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Generalized inductive limits of finite-dimensional C^*-algebras

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1 Introduction

The inductive limit construction for C^*-algebras has been of great importance over the years. It has not only served as a source of many interesting examples, but has also shed light on the structure of many naturally occurring C^*-algebras. In recent years, there has been an explosion of results concerned with writing C^*-algebras as inductive limits of well understood "building blocks," and obtaining complete isomorphism invariants of a K-theoretic nature, which are simple and computable, for such algebras.

Although these results are impressive by any standard, there are some drawbacks to the theory and the approach taken, which has been primarily to consider inductive limits of direct sums of matrix algebras over commutative C^*-algebras (approximately homogeneous or AH algebras), or more generally inductive limits of subhomogeneous C^*-algebras (approximately subhomogeneous or ASH algebras.) The first problem is that such an inductive system for a given C^*-algebra is not at all canonical, and an isomorphism between the limits of two such systems need not be induced in any reasonable way from an (ordinary) intertwining of the inductive systems when the base spaces are more than one dimensional. Secondly, one quickly runs into difficult and probably intractable topology problems with higher-dimensional base spaces. And third and most important, there is (so far) no reasonable intrinsic characterization of which C^*-algebras can be written as inductive limits of this form. This is even a problem for AF algebras (inductive limits of finite-dimensional C^*-algebras), which also have no good intrinsic description (by this we mean a characterization sufficient to immediately yield the results of [BtK], for example.)

This paper begins an attempt to bypass some of these difficulties while at the same time expanding the class of C^*-algebras considered. One of the principles coming out of the work on AH algebras is that in considering inductive systems of C*-algebras, asymptotic behavior is all that matters; exact good behavior at each step is not necessary. (This is also very much in the spirit of the E-theory of Connes and Higson.) In the AH algebra case this principle is primarily applied to intertwinings of inductive systems. However, it is also possible, and interesting, to relax the requirements on the connecting maps themselves, requiring them only to be asymptotically additive, *-preserving, and multiplicative. We therefore consider *generalized inductive systems* of C*-algebras, where the connecting maps only asymptotically preserve the structure of the algebras (see Sect. 2 for a precise definition.) It then turns out that a large and natural class of C*-algebras can be written as generalized inductive limits of sequences of finite-dimensional C*-algebras; we call such algebras *MF algebras*. We show that a separable C*-algebra is MF if and only if it has a quasidiagonal extension by the compact operators K. The MF algebras are also precisely the separable C*-algebras which can be placed at infinity in a continuous field over $\mathbb{N}\cup\{\infty\}$, with the fibers at the points of N finite-dimensional matrix algebras. (This is the inspiration for the name "MF algebra," which stands for "matricial field" or "M. Fell," who was the first to consider such algebras and propose that they are interesting objects for study. See also [Le].) Every C*-subalgebra of an MF algebra is MF, and every separable C*-algebra is a quotient of an MF algebra.

An even more interesting class of algebras is obtained by considering generalized inductive limits of sequences of finite-dimensional C*-algebras with completely positive (and asymptotically multiplicative) contractive connecting maps. It is shown that these algebras, called *NF algebras*, are precisely the nuclear MF algebras, and therefore the NF algebras are exactly the separable nuclear (weakly) quasidiagonal C*-algebras. There is good cause to believe that every sufficiently finite separable nuclear C*-algebra is NF, further justifying the terminology NF as "nuclear finite." ("Sufficiently finite" may mean stably finite or the existence of a faithful densely defined semifinite trace.) Every nuclear C*-subalgebra of an NF algebra is NF, and every separable nuclear C*-algebra is a quotient of an NF algebra (in this respect they are quite unlike AF algebras). Every inductive limit (even in the generalized sense of systems with completely positive asymptotically multiplicative connecting maps) of NF algebras is NF. In particular, every ASH algebra is NF.

There is another characterization of NF algebras which is closely related to the concept of nuclearity. Recall that a C*-algebra A is *nuclear* if the identity map on A can be approximately factored by completely positive contractions through matrix algebras (see 5.1.1). We show that a separable C*-algebra is an NF algebra if and only if the identity map on A can be approximately factored by completely positive almost multiplicative contractions through matrix algebras. Thus the NF algebras form a very natural class of nuclear C*-algebras, the ones in which not only the complete order structure but also the multiplication can be approximately modeled in finite-dimensional C*-algebras. (In fact, it is a bit surprising that such algebras were not studied in the 1970's when most of the structure work on nuclear C*-algebras was done.)

We also consider a slightly more restrictive class of C*-algebras, the *strong NF algebras*, which are inductive limits of sequences of finite-dimensional C*-algebras with connecting maps complete order embeddings (and, of course, asymptotically multiplicative.) We show that a C*-algebra is a strong NF algebra if and only if the identity map on A can be approximated by completely positive idempotent finite-rank maps from A to A. We will show in a subsequent paper [BKb] that most, but not all, NF algebras are strong NF algebras, including all simple NF algebras and all ASH algebras.

MF, NF, and strong NF inductive systems are also canonical or intrinsic in a certain sense: any two MF [resp. NF, strong NF] inductive systems for a C*-algebra are asymptotically equivalent, in a sense made precise in Sect. 2. Thus these systems can be used to calculate invariants in a nice way for the limit algebras. Unfortunately, it is much more difficult to calculate the K-theory, and even the trace space and ideal structure, for generalized inductive limits than for usual limits. Nonetheless, the point of view of generalized systems of finite-dimensional C*-algebras is a potentially useful one in describing and proving completeness of K-theoretic invariants. Such an inductive system provides a "combinatorial" description of the algebra (as opposed to the "topological" description of AH algebras now being used). One can even use this combinatorial description to define a version of Čech cohomology. In fact, a nice way of doing Čech cohomology for a space (commutative C*-algebra) is to write the algebra as an inductive limit of a suitable NF system and apply a certain algorithm to the system. We will investigate K-theory invariants and Čech cohomology for NF algebras in future work.

The paper is organized as follows. Sect. 2 gives the definition of a generalized inductive system and construction of the inductive limit, along with some general results about such limits and intertwinings between them. In Sect. 3, the principal results about MF algebras are established. Sect. 4 contains facts about completely positive maps and complete order embeddings, which are used in Sect. 5 to examine the structure of NF algebras and in Sect. 6 to prove the results about strong NF algebras. Section 7 contains some additional results about ideals and traces, along with some open problems.

2 Generalized inductive systems

2.1 Definitions and basic constructions

Definition 2.1.1. *A* **generalized inductive system of C*-algebras** *is a sequence* (A_n) *of C*-algebras, with coherent maps* $\phi_{m,n}: A_m \to A_n$ *for* $m < n$, *such that for all* k *and all* $x, y \in A_k, \lambda \in \mathbb{C}$, *and all* $\varepsilon > 0$, *there is an* M *such that, for all* $M \leqq m < n$,

(1) $\|\phi_{m,n}(\phi_{k,m}(x) + \phi_{k,m}(y)) - (\phi_{k,n}(x) + \phi_{k,n}(y))\| < \varepsilon$

(2) $\|\phi_{m,n}(\lambda\phi_{k,m}(x)) - \lambda\phi_{k,n}(x)\| < \varepsilon$

(3) $\|\phi_{m,n}(\phi_{k,m}(x)^*) - \phi_{k,n}(x)^*\| < \varepsilon$
(4) $\|\phi_{m,n}(\phi_{k,m}(x)\phi_{k,m}(y)) - \phi_{k,n}(x)\phi_{k,n}(y)\| < \varepsilon$
(5) $\sup_r \|\phi_{k,r}(x)\| < \infty$.

A system satisfying (1) [*resp.* (4)] *is called* **asymptotically additive** [*resp.* **asymptotically multiplicative**]. *A generalized inductive system in which all* $\phi_{m,n}$ *are linear is called a* **linear generalized inductive system**; *if all the* $\phi_{m,n}$ *also preserve adjoints*, *the system is called* ***-linear**. *A system is* **contractive** *if all the connecting maps are contractions.*

We will see that at least if all the A_n are finite-dimensional, there is no loss of generality in assuming that all the connecting maps are *-linear.

In this paper, we will consider only generalized inductive systems indexed by N; however, an obvious modification of the definition of a generalized inductive system and the construction of the inductive limit will work over an arbitrary directed index set. We will also only consider generalized inductive systems of C*-algebras, but in the same way one can define generalized inductive systems and limits of Banach spaces, Banach algebras, or Banach *-algebras by leaving out conditions (3) and/or (4) where irrelevant.

2.1.2. Suppose $(A_n, \phi_{m,n})$ is a generalized inductive system of C*-algebras. To define the inductive limit C*-algebra A, first form the set-theoretic direct limit L (recall that L is the quotient of the disjoint union of the A_n where an element of A_m is identified with its image in later A_n's.) If $x \in A_m$, denote its image in L by $\phi_m(x)$. Set $d(\phi_m(x), \phi_m(y)) = \limsup_n \|\phi_{m,n}(x) - \phi_{m,n}(y)\|$. It is easy to verify that d is a pseudometric; let A be the completion of L with respect to d. Then A has a natural induced structure as a C*-algebra. For example, to define $\phi_m(x) + \phi_m(y)$ in A (we use ϕ_m also to denote the natural map from A_m into A), note that $\langle \phi_n(\phi_{m,n}(x) + \phi_{m,n}(y)) \rangle$ is a Cauchy sequence in A as $n \to \infty$; let $\phi_m(x) + \phi_m(y)$ be the limit. Multiplication, scalar multiplication, and adjoint are defined similarly; the norm is defined by $\|\phi_m(x)\| = \limsup_n \|\phi_{m,n}(x)\|$. One readily checks that A is a C*-algebra. The construction is somewhat simpler if the system is linear, since then as a Banach space $A = \varinjlim (A_n, \phi_{m,n})$ in the usual sense. If the system is contractive, then $\|\phi_m(x)\| = \lim_n \|\phi_{m,n}(x)\| = \inf_n \|\phi_{m,n}(x)\|$ for all $x \in A_m$.

2.1.3. A more concrete description of the construction of the inductive limit of $(A_n, \phi_{m,n})$ is as follows. Let $\prod A_n$ be the full C*-direct product of the A_n, i.e. the set of bounded sequences $\langle x_n \rangle$, with $x_n \in A_n$, with pointwise operations and sup norm; and let $\oplus A_n$ be the C*-direct sum, the set of sequences converging to zero in norm. Then $\prod A_n$ is a C*-algebra and $\oplus A_n$ is a closed two-sided ideal; let π be the quotient map from $\prod A_n$ to $(\prod A_n)/(\oplus A_n)$. Each element x of A_m naturally defines an element $\phi_m(x) = \pi(\langle \phi_{m,n}(x) \rangle)$ of $(\prod A_n)/(\oplus A_n)$. The closure of the set of all such elements (for all m) is a C*-subalgebra of $(\prod A_n)/(\oplus A_n)$ naturally isomorphic to $A = \varinjlim (A_n, \phi_{m,n})$. Thus a C*-algebra which is an inductive limit of a generalized inductive system $(A_n, \phi_{m,n})$ can be embedded in $(\prod A_n)/(\oplus A_n)$.

Remark 2.1.4. An even slightly more general type of inductive system can be considered, where the connecting maps are only asymptotically coherent. The abstract inductive limit construction in this setting is a bit obscure, but the inductive limit can be readily constructed as in 2.1.3.

Recall that a unital C*-algebra A is *finite* if for all $x \in A$, $x^*x = 1$ implies $xx^* = 1$. This is equivalent to left invertible elements being (right) invertible. A is *stably finite* if $M_k(A)$ is finite for all k. If A is nonunital, we say A is finite [resp. stably finite] if its unitization A^+ is finite [resp. stably finite].

Proposition 2.1.5. *Let* $(A_n, \phi_{m,n})$ *be a generalized inductive system, with* $A = \varinjlim (A_n, \phi_{m,n})$. *If each* A_n *is commutative* [*resp. finite, stably finite*], *then* A *is commutative* [*resp. finite, stably finite*].

Proof. If each A_n is commutative, then $(\prod A_n)/(\oplus A_n)$ is obviously commutative. If each A_n is finite, we may assume A_n is unital without loss of generality, since $(\prod A_n)/(\oplus A_n)$ imbeds in $(\prod A_n^+)/(\oplus A_n^+)$. Suppose $x \in (\prod A_n)/(\oplus A_n)$ is left invertible, with left inverse y. If $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences in $\prod A_n$ representing x and y respectively, then $\| y_n x_n - 1 \| \to 0$, and so x_n is left invertible for sufficiently large n, and its left inverse z_n is actually x_n^{-1} by finiteness of A_n. Furthermore, $\langle z_n \rangle$ is a bounded sequence which represents y, so $xy = 1$ and $(\prod A_n)/(\oplus A_n)$ is finite. If $M_k(A_n)$ is finite for all n, then the same argument shows that $M_k((\prod A_n)/(\oplus A_n)) \cong (\prod M_k(A_n))/(\oplus M_k(A_n))$ is finite. □

2.2 Continuous fields

For future reference, we digress to note a close connection between the "corona" algebras used above and continuous fields of C*-algebras (cf. [Dx, Sect. 10]). We first need two definitions.

Definition 2.2.1. *Let* P *be the set of all polynomials in a sequence of noncommuting variables* $\langle X_n \rangle$ *and their formal adjoints* X_n^*, *with coefficients in* $\mathbb{Q} + \mathbb{Q}i$. (*Each element of* P *contains only finitely many* X_n). P *is countable*; *let* $\langle f_1, f_2, \ldots \rangle$ *be a fixed enumeration. If* A *is a complex* **-algebra and* $\langle x_n \rangle$ *is a sequence of elements of* A, *denote by* $f_j(\langle x_n \rangle_n)$ *the element of* A *obtained by evaluating* f_j *on the elements* $\{x_n, x_n^* : n \in \mathbb{N}\}$.

Definition 2.2.2. *A product* $\prod_{n=r}^{\infty} A_n$ *is called a* **tail** *of* $\prod_{n=1}^{\infty} A_n$; *denote the natural homomorphism from* $\prod_{n=1}^{\infty} A_n$ *to* $\prod_{n=r}^{\infty} A_n$ *by* ρ_r. *A product* $\prod_{n=r}^{s} A_n$ *is called a* **segment** *or* **truncated tail** *of the product*; *denote the natural homomorphism from* $\prod_{n=1}^{\infty} A_n$ *to* $\prod_{n=r}^{s} A_n$ *by* ρ_r^s.

Proposition 2.2.3. *Let* (A_n) *be a sequence of* C^**-algebras, and let* A *be a separable* C^**-algebra. Then the following are equivalent*:

(i) *A can be embedded in* $(\prod A_n)/(\oplus A_n)$

(ii) *There is a continuous field of* C^**-algebras* $\langle B(t) \rangle$ *over* $\mathbb{N} \cup \{\infty\}$, *with* $B(t)$ *a segment of* $\prod A_n$ *for* $t \in \mathbb{N}$, *with disjoint segments for different* t, *and with* $B(\infty) \cong A$.

(iii) *A can be embedded in $(\prod B(t))/(\oplus B(t))$, where each $B(t)$ is a segment of $\prod A_n$, with disjoint segments for different t, and such that, for every $x \in A$ and sequence $\langle x(t)\rangle \in \prod B(t)$ representing x, $\lim_{t\to\infty} \|x(t)\|$ exists and equals $\|x\|$.*

Proof. $(i) \Rightarrow (ii)$: Suppose A is embedded as a subalgebra of $(\prod A_n)/(\oplus A_n)$. Let $\{x_i\}$ be a countable dense set in A. For each i, let $\langle x_{in}\rangle$ be a sequence in $\prod A_n$ representing x_i. Then there are disjoint segments $[r_m, s_m]$ in N such that, for $f_j \in P$ (2.2.1), $|\,\|f_j(\langle x_i\rangle_i)\| - \|\rho_{r_m}^{s_m}(\langle f_j(\langle x_{in}\rangle_i)\rangle_n)\|\,| < m^{-1}$ for $1 \leqq j \leqq m$. Set $B(m) = \prod_{n=r_m}^{s_m} A_n$. The fields $\langle y_j\rangle$, where $y_j(m) = \rho_{r_m}^{s_m}(\langle f_j(\langle x_{in}\rangle_i)\rangle_n)$ and $y_j(\infty) = f_j(\langle x_i\rangle_i)$, along with the sequences converging to zero, generate a continuous field with the desired properties.

$(ii) \Rightarrow (iii)$: Given a field as in (ii), for each $x \in A = B(\infty)$, any continuous cross section $x(t)$ with $x(\infty) = x$ gives an element of $\prod_{t\in \mathrm{N}} B(t)$, and any two such sequences for x differ by an element of $\oplus B(t)$, so x gives a well-defined element of $(\prod_{t\in\mathrm{N}} B(t))/(\oplus_{t\in\mathrm{N}} B(t))$; this map is an isometric *-isomorphism on A.

$(iii) \Rightarrow (i)$: Note that $\prod B(t)$ [resp. $\oplus B(t)$] is isomorphic to $\prod_{n\in S} A_n$ [resp. $\oplus_{n\in S} A_n$] for some infinite subset S of N. $\prod_{n\in S} A_n$ embeds in $\prod_{n\in\mathrm{N}} A_n$ by setting the other coordinates to zero; this embedding drops to an embedding of $(\prod_{n\in S} A_n)/(\oplus_{n\in S} A_n)$ into $(\prod_{n\in\mathrm{N}} A_n)/(\oplus_{n\in\mathrm{N}} A_n)$. □

2.3 Uniform boundedness

We next record some simple facts about uniform boundedness and the behavior of finite-dimensional subspaces under linear generalized inductive limits.

Proposition 2.3.1. *Let $A = \varinjlim(A_n, \phi_{m,n})$, with each $\phi_{m,n}$ linear and bounded. Then $\sup_n \|\phi_{m,n}\| < \infty$ for all m, and $\phi_m\colon A_n \to A$ is bounded.*

Proof. Follows easily from the Banach-Steinhaus Theorem, regarding the maps $\phi_{m,n}$ as maps from A_m into $\prod A_n$. □

Proposition 2.3.2. *Let S be a finite-dimensional normed vector space, and $\varepsilon > 0$. Then there exist a finite number of elements $\{x_1, \dots, x_n\} \in S$ of norm 1 such that, if $\phi\colon S \to Z$ is a linear map from S to a normed vector space Z, then $\|\phi\| < (1+\varepsilon)[\max \|\phi(x_j)\|]$.*

Proof. Follows easily from compactness of the unit sphere of S. □

Definition 2.3.3. *Let A and B be C^*-algebras, and S a subset of A, and $\varepsilon > 0$. A map $\phi\colon A \to B$ is* **approximately multiplicative within** ε **on** *S if, for every $x, y \in S$,*

$$\|\phi(x)\phi(y) - \phi(xy)\| < \varepsilon\|x\|\,\|y\|\,.$$

ϕ is **approximately isometric within** ε **on** *S if $(1-\varepsilon)\|x\| < \|\phi(x)\| < (1+\varepsilon)\|x\|$ for all $x \in S$.*

Proposition 2.3.4. *Let $(A_n, \phi_{m,n})$ be a contractive linear generalized inductive system, S a finite-dimensional subspace of A_k, and $\varepsilon > 0$. Then there is an $M \geqq k$ such that, for all $M \leqq m < n$, $\phi_{m,n}$ is approximately isometric and approximately multiplicative within ε on $\phi_{k,m}(S)$.*

Proof. This follows easily from 2.3.2. □

2.4 Asymptotic intertwinings

Finally, we discuss asymptotic intertwinings between generalized inductive systems.

Definition 2.4.1. *Let $(A_n, \phi_{m,n})$ and $(B_n, \psi_{m,n})$ be generalized inductive systems of C^*-algebras. An* **asymptotic map** *from $(A_n, \phi_{m,n})$ to $(B_n, \psi_{m,n})$ is a pair of increasing sequences $\langle r_n \rangle$ and $\langle s_n \rangle$ and maps $\alpha_n \colon A_{r_n} \to B_{s_n}$, such that*

(1) *for every k, every $x \in A_k$, and $\varepsilon > 0$ there exists an M such that for all $M \leqq m < n$,*

$$\|\psi_{s_m,s_n} \circ \alpha_m \circ \phi_{k,r_m}(x) - \alpha_n \circ \phi_{k,r_n}(x)\| < \varepsilon .$$

(2) *The induced map from $\cup\phi_n(A_n)$ to $\cup\psi_n(B_n)$ extends to a bounded *-homomorphism α from $A = \varinjlim(A_n, \phi_{m,n})$ to $B = \varinjlim(B_n, \psi_{m,n})$.*

An **asymptotic intertwining** *between $(A_n, \phi_{m,n})$ and $(B_n, \psi_{m,n})$ is a pair of increasing sequences $\langle r_n \rangle$ and $\langle s_n \rangle$ and maps $\alpha_n \colon A_{r_n} \to B_{s_n}$ and $\beta_n \colon B_{s_n} \to A_{r_{n+1}}$ such that both the α_n and β_n are asymptotic maps, and for every k, every $x \in A_k$, $y \in B_k$, and $\varepsilon > 0$ there is an M such that, for all $M \leqq m < n$,*

$$\|\beta_n \circ \psi_{s_m,s_n} \circ \alpha_m \circ \phi_{k,r_m}(x) - \phi_{k,r_{n+1}}(x)\| < \varepsilon \tag{3}$$

$$\|\alpha_n \circ \phi_{r_{m+1},r_n} \circ \beta_m \circ \psi_{k,s_m}(x) - \psi_{k,s_n}(x)\| < \varepsilon \tag{4}$$

In condition (2), the requirement that α be a *-homomorphism just means that the α_n are asymptotically *-linear and multiplicative in the obvious sense. The requirement that α be bounded, however, is awkward to state directly in the general case; but under special circumstances this condition is easily checked (cf. 2.4.2, 5.1.6). Sometimes asymptotic multiplicativity is automatic (5.1.6).

The *-homomorphisms induced by an asymptotic intertwining are *-isomorphisms which are mutual inverses.

Proposition 2.4.2. *Let $(A_n, \phi_{m,n})$ and $(B_n, \psi_{m,n})$ be *-linear generalized inductive systems. Let $\langle \alpha_n \rangle$ be a sequence of *-linear maps satisfying condition* (1) *of* 2.4.1. *Suppose there are subspaces S_n of A_{r_n} with $\cup\phi_{r_n}(S_n)$ dense in A, and a constant K, such that, for each m,*

(1) $\|\alpha_n \circ \phi_{r_m,r_n}|_{S_m}\| \leqq K$ *for all sufficiently large n*

(2) *The α_n are asymptotically multiplicative in the sense that, for each m and each $x, y \in S_m$, and $\varepsilon > 0$, there is an N such that, whenever $N \leqq n$,*

$$\|\alpha_n([\phi_{m,r_n}(x)][\phi_{m,r_n}(y)]) - [\alpha_n \circ \phi_{m,r_n}(x)][\alpha_n \circ \phi_{m,r_n}(y)]\| < \varepsilon$$

Then $\langle \alpha_n \rangle$ is an asymptotic map.

Proposition 2.4.3. *Let $(A_n, \psi_{m,n})$ be a linear generalized inductive system, with each $\psi_{m,n}$ bounded. Let $\phi_{n,n+1}: A_n \to A_{n+1}$ be a bounded linear map for each n, and define $\phi_{m,n}: A_m \to A_n$ by composition. If $\varepsilon_n = \|\phi_{n,n+1} - \psi_{n,n+1}\|$ satisfies $\sum \varepsilon_n < \infty$, then $(A_n, \phi_{m,n})$ is a generalized inductive system, and $\varinjlim(A_n, \phi_{m,n})$ and $\varinjlim(A_n, \psi_{m,n})$ are isomorphic.*

Proof. It is routine to verify that $(A_n, \phi_{m,n})$ is a generalized inductive system (use the fact that $\|\phi_{m,n}\|$ and $\|\psi_{m,n}\|$ are uniformly bounded – cf. 2.3.1), and it is easily checked that $r_n = s_n = n, \alpha_n = id_{A_n}, \beta_n = \phi_{n,n+1}$ give an asymptotic intertwining. □

3 MF algebras

3.1 Quasidiagonality

We briefly review the notion of quasidiagonality for sets of operators and for C*-algebras. For a more complete discussion of this important and somewhat mysterious concept, see [Hm] and [Vo3].

Definition 3.1.1. *A set S of operators on a (separable, infinite-dimensional) Hilbert space is* **quasidiagonal** *if there is an increasing sequence $\langle p_n \rangle$ of finite-rank projections with $p_n \to 1$ strongly and $\| p_n a - a p_n \| \to 0$ for all $a \in S$.*

If S is quasidiagonal, then so is $C^*(S) + K$ using the same sequence of projections. Any subset of S is also quasidiagonal. A C*-algebra B containing K is quasidiagonal if and only if there is an approximate identity of projections for K which is quasicentral for B.

Definition 3.1.2. *A C^*-algebra A is* **quasidiagonal** *(called* **weakly quasidiagonal** *in some references such as [Hw]) if it has a faithful representation as a quasidiagonal algebra of operators. A is* **strongly quasidiagonal** *if every representation of A is quasidiagonal.*

By Voiculescu's Theorem [Vo1], if A is separable and quasidiagonal, then every faithful representation of A on a separable Hilbert space not containing any nonzero compact operator is quasidiagonal.

A C*-algebra A has a quasidiagonal essential extension by K if there is a quasidiagonal C*-algebra of operators B, containing K, with $B/\mathrm{K} \cong A$. A is a quasidiagonal C*-algebra if and only if it has a split quasidiagonal essential extension by K.

Proposition 3.1.3. *Let A be a C^*-algebra. Then A has an essential quasidiagonal extension by* K *if and only if A can be embedded in $(\prod \mathrm{M}_{k_n})/(\oplus \mathrm{M}_{k_n})$ for some sequence $\langle k_n \rangle$.*

Proof. If B is a C*-algebra of B(H) containing K with $B/\mathrm{K} \cong A$, and B is a quasidiagonal set of operators with respect to projections $\langle p_n \rangle$, set $q_n = p_n - p_{n-1}$ and $k_n = rank\, q_n$. If $x \in B$, set $\rho(x) = \langle q_n x q_n \rangle \in \prod \mathrm{M}_{k_n}$. ρ drops to a *-homomorphism from B to $(\prod M_{k_n})/(\oplus M_{k_n})$ whose kernel is exactly K. Conversely, if A is a C^*-subalgebra of $(\prod M_{k_n})/(\oplus M_{k_n})$, choose a sequence of orthogonal projections $\langle q_n \rangle$ on a Hilbert space H, adding up to the identity, with $rank\, q_n = k_n$. If $\sigma : A \to \prod M_{k_n}$ is a (set-theoretic) cross section for the identity map on A, for $x \in A$ let $\alpha(x)$ be the block diagonal operator which on $q_n H$ is $\sigma(x)_n$. Then $B = \alpha(A) + K$ is a C^*-subalgebra of B(H) which is quasidiagonal with respect to $\langle p_n \rangle$, where $p_n = \sum_{j=1}^n q_j$, and $B/K \cong A$. $\square$

Proposition 3.1.4. [Vo2, Theorem 1] *Let A be a C^*-algebra. Then A is quasidiagonal if and only if, for every $x_1, \ldots, x_n \in A$ and $\varepsilon > 0$, there is a finite-dimensional C^*-algebra B and a completely positive contraction $\alpha\colon A \to B$ such that $\|\alpha(x_i)\| \geqq \|x_i\| - \varepsilon$ and $\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| < \varepsilon$ for $1 \leqq i, j \leqq n$.*

See [Sa] for a generalization.

3.2 Definition and the main theorem

In this section, we prove the theorem giving a number of equivalent characterizations of inductive limits of generalized inductive systems of finite-dimensional C^*-algebras.

Definition 3.2.1. *A separable C^*-algebra is an* **MF algebra** *if it can be written as the inductive limit of a generalized inductive system of finite-dimensional C^*-algebras.*

Theorem 3.2.2. *Let A be a separable C^*-algebra. Then the following are equivalent*:
(i) *A is an MF algebra*
(ii) *A is isomorphic to $\varinjlim (A_n, \phi_{m,n})$ for a *-linear generalized inductive system of finite-dimensional C^*-algebras*
(iii) *A can be embedded as a C^*-subalgebra of $(\prod M_{k_n})/(\oplus M_{k_n})$ for some sequence $\langle k_n \rangle$*
(iv) *A has an essential quasidiagonal extension by the compact operators* K
(v) *There is a continuous field of C^*-algebras $\langle B(t) \rangle$ over $\mathbb{N} \cup \{\infty\}$ with $B(\infty) \cong A$ and $B(n)$ finite-dimensional for $n < \infty$.*
(vi) *There is a continuous field of C^*-algebras $\langle B(t) \rangle$ over $\mathbb{N} \cup \{\infty\}$ with $B(\infty) \cong A$ and $B(n) = M_{k_n}$ for $n < \infty$.*

The proof of 3.2.2 requires the following extension of the Hahn-Banach theorem:

Lemma 3.2.3. *Let X be a (real or complex) normed vector space, and Y a finite-dimensional subspace, of dimension d. Then there is a bounded projection P from X onto Y, of norm $\leqq d$.*

Proof. If $\{\xi_1, \ldots, \xi_d, f_1, \ldots, f_d\}$ is an Auerbach system for Y [LT, 1.c.3], i.e. $\xi_j \in Y$, $f_k \in Y^*$ with $\|\xi_j\| = \|f_j\| = 1$ and $f_j(\xi_k) = \delta_{jk}$ for all j, k, extend f_j to $g_j \in X^*$ with $\|g_j\| = 1$, and set $P(x) = \sum_j g_j(x)\xi_j$. $\square$

Actually, there is always a projection of norm $\leqq \sqrt{d}$, but this is much harder to prove (cf. [Ps, 1.14].)

Proof of 3.2.2. The only implication we must prove which is not almost immediate is (iii) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (v) are trivial; (i) $\Rightarrow$ (iii) is the construction of 2.1.3, and (iii) $\Leftrightarrow$ (v) is a special case of 2.2.3. (iii) $\Leftrightarrow$ (iv) is 3.1.3.

(v) $\Rightarrow$ (vi) is easy since in a continuous field the fibers at any isolated points may be enlarged without destroying the continuity of the field (throw in sections nonzero only at those points), and any finite-dimensional C*-algebra can be embedded in a full matrix algebra.

For the proof of (iii) $\Rightarrow$ (ii), suppose A is a C*-subalgebra of $(\prod \mathrm{M}_{k_n})/(\oplus \mathrm{M}_{k_n})$. To begin with, choose a dense sequence $\{x_1, x_2, \ldots\}$ in A, and a *-linear (not necessarily bounded) map σ from A to $\prod \mathrm{M}_{k_n}$ such that $\pi \circ \sigma$ is the identity map on A. [σ exists by elementary linear algebra; it can be chosen *-preserving by replacing σ by $(\sigma + \sigma^*)/2$, where $\sigma^*(x) = (\sigma(x^*))^*$]. Inductively define finite-dimensional C*-algebras A_n and *-linear maps $\alpha_n : A_n \to A$ and $\beta_n : A \to A_{n+1}$ as follows. Let $A_1 = \mathrm{M}_{k_1}$, and α_1 any *-linear map from A_1 to A. Suppose $A_1, \ldots, A_m$, $\alpha_1, \ldots, \alpha_m$, and $\beta_1, \ldots, \beta_{m-1}$ have been defined. Let S_m be the *-subspace of A generated by $\alpha_m(A_m), \{\alpha_m(x)\alpha_m(y) : x, y \in A_m\}$, and $\{x_1, \ldots, x_m\}$. S_m is finite-dimensional; let $d_m = dim\, S_m$. Since σ is isometric and multiplicative modulo sequences converging to zero, there is an r_m such that $\rho_{r_m} \circ \sigma$ (cf. 2.2.2) is approximately isometric and approximately multiplicative within $2^{-m-1}/d_m$ on S_m. By the same argument, there is then an s_m such that $\rho_{r_m}^{s_m} \circ \sigma$ (cf. 2.2.2) is approximately isometric and approximately multiplicative within $2^{-m}/d_m$ on S_m. (In particular, it is injective.) Set $A_{m+1} = \prod_{n=r_m}^{s_m} \mathrm{M}_{k_n}$ and $\beta_m = \rho_{r_m}^{s_m} \circ \sigma$. Since $dim\, \beta_m(S_m) \leqq d_m$, there is a *-linear projection ω_m of norm $\leqq d_m$ from A_{m+1} onto $\beta_m(S_m)$ (3.2.3); let $\alpha_{m+1} = \beta_m^{-1} \circ \omega_m$. Then $\|\alpha_{m+1}\| \leqq (1 - 2^{-m})^{-1} d_m \leqq 2d_m$.

Now set $\phi_{m,m+1} = \beta_m \circ \alpha_m : A_m \to A_{m+1}$, and $\phi_{m,n}$ by composition. Then $\phi_{m,n} = \beta_{n-1} \circ \alpha_m$. We must show that $(A_n, \phi_{m,n})$ is a generalized inductive system. Since the $\phi_{m,n}$ are *-linear, we only need to show conditions (4) and (5) of 2.1.1. For (5), note that by construction $\|\phi_{m,n}(x)\| \leqq (1 + 2^{-n})\|\alpha_m(x)\|$, so for each m, n we have $\|\phi_{m,n}\| < 2\|\alpha_m\|$. For (4), let $x, y \in A_k$. If $k < m < n$, then

$$\|\phi_{k,m+1}(x)\phi_{k,m+1}(y) - \beta_m(\alpha_m(\phi_{k,m}(x))\alpha_m(\phi_{k,m}(y))\|$$
$$< 2^{-m} d_m^{-1} \|\alpha_m(\phi_{k,m}(x))\| \, \|\alpha_m(\phi_{k,m}(y))\|$$

$$\|\phi_{k,n+1}(x)\phi_{k,n+1}(y) - \beta_n(\alpha_n(\phi_{k,n}(x))\alpha_n(\phi_{k,n}(y))\|$$
$$< 2^{-n} d_n^{-1} \|\alpha_n(\phi_{k,n}(x))\| \, \|\alpha_n(\phi_{k,n}(y))\|$$

and $\alpha_n(\phi_{k,n}(x)) = \alpha_m(\phi_{k,m}(x)) = \alpha_k(x)$, $\alpha_n(\phi_{k,n}(y)) = \alpha_m(\phi_{k,m}(y)) = \alpha_k(y)$, and $\|\alpha_{m+1}\| \leqq 2d_m$, so we have

$$\|\alpha_{m+1}(\phi_{k,m+1}(x)\phi_{k,m+1}(y)) - \alpha_k(x)\alpha_k(y)\| < 2^{-m+1}\|\alpha_k(x)\| \, \|\alpha_k(y)\|$$

and therefore

$$\|\phi_{m+1,n+1}(\phi_{k,m+1}(x)\phi_{k,m+1}(y)) - \phi_{k,n+1}(x)\phi_{k,n+1}(y)\|$$
$$< [(1+2^{-n})2^{-m+1} + 2^{-n}/d_n]\|\alpha_k(x)\|\,\|\alpha_k(y)\|$$

which suffices to show (4).

The maps $\langle\alpha_n\rangle$ obviously give an asymptotic map from the generalized inductive system $(A_n, \phi_{m,n})$ to the constant system (A, id). The maps $\langle\beta_n\rangle$ give an asymptotic map in the reverse direction by 2.4.2, taking the subspace S_n in the n'th copy of A. The maps $\langle\alpha_n\rangle$ and $\langle\beta_n\rangle$ clearly give an asymptotic intertwining between $(A_n, \phi_{m,n})$ and (A, id). Thus $A \cong \varinjlim (A_n, \phi_{m,n})$.

This completes the proof of Theorem 3.2.2. □

3.3 Properties of MF algebras

Corollary 3.3.1. *Every C^*-subalgebra of an MF algebra is MF.*

Recall that a C*-algebra A is *residually finite-dimensional* if A has a separating family of finite-dimensional representations. A C*-algebra in which every irreducible representation is finite-dimensional (in particular, every commutative C*-algebra and every unital subhomogeneous C*-algebra) is residually finite-dimensional. But there are many other such C*-algebras: for example, C$^*(G)$ is residually finite-dimensional for many (discrete) groups G, including free groups [Ch2] and more generally free products of abelian groups [GM], [ExL].

Proposition 3.3.2. *Every residually finite-dimensional C^*-algebra is an MF algebra.*

Proof. If A embeds in $B = \prod \mathrm{M}_{k_n}$, it also embeds in $\prod_{n=1}^{\infty} B_n$, where $B_n \cong B$, and therefore in the corona algebra of a product of matrix algebras. □

In fact, a residually finite-dimensional C*-algebra is quasidiagonal by 3.1.4.

Proposition 3.3.3. *Every separable C^*-algebra is a quotient of an MF algebra.*

Proof. By [GM], every separable C*-algebra is a quotient of a residually finite-dimensional C*-algebra. □

Not every MF algebra is residually finite-dimensional – many AF algebras are not residually finite-dimensional.

The next proposition gives an alternate proof for 3.3.3:

Proposition 3.3.4. *If A is any separable C^*-algebra, then $CA = C_0((0,1], A) \cong A \otimes C_0((0,1])$ and $SA = C_0((0,1), A) \cong A \otimes C_0(\mathrm{R})$ are MF algebras.*

Proof. CA is quasidiagonal by [Vo2], and SA is a C*-subalgebra of CA. □

Proposition 3.3.5. *If A is an MF algebra, then A can be embedded in $(\prod A_n)/(\oplus A_n)$, where each A_n is finite-dimensional and, for every $x \in A$, every representing sequence $\langle x_n\rangle$ for x satisfies $\lim_{n\to\infty}\|x_n\| = \|x\|$.*

Proof. This is a special case of 2.2.3(iii). □

Proposition 3.3.6. *If A and B are MF algebras, then $A \otimes_\alpha B$ is an MF algebra for some cross norm α. If one of them is nuclear, then $A \otimes B = A \otimes_{\min} B$ is an MF algebra.*

Proof. Let $A \subseteq (\prod A_n)/(\oplus A_n)$ and $B \subseteq (\prod B_n)/(\oplus B_n)$ as in 3.3.5. If $x \in A$ and $y \in B$, choose representing sequences $\langle x_n \rangle \in \prod A_n$ and $\langle y_n \rangle \in \prod B_n$, and let $\theta(x \otimes y)$ be the image of $\langle x_n \otimes y_n \rangle$ in $[\prod(A_n \otimes B_n)]/[\oplus(A_n \otimes B_n)]$. Then $\theta(x \otimes y)$ is independent of the representing sequences chosen for x and y, and θ extends to a *-homomorphism of the algebraic tensor product $A \odot B$ into $[\prod(A_n \otimes B_n)]/[\oplus(A_n \otimes B_n)]$. Furthermore,

$$\|\theta(x \otimes y)\| = \limsup \|x_n \otimes y_n\| = \lim \|x_n\| \, \|y_n\| = \|x\| \, \|y\|$$

by 3.3.5, so the induced seminorm on $A \odot B$ is a C*-cross seminorm. The fact that a C*-cross seminorm is a norm follows from the argument in [Tk, Chapter IV, Sects. 1–4]; if A or B is nuclear, this also follows easily from [Bl1, 3.3]. □

Corollary 3.3.7. *If A is MF, then every separable C*-algebra strong Morita equivalent to A is MF.*

Proof. Any separable C*-algebra strong Morita equivalent to A is stably isomorphic to A, and hence can be embedded in $A \otimes \mathbb{K}$, which is MF by 3.3.6 (actually, if A is MF, it is easily seen directly that $A \otimes \mathbb{K}$ is MF). □

Not every separable C*-algebra is an MF algebra:

Proposition 3.3.8. *Every MF algebra is stably finite.*

Proof. This is a special case of 2.1.5. □

So there are even type I C*-algebras (e.g. the Toeplitz algebra) which are not MF. It is unknown whether every stably finite type I C*-algebra is MF. Every residually stably finite type I C*-algebra is MF [Sp].

There seem to be no known examples of a stably finite separable C*-algebra which is not MF, but $C_r^*(\mathbb{F}_2)$ is a candidate [Vo3]. $C_r^*(G)$ for a property (T) group G which is not residually finite dimensional (such groups have apparently been constructed by Gromov) is perhaps a better candidate.

An MF algebra can fail to be strongly quasidiagonal: the examples of [Bn] and [BD] are MF but not quasidiagonal in their natural representations. An MF algebra can even fail to be (weakly) quasidiagonal [Ws, Prop. 5].

3.4 Inductive limits and continuous fields of MF algebras

Proposition 3.4.1. *Let $\langle B(t) \rangle$ be a continuous field of separable C*-algebras over $\mathbb{N} \cup \{\infty\}$. If each $B(t)$ for $t \in \mathbb{N}$ is an MF algebra, then $B(\infty)$ is an MF algebra.*

Proof. Choose a dense sequence $\langle x_k \rangle$ in $B(\infty)$. For each $t \in \mathrm{N}$ choose a sequence $\langle B_m(t) \rangle$ of finite-dimensional C*-algebras such that $B(t)$ embeds in $(\prod_m B_m(t))/(\oplus_m B_m(t))$ as in 3.3.5. For each k, choose a continuous field $\langle y_k(t) \rangle$ in $\langle B(t) \rangle$ with $y_k(\infty) = x_k$, and then for each k and t in N choose a representing sequence $\langle y_{km}(t) \rangle \in \prod_m B_m(t)$ for $y_k(t)$. For f_j as in 2.2.1, for each n choose a $t_n \in \mathrm{N}$ such that

$$| \, \| f_j(\langle y_k(t_n) \rangle_k) \| - \| f_j(\langle x_k \rangle_k) \| \, | < n^{-1}$$

for $1 \leqq j \leqq n$, and then choose m_n such that

$$| \, \| f_j(\langle y_{km_n}(t_n) \rangle_k) \| - \| f_j(\langle y_k(t_n) \rangle_k) \| \, | < n^{-1}$$

for $1 \leqq j \leqq n$. Set $A(n) = B_{m_n}(t_n)$ for $n \in \mathrm{N}$ and $A(\infty) = B(\infty)$. Then the fields $\langle z_j(n) \rangle$, where $z_j(n) = f_j(\langle y_{km_n}(t_n) \rangle_k)$ for $n \in \mathrm{N}$ and $z_j(\infty) = f_j(\langle x_k \rangle_k)$, along with the sequences converging to zero, generate a continuous field as in 3.2.2(v). □

Corollary 3.4.2. *Let $\langle B(t) \rangle$ be a continuous field of separable C*-algebras over a first countable locally compact Hausdorff space X. Then*

$$\{t \in X : B(t) \text{ is an MF algebra}\}$$

is a closed subset of X.

Corollary 3.4.3. *Let $\langle A_n \rangle$ be a sequence of MF algebras. Then any separable C*-subalgebra of $(\prod A_n)/(\oplus A_n)$ is an MF algebra.*

Proof. Combine 3.4.1 and 2.2.3. □

Corollary 3.4.4. *Let $(A_n, \phi_{m,n})$ be a generalized inductive system of C*-algebras, with $A = \varinjlim(A_n, \phi_{m,n})$. If each A_n is an MF algebra, then A is an MF algebra.*

Although extensions of MF algebras are not in general MF (e.g. the Toeplitz algebra), continuous fields of MF algebras are MF:

Corollary 3.4.5. *Let $\langle B(t) \rangle$ be a continuous field of separable C*-algebras over a second countable locally compact Hausdorff space X, and let A be the C*-algebra defined by the field. If $B(t)$ is an MF algebra for a dense set of t, then $B(t)$ is MF for all t and A is an MF algebra.*

Proof. A is separable since X is second countable and each $B(t)$ is separable. Suppose $B(t_n)$ is MF for a dense sequence $\langle t_n \rangle$ in X (with isolated points repeated infinitely often). Then A embeds in $(\prod_n B(t_n))/(\oplus_n B(t_n))$. The result now follows from 3.4.3. It also follows from 3.4.2 that each $B(t)$ is MF. □

Note that for A to be MF in 3.4.5, it is not necessary for the $B(t)$ to be MF (3.3.4).

4 Completely positive maps and complete order embeddings

In this section, we recall and/or establish some general facts about completely positive maps and complete order embeddings which will be needed in Sects. 5 and 6. A good general reference is [Pa].

4.1 Completely positive maps

Recall that if $\phi\colon\ A \to B$ is a map between C*-algebras (or self-adjoint subspaces of C*-algebras), then ϕ induces a map $\phi^{(n)}$ from $M_n(A)$ to $M_n(B)$ in the obvious way. ϕ is *positive* if $\phi(A_+) \subseteq B_+, \phi$ is *n-positive* if $\phi^{(n)}$ is positive, and ϕ is *completely positive* if ϕ is n-positive for all n. A unital 2-positive map is automatically a contraction. Compositions and pointwise limits of n-positive [resp. completely positive] maps are n-positive [resp. completely positive]. If A and B are C*-algebras, with B unital, and ϕ is a completely positive contraction from A to B, then ϕ extends (uniquely) to a unital completely positive map from A^+ to B.

A *complete order embedding* from A to B is a completely positive isometry ϕ from A into B such that ϕ^{-1} is a completely positive map from $\phi(A)$ to A. A surjective complete order embedding is a *complete order isomorphism*.

Note that for a completely positive map ϕ to be a complete order embedding, it is not sufficient in general that ϕ be isometric (we are indebted to V. Paulsen for this observation.) It is sufficient (and also necessary) that ϕ be completely isometric.

Completely positive maps are natural and important, largely because of the characterization given by Stinespring's Theorem, which we recall in the form we will use. First we establish some useful notation:

Definition 4.1.1. *Let $\phi\colon\ A \to B$ be a map, and p a projection in B. Write ϕ_p for the map from A to pBp defined by $\phi_p(x) = p\phi(x)p$. ϕ_p is the* **compression** *of ϕ to p (or to pBp). If ϕ is n-positive* [*resp. completely positive*], *then ϕ_p is n-positive* [*resp. completely positive*].

Theorem 4.1.2. (Stinespring [St]) *Let A be a unital C^*-algebra, and ϕ a completely positive unital map from A to* B(H)*. Then ϕ is the compression of a homomorphism, i.e. there is a Hilbert space $\tilde{\mathrm{H}}$ containing* H *and a *-homomorphism $\pi\colon\ A \to \mathrm{B}(\tilde{\mathrm{H}})$ such that $\phi = \pi_p$, where p is the projection of $\tilde{\mathrm{H}}$ to* H*. $\tilde{\mathrm{H}}$ and π are canonically associated with ϕ, and are minimal in the sense that if $\hat{\mathrm{H}}$ is any Hilbert space containing* H, *and $\rho\colon\ A \to \mathrm{B}(\hat{\mathrm{H}})$ is a *-homomorphism whose compression to* H *is ϕ, then there is an isometric map of $\tilde{\mathrm{H}}$ onto the smallest subspace of $\hat{\mathrm{H}}$ which contains* H *and is invariant under ρ, such that the restriction of ρ to $\tilde{\mathrm{H}}$ is π. If A and* H *are finite-dimensional, then $\tilde{\mathrm{H}}$ is finite-dimensional.*

Outline of proof. Define a pre-inner product on the algebraic tensor product $A \otimes \mathrm{H}$ by $\langle a\otimes\xi, b\otimes\eta\rangle = \langle\phi(b^*a)\xi,\eta\rangle$. Let $\tilde{\mathrm{H}}$ be the completion, and embed H into $\tilde{\mathrm{H}}$ by sending ξ to $1\otimes\xi$. Let $\pi(x)(a\otimes\xi) = xa\otimes\xi$. If $(\hat{\mathrm{H}},\rho)$ is as above, map $\tilde{\mathrm{H}}$ to $\hat{\mathrm{H}}$ by sending $a\otimes\xi$ to $\rho(a)\xi$. $\square$

Corollary 4.1.3. *Let ϕ be a completely positive contraction between C^*-algebras A and B. If $x \in A$, then $\phi(x^*x) \geqq \phi(x)^*\phi(x)$, and if $\phi(x^*x) = \phi(x)^*\phi(x)$, then $\phi(yx) = \phi(y)\phi(x)$ for all $y \in A$.*

Proof. We may assume A is unital, $B = \mathrm{B(H)}$, and ϕ is unital. $\phi(x^*x) = \phi(x)^*\phi(x)$ if and only if $\pi(x)p = p\pi(x)p$. □

Corollary 4.1.4. *A complete order isomorphism between C^*-algebras is a *-isomorphism.*

Proof. Let $\phi\colon A \to B$ be a completely positive surjective isometry, with completely positive inverse ϕ^{-1}. Then if $x \in A$ and $y = \phi(x)$, we have

$$y^*y = \phi(\phi^{-1}(y^*y)) \geqq \phi(\phi^{-1}(y)^*\phi^{-1}(y)) = \phi(x^*x) \geqq \phi(x)^*\phi(x) = y^*y$$

so $\phi(x^*x) = \phi(x)^*\phi(x)$. The result now follows from 4.1.3. □

4.1.3 and 4.1.4 are actually true for 2-positive maps [Ch1].

Corollary 4.1.5. *Let $\phi : A \to B$ be a completely positive contraction, and $x \in A$, $\|x\| = 1$. If $\phi(x)$ is unitary in B, or if $x \geqq 0$ and $\phi(x)$ is a projection in B, then $\phi(xy) = \phi(x)\phi(y)$ and $\phi(yx) = \phi(y)\phi(x)$ for all $y \in A$.*

Proof. If $0 \leqq x \leqq 1$ and $\phi(x)$ is a projection, then $\phi(x) \geqq \phi(x^2) \geqq \phi(x)^2 = \phi(x)$. If $\phi(x)$ is unitary, then $1 = \phi(x)^*\phi(x) \leqq \phi(x^*x) \leqq 1$. So in both cases $\phi(yx) = \phi(y)\phi(x)$ for all y. In both cases we also have $\phi(x)\phi(x^*) = \phi(xx^*)$, so $\phi(xy) = [\phi(y^*x^*)]^* = [\phi(y^*)\phi(x^*)]^* = \phi(x)\phi(y)$ for all y. □

We then get the following variant of 4.1.4.

Corollary 4.1.6. *Let A and B be C^*-algebras. Assume B is unital and that every extreme point of the unit ball of B is unitary* (*cf.* [*Pd*, 1.4.7].) *If ϕ is a completely positive isometry from A onto B, then A is unital and ϕ is a *-isomorphism.* (*In particular, ϕ^{-1} is completely positive.*)

Proof. ϕ and ϕ^{-1} send extreme points of the unit ball to extreme points of the unit ball, so the unit ball of A contains extreme points and hence A is unital [Pd, 1.4.7]. Let x be a unitary in A. Then $\phi(x)$ is extreme in the unit ball of B, hence unitary, and so ϕ is multiplicative on x in the sense of 4.1.5. Thus ϕ is multiplicative on all of A since the unitaries in A span A. □

Corollary 4.17 (cf. [Cr]). *For any $d > 0$ and $\varepsilon > 0$ there is a $\delta > 0$ such that, whenever A is a C^*-algebra, D is a finite-dimensional C^*-subalgebra of A with $\dim(D) \leqq d$,* H *is a Hilbert space, q is a projection in* B(H), *and $\phi\colon A \to$* B(H) *is a completely positive contraction with ϕ_q approximately multiplicative within δ on D* (2.3.3), *then there is a completely positive contraction ψ from A to* B(H) *with ψ_q exactly multiplicative on D and $\|\phi - \psi\| < \varepsilon$.*

Proof. By replacing A, D by A^+, D^+ if necessary, we may assume A is unital, D contains the unit of A, and ϕ is unital. If ϕ_q is approximately multiplicative within δ, then $\pi(D)$ approximately commutes within δ with q. There is then a projection $q' \in \mathrm{B}(\tilde{\mathrm{H}})$, commuting with $\pi(D)$, with $\|q - q'\| < \delta'$, with δ' depending on d and δ [BKR, Sect. 2]. Then there is a unitary $u \in \mathrm{B}(\tilde{\mathrm{H}})$ with $\|u - 1\| < 2\delta'$ and $q' = uqu^*$ [Bl 3, 4.3.2 and 4.6.5]. Set $\rho(x) = u^*\pi(x)u$ and $\psi = \rho_p$. $\|\phi - \psi\| < 4\delta' < \varepsilon$ if δ is small enough (depending on d.) $\rho(D)$ commutes with q, and hence ρ_q is multiplicative on D. $\psi_q = \rho_q$ since q commutes with p (because $q \leqq p$). □

Finally, we have Arveson's extension theorem (a generalization of the Hahn-Banach Theorem for states) which gives a strong form of injectivity for finite-dimensional C*-algebras (and, more generally, type I factors.) For another generalization of the Hahn-Banach Theorem, see 3.2.3.

Proposition 4.1.8. [Ar] *Let B be a finite-dimensional C^*-algebra, and let $Y \subseteq X$ be operator systems. They any completely positive contraction from Y to B can be extended to a completely positive contraction from X to B. In particular, if ϕ is a complete order embedding from B into an operator system X, then there is a completely positive idempotent contraction from X onto $\phi(B)$.*

4.2 Complete order embeddings

Proposition 4.2.1. [CE 2, 3.1] *Let A be a C^*-algebra, ω a completely positive idempotent map from A to A. Then $\omega(A)$ is completely order isomorphic to a C^*-algebra B, with multiplication given by $x \cdot y = \omega(xy)$, where xy is the product in A. If C is the C^*-subalgebra of A generated by $\omega(A)$, then $\omega|_C$ is a *-homomorphism from C onto B.*

Proposition 4.2.2. [CE 1, 4.1] *Let A and B be C^*-algebras, and ϕ a complete order embedding of B into A. If C is the C^*-subalgebra of A generated by $\phi(B)$, then the map $\phi^{-1}: \phi(B) \to B$ extends to a *-homomorphism from C onto B.*

Proposition 4.2.3. *Let A be an operator space, $\phi: A \to A$ a completely positive finite-rank contraction. Set $S = \{x \in A : \phi(x) = x\}$. Then there is an idempotent completely positive contraction from A onto S. So if A is a C^*-algebra, then S is completely order isomorphic to a (finite-dimensional) C^*-algebra.*

Proof. Let $C = \phi(A)$, and let Ω be the set of all completely positive contractions from A to C which are the identity on S. Then Ω is a nonempty compact convex set in the topology of pointwise convergence (since C is finite-dimensional.) Let $T: \Omega \to \Omega$ be defined by $T(\omega) = \phi \circ \omega$. Then T has a fixed point ψ by the Schauder fixed-point theorem. If $x \in A$, then $\phi(\psi(x)) = \psi(x)$, so $\psi(x) \in S$, and hence $\psi(\psi(x)) = \psi(x)$ since ψ is the identity on S. The last statement follows from 4.2.1. □

Definition 4.2.4. *A map ϕ between C^*-algebras A and B is* **supermultiplicative** *if, for all $x_1,\ldots,x_n \in A$, $\|\phi(x_1)\phi(x_2)\ldots\phi(x_n)\| \geqq \|\phi(x_1x_2\ldots x_n)\|$.*

Proposition 4.2.5. *A complete order embedding of one C^*-algebra into another is supermultiplicative.*

Proof. Follows immediately from 4.2.2, since the *-homomorphism extending ϕ^{-1} is norm-decreasing. □

The converse is true in the finite-dimensional case, and there is also an approximate version:

Proposition 4.2.6. *Let A and B be finite-dimensional C^*-algebras, and $\phi: A \to B$ a completely positive supermultiplicative contraction. Then ϕ is a complete order embedding.*

Proof. The proof is a simplified version of the proof of 4.2.8 below, taking $\varepsilon = \delta = 0$. □

Corollary 4.2.7. *If $B_1,\ldots,B_m$ are finite-dimensional C^*-algebras with $B = B_1 \oplus \ldots \oplus B_{m'}$ and $\phi = \phi_1 \oplus \ldots \oplus \phi_m$ is a complete order embedding of a matrix algebra into B, then at least one ϕ_j is a complete order embedding.*

Proposition 4.2.8. *Let $d > 0$ and $\varepsilon > 0$. Then there is a $\delta > 0$ such that, whenever A is a finite-dimensional C^*-algebra with $dim(A) \leqq d$ and matrix units $\{e^r_{ij}: 1 \leqq i,j \leqq k_r, 1 \leqq r \leqq n\}$, B is a finite-dimensional C^*-algebra, and ϕ is a completely positive contraction from A to B with $\|\phi(e^r_{12})\phi(e^r_{23})\ldots\phi(e^r_{k_r-1,k_r})\| \geqq 1-\delta$ for all r, then there is a complete order embedding ψ from A to B with $\|\psi - \phi\| < \varepsilon$.*

Proof. Fix A and $\varepsilon > 0$, and let ϕ be as above for some $\delta, 0 < \delta < 1$. To simplify notation, set $y^r_{ij} = \phi(e^r_{ij})$ for each i,j,r; then $(y^r_{ij})^* = y^r_{ji}$. For each r let p_r be the spectral projection of $y^r_{12}y^r_{23}\ldots y^r_{k_r-1,k_r}y^r_{k_r,k_r-1}\ldots y^r_{32}y^r_{21}$ corresponding to the largest eigenvalue $[\geqq(1-\delta)^2]$. Then $p_r \neq 0$, and

$$(1-\delta)^2 p_r \leqq p_r y^r_{12}y^r_{23}\ldots y^r_{j-1,j}y^r_{j,j-1}\ldots y^r_{21}p_r \leqq p_r$$

for $1 < j \leqq k_r$, since by 4.1.3

$$\begin{aligned} y^r_{12}y^r_{23}\ldots y^r_{k_r-1,k_r}y^r_{k_r,k_r-1}\ldots y^r_{21} &\leqq y^r_{12}\ldots y^r_{k_r-2,k_r-1}y^r_{k_r-1,k_r-2}\ldots y^r_{21} \\ &\leqq \ldots \leqq y^r_{12}y^r_{21} \leqq (y^r_{11})^2 \end{aligned}$$

So if $z^r_{11} = p_r y^r_{11}, z_{1j} = p_r y^r_{12}y^r_{23}\ldots y^r_{j-1,j}$ for $1 < j \leqq k_r$, then z^r_{1j} is approximately a partial isometry with range projection p_r. The support projections are also approximately orthogonal since

$$z^{r*}_{1j}z^r_{1j} \leqq y^r_{j,j-1}y^r_{j-2,j-1}\ldots y^r_{21}y^r_{12}\ldots y^r_{j-1,j} \leqq (y^r_{jj})^2 \leqq y^r_{jj}$$

by 4.1.3, and $\sum_{j,r} y^r_{jj} \leqq 1$. Thus, if $z^r_{ij} = z^{r*}_{1i}z^r_{1j}$, then $\{z^r_{ij}, 1-\sum_{j,r} z^r_{jj}\}$ forms a set of approximate matrix units of type $A \oplus C$ in B within δ_1 [BKR, 2.2], where

δ_1 may be made as small as desired by making δ sufficiently small. Then by [BKR, 2.3] this set of approximate matrix units can be moved by not more than a small δ_2 to yield a set of exact matrix units $\{f_{ij}^r, q\}$ in B generating a unital subalgebra isomorphic to $A \oplus C$. q approximately commutes with $\phi(A)$ within δ_3, and $\|f_{ij}^r - (1-q)y_{ij}^r(1-q)\| < \delta_3$ for all i, j, r, where δ_3 can also be made small by choosing δ sufficiently small. If we set $\rho(e_{ij}^r) = f_{ij}^r$, then ρ is an injective *-homomorphism from A to $(1-q)B(1-q)$, and if $\psi = \rho + \phi_q$, then ψ is a complete order embedding which can be made as close as desired to ϕ by choosing δ small enough. □

Remark 4.2.9. In 4.2.8, the requirement that B be finite-dimensional can be relaxed to only require that B have real rank zero, by making a slight modification in the selection of the projection p_r. It appears that the result may be true with no restrictions on B, but the proof would be considerably more complicated.

Recall that every complete order embedding of one finite-dimensional C*-algebra into another is a direct sum of an injective, *-homomorphism and another completely positive contraction:

Proposition 4.2.10. [CE 2, 7.1] *Let A and B be finite-dimensional C*-algebras, and ϕ a complete order embedding of A into B. Then there is a projection h in B such that $\phi = \phi_h + \phi_{1-h}$ and such that ϕ_h is an injective *-homomorphism from A to hBh. (Conversely, any completely positive contraction with an injective *-homomorphism as a direct summand is a complete order embedding.)*

As a corollary, we get a version of 4.1.7 for complete order embeddings:

Corollary 4.2.11. *For any $d > 0$ and $\varepsilon > 0$ there is a $\delta > 0$ such that, whenever A is a finite-dimensional C*-algebra, D is a C*-subalgebra of A with $dim(D) \leqq d$, H is a finite-dimensional Hilbert space, q is a projection in* B(H), *and $\phi\colon A \to$ B(H) is a completely positive contraction with ϕ_q a complete order embedding on A which is approximately multiplicative within δ on D, then there is a completely positive contraction ψ from A to* B(H) *with ψ_q a complete order embedding on A which is exactly multiplicative on D and $\|\phi - \psi\| < \varepsilon$.*

Proof. Let $d > 0$ and $\varepsilon > 0$, and choose δ as in 4.1.7. Suppose A, D, H, q, and ϕ are as above. Let $h \leqq q$ be such that $\phi_q = (\phi_q)_h + (\phi_q)_{1-h}$ and $(\phi_q)_h = \phi_h$ is an injective *-homomorphism. Then $\phi = \phi_h + \phi_{1-h}$, and ϕ_{1-h} is a completely positive contraction from A to $(1-h)\mathrm{B(H)}(1-h) \cong \mathrm{B}((1-h)\mathrm{H})$ such that $(\phi_{1-h})_{q-h} = \phi_{q-h} = (\phi_q)_{q-h}$ is approximately multiplicative within δ on D, so there is a completely positive contraction θ from A to $(1-h)\mathrm{B(H)}(1-h)$ such that θ_{q-h} is exactly multiplicative on D, with $\|\theta - \phi_{1-h}\| < \varepsilon$. Then $\psi = \phi_h + \theta$ has the desired properties. □

4.3 CP-maps and states

There is a one-one correspondence between completely positive maps from a C*-algebra A to M_n and positive linear functionals on $M_n(A)$ [Pa, Sect. 5], given as follows: if $\phi\colon A \to \mathrm{M}_n$, set $s_\phi((a_{ij})) = n^{-1}\sum_{i,j}(\phi(a_{ij})_{ij})$, and if $s\colon M_n(A) \to C$, set $(\phi_s(a))_{ij} = s(a \otimes e_{ij})$. (If A has a unit and ϕ is unital, then s_ϕ is unital, but if s is a state then ϕ_s is not necessarily unital.) This correspondence is a homeomorphism for the weak-* topologies (topology of pointwise convergence.) As a consequence, many results about states have analogs for completely positive maps into matrix algebras. We will need one such result. We first describe the result for states, which is probably well known but does not seem to be explicitly written in the literature.

Proposition 4.3.1. *Let A be a C^*-algebra, J a closed 2-sided ideal. Let $R = \{s \in S(A)\colon s|_J$ is a state on $J\}$. Then the weak-* closure of R in $S(A)$ is $\{s\colon s(J^\perp) = 0\}$. If J is an essential ideal in A, then R is dense in $S(A)$.*

Proof. This follows immediately from [Dx, 3.4.2], taking the set (π_i) to be the set of irreducible representations of A which are nonzero on J. □

Definition 4.3.2. *An increasing approximate identity $\langle p_i\rangle$ for a C^*-algebra is* **almost idempotent** *if $p_i p_j = p_i$ for $i < j$.*

An increasing approximate identity of projections is almost idempotent. Every σ-unital C*-algebra has an almost idempotent approximate identity, which can be easily constructed from a strictly positive element by functional calculus.

Proposition 4.3.3. *Let $\langle p_i\rangle$ be an almost idempotent approximate identity in a C^*-algebra B. Then $\{s \in S(B)\colon s(p_i) = 1$ for sufficiently large $i\}$ is weak-* dense in $S(B)$.*

Proof. If $s \in S(B)$, then $s(p_i) \to 1$. If $s(p_i) > 0$, define $s_i \in S(B)$ by $s_i(x) = s(p_i)^{-1}s(p_i^{1/2}xp_i^{1/2})$. Then $s_i \to s$, and $s_i(p_j) = 1$ for $i < j$. □

Proposition 4.3.4. *Let J be an essential ideal in a C^*-algebra A, and let $\langle p_i\rangle$ be an almost idempotent approximate identity for J. Then $\{s \in S(A)\colon s(p_i) = 1$ for some $i\}$ is weak-* dense in $S(A)$.*

Proof. Combine 4.3.1 and 4.3.3. □

Corollary 4.3.5. *Let J be an essential ideal in a unital C^*-algebra A, and $\langle p_i\rangle$ an almost idempotent approximate identity for J. If B is a finite-dimensional C^*-algebra, let $CP_1(A,B)$ be the set of all completely positive unital maps from A to B. Then $\{\phi \in CP_1(A,B)\colon \phi(p_i) = 1$ for some $i\}$ is dense in $CP_1(A,B)$ in the topology of pointwise convergence.*

Proof. This almost follows immediately from 4.3.4. We may assume B is a matrix algebra M_n by considering each central summand separately. If $\phi \in CP_1(A,B)$, write $s_\phi \in S(M_n(A))$ as a limit of s_i, where $s_i(p_i \otimes 1) = 1$. Then $\phi_{s_i} \to \phi$. However, ϕ_{s_i} is not necessarily unital. But $\phi_{s_i}(p_i) = \phi_{s_i}(1) \to 1$, so for sufficiently large $i, \phi_{s_i}(p_i)$ is invertible, and if ψ_i is defined by $\psi_i(x) = \phi_{s_i}(p_i)^{-1/2}\phi_{s_i}(x)\phi_{s_i}(p_i)^{-1/2}$, then $\psi_i \in CP_1(A,B)$ and $\psi_i \to \phi$. □

Corollary 4.3.6. *Let J be an essential ideal in a C^*-algebra A, and $\langle p_i \rangle$ an almost idempotent approximate identity for J. If B is a finite-dimensional C^*-algebra, let $CP_c(A,B)$ be the set of all completely positive contractions from A to B. Then $\{\phi \in CP_c(A,B)\colon \phi(p_i) = 1$ for some $i\}$ is dense in $CP_c(A,B)$ in the topology of pointwise convergence.*

Proof. $CP_c(A,B)$ is in natural 1-1 correspondence with $CP_1(A^+,B)$. □

5 Nuclearity and NF algebras

5.1 Generalized inductive systems and nuclearity

We first recall the definition and most important properties of nuclear C*-algebras, summarized in the following proposition, which is an amalgamation of several important and deep theorems:

Proposition 5.1.1. *Let A be a C^*-algebra. The following are equivalent*:

(i) The identity map on A can be approximated in the point-norm topology by completely positive finite-rank contractions from A to A, i.e. given $x_1, \ldots, x_n \in A$ and $\varepsilon > 0$, there is a completely positive finite-rank contraction $\phi\colon A \to A$ with $\|x_i - \phi(x_i)\| < \varepsilon$ for $1 \leqq i \leqq n$

(ii) The identity map on A can be approximated in the point-norm topology by completely positive contractions through finite-dimensional C^*-algebras [matrix algebras], i.e. given $x_1, \ldots, x_n \in A$ and $\varepsilon > 0$, there is a finite-dimensional C^*-algebra [matrix algebra] B and completely positive contractions $\alpha\colon A \to B$ and $\beta\colon B \to A$ such that $\|x_i - \beta \circ \alpha(x_i)\| < \varepsilon$ for $1 \leqq i \leqq n$

(iii) For every C^*-algebra B, the algebraic tensor product $A \odot B$ has a unique C^*-cross norm

(iv) A^{**} is an injective von Neumann algebra

(v) A is C^*-amenable, i.e. every derivation from A into a dual normal A-bimodule is inner.

A C*-algebra with these properties is called *nuclear*.

The class of nuclear C*-algebras includes all type I C*-algebras and is closed under inductive limits, tensor products, and crossed products by amenable groups. For more about nuclear C*-algebras, and attributions and proofs of the various parts of 5.1.1, see [La] and its references.

In order to deal with inductive limits of NF algebras, we need the next two facts:

Lemma 5.1.2. *Let $(A_n, \phi_{m,n})$ be a generalized inductive system of C^*-algebras, with the $\phi_{m,n}$ complete order embeddings. If each A_n is nuclear, then A is nuclear.*

Proof. Let $x_1, \ldots, x_k \in A$ and $\varepsilon > 0$. Choose n large enough that there are $y_1, \ldots, y_k \in A_n$ with $\|x_j - \phi_n(y_j)\| < \varepsilon/3$ for $1 \leqq j \leqq k$. Choose a finite-dimensional C*-algebra B and completely positive contractions $\alpha\colon A_n \to B$ and $\beta\colon B \to A_n$ such that $\|y_j - \beta \circ \alpha(y_j)\| < \varepsilon/3$ for all j. Extend the completely positive contraction $\alpha \circ \phi_n^{-1}\colon \phi_n(A_n) \to B$ to a completely positive contraction $\omega\colon A \to B$ by 4.1.8. Then $\theta = \phi_n \circ \beta \circ \omega$ is a completely positive finite-rank contraction from A to A, and

$$\|x_j - \theta(x_j)\| \leqq \|x_j - \phi_n(y_j)\| + \|y_j - \beta \circ \alpha(y_j)\| + \|\beta \circ \alpha(y_j) - \beta \circ \omega(x_j)\| < \varepsilon$$

for all j, so A is nuclear. □

Proposition 5.1.3. *Let $(A_n, \phi_{m,n})$ be a generalized inductive system of C^*-algebras, with the $\phi_{m,n}$ completely positive contractions. If each A_n is nuclear, then A is nuclear.*

Proof. Let $B_n = A_1 \oplus \ldots \oplus A_n$, $\psi_{n,n+1} = id_{B_n} \oplus \phi_{n,n+1}\colon B_n \to B_{n+1}$, and define $\psi_{m,n}$ for $m < n$ by composition; then $(B_n, \psi_{m,n})$ is a generalized inductive system with each B_n nuclear and each $\psi_{m,n}$ a complete order embedding, so $B = \varinjlim(B_n, \psi_{m,n})$ is nuclear, and A is a quotient of B. □

Using nuclearity, we can obtain approximate cross sections for the embedding maps in a generalized inductive system of nuclear C*-algebras:

Proposition 5.1.4. *Let $(A_n, \phi_{m,n})$ be a generalized inductive system with the $\phi_{m,n}$ completely positive contractions, and $A = \varinjlim(A_n, \phi_{m,n})$. Suppose A is nuclear (e.g. suppose each A_n is nuclear). Then there are completely positive contractions $\gamma_n\colon A \to A_n$ such that $\phi_n \circ \gamma_n$ converges to id_A in the point-norm topology, and such that the γ_n are asymptotically multiplicative, i.e. given $\{x_1, \ldots, x_k\} \subseteq A$ and $\varepsilon > 0$, then γ_n is approximately multiplicative within ε on $\{x_1, \ldots, x_k\}$ for all sufficiently large n.*

Proof. Regard A as a C*-subalgebra of $(\prod A_n)/(\oplus A_n)$ in the standard way, and let $\sigma\colon A \to \prod A_n$ be a completely positive contractive cross section for the quotient map $\pi\colon \prod A_n \to (\prod A_n)/(\oplus A_n)$ [CE 1]. Let γ_m be σ followed by projection of $\prod A_n$ onto the m'th coordinate A_m. Since σ is isometric and multiplicative modulo sequences converging to zero, the properties of the γ_n follow. □

Proposition 5.1.5. *Let $(A_n, \phi_{m,n})$ and $(B_n, \psi_{m,n})$ be generalized inductive systems for a nuclear C^*-algebra A, with each A_n and B_n separable and each $\phi_{m,n}$ and $\psi_{m,n}$ a completely positive contraction. Then the systems are asymptotically equivalent as in* 2.4.1, *with the maps α_n and β_n completely positive contractions.*

Proof. Let $\gamma_n\colon A \to A_n$ and $\theta_n\colon A \to B_n$ be cross sections as in 5.1.4. One can then inductively define $\alpha_n\colon A_{r_n} \to B_{s_n}$ to be $\theta_{s_n} \circ \phi_{r_n}$ for sufficiently large s_n, and then $\beta_n\colon B_{s_n} \to A_{r_{n+1}}$ by $\beta_n = \gamma_{r_{n+1}} \circ \psi_{s_n}$ for sufficiently large r_{n+1}. If s_n and r_{n+1}

are chosen sufficiently large at each stage, the α_n and β_n give an asymptotic intertwining between the systems. The technical details are straightforward but messy, and are left to the reader. □

It is worth noting that when intertwining generalized inductive systems with completely positive contractions, asymptotic multiplicativity is automatic:

Proposition 5.1.6. *Let $(A_n, \phi_{m,n})$ and $(B_n, \psi_{m,n})$ be *-linear generalized inductive systems of C^*-algebras, with all $\phi_{m,n}$ and $\psi_{m,n}$ completely positive contractions. If (α_n) and (β_n) are sequences of completely positive linear contractions satisfying* (1), (3), *and* (4) *of* 2.4.1, *then the α_n and β_n are automatically asymptotically multiplicative and form an asymptotic intertwining.*

Proof. The induced maps α and β are complete order isomorphisms between A and B, hence are C*-isomorphisms by 4.1.4. □

5.2 Characterization of NF algebras

Definition 5.2.1. *A separable C^*-algebra A is an NF algebra if A is isomorphic to the inductive limit of a generalized inductive system $(A_n, \phi_{m,n})$ where the A_n are finite-dimensional and the $\phi_{m,n}$ are completely positive contractions. Such a system is called an* **NF system** *for A. If each $\phi_{m,n}$ is a complete order embedding, the system is called a* **strong NF system** *and A is called a* **strong NF algebra**.

Theorem 5.2.2. *Let A be a separable C^*-algebra. The following are equivalent:*

(i) *A is an NF algebra*
(ii) *A is a nuclear MF algebra*
(iii) *A is nuclear and can be embedded as a C^*-subalgebra of $(\prod \mathrm{M}_{k_n})/(\oplus \mathrm{M}_{k_n})$ for some sequence $\langle k_n \rangle$*
(iv) *A is nuclear and has an essential quasidiagonal extension by the compact operators* K
(v) *A is nuclear and (weakly) quasidiagonal*
(vi) *The identity map on A can be approximated in the point-norm topology by completely positive approximately multiplicative contractions through finite-dimensional C^*-algebras, i.e. given $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$, there is a finite-dimensional C^*-algebra B and completely positive contractions $\alpha\colon A \to B$ and $\beta\colon B \to A$ such that $\|x_i - \beta \circ \alpha(x_i)\| < \varepsilon$ and $\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| < \varepsilon$ for all i, j.*
(vii) *A is nuclear and there is a continuous field of C^*-algebras $\langle B(t) \rangle$ over $\mathbb{N} \cup \{\infty\}$ with $B(\infty) \cong A$ and $B(n)$ finite-dimensional for $n < \infty$.*
(viii) *A is nuclear and there is a continuous field of C^*-algebras $\langle B(t) \rangle$ over $\mathbb{N} \cup \{\infty\}$ with $B(\infty) \cong A$ and $B(n) = \mathrm{M}_{k_n}$ for $n < \infty$.*

Proof. $(i) \Rightarrow (ii)$: An NF algebra is obviously MF, and is nuclear by 5.1.2.

$(ii), (iii), (iv), (vii)$, and $(viii)$ are equivalent by 3.2.2.

$(iv) \Rightarrow (v)$: A nuclear C*-algebra which has a quasidiagonal extension by the compacts is itself quasidiagonal by [DHS, 4.2].
(One could also use 3.1.4 to show (vi) $\Rightarrow$ (v).)

$(v) \Rightarrow (iv)$ is trivial.

$(iii) \Rightarrow (vi)$ and $(vi) \Rightarrow (i)$ are the most involved steps.

$(iii) \Rightarrow (vi)$: Suppose A is a C*-subalgebra of $(\prod \mathrm{M}_{k_n})/(\oplus \mathrm{M}_{k_n})$ for some sequence $\langle k_n \rangle$, and let $\pi\colon \prod \mathrm{M}_{k_n} \to (\prod \mathrm{M}_{k_n})/(\oplus \mathrm{M}_{k_n})$ be the quotient map. π has a completely positive contractive cross section $\sigma : A \to \prod \mathrm{M}_{k_n}$ by nuclearity of A (σ is actually a complete order embedding). Let $x_1, \ldots, x_m \in A$ and $\varepsilon > 0$. By nuclearity of A, choose a finite-dimensional C*-algebra D and completely positive contractions $\phi\colon A \to D$ and $\psi\colon D \to A$ such that $\|x_j - \psi \circ \phi(x_j)\| < \varepsilon/2$ for $1 \leqq j \leqq m$. Since σ is multiplicative modulo sequences converging to zero, there is an r such that $\rho_r \circ \sigma\colon A \to \prod_{n=r}^{\infty} \mathrm{M}_{k_n}$ (2.2.2) is approximately multiplicative within ε on $\{x_1, \ldots, x_m\}$. $\gamma = \rho_r \circ \sigma$ is a complete order embedding from A into $\prod_{n=r}^{\infty} \mathrm{M}_{k_n}$. By 4.1.8 $\phi \circ \gamma^{-1}\colon \gamma(A) \to D$ extends to a completely positive contraction θ from $\prod_{n=r}^{\infty} \mathrm{M}_{k_n}$ to D. Let p_i be the projection in $\prod_{n=r}^{\infty} \mathrm{M}_{k_n}$ which is 1 in the first i places and 0 elsewhere; then $\langle p_i \rangle$ is an approximate identity for the essential ideal $\bigoplus_{n=r}^{\infty} \mathrm{M}_{k_n}$ of $\prod_{n=r}^{\infty} \mathrm{M}_{k_n}$. Thus by 4.3.6 there is a completely positive contraction from $\prod_{n=r}^{\infty} \mathrm{M}_{k_n}$ to D supported on some p_i and approximating θ, i.e. there is an $s > r$ and a completely positive (unital) contraction ω from $B = \prod_{n=r}^{s} \mathrm{M}_{k_n}$ to D such that $\|\omega \circ \rho_r^s \circ \sigma(x_j) - \phi(x_j)\| < \varepsilon/2$ for $1 \leqq j \leqq m$. Set $\alpha = \rho_r^s \circ \sigma\colon A \to B$ and $\beta = \psi \circ \omega\colon B \to A$. Then

$$\|\beta \circ \alpha(x_j) - x_j\| \leqq \|\psi \circ \omega \circ \rho_r^s \circ \sigma(x_j) - \psi \circ \phi(x_j)\| + \|\psi \circ \phi(x_j) - x_j\| < \varepsilon$$

for $1 \leqq j \leqq m$, and α is approximately multiplicative within ε on $\{x_1, \ldots, x_m\}$ since $\rho_r \circ \sigma$ is.

$(vi) \Rightarrow (i)$: Define a sequence A_n of finite-dimensional C*-algebras and completely positive contractions $\phi_{n,n+1} : A_n \to A_{n+1}$ inductively as follows. Let $\{x_1, x_2, \ldots\}$ be a dense set in A; let $\alpha_1 : A \to A_1$ and $\beta_1 : A_1 \to A$ be completely positive contractions with $\|x_1 - \beta_1 \circ \alpha_1(x_1)\| < 1/2$, for some finite-dimensional C*-algebra A_1. Suppose $A_1, \ldots, A_n, \phi_{1,2}, \ldots, \phi_{n-1,n}$ have been defined, with $\alpha_m : A \to A_m$ and $\beta_m : A_m \to A$. Let $\alpha_{n+1} : A \to A_{n+1}$ and $\beta_{n+1} : A_{n+1} \to A$ be completely positive contractions with A_{n+1} a finite-dimensional C*-algebra, such that $\beta_{n+1} \circ \alpha_{n+1}$ is within 2^{-n} of the identity on $\{x_1, \ldots, x_n\}$ and on the unit ball of $\beta_n(A_n)$ and the subspace spanned by $\{xy : x, y \in \beta_n(A_n)\}$, and such that α_{n+1} is approximately multiplicative within 2^{-n} on this set. Set $\phi_{n,n+1} = \alpha_{n+1} \circ \beta_n : A_n \to A_{n+1}$, and if $m < n$, define $\phi_{m,n} : A_m \to A_n$ by composition. We have the following inequalities:

$$\|\beta_j - \beta_{j+1} \circ \phi_{j,j+1}\| = \|\beta_j - (\beta_{j+1} \circ \alpha_{j+1}) \circ \beta_j\| \leqq 2^{-j}$$

for any j, and so for $m < n$,

$$\begin{aligned}
&\|\beta_{m-1} - \beta_{n-1} \circ \phi_{m-1,n-1}\| \\
&\quad \leqq \|\beta_{m-1} - \beta_m \circ \phi_{m-1,m}\| + \|\beta_m \circ \phi_{m-1,m} - \beta_{m+1} \circ \phi_{m,m+1} \circ \phi_{m-1,m}\| \\
&\qquad + \ldots + \|\beta_{n-2} \circ \phi_{n-3,n-2} \circ \ldots \circ \phi_{m-1,m} \\
&\qquad - \beta_{n-1} \circ \phi_{n-2,n-1} \circ \cdots \circ \phi_{m-1,m}\| \\
&\quad \leqq \sum_{j=0}^{n-m+1} 2^{-m-j+1} < 2^{-m+2}
\end{aligned}$$

and thus for $k < m < n$,

$$\begin{aligned}
&\|\beta_{m-1} \circ \phi_{k,m-1} - \beta_{n-1} \circ \phi_{k,n-1}\| \\
&\quad = \|\beta_{m-1} \circ \phi_{k,m-1} - \beta_{n-1} \circ \phi_{m-1,n-1} \circ \phi_{k,m-1}\| < 2^{-m+2} .
\end{aligned}$$

Also, if ψ_1, ψ_2 are any contractive maps between normed algebras B and C, and $u, v \in B$, then

$$\begin{aligned}
&\|\psi_1(u)\psi_1(v) - \psi_2(u)\psi_2(v)\| \\
&\quad \leqq \|\psi_1(u)[\psi_1(v) - \psi_2(v)]\| + \|[\psi_1(u) - \psi_2(u)]\psi_2(v)\| \\
&\quad \leqq 2\|\psi_1 - \psi_2\| \, \|u\| \, \|v\| .
\end{aligned}$$

We claim that $(A_n, \phi_{m,n})$ is a generalized inductive system, i.e. is asymptotically multiplicative. For any k and $x, y \in A_k$, and $k < m < n$, we have, setting $z = \phi_{k,m-1}(x)$, $w = \phi_{k,m-1}(y)$ for notational simplicity:

$$\begin{aligned}
&\|\phi_{k,n}(x)\phi_{k,n}(y) - \phi_{m,n}(\phi_{k,m}(x)\phi_{k,m}(y))\| \\
&\quad \leqq \|[\alpha_n \circ \beta_{n-1}(\phi_{k,n-1}(x))][\alpha_n \circ \beta_{n-1}(\phi_{k,n-1}(y))] \\
&\qquad\quad - \alpha_n([\beta_{n-1} \circ \phi_{k,n-1}(x)][\beta_{n-1} \circ \phi_{k,n-1}(y)])\| \\
&\qquad + \|\alpha_n([\beta_{n-1} \circ \phi_{k,n-1}(x)][\beta_{n-1} \circ \phi_{k,n-1}(y)]) \\
&\qquad\quad - \alpha_n([\beta_{m-1} \circ \phi_{k,m-1}(x)][\beta_{m-1} \circ \phi_{k,m-1}(y)])\| \\
&\qquad + \|\alpha_n([\beta_{m-1}(z)][\beta_{m-1}(w)]) - \alpha_n(\beta_m \circ \alpha_m([\beta_{m-1}(z)][\beta_{m-1}(w)]))\| \\
&\qquad + \|\alpha_n(\beta_m \circ \alpha_m([\beta_{m-1}(z)][\beta_{m-1}(w)])) \\
&\qquad\quad - \alpha_n(\beta_m([\alpha_m \circ \beta_{m-1}(z)][\alpha_m \circ \beta_{m-1}(w)]))\| \\
&\qquad + \|\alpha_n(\beta_m(\phi_{k,m}(x)\phi_{k,m}(y))) - \alpha_n(\beta_{n-1} \circ \phi_{m,n-1}(\phi_{k,m}(x)\phi_{k,m}(y)))\| \\
&\quad \leqq (2^{-n+1} + 2 \cdot 2^{-m+2} + 2^{-m+1} + 2^{-m+1} + 2^{-m+2})\|x\| \, \|y\| \\
&\quad \leqq 9 \cdot 2^{-m+1}\|x\| \, \|y\| .
\end{aligned}$$

This suffices to show that the $\phi_{m,n}$ are asymptotically multiplicative.

The maps α_n and β_n define an asymptotic intertwining in the sense of 2.4.2 between the system $(A_n, \phi_{m,n})$ and the constant system (A, id), taking the subspace S_n in the n'th copy of A to be $\beta_n(A_n)$. Thus the inductive limits are isomorphic. □

Contrast condition 5.2.2(vi) with the condition of 3.1.4. The separable quasidiagonal C*-algebras form a class containing the NF algebras and contained in the MF algebras, and distinct from either (for example, $C^*(\mathrm{F}_2)$ is quasidiagonal but not NF, and the example of [Ws, Prop. 5] is MF but not quasidiagonal.)

5.3 Properties of NF algebras

Although a C*-subalgebra of an NF algebra need not be NF because it is not necessarily nuclear, we have:

Corollary 5.3.1. *A nuclear C*-subalgebra of an NF algebra is NF.*

Corollary 5.3.2. *Every nuclear residually finite-dimensional C*-algebra* (*in particular*, *every subhomogeneous C*-algebra*) *is an NF algebra.*

Corollary 5.3.3. *If A is any separable nuclear C*-algebra*, *then CA and SA* (3.3.4) *are NF algebras.*

Corollary 5.3.4. *Every separable nuclear C*-algebra is a quotient of an NF algebra.*

Proof. A is a quotient of CA. As an alternate proof, by [Kb] A is a quotient of a C*-subalgebra B of a UHF algebra by an AF ideal J; B is MF by 3.3.1 and nuclear since it is an extension of A by J. □

Corollary 5.3.5. *Let $(A_n, \phi_{m,n})$ be a generalized inductive system of C*-algebras*, *with each $\phi_{m,n}$ a completely positive contraction. If each A_n is an NF algebra*, *then A is an NF algebra.*

Proof. Combine 5.1.3 and 3.4.4. □

Remark 5.3.6. 5.1.4 can be used to give a simple proof of 5.3.5 from 5.1.3 without using 3.4.4: given $x_1, \dots, x_k \in A$ and $\varepsilon > 0$, choose n so that $\phi_n \circ \gamma_n$ is almost the identity and γ_n is almost multiplicative on $x_1, \dots, x_k$, and choose $\alpha\colon A_n \to B$ and $\beta\colon B \to A_n$ to be completely positive contractions with $\beta \circ \alpha$ almost the identity and α almost multiplicative on $\gamma_n(x_1), \dots, \gamma_n(x_k)$; then $\alpha \circ \gamma_n\colon A \to B$ and $\phi_n \circ \beta\colon B \to A$ are maps as in 5.2.2(vi).

Corollary 5.3.7. *Every ASH algebra is NF.*

Not every NF algebra is ASH, since an ASH algebra is residually stably finite and an NF algebra is not necessarily residually stably finite by 5.3.4. The examples of [Bn], [BD], and [DL] are residually stably finite and NF, but not ASH; the ones from [Bn] and [BD] are also not quasidiagonal in their natural representations, hence are not strongly quasidiagonal C*-algebras. See [Sa] for related examples.

Corollary 5.3.8. *Let* $(A_n, \phi_{m,n})$ *be an NF system, and* $A = \varinjlim(A_n, \phi_{m,n})$. *Then there are asymptotic cross sections* γ_n *as in* 5.1.4.

Proposition 5.3.9. *Any two NF systems for an NF algebra are asymptotically equivalent as in* 2.4.1, *with the maps* α_n *and* β_n *completely positive contractions.*

Proof. This is a special case of 5.1.5. □

The next proposition is of interest in constructing "good" NF systems with respect to AF C*-subalgebras.

Proposition 5.3.10. *Let* A *be an NF algebra, and let* $\langle B_n \rangle$ *be an increasing sequence of finite-dimensional* C^**-subalgebras of* A. *Then there is an NF system* $(A_n, \phi_{m,n})$ *for* A *and* C^**-subalgebras* C_n *of* A_n, *with* $C_n \cong B_n$, *such that* $\phi_{n,n+1}$ *is exactly multiplicative on* C_n, $\phi_{n,n+1}(C_n) \subseteq C_{n+1}$, *and* $\phi_n(C_n) = B_n$.

Proof. Fix an NF system $(D_n, \psi_{m,n})$ for A, and let $\gamma_n \colon A \to D_n$ be as in 5.1.4.

First do the following construction inductively on n. Let $\{e^r_{ij}(n)\}$ be a set of matrix units for B_n, and set $d_n = dim\, B_n$. Choose $\eta_n \leqq 2^{-n}$ such that, whenever $\{f^r_{ij}\}$ and $\{g^r_{ij}\}$ are matrix units for copies of B_n inside a unital C*-algebra D, with $\|f^r_{ij} - g^r_{ij}\| < 4\eta_n$ for all i, j, r, there is a unitary $u \in D$ with $\|u - 1\| < 2^{-n}$ and $g^r_{ij} = u^* f^r_{ij} u$ for all i, j, r.

Then choose $\varepsilon_n \leqq \eta_n$, $\varepsilon_n \leqq \varepsilon_{n-1}$, such that whenever C and D are finite-dimensional C*-algebras with C containing (a copy of) B_n, and $\phi \colon C \to D$ is a completely positive contraction which is approximately multiplicative within ε_n on B_n, there is a completely positive contraction $\theta \colon C \to D$ with $\|\theta - \phi\| < \eta_n$ and θ exactly multiplicative on B_n (4.1.7).

Next choose $\delta_n \leqq \varepsilon_n/2$ such that whenever $\{æ^r_{ij}\}$ are approximate matrix units within δ_n of type B_n in a C*-algebra $Æ$, then there are exact matrix units $\{f^r_{ij}\}$ in $Æ$ with $\|æ^r_{ij} - f^r_{ij}\| < \varepsilon_n/2d_n$ [BKR, 2.3].

Then choose $k_n > k_{n-1}$ such that $\{\psi_{k,m} \circ \gamma_k(e^r_{ij}(n))\}$ is a set of approximate matrix units within δ_n, and that $\|\psi_{k,m} \circ \gamma_k(e^r_{ij}(n)) - \gamma_m(e^r_{ij}(n))\| < \eta_n$, for all $k_n \leqq k < m$. By replacing k_n by $k > k_n$ and γ_{k_n} by $\psi_{k_n,k} \circ \gamma_{k_n}$ if necessary, we may assume that $\psi_{k_n,m}$ is approximately multiplicative within $\varepsilon_n/2d_n$ on $\{\gamma_{k_n}(e^r_{ij}(n))\}$ for all $m > k_n$.

Set $A_n = D_{k_n}$. By the choice of δ_n there are exact matrix units $\{f^r_{ij}(n)\}$ in A_n, of type B_n, with $\|f^r_{ij}(n) - \gamma_{k_n}(e^r_{ij}(n))\| < \varepsilon_n/2d_n$. Let C_n be the span of $\{f^r_{ij}(n)\}$. This completes the first inductive construction.

We now do a second inductive construction. Fix n. $\psi_{k_n,k_{n+1}}$ is approximately multiplicative on $\{\gamma_{k_n}(e^r_{ij}(n))\}$ within $\varepsilon_n/2d_n$, and since $\|f^r_{ij}(n) - \gamma_{k_n}(e^r_{ij}(n))\| < \varepsilon_n/2d_n$, it follows that $\psi_{k_n,k_{n+1}}$ is approximately multiplicative on C_n within ε_n. Thus there is a completely positive contraction $\theta_{n,n+1} \colon A_n \to A_{n+1}$ with $\|\theta_{n,n+1} - \psi_{k_n,k_{n+1}}\| < \eta_n$, and $\theta_{n,n+1}$ exactly multiplicative on C_n. Let $\{f^r_{ij}(n)\}$ be the matrix units for C_n defined above. Let ω be the embedding of B_n into C_{n+1} which is the composition of the inclusion of B_n into B_{n+1} and the map sending $e^r_{ij}(n+1)$ to $f^r_{ij}(n+1)$, and set $g^r_{ij}(n) = \omega(e^r_{ij}(n))$. Then

$$\begin{aligned}
&\|\theta_{n,n+1}(f^r_{ij}(n)) - g^r_{ij}(n)\| \\
&\quad \leqq \|\theta_{n,n+1}(f^r_{ij}(n)) - \psi_{k_n,k_{n+1}}(f^r_{ij}(n))\| \\
&\qquad + \|\psi_{k_n,k_{n+1}}(f^r_{ij}(n)) - \psi_{k_n,k_{n+1}}(\gamma_{k_n}(e^r_{ij}(n)))\| \\
&\qquad + \|\psi_{k_n,k_{n+1}}(\gamma_{k_n}(e^r_{ij}(n))) - \gamma_{k_{n+1}}(e^r_{ij}(n))\| \\
&\qquad + \|\gamma_{k_{n+1}}(e^r_{ij}(n)) - g^r_{ij}(n)\| < \eta_n + \frac{\varepsilon_n}{2} + \eta_n + \frac{\varepsilon_{n+1}}{2} \leqq 4\eta_n
\end{aligned}$$

So there is a $u_n \in A_{n+1}$ such that $\|u_n - 1\| < 2^{-n}$ and $g^r_{ij}(n) = u_n^* \theta_{n,n+1}(f^r_{ij}(n)) u_n$ for all i, j, r. Set $\phi_{n,n+1} = (ad\, u_n) \circ \theta_{n,n+1}$. Then $\phi_{n,n+1}$ is a completely positive contraction from A_n to A_{n+1}, and $\|\phi_{n,n+1} - \psi_{k_n,k_{n+1}}\| < 3 \cdot 2^{-n}$. This completes the second inductive construction.

Define $\phi_{m,n}\colon A_m \to A_n$ for $m < n$ by composition. Then $(A_n, \phi_{m,n})$ is an NF system with inductive limit isomorphic to A by 2.4.3, and the C_n have the right properties with respect to the $\phi_{m,n}$. □

6 Strong NF algebras

6.1 Properties of strong NF algebras

Recall the definition of a strong NF algebra from 5.2.1.

Theorem 6.1.1. *Let A be a separable C^*-algebra. The following are equivalent*:

(i) *A is a strong NF algebra.*

(ii) *There is an increasing sequence $\langle S_n \rangle$ of finite-dimensional *-subspaces of A, each completely order isomorphic to a (finite-dimensional) C^*-algebra, with dense union.*

(iii) *Given $x_1, \dots, x_n \in A$ and $\varepsilon > 0$, there is a finite-dimensional C^*-algebra B, a complete order embedding ϕ of B into A, and elements $b_1, \dots, b_n \in B$ with $\|x_i - \phi(b_i)\| < \varepsilon$ for $1 \leqq i \leqq n$.*

(iv) *The identity map on A can be approximated in the point-norm topology by idempotent completely positive finite-rank contractions from A to A, i.e. given $x_1, \dots, x_n \in A$ and $\varepsilon > 0$, there is an idempotent completely positive finite-rank contraction $\omega\colon A \to A$ with $\|x_i - \omega(x_i)\| < \varepsilon$ for $1 \leqq i \leqq n$.*

(v) *The identity map on A can be approximated in the point-norm topology by completely positive approximately multiplicative retractive contractions through finite-dimensional C^*-algebras, i.e. given $x_1, \dots, x_n \in A$ and $\varepsilon > 0$, there is a finite-dimensional C^*-algebra B and completely positive contractions $\alpha\colon A \to B$ and $\beta\colon B \to A$ with $\alpha \circ \beta = id_B$ (β is then automatically a complete order embedding), such that $\|x_i - \beta \circ \alpha(x_i)\| < \varepsilon$ and $\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| < \varepsilon$ for all i, j.*

(vi) *Same as* (v) *with the* "*approximately multiplicative*" *condition on* α *deleted.*

(vii) *Given* $x_1, \ldots, x_n \in A$ *and* $\varepsilon > 0$, *there is a finite-dimensional* C^**-algebra* B *and completely positive contractions* $\alpha\colon A \to B$ *and* $\beta\colon B \to A$ *with* β *a complete order embedding, such that* $\|x_i - \beta \circ \alpha(x_i)\| < \varepsilon$ *and* $\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| < \varepsilon$ *for all* i, j.

(viii) *Same as* (vii) *with the* "*approximately multiplicative*" *condition on* α *deleted.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (v) $\Rightarrow$ (vi) $\Rightarrow$ (viii), and (v) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (iv): Let $x_1, \ldots, x_n \in A$ and $\varepsilon > 0$. There is a finite-dimensional C*-algebra B, elements $b_1, \ldots, b_n \in B$, and a complete order embedding ϕ of B into A with $\|x_i - \phi(b_i)\| < \varepsilon/2$ for $1 \leqq i \leqq n$ by (iii). There is an idempotent completely positive contraction ω from A onto $\phi(B)$ by 4.1.8. Then, for $1 \leqq i \leqq n$,

$$\|\omega(x_i) - \phi(b_i)\| = \|\omega(x_i - \phi(b_i))\| \leqq \|x_i - \phi(b_i)\| < \varepsilon/2$$

and so $\|\omega(x_i) - x_i\| < \varepsilon$.

(iv) $\Rightarrow$ (v): Let $x_1, \ldots, x_n \in A$ and $\varepsilon > 0$. Fix $K \geqq \max(\|x_i\|), K \geqq 1/2$. Let $\omega\colon A \to A$ be an idempotent completely positive finite-rank contraction such that $\|x_i - \omega(x_i)\| < \varepsilon/2K \leqq \varepsilon$ for all i. $\omega(A)$ becomes a finite-dimensional C*-algebra B with the multiplication defined in 4.2.1; let β be the identity map on B, regarded as a complete order isomorphism of B onto $\omega(A)$, and $\alpha = \beta^{-1} \circ \omega$. By the way the multiplication in B is defined in 4.2.1, we have $\alpha(x_i)\alpha(x_j) = \beta^{-1}(\omega(\omega(x_i)\omega(x_j)))$, and since β^{-1} is an isometry and ω is a contraction we have

$$\begin{aligned}
&\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| \\
&\quad = \|\omega(x_i x_j) - \omega(\omega(x_i)\omega(x_j))\| \\
&\quad \leqq \|x_i x_j - \omega(x_i)\omega(x_j)\| \\
&\quad \leqq \|x_i(x_j - \omega(x_j))\| + \|(x_i - \omega(x_i))\omega(x_j)\| < K\frac{\varepsilon}{2K} + \frac{\varepsilon}{2K}K = \varepsilon\,.
\end{aligned}$$

(vii) $\Rightarrow$ (i): This is almost identical to the proof of 5.2.2((vi) $\Rightarrow$ (i)). The only change is that $\phi_{n,n+1}$ cannot be chosen to be $\alpha_{n+1} \circ \beta_n$ since this is not necessarily a complete order embedding. However, if each β_n is chosen to be a complete order embedding, then by 4.2.8 α_{n+1} can be chosen so that $\alpha_{n+1} \circ \beta_n$ is so close to being supermultiplicative on a set of matrix units that there is a complete order embedding $\phi_{n,n+1}$ with $\|\phi_{n,n+1} - \alpha_{n+1} \circ \beta_n\| < 2^{-n}$. The final estimates in the proof must be slightly modified. Details are left to the reader. □

Remarks 6.1.2. (*a*) Condition 6.1.1(iii) could be taken as the definition of a "strong NF algebra in the local sense." One then has that a strong NF algebra in the local sense is a strong NF algebra. Similarly, one could define an AF [resp. AH, ASH] algebra in the local sense to be a C*-algebra A with the property that, for any $x_1, \ldots, x_n \in A$ and $\varepsilon > 0$, there is a finite-dimensional [resp. locally homogeneous, subhomogeneous] C*-algebra B, elements $b_1, \ldots, b_n \in B$, and a *-homomorphism $\phi\colon B \to A$ with $\|x_i - \phi(b_i)\| < \varepsilon$ for $1 \leqq i \leqq n$. An AF algebra in the local sense is an AF algebra [Bt, 2.2], and this is true also for AH algebras over the circle [El, 4.3]; but it is not known in general whether an AH [ASH] algebra in the local sense is an AH [ASH] algebra. (See [BtK] and [DL, 1.2] for other partial results.)

(b) The proof of 6.1.1((iv) $\Rightarrow$ (v)) shows that if $\{\alpha_n, \beta_n\}$ is a sequence of maps as in (vi), with $\beta_n \circ \alpha_n \to id_A$ in the point-norm topology, then the α_n are automatically asymptotically multiplicative in the sense of (v).

(c) The map β in (v) is not in general approximately multiplicative; in fact, it appears likely that if the β can be chosen approximately multiplicative in (v), then A is an AF algebra (this is easily seen to be true if A is commutative.)

Corollary 6.1.3. *Let $(A_n, \phi_{m,n})$ be a generalized inductive system, where each $\phi_{m,n}$ is a complete order embedding. If each A_n is a strong NF algebra, then $A = \varinjlim(A_n, \phi_{m,n})$ is a strong NF algebra.*

Proof. A clearly satisfies 6.1.1(iii). □

An important feature implicit in the proof of 6.1.1 is the existence of cross sections for a strong NF system in a stronger sense than in 5.1.4 or 5.3.9:

Proposition 6.1.4. *Let $(A_n, \phi_{m,n})$ be a strong NF system, and $A = \varinjlim(A_n, \phi_{m,n})$. Denote the induced map from A_n to A by ϕ_n; ϕ_n is a complete order embedding. There is a sequence $\gamma_n : A \to A_n$ of completely positive contractions such that $\gamma_n \circ \phi_n = id_{A_n}$. $(\phi_n \circ \gamma_n)$ converges to id_A in the point-norm topology, and the γ_n are asymptotically multiplicative on A. The γ_n can be chosen to be coherent in the sense that there are completely positive contractions $\gamma_{n,m} : A_n \to A_m$ for $m < n$, with $\gamma_{n,m} \circ \phi_{m,n} = id_{A_m}, \gamma_{n,m} \circ \gamma_{m,k} = \gamma_{n,k}$, and $\gamma_m = \gamma_{n,m} \circ \gamma_n$ for all $k < m < n$.*

Proof. By 4.1.8 there is an idempotent completely positive contraction $\omega_{n+1,n}$ from A_{n+1} onto $\phi_{n,n+1}(A_n)$ for each n; set $\gamma_{n+1,n} = \phi_{n,n+1}^{-1} \circ \omega_{n+1,n}$. For $m < n$, set $\gamma_{n,m} = \gamma_{m+1,m} \circ \gamma_{m+2,m+1} \circ \ldots \circ \gamma_{n,n-1}$. For fixed m, define γ_m to be $\phi_n^{-1} \circ \gamma_{n,m}$ for each $n > m$; this defines γ_m unambiguously on the dense subspace $\bigcup_n \phi_n(A_n)$ of A. γ_m is a contraction, hence extends to A, and the extension is a completely positive idempotent contraction, also denoted γ_m. $(\phi_n \circ \gamma_n)$ converges to id_A since the $\phi_n(A_n)$ are nested and their union is dense in A. Asymptotic multiplicativity comes from the way the multiplication is defined in 4.2.1. □

Proposition 6.1.5. *Any two strong NF systems for a strong NF algebra are asymptotically equivalent as in* 2.4.1, *with the maps α_n and β_n complete order embeddings.*

Proof. This is almost the same as the proof of 5.1.5. However, the α_n and β_n defined in that proof are not necessarily complete order embeddings; they are only completely positive contractions. But the α_n and β_n, if chosen properly, are almost supermultiplicative in the sense of 4.2.8, and can thus be slightly perturbed to complete order embeddings, with the perturbed maps still giving an asymptotic intertwining. Details are left to the reader. □

Proposition 6.1.6. *Let A be a strong NF algebra. Then there is an increasing sequence (C_k) of C^*-subalgebras of A, with dense union, such that each C_k is a residually finite-dimensional strong NF algebra. If A is not residually finite-dimensional* (*in particular, if A is infinite-dimensional and simple*), *then the sequence (C_k) can be chosen to be strictly increasing.*

Proof. Let $(A_n, \phi_{m,n})$ be a strong NF system for A. Fix k. For $n \geqq k$, inductively define $C_{k,n}$ by taking $C_{k,k} = A_k$ and $C_{k,n}$ the C*-subalgebra of A_n generated by $\phi_{n-1,n}(C_{k,n-1})$ for $n > k$. Then $C_k = [\bigcup_{n>k} \phi_n(C_{k,n})]^-$ is a C*-subalgebra of A which is a strong NF algebra. For $k < n$, the map $\gamma_{n,n-1}|_{C_{k,n}}$ is a homomorphism from $C_{k,n}$ onto $C_{k,n-1}$ by 4.2.1; so by composition, if $k \leqq m < n, \gamma_{n,m}|_{C_{k,n}}$ is a homomorphism from $C_{k,n}$ onto $C_{k,m}$. Thus, by letting $n \to \infty$, for $k \leqq m$, the map $\gamma_m|_{C_k}$ is a homomorphism from C_k onto $C_{k,m}$. Thus C_k is residually finite-dimensional. $\phi_k(A_k) \subseteq C_k \subseteq C_{k+1}$ for all k, so $\cup C_k$ is dense in A. □

Note that $C_{k,n}$ is strictly larger than $D_{k,n} = \mathrm{C}^*(\phi_{k,n}(A_k))$ in general, so C_k is strictly larger than $D_k = \mathrm{C}^*(\phi_k(A_k))$. D_k is also residually finite-dimensional (since it is a C*-subalgebra of C_k), but it is not obvious that it is nuclear. If it is, it is a strong NF algebra by [BKb]. It appears that D_k should be a generalized inductive limit of the $D_{k,n}$, where the connecting map from $D_{k,n}$ to $D_{k,n+1}$ is $\phi_{n,n+1}|_{D_{k,n}}$ followed by a conditional expectation of $C_{k,n+1}$ onto $D_{k,n+1}$.

Although a C*-subalgebra of a strong NF algebra is not necessarily a strong NF algebra, even if it is nuclear (it will be shown in [BKb] that every NF algebra can be embedded in a strong NF algebra, but not every NF algebra is strong NF), we have the following two results:

Proposition 6.1.7. *A hereditary C^*-subalgebra of a strong NF algebra is strong NF.*

Proof. Let A be a strong NF algebra, and B a hereditary C*-subalgebra. Let $\langle p_i \rangle$ be an almost idempotent approximate identity for B (4.3.2). If $(A_n, \phi_{m,n})$ is a strong NF system for A, with approximate cross sections γ_n as in 6.1.4, choose n_1 so that $\phi_{n_1} \circ \gamma_n$ is approximately the identity and

γ_{n_1} is approximately multiplicative within ε_1 on p_1, and let q_1 be the support projection of $\gamma_1(p_1)$ in A_{n_1} and $B_1 = q_1 A_1 q_1$. Now choose n_2 such that $\phi_{n_2} \circ \gamma_{n_2}$ is approximately the identity and γ_{n_2} is approximately multiplicative within ε_2 on $\phi_{n_1}(A_{n_1}) \cup \{p_2\}$, and define q_2 and B_2 as above. Then if $\theta = \phi_{n_1,n_2}|_{B_1}$, then $\theta_{q_2} : B_1 \to B_2$ (4.1.1) is very close to being supermultiplicative, and so can be slightly perturbed to a complete order embedding $\psi_{1,2}$ by 4.2.8. Continue inductively to obtain a system $(B_n, \psi_{m,n})$, which is a strong NF system with inductive limit isomorphic to B if $\varepsilon_n \to 0$ rapidly enough. □

Proposition 6.1.8. *Every NF algebra is a quotient of a strong NF algebra. Hence every separable nuclear C*-algebra is a quotient of a strong NF algebra.*

Proof. This is similar to the argument of 5.1.3. Let A be an NF algebra, and $(A_n, \phi_{m,n})$ an NF system for A. Let $B_n = A_1 \oplus \ldots \oplus A_n = B_{n-1} \oplus A_n$, and $\psi_{n,n+1} = id_{B_n} \oplus \phi_{n,n+1} : B_n \to B_{n+1}$. Then $\psi_{n,n+1}$ is a complete order embedding. If $\psi_{m,n}$ is defined by composition for $m < n$, then $(B_n, \psi_{m,n})$ is asymptotically multiplicative, and hence is a strong NF system. A is a quotient of the inductive limit in the obvious way. The last statement follows from 5.3.4. □

We will give an improved version of 6.1.8 in [BKb].

The next proposition might be helpful in showing that certain NF algebras are strong NF:

Proposition 6.1.9. *Let $(A_n, \phi_{m,n})$ be an NF system for an (NF) C*-algebra A. Suppose that for all $n > 1, \phi_{n,n+1}$ is a complete order embedding on the image of $\phi_{n-1,n}$. Then A is a strong NF algebra.*

Proof. It follows that ϕ_n gives a complete order embedding θ_n of $\phi_{n-1,n}(A_{n-1})$ into A, whose image is $\phi_{n-1}(A_{n-1})$. $\theta_n^{-1} : \phi_{n-1}(A_{n-1}) \to A_n$ extends to a completely positive contraction $\alpha_n : A \to A_n$ by 4.1.8; set $\beta_n = \phi_n \circ \alpha_n$. Then β_n is a completely positive finite rank contraction from A to A. If $S_n = \{x \in A : \beta_n(x) = x\}$, then $\phi_{n-1}(A_{n-1}) \subseteq S_n$, and S_n is completely order isomorphic to a finite-dimensional C*-algebra by 4.2.3, so A is strong NF by 6.1.1(iii). □

Finally, we have a version of 5.3.10 for strong NF algebras:

Proposition 6.1.10. *Let A be a strong NF algebra, and let $\langle B_n \rangle$ be an increasing sequence of finite-dimensional C*-subalgebras of A. Then there is a strong NF system $(A_n, \phi_{m,n})$ for A and C*-subalgebras C_n of A_n, with $C_n \cong B_n$, such that $\phi_{n,n+1}$ is exactly multiplicative on C_n and $\phi_n(C_n) = B_n$.*

The proof is identical to the proof of 5.3.10 with 4.2.11 used in place of 4.1.7 and 6.1.4 in place of 5.1.4. □

6.2 Approximately homogeneous C^*-algebras

Although the results of this section will be improved in [BKb], it is appropriate to give direct proofs in the cases of greatest interest.

Proposition 6.2.1. *Every separable commutative C^*-algebra is a strong NF algebra. Every separable continuous trace C^*-algebra (in particular, every separable homogeneous C^*-algebra) over a compact space is a strong NF algebra.*

Proof. Because of 6.1.7 it suffices to show that if A is a stable homogeneous C*-algebra over a compact metrizable space X, then A is strong NF. We will show that A satisfies 6.1.1(iii). First, such an algebra is defined by a locally trivial continuous field of elementary C*-algebras over X, with fibers isomorphic to K [Dx, 10.10.10]. Choose a finite open cover U of X such that the field is trivial on each open set in U, and fix a trivialization for each open set. If $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$, choose a finite open cover $\mathrm{V} = \{V_1, \ldots, V_m\}$ which refines U and such that $a_1, \ldots, a_n$ vary in value (in K) in norm by less than $\varepsilon/4$ on each V_k, for each of the trivializations chosen for the sets in U. We may assume no V_k is contained in the union of the others; let $x_k \in V_k \setminus \bigcup_{j \neq k} V_j$. Fix a trivialization θ_k for each V_k, as the restriction of one of the trivializations from U. θ_k may be regarded as a *-isomorphism from $C_0(V_k, \mathrm{K})$ onto the corresponding ideal of A. Choose a partition of unity $\{f_k\}$ subordinate to V; then $f_k(x_k) = 1$. Then the element $c_{jk} = f_k a_j$ is well defined in A, and is in the ideal corresponding to V_k. If $d_{jk} = \theta_k^{-1}(c_{jk})$, we can choose a finite-dimensional C*-subalgebra B_k of K and elements $b_{1k}, \ldots, b_{nk} \in B_k$ such that $\|b_{jk} - d_{jk}(x_k)\| < \varepsilon/4$ for $1 \leqq j \leqq n$. Set $B = B_1 \oplus \ldots \oplus B_m$. B is a finite-dimensional C*-algebra. Define $\phi\colon B \to A$ by $\phi(z_1, \ldots, z_m) = \sum_{k=1}^m \theta_k(f_k z_k)$, where $f_k z_k$ is the function in $C_0(V_k, \mathrm{K})$ whose value at x is $f_k(x) z_k$. ϕ is a complete order embedding since each f_k attains the value 1, and we have $\|a_j - \phi(b_{j1}, \ldots, b_{jm})\| < \varepsilon$ for $1 \leqq j \leqq n$. □

This argument can be somewhat simplified in the commutative case.

Corollary 6.2.2. *Every AH algebra is a strong NF algebra.*

Proof. The class of strong NF algebras is closed under taking direct sums. Every AH algebra, defined to be a usual inductive limit of a sequence of C*-algebras, each of which is a finite direct sum of homogeneous C*-algebras, can be written as a usual inductive limit of such algebras with injective connecting maps, so the result follows from 6.1.3. □

Remarks 6.2.3. (*a*) In [BKb], we will show that there are NF algebras which are not strong NF (in particular, the examples of [Bn] and [BD, Ex. 20, 23] are NF but not strong NF), but that many types of NF algebras, including all prime antiliminal NF algebras and all ASH algebras, are strong NF.

(b) Not every strong NF algebra is ASH, since an ASH algebra is residually stably finite, and in fact residually strong NF since a quotient of an ASH algebra is ASH. There is a strong NF algebra with O_2 as a quotient (6.1.8), and there is a residually stably finite strong NF algebra which is not residually strong NF [BKb]. The example in [DL] is a sub-AF algebra, hence NF, and therefore strong NF by [BKb]; it is easily seen to be residually strong NF also, although it is not ASH.

(c) We are still lacking a complete description of which NF algebras are strong NF. In particular, we do not know for which A we have that CA and/or SA are strong NF (cf. 5.3.3.)

7 Other structure and open problems

7.1 Ideals

In this section, "ideal" means "closed two-sided ideal."

If $(A_n, \phi_{m,n})$ is a generalized inductive system with inductive limit A, it is desirable to describe the ideals of A in terms of the inductive system. Unfortunately, this is not as easy, even for strong NF systems, as it is in the case of ordinary (*-linear and multiplicative) systems.

One observation is easy: if J_n is an ideal in A_n, and $\phi_{n,n+1}$ maps J_n into J_{n+1} for each n, then it is easily verified that the closure of $\cup\phi_n(J_n)$ is an ideal of A. These are the ideals which are well behaved with respect to the inductive system. To work backwards from ideals of A is more difficult since not every ideal arises this way.

Definition 7.1.1. *If* $(A_n, \phi_{m,n})$ *is a generalized inductive system with inductive limit* A, *and* J *is an ideal of* A, *then* J *is* **compatible** *with* $(A_n, \phi_{m,n})$ *if*

(1) $J_n = \phi_n^{-1}(\phi_n(A_n) \cap J)$ *is an ideal of* A_n *for each* (*or for all sufficiently large*) n *and*

(2) $\cup\phi_n(J_n)$ *is dense in* J.

If A *is a* [*strong*] *NF algebra, then* J *is a* [**strong**] **NF ideal** *if it is compatible with some* [*strong*] *NF system for* A.

If $(A_n, \phi_{m,n})$ is an ordinary inductive system, then it is well known that every ideal of A is compatible with $(A_n, \phi_{m,n})$ (cf. [Bt, 3.1], [Bl 1, 4.5]). For generalized inductive systems this is far from the case. In fact, every NF algebra (e.g. every commutative C*-algebra) has an NF system $(B_n, \psi_{m,n})$ with B_n a single full matrix algebra. For if $(A_n, \phi_{m,n})$ is an NF system for an NF algebra A, then there is an embedding ι_n of A_n into a matrix algebra $B_n = \mathrm{M}_{k_n}$ and a conditional expectation ω_n from B_n onto $\iota_n(A_n)$. If $\psi_{m,n} = \iota_n \circ \phi_{m,n} \circ \iota_m^{-1} \circ \omega_m : B_m \to B_n$, then $(B_n, \psi_{m,n})$ is also an NF system for A.

More interestingly, this phenomenon can also occur among strong NF algebras:

Example 7.1.2. Let $A_n = \mathrm{M}_n$, with

$$\phi_{n,n+1}\left(\begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix}\right) = \begin{bmatrix} t_{11} & \cdots & t_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ t_{n1} & \cdots & t_{nn} & 0 \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$$

This system is asymptotically multiplicative, and the limit is isomorphic to $\mathrm{K} + \mathrm{C}1$.

We will show in [BKb] that every prime (primitive) strong NF algebra has a strong NF system of the form $(\mathrm{M}_{k_n}, \phi_{m,n})$.

Note that even though every ideal in an NF algebra is itself an NF algebra, not every ideal is an NF ideal. For if J is an NF ideal in A, compatible with the NF system $(A_n, \phi_{m,n})$, then $\phi_{m,n}$ induces a completely positive contraction $\psi_{m,n}$ from A_m/J_m to A_n/J_n, and it is easily checked that $(A_n/J_n, \psi_{m,n})$ is an NF system with inductive limit A/J, so A/J is an NF algebra. This proves one direction of the following conjecture, and with 5.3.4 shows that not every ideal in an NF algebra is an NF ideal (since e.g. O_2 is a quotient of an NF algebra).

Conjecture 7.1.3. *An ideal J in a* [*strong*] *NF algebra A is a* [*strong*] *NF ideal if and only if A/J is a* [*strong*] *NF algebra.*

The following conjecture seems likely, especially if 7.1.3 is true:

Conjecture 7.1.4. *If A is a* [*strong*] *NF algebra, then the* [*strong*] *NF ideals of A form a lattice, and there is a* [*strong*] *NF system $(A_n, \phi_{m,n})$ for A such that every* [*strong*] *NF ideal of A is compatible with $(A_n, \phi_{m,n})$.*

Such a system could be called an "ideal" [strong] NF system. We can show that every commutative C*-algebra has an ideal strong NF system (and that every ideal is a strong NF ideal).

We say that A is *residually* [*strong*] NF if every quotient of A is [strong] NF. It would follow from 7.1.3 and 7.1.4 that a residually [strong] NF algebra has a good [strong] NF system compatible with every ideal of A, as for commutative C*-algebras. We show in [BKb] that a separable nuclear C*-algebra is residually strong NF if and only if it is strongly quasidiagonal (3.1.2).

An extension of NF algebras is not necessarily NF (e.g. the Toeplitz algebra). It appears that a split extension of [strong] NF algebras is [strong] NF. One instance is almost trivial to prove: if A is [strong] NF, then so is A^+.

Definition 7.1.5. *An extension $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \to 0$ is* **approximately split** *if, for every $x_1, \ldots, x_n \in A/J$ and $\varepsilon > 0$, there is a completely positive contractive cross section $\sigma : A/J \to A$ for π which is approximately multiplicative within ε on $\{x_1, \ldots, x_n\}$.*

Conjecture 7.1.6. *An approximately split extension of* [*strong*] *NF algebras is* [*strong*] *NF.*

An extension does not have to be approximately split to be NF: for example, $0 \to C_0((0,1)) \to C([0,1]) \to C^2 \to 0$.

The following simple fact about approximately split extensions is of interest, since quotients of NF algebras are not in general NF.

Proposition 7.1.7. *Let* $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \to 0$ *be an approximately split extension of C*-algebras. If A is an NF algebra, then so is A/J.*

Proof. Let $x_1, \ldots, x_m \in A/J$, of norm 1, and $\varepsilon > 0$, and let σ be a completely positive contractive cross section for π which is approximately multiplicative within $\varepsilon/2$ on $\{x_1, \ldots, x_n\}$. Let B be a finite-dimensional C*-algebra and $\alpha\colon A \to B$ and $\beta\colon B \to A$ completely positive contractive maps with $\beta \circ \alpha$ approximately the identity within ε and α approximately multiplicative within $\varepsilon/2$ on $\{\sigma(x_1), \ldots, \sigma(x_n)\}$. Then $\alpha \circ \sigma : A/J \to B$ and $\pi \circ \beta : B \to A/J$ satisfy the conditions of 5.2.2(vi). □

The conclusion of 7.1.7 is trivial (from 5.3.1) if the exact sequence splits, for then A/J is isomorphic to a subalgebra of A.

7.2 Traces

As with ideals, identifying the tracial states on an NF algebra from an NF system is more difficult than for usual inductive systems. In this section, for simplicity we consider only unital NF systems and algebras. (Note that any unital NF algebra has a unital NF system.)

Suppose $(A_n, \phi_{m,n})$ is a unital NF system with inductive limit A. Let γ_n be the approximate cross section defined in 5.1.4; γ_n may be taken to be unital. If τ_n is a tracial state on A_n, let $\sigma_n = \tau_n \circ \gamma_n$. Then σ_n is a state on A which is not in general tracial; but if τ is any weak-* limit of the sequence $\langle \sigma_n \rangle$, then τ is a tracial state on A. Such a trace is said to be *induced* from the inductive system. Since finite-dimensional C*-algebras have tracial states, we have proved.

Proposition 7.2.1. *Every unital NF algebra has at least one tracial state.*

This also follows from 5.2.2(iii), or [Vo3, 2.4], or from 3.3.8 and the deep results of [Ha].

If $(A_n, \phi_{m,n})$ is an ordinary system, then every tracial state on A is induced from the system. It is not all obvious that this is true for a generalized inductive system, since if τ is a trace on A, then $\tau \circ \phi_n$ is not in general a trace on A_n. It is not clear that a general NF algebra has a single inductive system which induces all traces, or even that every tracial state on an NF algebra is induced from some NF system. These are problems requiring study because of the relevance of traces in determining orderings on K-theoretic invariants.

7.3 Open questions

We summarize the principal open questions arising from the work of this paper.

Question 7.3.1. *Is every stably finite separable nuclear C*-algebra an NF algebra?*

The best way to approach this problem is to gain an understanding of how the multiplication in a general nuclear C*-algebra can be recovered from the factorizations of 5.1.1(ii). These factorizations determine the complete order structure in an evident way [in fact, any separable nuclear C^*-algebra can be written as a complete order space as an inductive limit of finite-dimensional C^*-algebras with completely positive connecting maps by the method of the proof of 5.2.2((vi)$\Rightarrow$(i))], and hence in theory determine the multiplication by 4.1.4; however, it is not well understood how to do this in practice. Such an understanding would be very important also in other aspects of the study of nuclear C^*-algebras.

A less ambitious question which is perhaps simpler is:

Question 7.3.2. *Is every separable nuclear C^*-algebra which has a faithful densely defined semifinite trace an NF algebra*?

To prove this, it would be sufficient (and also necessary) to show that every separable nuclear C^*-subalgebra of the hyperfinite II_1 factor, in its natural representation, is a quasidiagonal algebra of operators. (It is known that the hyperfinite II_1 factor itself is not quasidiagonal, and has non-quasidiagonal separable C^*-subalgebras which are not nuclear.)

Question 7.3.3. *Can every NF algebra be embedded in an AF algebra*?

It would suffice to show this for strong NF algebras, since by [BKb] every NF algebra can be embedded in a strong NF algebra. Many special cases are known, e.g. [Pm], [Sp]. There is an obvious strategy for approaching this question using Stinespring's theorem, which unfortunately seems to lead to difficult technical problems.

Question 7.3.4. *Are there computable K-theoretic type invariants which classify* [*strong*] *NF algebras*, *or natural subclasses*, *up to isomorphism or up to shape equivalence*?

Obvious subclasses to consider are the simple algebras and/or the algebras of real rank zero.

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References

[Ar] Arveson, W.: Notes on extensions of C^*-algebras. Duke Math. J. **44** (1977), 329–355

[Bl1] Blackadar, B.: Infinite tensor products of C^*-algebras. Pacific J. Math. **72** (1977), 313–334

[Bl2] Blackadar, B.: A simple unital projectionless C^*-algebra. J. Operator Theory **5** (1981), 63–71

[Bl3] Blackadar, B.: *K*-Theory for Operator Algebras. Springer-Verlag, New York, Heidelberg, Berlin, Tokyo, 1986

[BKb] Blackadar, B., Kirchberg, E.: Inner quasidiagonality and strong NF algebras (To appear)

[BKR] Blackadar, B., Kumjian, A., Rørdam, M.: Approximately central matrix units and the structure of noncommutative tori. *K*-Theory **6** (1992), 267–284

[Bt] Bratteli, O.: Inductive limits of finite-dimensional C^*-algebras. Trans. Amer. Math. Soc. **171** (1972), 195–234

[BtK] Bratteli, O., Kishimoto, A.: Non-commutative spheres III: irrational rotations (To appear)

[Bn] Brown, L.: The universal coefficient theorem for Ext and quasidiagonality. In: Operator Algebras and Group Representations. Pitman, Boston, London, Melbourne, 1983

[BD] Brown, L., Dadarlat, M.: Extensions of C^*-algebras and quasidiagonality (To appear)

[Ch1] Choi, M.-D.: A. Schwartz inequality for positive linear maps on C^*-algebras, Ill. J. Math. **18** (1974), 565–574

[Ch2] Choi, M.-D.: The full C^*-algebra of the free group on two generators. Pacific J. Math. **87** (1980), 41–48

[CE1] Choi, M.-D., Effros, E.: The completely positive lifting problem for C^*-algebras. Ann. Math. **104** (1976), 585–609

[CE2] Choi, M.-D., Effros, E.: Injectivity and operator spaces. J. Funct. Anal. **24** (1977), 156–209

[Cr] Christensen, E.: Derivations and their relation to perturbations of operator algebras. Operator Algebras and Applications. Proc. Symp. Pure Math. 38, v. 2. American Math. Soc., 1982

[DL] Dadarlat, M., Loring, T.: Extensions of certain real rank zero C^*-algebras (To appear)

[DHS] Davidson, K., Herrero, D., Salinas, N.: Quasidiagonal operators, approximation and C^*-algebras. Indiana Univ. Math. J. **38** (1989), 973–998

[Dx] Dixmier, J.: C^*-Algebras. Gauthier-Villars, Paris, 1964; North-Holland, Amsterdam, 1977

[El] Elliott, G.: On the classification of C^*-algebras of real rank zero. J. Reine Angew. Math. **443** (1993), 179–219

[ExL] Exel, R., Loring, T.: Finite-dimensional representations of free product C^*-algebras (To appear)

[GM] Goodearl, K., Menal, P.: Free and residually finite-dimensional C^*-algebras. J. Functional Anal. **90** (1990), 391–410

[Ha] Haagerup, U.: Quasitraces on exact C^*-algebras are traces (To appear)

[Hw] Hadwin, D.: Strongly quasidiagonal C^*-algebras. J. Operator Theory **18** (1987), 3–15

[Hm] Halmos, P.: A Hilbert Space Problem Book. Van Nostrand, Princeton, 1967

[Kb] Kirchberg, E.: On subalgebras of the CAR algebra. To appear in J. Funct. Anal.

[La] Lance, E.C.: Tensor products and nuclear C^*-algebras. Operator Algebras and Applications (Ed. R.V. Kadison), Proc. Symp. Pure Math. 38, part 1, 379–399

[Le] Lee, R.-Y.: On C^*-algebras which are approachable by finite-dimensional C^*-algebras. Bull. Inst. Acad. Math. Sinica **55** (1977), 265–283

[LT] Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces. Springer-Verlag, Berlin, Heidelberg, New York, 1977

[Pa] Paulsen, V.: Completely Bounded Maps and Dilations. Pitman Research Notes in Math. **146** (1986)

[Pd] Pedersen, G.: C^*-algebras and their Automorphism Groups. Academic Press, London, New York, San Francisco, 1979

[Pm] Pimsner, M.: Embedding some transformation group C^*-algebras into AF algebras. Ergodic Theory Dyn. Systems **3** (1983), 613–626

[Ps] Pisier, G.: Factorization of Linear Operators and Geometry of Banach Spaces. CBMS Lecture Series 60 (1986), Amer. Math. Society

[Ro] Rosenberg, J.: Appendix to “Strongly quasidiagonal C^*-algebras” by D. Hadwin. J. Operator Theory **18** (1987), 15–18

[Sa] Salinas, N.: Relative quasidiagonality and KK-theory. Houston J. Math. **18** (1992), 97–116

[Sp] Spielberg, J.: Embedding C^*-algebra extensions into AF algebras. J. Functional Anal. **81** (1989), 325–344

[St] Stinespring, W.: Positive functions on C^*-algebras. Proc. Amer. Math. Soc. **6** (1955), 211–216

[Tk] Takesaki, M.: Theory of Operator Algebras I. Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1979

[Vo1] Voiculescu, D.: A noncommutative Weyl-von Neumann theorem. Rev. Roum. Math. Pures Appl. **21** (1976), 97–113

[Vo2] Voiculescu, D.: A note on quasidiagonal C^*-algebras and homotopy. Duke Math. J. **62** (1991), 267–272

[Vo3] Voiculescu, D.: Around quasidiagonal operators. Integral Equat. Operator Theory **17** (1993), 137–148

[Ws] Wassermann, S.: C^*-Algebras associated with groups with Kazhdan's property T. Ann. Math. **134** (1991), 423–431