma \cup \beta$, where β is an arc properly embedded in Σ^C and parallel to an arc in Σ. As stabilization is unique up to isotopy of the handlebody, the splitting determined by stabilization is independent of the choice of α or β. Following the notation in [SC] the stabilized spine $\Sigma \cup \beta$ will be denoted $S(\Sigma)$. Denote $S(S(\Sigma))$ by $S^2(\Sigma)$ and so forth.

Let Σ and Σ' be spines of splittings of genus p and q respectively, $p \leq q$. We say that the splittings represented by Σ and Σ' are *equivalent after one stabilization* if $S^{q-p+1}(\Sigma) \sim S^1(\Sigma')$.

For more information regarding Heegaard splittings see [RS] or [ST2].

2. Twisting Surfaces

Let F be an orientable spanning surface for a knot K in S^3. If there exists an embedded sphere S in S^3 which intersects F in two non-separating arcs, α and β, we can form a new spanning surface for the knot K.

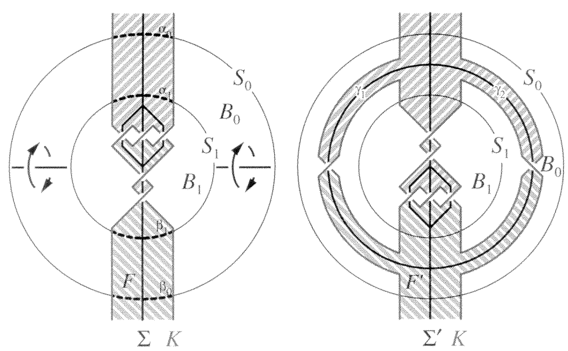

Fig. 1a,b. The original and twisted surfaces

First choose a homeomorphism h from S to the standard sphere S_0 such that α and β are sent to α_0 and β_0 as pictured in Fig. 1a. This homeomorphism must be chosen so that opposite faces of F are facing upwards at α_0 and β_0. Form the unique (up to isotopy) extension of h to the rest of S^3. This fills in the inside of S_0, a ball we will call B_0. Now form a sphere, S_1, parallel to and inside S_0. Let the ball inside S_1 be called B_1. We will consider all manipulations to occur in the original S^3 by conjugating changes we make with the homeomorphism h.

We now form the new surface. Sever F along α_1 and β_1. Rotate B_1 through 180° along a horizontal axis as indicated. Position the knot as indicated in Fig. 1b and form the new surface by attaching the piece of F inside B_1 to the piece of F on the outside of B_0 by using the new piece in $B_0 - B_1$ as indicated in the figure.

We have now twisted the surface F to a new surface F'. Note that the new surface is orientable as we required opposite faces of F to appear at α_0 and β_0. Also note that we have increased the genus of the surface by two.

We can choose a spine Σ for the surface F to intersect each of the curves α and β exactly once. A spine Σ' for F' is indicated in Fig. 1b. Σ' is assembled by flipping $\Sigma \cap B_1$ through 180°, reattaching it to $\Sigma \cap B_1^C$ and adding the arcs γ_1 and γ_2 which run along the twisted bands.

Note that the construction depended on the homeomorphism h for which there is no canonical choice. Any two such homeomorphisms will differ by an automorphism of S^2 that fixes $N(\alpha \cup \beta)$. As the group (of isotopy classes) of such automorphisms is isomorphic to $\mathbb{Z}$ we note that this process actually generates an infinite family of surfaces all of the same genus. Also, α and β are now non-separating arcs on F'. Thus, we can repeat the process in order to generate an infinite family of surfaces. Each of these surfaces will have genus two greater than its predecessor.

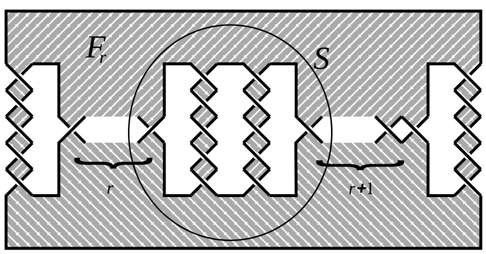

Fig. 2. The standard projection $(5, -1_1, .., -1_r, 5, 5, 1_1, .., 1_{r+1}, 5), r \geq 0$

It may appear that the above process is a trivial manipulation of the surface, but, the work of Parris demonstrates that this is not the case. Parris considers pretzel knots with a standard projection $(p_1, .., p_n)$ such that each p_i is odd and n is odd. The standard projection determines an obvious Seifert surface F for the given knot. Parris [PA] was able to give conditions to determine whether or not this surface is incompressible in the knot complement. For a specific set of examples, examine the pretzel knot projection $(p_1, p_2, p_3, 1, p_4)$ where $|p_i| \geq 5$. By twisting the surface according to the above process we obtain the projections $(p_1, -1_1, .., -1_r, p_2, p_3, 1_1, .., 1_{r+1}, p_4), r \geq 0$ (see Fig. 2). The conditions of Parris guarantee that, for each value of r, the surface F_r is incompressible in the knot complement. Thus, each of these knots possesses an infinite number of incompressible surfaces of arbitrarily high genus.

3. Forming Heegaard splittings

Fix a knot K and consider the knot complement $X = S^3 - N(K)$. Let F be a spanning surface for K. Then $N(F)$ is always a handlebody. Additionally assume that $N(F)^C$ is also a handlebody. Let $M = K(a/b)$ be the manifold obtained by an a/b Dehn filling on X. Call the filling solid torus T. We can think of the manifold as being decomposed into two pieces, $N(F)$ and $N(F)^C \cup T$, where T is glued to $N(F)^C$ along an annulus. It is well known (see for example [MS]) that after gluing T to $N(F)^C$ along an annulus, $N(F)^C \cup T$ will also be a handlebody if the filling slope is $1/q$ for some q.

Then, for any spanning surface F for which $N(F)^C$ is a handlebody we obtain a Heegaard splitting, $M = N(F) \cup (N(F)^C \cup T)$, of the manifold $M = K(1/q)$. Casson and Gordon ([CG],[KO]) proved that if F is an incompressible spanning surface that induces a Heegaard splitting, $M = N(F) \cup (N(F)^C \cup T)$, of $M = K(1/q)$ and $|q| \geq 6$ [MS], then the induced splitting is irreducible.

Let K be the pretzel knot with standard presentations $(p_1, -1_1, .., -1_r, p_2, p_3, 1_1, .., 1_{r+1}, p_4), r \geq 0, |p_i| \geq 5$. Then the surface F_r is incompressible and as seen in Fig. 2, $N(F_r)^C$ is a handlebody. Thus each of the manifolds $M = K(1/q)$ has an infinite number of irreducible Heegaard splittings of arbitrarily high genus.

We now examine the general case. Let F' be a surface obtained from a spanning surface F as in Sect. 2. We show that F' induces a Heegaard splitting if F does. A technical lemma is needed first.

Lemma 1. *Let H be a handlebody and A a separating annulus embedded in H. Let A separate H into two pieces, H_1 and H_2. If A is incompressible then H_1 and H_2 are handlebodies.*

Proof. As H is a handlebody, it contains a complete collection of compressing disks Δ which intersect the annulus A minimally. A standard innermost loop argument shows that Δ must intersect A either in essential arcs, essential loops, or not at all.

If it intersects in essential arcs (Fig. 3a) then Δ cut along A will contain a complete collection of compressing disks for each of H_1 and H_2 showing that they are handlebodies . If it intersects A in essential loops (Fig. 3b), then an innermost loop on Δ will contain a compressing disk for A. If they are disjoint (Fig. 3c), then A is compressible being properly embedded in one of the balls resulting after cutting along Δ.

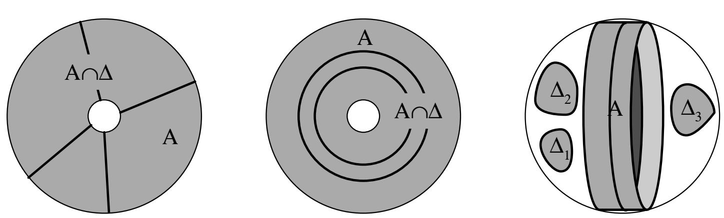

Fig. 3a–c.

Lemma 2. *Let F be a spanning surface for a knot K. Let F' be obtained from F by twisting as in Sect. 2. If $N(F)^C$ is a handlebody then $N(F')^C$ is a handlebody.*

Proof. $N(F')^C$ is obtained from $N(F)^C$ by replacing $B_0 - B_1 - N(F)$ by $B_0 - B_1 - N(F')$. Temporarily use edge slides to move γ_1 and γ_2 onto S_0. Notice that $S_0 - N(\alpha_0 \cup \beta_0)$ is a separating annulus and that $S_0 - N(\alpha_0 \cup \beta_0) - N(\gamma_1 \cup \gamma_2)$ is two disks (see Fig. 4).

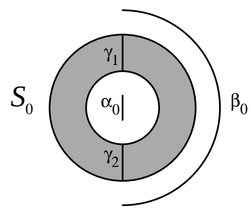

Fig. 4. The annulus $S_0 - N(\alpha_0 \cup \beta_0)$

Forming $N(F')^C$ corresponds to cutting $N(F)^C$ along this separating annulus and gluing the resulting two pieces together along the two disks. The annulus is incompressible, if not it would compress to one one side of S_0 implying that α and β separate, contrary to our assumption . Thus, according to Lemma 1, after cutting along the annulus the two pieces will be handlebodies, and $N(F')^C$, the result of gluing these two handlebodies along two disks is also a handlebody.

4. Stabilizing the Heegaard splittings

In this section K will be a knot in S^3, F a spanning surface for K such that $N(F)^C$ is a handlebody, and $M = K(1/q)$ the manifold obtained from a $1/q$ Dehn filling on the knot complement. We have seen that F induces a Heegaard splitting, $M = N(F) \cup (N(F)^C \cup T)$. If F' is obtained from F by twisting then $N(F')^C$ is a handlebody and we obtain another Heegaard splitting, $M = N(F') \cup (N(F')^C \cup T)$. In this section we show that these splittings are equivalent after one stabilization. As stabilization is unique it follows that any two splittings obtained in this manner are equivalent after one stabilization.

Note that these splittings are represented by Σ and Σ' which are the spines of the surfaces F and F', respectively. We now prove the main theorem:

Theorem 1. $S^1(\Sigma') \sim S^3(\Sigma)$.

But first a lemma:

Lemma 4. *Let Γ and Γ' be spines of Heegaard splittings of genus p and q respectively, $p \leq q$. If $\Gamma' \supseteq \Gamma$ then $S^{q-p}(\Gamma) \sim \Gamma'$.*

Proof. $\partial N(\Gamma')$ is a splitting surface for the handlebody $N(\Gamma)^C$. As any splitting of a handlebody is just a stabilization of its boundary [ST], the boundary in this case being $\partial N(\Gamma)$, Γ' represents a stabilization of Γ. Thus $\Gamma' \sim S^{q-p}(\Gamma)$.

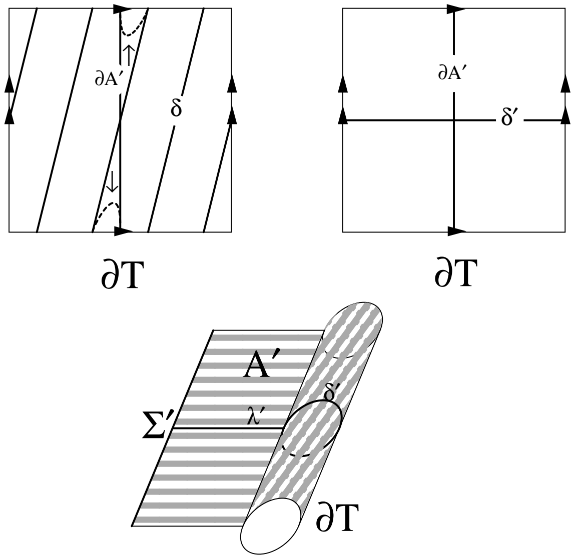

Fig. 5a–c. Stabilizing and unwinding

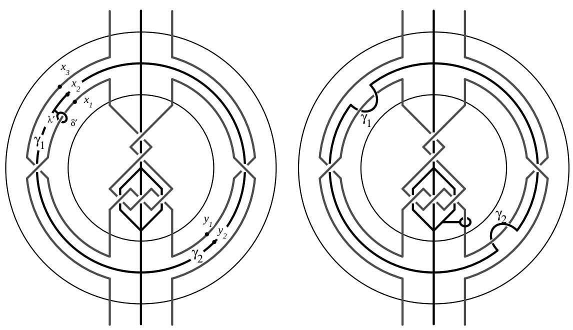

Fig. 6a,b. Changing crossings

Proof (Proof of Theorem 3). We show that $S^1(\Sigma') \supseteq \Sigma$. The proof then follows from Lemma 4. We proceed in three steps:

Step 1 - Stabilizing. Remember that F' cut along Σ' is an annulus, call this annulus A'. View ∂T in filling coordinates. One component of $\partial A'$ is a curve representing the longitude. The meridian of ∂T is given by the filling that yields S^3. Let δ be a standard $1/q$ curve on ∂T; we performed a $1/q$ surgery thus δ bounds a disk in T . Also δ intersects the chosen component of $\partial A'$ exactly once. (See Fig. 5a). Add $N(\delta)$ to the handlebody $N(F')$; this represents a stabilization of the splitting determined by $N(F')$.

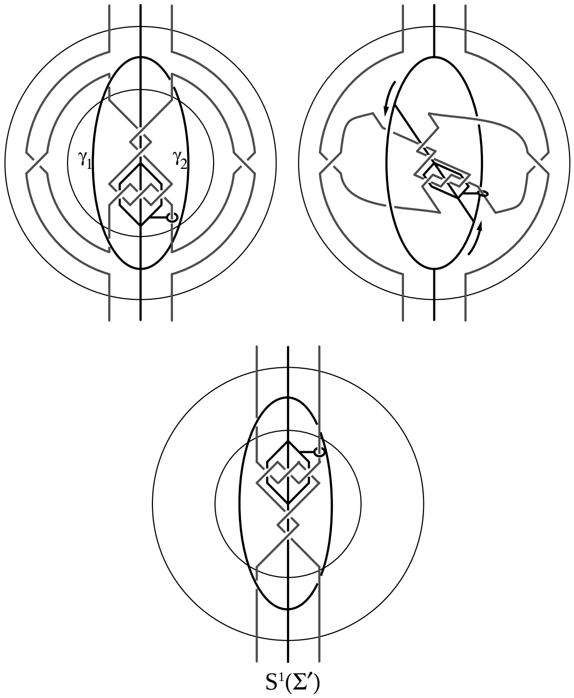

Fig. 6c–e. Rotating

Step 2 - Unwinding Now slide the foot of δ along the curve $\partial F'$ in order to obtain δ', a $1/0$ curve in the surgery coordinates (See Fig. 5b). This represents an isotopy of the stabilized handlebody $N(F') \cup N(\delta)$. The spine $S^1(\Sigma')$ which represents the splitting now appears as $\Sigma' \cup \delta' \cup \lambda'$, where λ' is an arc in A' with one endpoint at δ' and the other at Σ'. Call the endpoint at Σ' the foot of λ' (Fig. 5c).

Step 3 - Changing Crossings and Rotating We can use A' to guide edgeslides. Slide δ' along K to x_1 and the foot of λ' along Σ' to x_2 as marked in Fig. 6a. Now perform an edgeslide that wraps γ_1 around K as pictured in Fig. 7a–c. Notice that we have disrupted the annulus A' in a neighborhood of the point x_3. We now slide δ' to y_1 and the foot of λ' to y_2 as marked in Fig. 6a. We cannot use A' to guide the slide in a neighborhood of x_3 but we can go in the opposite direction, see Fig. 8. Now wrap γ_2 around K to result in Fig. 6b. Pull γ_1 towards the front of S_0 and γ_2 towards the back. We can now rotate the ball B_1 through $180°$ back to its original position. The feet of Σ', protruding from B_1, will slide along the arcs γ_1 and γ_2, (see Fig. 6c–e).

We have manipulated $S^1(\Sigma')$ by isotopy and edgeslides. Note that $S^1(\Sigma') \supseteq \Sigma$. Thus, by Lemma 4, $S^1(\Sigma') \sim S^3(\Sigma)$.

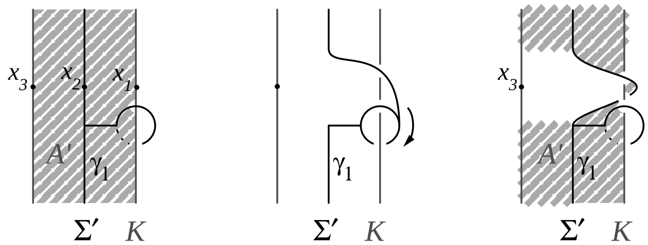

Fig. 7a–c. Changing crossings

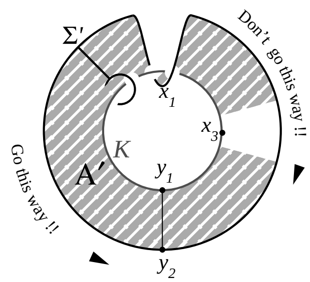

Fig. 8. Using A' to guide edgeslides

In particular, we have shown that the infinite family of irreducible Heegaard splittings due to Casson and Gordon are equivalent after one stabilization.

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