Math. Ann. 307, 93–108 (1997)

Mathematische Annalen

Toric structures on the moduli space of flat connections on a Riemann surface II: Inductive decomposition of the moduli space

Lisa Jeffrey[1,★] **Jonathan Weitsman**[2,★★]

[1] Mathematics Department, Mc Gill University, Montréal, Quebec, Canada, H3A 2K 6, USA (e-mail: jeffrey@gauss.math.mcgill.ca)
[2] Mathematics Department, University of California Santa Cruz, Santa Cruz, CA 95064, USA

Received: 16 August 1995/Revised version: 8 November 1995

Mathematics Subject Classification (1991): 58F05, 58D27, 32G13

1 Introduction

Let Σ^g be a compact two-manifold of genus g, with $\partial\Sigma^g = \emptyset$ and fundamental group $\pi = \pi_1(\Sigma^g)$. Let K be a compact Lie group, which in this paper will be $SU(2)$. We define $\bar{\mathscr{S}}_g = \mathrm{Hom}(\pi, K)/K$, the space of conjugacy classes of representations of π in K. This space may be identified [AB] with the space $\mathscr{A}_F(\Sigma^g)/\mathscr{G}(\Sigma^g)$, where $\mathscr{A}_F(\Sigma^g)$ are the flat connections on a (trivial) principal K bundle over Σ^g, and $\mathscr{G}(\Sigma^g)$ is the gauge group. The space $\bar{\mathscr{S}}_g$ is compact, and has an open dense subset $\mathscr{S}_g$ which has the structure of a smooth symplectic manifold of dimension $6g-6$: we denote the symplectic form by ω. (See [AB] and [G1] for the construction of the symplectic structure on $\mathscr{S}_g$.) The choice of a complex structure on Σ^g endows $\bar{\mathscr{S}}_g$ with the structure of a Kähler variety: it is identified [NS] with a moduli space of holomorphic vector bundles on Σ^g.

We shall also consider two-manifolds $\Sigma^g_r = \Sigma^g - (D_1 \sqcup \ldots \sqcup D_r)$ with genus g and r oriented boundary components $S_1, \ldots, S_r$, obtained from Σ^g by deleting r disjoint disks $D_1, \ldots, D_r$ from Σ^g. By choosing a base point $*$ in Σ^g and for each j an arc σ_j joining $*$ to a point on S_j, we obtain an element $[S_j] = [\sigma_j \circ S_j \circ \sigma_j^{-1}]$ in $\pi_1(\Sigma^g_r)$, whose conjugacy class is well defined (independent of the choice of the σ_j). Let $\mathbf{t} = (t_1, \ldots, t_r) \in [0,1]^r$. We may then define

$$\bar{\mathscr{S}}_g(\mathbf{t}) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma^g_r), K):\ \rho([S_j]) \text{ is conjugate to } \mathrm{diag}(e^{i\pi t_j}, e^{-i\pi t_j})\}/K\,, \tag{1}$$

★This material is based on work supported by the National Science Foundation Under Grant No. DMS-9306029
★★Supported in part by NSF grant DMS-9403567, and by NSF Young Investigator grant 9457821

where diag(a,b) is the 2×2 diagonal matrix $(M_{\alpha\beta})$ with $M_{11}=a$, $M_{22}=b$. The parameters t_j thus specify the conjugacy class in K to which representations send the elements $[S_j]\in\pi_1(\Sigma_r^g)$.

The space $\bar{\mathscr{S}}_g(\mathbf{t})$ has an open dense set $\mathscr{S}_g(\mathbf{t})$ with the structure of a smooth symplectic manifold of dimension $6g-6+2r'$, where $r'=\#\{j\colon\ t_j\in(0,1)\}$. Just as $\bar{\mathscr{S}}_g$ may be identified with a space of holomorphic vector bundles on Σ^g, the spaces $\bar{\mathscr{S}}_g(\mathbf{t})$ appear in algebraic geometry as moduli spaces of holomorphic vector bundles with parabolic structure (see [MS]).

In this paper we shall describe some topological results about the spaces $\bar{\mathscr{S}}_g$ and $\bar{\mathscr{S}}_g(\mathbf{t})$ which follow from their properties in symplectic geometry. We shall primarily be concerned with the cases $r=0,1,2$. The central idea in this paper is that the moduli space $\bar{\mathscr{S}}_g(\mathbf{t})$ has a decomposition into components given by moduli spaces corresponding to surfaces of lower genus. This decomposition resembles Grothendieck's decomposition [Gr] of the moduli space of curves: it is obtained by applying the results of [JW3], where we consider a simple closed curve C in a 2-manifold Σ_r^g such that Σ_r^g-C decomposes into two 2-manifolds $\Sigma_{r_1}^{g_1}$ and $\Sigma_{r_2}^{g_2}$ of genus g_1 and g_2. There is then a function $\mu_C:\bar{\mathscr{S}}_g(\mathbf{t})\to[0,\,1]$ specifying the holonomy of flat connections around C. Each level set $\mu_C^{-1}(s)$ of μ_C fibers over $\bar{\mathscr{S}}_{g_1}(t_1,\dots,t_{r_1},s)\times\bar{\mathscr{S}}_{g_2}(t_{r_1+1},\dots,t_{r_1+r_2},s)$ with generic fiber S^1. (See Theorem 6.)

On the other hand, the results of [JW3] exhibit the structure of a (noncompact) toric variety on an open dense subset of $\bar{\mathscr{S}}_g(\mathbf{t})$. To this toric variety is associated a polyhedron $B_g(\mathbf{t})$ (the closure of the image of the moment map $\mu=(\mu_{C_1},\dots,\mu_{C_{3g-3+r}})$ where the C_j are boundary circles corresponding to a pants decomposition of Σ_r^g). We see that

$$B_g(\mathbf{t})=\bigcup_{0\leqq s\leqq 1}B_{g_1}(s,t_1,\dots,t_{r_1})\times B_{g_2}(s,t_{r_1+1},\dots,t_{r_1+r_2})\,. \tag{2}$$

We may apply these results in particular when $g_2=1$ and $r_2=1$: one finds easily that $B_1(s,t)$ is a rectangle, so the usual methods of toric geometry then show (for generic values of s and t) that $\bar{\mathscr{S}}_1(s,t)$ is $S^2\times S^2$. An immediate application of this yields formulas for the symplectic volume $S_g(t)$ of the moduli space $\bar{\mathscr{S}}_g(t)$ of representations into $SU(2)$ of the fundamental group of Σ_1^g for which the element $[S_1]$ in $\pi_1(\Sigma_1^g)$ represented by a loop around the boundary is sent to the conjugacy class of 2×2 matrices with eigenvalues $e^{i\pi t}$ and $e^{-i\pi t}$. The equation (2) leads immediately to

Theorem 7.

$$S_g^e(t)=\int S_{g-1}^e(s)\Phi(s,t)\,ds\ ,$$

where we have introduced $S_g^e(t)=\mathrm{vol}_{\mathrm{eucl}}\,\mu(\bar{\mathscr{S}}_g(t))$, *and* $\Phi(s,t)$ *is the Euclidean volume of* $B_1(s,t)=\mu(\bar{\mathscr{S}}_1(s,t))$ *in an appropriate normalization.*

Now $B_1(s,t)$ is a rectangle with side lengths proportional to s and $1-t$ (when $s<t$), so $\Phi(s,t)$ is proportional to $s(1-t)$. One sees immediately from Theorem 7 (starting from an easy computation showing that $S_1(t)$ is linear in t) that $S_g(t)$ is piecewise polynomial in t and that the degree can be no greater

than $2g-1$. This is less than the degree one might have expected on strictly dimensional grounds. The term **lacunarity** has been used in the mathematical literature to refer to the fact that functions of this type have lower degrees than expected (see Sect. 4).[1]

For $t \in (0,1)$ the moduli space $\bar{\mathscr{S}}_g(t)$ (which has dimension $6g-4$) fibres over $\bar{\mathscr{S}}_g(1)$ (which has dimension $6g-6$) with fibre S^2. As was described in [Don] (Sects. 1 and 6) following observations of Witten, the formula for the symplectic volume of $\bar{\mathscr{S}}_g(t)$ encodes the formulas for the intersection numbers in the moduli space $\bar{\mathscr{S}}_g(1)$, because of the properties of this fibration.[2] In particular the fact that the degree of $S_g(t)$ can be no greater than $2g-1$ immediately implies the *Newstead conjecture* [Ne] that the g-th power of a certain cohomology class of degree 4 in $\bar{\mathscr{S}}_g(1)$ vanishes (see Sect. 4). As noted above, the limit on the degree of the piecewise polynomial function $S_g(t)$ follows from the decomposition of the moduli space by induction on the genus, and does not require full knowledge of the explicit formula for $S_g(t)$. In fact one may proceed further and use Theorem 7 to extract an explicit formula for $S_g(t)$, which was first found by Donaldson [Don]: the formula specifies $S_g(t)$ as a Bernoulli polynomial of degree $2g-1$.

This paper is organized as follows: in Sect. 2 we recall some fundamental results on torus actions and moment maps, which are then applied to moduli spaces. We also recall the explicit description of inequalities that give the image of the moment map of torus actions on moduli spaces as a convex polyhedron. In Sect. 3 we develop the most important idea of the paper: that a decomposition of the surface into surfaces of lower genus gives rise to a description of the moduli space of flat connections in terms of moduli spaces corresponding to surfaces of lower genus. This leads inductively to formulas for the symplectic volumes of these moduli spaces, which are developed explicitly in Sect. 4. We then give simple proofs of formulas of Donaldson [Don] for volumes of certain moduli spaces: these formulas are given in terms of Bernoulli numbers.

An announcement of the results of this paper is to be found in [JW6].

2 Torus actions on moduli spaces

Our results for the spaces $\bar{\mathscr{S}}_g(\mathbf{t})$ will arise from the existence of several commuting Hamiltonian S^1 actions on open dense sets in these spaces. Suppose

[1]More precisely, the term *lacunarity* has been applied to the function $S(\mathbf{t})$ which gives the symplectic volume of a family of symplectic quotients $M_{\mathbf{t}} = \mu^{-1}(\mathbf{t})/T$ with respect to the Hamiltonian action of a torus T on a symplectic manifold M: here the parameter $\mathbf{t}$ specifies the value of the moment map μ for the T action. The properties of such functions (notably including the phenomenon of lacunarity) have been studied extensively in recent work of Guillemin, Lerman and Sternberg [GLS]. In our case $T = U(1)$

[2]The space $\bar{\mathscr{S}}_g(1)$ appears in algebraic geometry as the moduli space $N(2,1)$ of (semi) stable holomorphic vector bundles of rank 2 and degree 1 with fixed determinant over a closed Riemann surface of genus g

S^1 acts on a symplectic manifold (M, ω) preserving the symplectic form. The action is called *Hamiltonian* if there is a C^∞ map $\mu\colon M \to \mathbb{R}$ such that

$$\iota_\eta \omega = d\mu \tag{3}$$

(where η is the vector field on M corresponding to the S^1 action and ι_η is the interior product, in other words $\iota_\eta\omega(Y) = \omega(\eta, Y)$.) Moment maps for actions of a torus $T = (S^1)^r$ on a symplectic manifold are defined as $\mu = (\mu_1, \dots, \mu_r)\colon M \to \mathbb{R}^r$, where each μ_j satisfies $d\mu_j = \iota_{\eta_j}\omega$ if η_j is the vector field corresponding to the action of the j'th copy of S^1. The action of T then fixes the value of μ: in other words, $\mu(x) = \mu(\tau x)$ for all $\tau \in T$.

Theorem 1. [A], [GS1] *If M is compact and connected then $\mu(M)$ is a convex polyhedron.*

A particularly attractive situation arises when a symplectic manifold M of dimension $2N$ is acted on by a torus of dimension N and the vector fields generating the torus action are linearly independent almost everywhere. In this case the following result is well known:

Theorem 2. (see [JW3]) *The symplectic volume* $\mathrm{vol}_\omega(M)$ *of M is equal to the Euclidean volume* $\mathrm{vol}_\mu(\mu(M))$ *of the image $\mu(M) \subset \mathbb{R}^N$, where vol_μ is an appropriately normalized Euclidean volume on $\mathbb{R}^N$.*[3]

We have the following theorem:

Theorem 3. [JW3] *Let Γ be a pants decomposition of Σ_r^g, in other words a choice of $N = 3g - 3 + r$ disjoint oriented simple closed curves $C_1, \dots, C_N$ in the interior of Σ_r^g, such that $\Sigma_r^g - (C_1 \sqcup \dots \sqcup C_N)$ is a disjoint union of 3-punctured spheres or "pairs of pants". Let $N' = 3g - 3 + r' = \dim \bar{\mathcal{S}}_g(\mathbf{t})/2$. Then*:
(*a*) *There exists a Hamiltonian action of $(S^1)^{N'}$ on an open dense set U in $\bar{\mathcal{S}}_g(\mathbf{t})$.*
(*b*) *The moment map $\mu = (\mu_1, \dots, \mu_{N'})\colon U \to \mathbb{R}^{N'}$ may be chosen to take its image in $(0,1)^{N'}$, and extends to a continuous function $\bar{\mathcal{S}}_g(\mathbf{t}) \to [0,1]^{N'}$ which will also be denoted μ. Furthermore, $U = \mu^{-1}((0,1)^{N'})$.*
(*c*) *The image $\mu(\bar{\mathcal{S}}_g(\mathbf{t}))$ is a convex polyhedron in $[0,1]^{N'}$, which will be specified explicitly below* (*Proposition* 8).

We remark that the functions μ_j and the associated S^1 actions depend on the choice of a pants decomposition Γ. These functions are given as follows: they are part of a class of functions whose role in the symplectic geometry of moduli spaces of representations of surface groups was studied by Goldman [G2]. We choose a basepoint $*$ in Σ_r^g, and for each j an arc $\tilde{\sigma}_j$ joining $*$ to a

[3] The volume vol_μ is the usual Euclidean volume $\mathrm{vol}_{\mathrm{eucl}}$ on $\mathbb{R}^N$ if the group $(S^1)^N$ acts *effectively*. Otherwise there is a finite subgroup Λ of $(S^1)^N$ such that the action of $(S^1)^N$ factors through an effective action $(S^1)^N/\Lambda$, and we have $\mathrm{vol}_\mu = \mathrm{vol}_{\mathrm{eucl}}/|\Lambda|$, where $|\Lambda|$ is the order of Λ: see [JW3]

point on C_j. Then the element $[C_j]$ in $\pi_1(\Sigma_r^g)$, defined as the homotopy class of the based loop $\tilde{\sigma_j} \circ C_j \circ \tilde{\sigma}_j^{-1}$, is well defined up to conjugation (independently of the choice of the arcs $\tilde{\sigma}_j$).

We may then define μ_j: $\bar{\mathscr{S}}_g(\mathbf{t}) \subset \mathrm{Hom}(\pi_1(\Sigma_r^g), K)/K \to [0,1]$ by

$$\mu_j([\rho]) = s \Leftrightarrow \rho([C_j]) \text{ is conjugate to } \mathrm{diag}\,(e^{i\pi s}, e^{-i\pi s})\,. \tag{4}$$

Here, $\rho \in \mathrm{Hom}(\pi_1(\Sigma_r^g), K)$, and its equivalence class in the quotient $\mathrm{Hom}\,(\pi_1(\Sigma_r^g), K)/K$ is denoted $[\rho]$. The function μ_j is continuous on all of $\bar{\mathscr{S}}_g(\mathbf{t})$, but it is smooth only on the open dense set $U_j = \mu_j^{-1}((0,1))$. It follows from the work of Goldman [G2] that $\mu_j|_{U_j}$ is the moment map for an S^1 action, and that these S^1 actions commute with each other.[4]

We now give an explicit description of the polyhedron $\mu(\bar{\mathscr{S}}_g(\mathbf{t}))$: for simplicity we shall assume from now on that the parameters t_j take values in the open interval $(0,1)$. This polyhedron arises in mathematical physics (conformal field theory): for suitable values of the t_j, the points of an appropriately defined integer lattice which are inside $\mu(\bar{\mathscr{S}}_g(\mathbf{t}))$ are in bijective correspondence with a basis for the vector space associated to the surface Σ_r^g by a conformal field theory, the $SU(2)$ Wess-Zumino-Witten model. As we showed in [JW1], the number of such points is the *Verlinde dimension* [V].

Our characterization of the polyhedron $B_g(\mathbf{t}) = \mu(\bar{\mathscr{S}}_g(\mathbf{t}))$ arises from the study of representations of the fundamental group $\tilde{\pi}$ of the 3-punctured sphere in $SU(2)$. This fundamental group is the free group on two generators, but it may also be written as a group with three generators and one relation, as follows:

$$\tilde{\pi} = \langle p_1, p_2, p_3 | p_1 p_2 p_3 = 1 \rangle \tag{5}$$

Let Δ be the tetrahedron in $[0,1]^3$ determined by the inequalities

$$(x_1, x_2, x_3) \in \Delta \Leftrightarrow |x_1 - x_2| \leqq x_3 \quad \text{and} \quad |x_1 - (1 - x_2)| \leqq 1 - x_3\,. \tag{6}$$

(Note that Δ is in fact invariant under permutations of the coordinates.) We have the following

Theorem 4. [JW1] *The point* (x_1, x_2, x_3) *is in* Δ *if and only if there exists a representation* $\rho\colon \tilde{\pi} \to K$ *such that* $\rho(p_j)$ *is conjugate to* $\mathrm{diag}\,(e^{i\pi x_j}, e^{-i\pi x_j})$. *Moreover, in this case such a representation is unique up to the action of* K *by conjugation.*

Now recall that the pants decomposition Γ of Σ_r^g was specified by N disjoint simple closed curves C_j, $j = 1, \ldots, N$ in the interior of Σ_r^g. In addition we have the components S_j, $j = 1, \ldots, r$ of the boundary of Σ_r^g, which we denote henceforth by C_{N+j}. Now each 3-punctured sphere γ in the pants decomposition

[4] If one works with connections on Σ_r^g rather than with representations of the fundamental group, a direct gauge theoretic construction of the S^1 action may be given (using 'gluing parameters', which arise in gauge theory when the stabilizer of the holonomy of the restriction of a connection to some submanifold is a Lie group of dimension $\geqq 1$): see Sect. 2 of [Don]

Γ determines a map $p_\gamma\colon [0,1]^{N+r} \to [0,1]^3$, where

$$p_\gamma(x_1,\ldots,x_{N+r}) = (x_{i_1(\gamma)}, x_{i_2(\gamma)}, x_{i_3(\gamma)}) \tag{7}$$

if $C_{i_1(\gamma)}, C_{i_2(\gamma)}, C_{i_3(\gamma)}$ (for $i_1(\gamma) \leqq i_2(\gamma) < i_3(\gamma)$) are the three boundary components of the 3-punctured sphere γ.

Thus we have the following

Theorem 5. [JW3] *Suppose* $0 < t_j < 1$ *for* $j = 1,\ldots,r$. *Then the polyhedron*

$$B = \left\{ (x_1,\ldots,x_N) \in [0,1]^N\colon\ (x_1,\ldots,x_N,t_1,\ldots,t_r) \in \bigcap_\gamma p_\gamma^{-1}(\Delta) \right\} \tag{8}$$

is the image $B_g(\mathbf{t}) = \mu(\bar{\mathscr{S}}_g(\mathbf{t}))$.

3 Decomposition of moduli spaces

The essential new idea we shall use to extract formulas for volumes of moduli spaces is that a decomposition of a 2-manifold into pairs of pants gives rise to a fibration of moduli spaces and hence to a fibration of the associated polyhedra. This is precisely analogous to the "Grothendieck decomposition" [Gr] of the moduli space of curves, which arises from the choice of a pants decomposition of a 2-manifold.

If we cut Σ_r^g along a separating curve C, so that $\Sigma_r^g = \Sigma_{r_1}^{g_1} \cup \Sigma_{r_2}^{g_2}$ with $r_1 + r_2 = r + 2$ and $g = g_1 + g_2$, then we find $\bar{\mathscr{S}}_g(t_1,\ldots,t_{r_1},s_1,\ldots,s_{r_2})$ maps to the fiber product $\bigcup_{w\in[0,1]} \mathscr{S}^{g_1}(t_1,\ldots,t_{r_1},w) \times \mathscr{S}^{g_2}(s_1,\ldots,s_{r_2},w)$, with generic fiber S^1. This follows from consideration of the S^1 action associated to the curve C (see the paragraph containing (4)). Applying this to the case when $g_2 = 1$, we obtain the following

Theorem 6. *Suppose* $0 < t < 1$. *We let* μ_0 *denote the moment map corresponding to the* (*exterior*) *boundary circle* S_1 *while* μ_1 *and* μ_2 *are the moment maps corresponding to the interior boundary circles* C_1 *and* C_2 (*see Fig.* 2); *then we define*

$$\bar{\mathscr{S}}_g^0(t) = \bar{\mathscr{S}}_g(t) - \mu_0^{-1}(0) - \mu_0^{-1}(1) - \mu_0^{-1}(t)\,,$$

$$\bar{\mathscr{S}}_g'(t) = \mu_0^{-1}(t) - (\mu_1^{-1}(0) \cup \mu_1^{-1}(1) \cup \mu_2^{-1}(0) \cup \mu_2^{-1}(1))\,,$$

$$\bar{\mathscr{S}}_g''(t) = \mu_0^{-1}(t) \cap (\mu_1^{-1}(0) \cup \mu_1^{-1}(1) \cup \mu_2^{-1}(0) \cup \mu_2^{-1}(1))\,,$$

and

$$\bar{\mathscr{S}}_g'''(t) = \mu_0^{-1}(0) \coprod \mu_0^{-1}(1)\,.$$

In terms of this notation, $\bar{\mathscr{S}}_g(t)$ *decomposes as follows*:

$$\bar{\mathscr{S}}_g(t) = \bar{\mathscr{S}}_g^0(t) \cup \bar{\mathscr{S}}_g'(t) \cup \bar{\mathscr{S}}_g''(t) \cup \bar{\mathscr{S}}_g'''(t)\,.$$

We note that $\mu_0^{-1}(s) \subset \bar{\mathscr{S}}_g^0(t)$ *fibers over* $\bar{\mathscr{S}}_{g-1}(s) \times S^2 \times S^2$ *with fiber* S^1, *while* $\bar{\mathscr{S}}_g'(t)$ *fibers over* $\bar{\mathscr{S}}_{g-1}(t) \times (S^2 - \{0,\infty\}) \times (S^2 - \{0,\infty\})$ *with fiber* S^1, *and* $\bar{\mathscr{S}}_g''(t)$ *fibers over* $\bar{\mathscr{S}}_{g-1}(t) \times S^2$ *with fiber* S^1.

Proof. We see by the methods of [JW1] (e.g. Theorem 2.5) or [JW3] (e.g. Lemma 3.3 and Proposition 3.7) that the map μ_0 is a submersion onto $[0,1]$; when $0 < s < 1$, each level set $\mu_0^{-1}(s)$ fibers over $\bar{\mathscr{S}}_{g-1}(s) \times \mathscr{S}_1(s,t)$ with fiber S^1. The space $\mathscr{S}_1(s,t)$ in turn fibers over the rectangle $\mu(\mathscr{S}_1(s,t))$. When $s \neq t$, we see from Fig. 3 that this rectangle does not intersect the lines $x_1 = 0,\ x_2 = 0,\ x_1 = 1,\ x_2 = 1$. The space $\mathscr{S}_1(s,t)$ is then associated to the rectangle by the usual results governing toric varieties, so that $\mathscr{S}_1(s,t) = S^2 \times S^2$, where the two copies of S^2 have diameter $\sqrt{2}s$ resp. $\sqrt{2}(1-t)$ (if $s < t$). When $s = t$, the vertices of the rectangle all satisfy either $x_1 = 0,\ x_2 = 0,\ x_1 = 1$ or $x_2 = 1$. We examine these extreme points: without loss of generality (considering Fig. 3) we may take $x_1 = 0$, which forces $x_2 = t$. Then we see (by the methods of Sect. 6 of [JW1]; see e.g. Lemma 6.4) that the preimage of the point $(x_1 = 0,\ x_2 = t)$ under $(\mu_1, \mu_2)\colon\ \mathscr{S}_1(t,t) \to [0,1]^2$ is given by $SU(2)/S^1 \cong S^2$, where S^1 acts on $SU(2)$ by right multiplication. For points $(x_1, x_2) \in [0,1]^2$ other than the four vertices, the preimage is $S^1 \times S^1$; thus, by the usual rules of toric geometry,

$$\begin{aligned}&\mathscr{S}_1(t,t) - \mu_1^{-1}(0) - \mu_1^{-1}(1) - \mu_2^{-1}(0) - \mu_2^{-1}(1)\\ &\quad\cong (S^2 - \{0,\infty\}) \times (S^2 - \{0,\infty\})\,. \qquad \square\end{aligned}$$

Remark. We note that by applying the result of Theorem 6 recursively, we eventually obtain a decomposition of $\bar{\mathscr{S}}_g(t)$ in terms of moduli spaces corresponding to genus 1.

A decomposition theorem generalizing Theorem 6 is true for $\bar{\mathscr{S}}_g(\mathbf{t})$ (when $r > 1$). The associated polyhedra fiber as follows:

$$\begin{aligned}&B_g^{r_1+r_2}(t_1,\ldots,t_{r_1},s_1,\ldots,s_{r_2})\\ &\quad= \bigcup_{w\in[0,1]} B_{g_1}^{r_1+1}(t_1,\ldots,t_{r_1},w) \times B_{g_2}^{r_2+1}(s_1,\ldots,s_{r_2},w)\,. \qquad (9)\end{aligned}$$

We thus get

$$S_g(\mathbf{t}) = \int_{w=0}^{1} dw S_{g_1}(t_1,\ldots,t_{r_1},w) \times S_{g_2}(s_1,\ldots,s_{r_2},w)\,. \qquad (10)$$

In fact, the formula may be used to derive the identification of $S_g(\mathbf{t})$ with the volume of the polyhedron determined by the inequalities (6): applying (6) inductively to the boundary circles of the pants decomposition, we find that

$$B_g(\mathbf{t}) = \int_{t_1,\ldots,t_{3g-3+r}} dt_1 \ldots dt_{3g-3+r} \prod_\gamma \chi(t_{i_1(\gamma)}, t_{i_2(\gamma)}, t_{i_3(\gamma)}) \qquad (11)$$

where $t_{i_r(\gamma)}$ gives the holonomy of the flat connection around the boundary circle $i_r(\gamma)$ and $C_{i_1(\gamma)},\ C_{i_2(\gamma)},\ C_{i_3(\gamma)}$ are the three boundary circles bounding the pair of pants γ. (Those of the $t_{i_r(\gamma)}$ corresponding to external boundary circles of Σ_r^g are defined to be the fixed values $s_1, \ldots, s_r$.) Here, $\chi(t_1, t_2, t_3)$ is the characteristic function of the tetrahedron determined by (6): in other words $\chi(t_1, t_2, t_3) = 1$ if (t_1, t_2, t_3) satisfy (6) and $\chi(t_1, t_2, t_3) = 0$ otherwise.

4 Application: Lacunarity and symplectic volumes of moduli spaces

In this section, we shall use the decomposition of the moduli space to rederive a formula of Donaldson ((27) below) for the symplectic volume $S_g(t)$ of the moduli space $\bar{\mathscr{S}}_g(t)$ of flat connections on a surface of genus g with one boundary component, for which the holonomy around the boundary is t; we shall also rederive the formula (33) for the symplectic volume of a related moduli space $\mathscr{Q}_g(t)$ of flat connections on a surface with two boundary components and holonomies $t, 1-t$ around these boundary components. Even before deriving these explicit formulas, we may note qualitatively that equation (10) (when $\Sigma_{r_1}^{g_1}$ is a twice-punctured torus and $\Sigma_{r_2}^{g_2}$ is a once punctured surface of genus g) allows one to show by induction on the genus that the function $S_g(t)$ is a piecewise polynomial function of degree $2g-1$. On dimensional grounds one would instead expect the piecewise polynomial function $S_g(t)$ to have degree $3g-2$ (since we know from [J] that $\bar{\mathscr{S}}_g(t)$ is the symplectic quotient of an action of $U(1)$ on a manifold of dimension $6g-2$, and by the Duistermaat-Heckman theorem, the symplectic form on such quotients is linear in the parameter t when t varies over regular values of the moment map). The phenomenon that the piecewise polynomial function specifying the symplectic volume of a family of reduced spaces of a manifold M with respect to the action of a torus T (*Duistermaat-Heckman polynomial*) should have lower degree than one would expect based on the dimensions of M and T is known as *lacunarity*, and has been extensively studied for actions of the maximal torus of a compact Lie group G on some of its coadjoint orbits by Guillemin, Lerman and Sternberg [GLS].

In particular we recover the *Newstead conjecture* [Ne] that $\beta^g = 0$ where β is a cohomology class of degree 4 on $\bar{\mathscr{S}}_g(1)$. We use the fibration (see [Don], Sect. 2) $\pi\colon\ \bar{\mathscr{S}}_g(t) \to \bar{\mathscr{S}}_g(1)$ with fiber S^2, which exists when $|1-t| \ll 1$. There is then a cohomology class e of degree 2 on $\bar{\mathscr{S}}_g(\mathbf{t})$, such that e restricts on the fiber of π to the generator of $H^2(S^2)$, and for which $\pi^*\beta = e^2$; furthermore, the cohomology class of the symplectic form ω_t on $\bar{\mathscr{S}}_g(t)$ is given in terms of e and the symplectic form ω_1 on $\bar{\mathscr{S}}_g(1)$ as

$$[\omega_t] = \pi^*[\omega_1] + (1-t)e\ .$$

This approach to volumes of moduli spaces was first suggested by Witten, and is described in [Don]. An alternative proof of the Newstead conjecture using a homology cycle Poincaré dual to β is given in [We2].

In previous work [JW1, JW3] we gave a formula for the volumes of the spaces $\bar{\mathscr{S}}_g(\mathbf{t})$ in terms of the numbers of integer points in polyhedra, by combining Theorem 2 with the explicit description of the polyhedron $\mu(\bar{\mathscr{S}}_g(\mathbf{t}))$ given

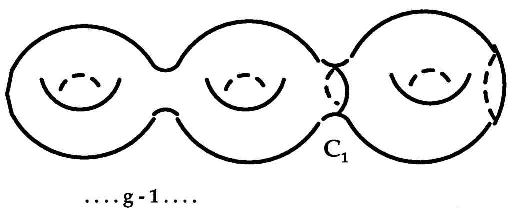

Fig. 1. A curve C_1 separating a surface of genus g into a surface of genus $g-1$ and a 2-punctured torus

in Theorem 5. (For a review along these lines see [JW4].) Here we shall instead apply the decomposition obtained in Sect. 3 to give a simple proof of the formula for the volume $S_g(t)$ of the moduli space $\bar{\mathscr{S}}_g(t)$ associated to a surface Σ_1^g of genus g with one boundary component, when $t \in (0,1)$. This formula is due to Donaldson ([Don], Sect. 6) and Witten ([Wit], (4.116)) and it may be used (as described in [Don](18)–(23)) to determine the ring structure of the cohomology $H^*(\bar{\mathscr{S}}_g(1))$. The approach to the ring structure via symplectic volumes relies on the fact (described in the previous paragraph) that for t sufficiently close to 1 the space $\bar{\mathscr{S}}_g(t)$ fibers over $\bar{\mathscr{S}}_g(1)$ with fiber S^2: it is an alternative to the approach of Thaddeus [T], who found the ring structure by applying the Riemann-Roch theorem to sections of powers of a holomorphic line bundle over $\bar{\mathscr{S}}_g(1)$. For related material see [BSz, Ki, NR, Sz].

We shall give an inductive formula relating $S_g(t)$ to $S_{g-1}(t)$, from which one may deduce the formula for $S_g(t)$. Recall that by Theorem 2,

$$S_g(t) = \mathrm{vol}_\mu(\mu(\bar{\mathscr{S}}_g(t))) , \tag{12}$$

where μ is the moment map for the action of $(S^1)^{3g-2}$ associated to any pants decomposition of Σ_1^g. We may choose the pants decomposition to contain one curve C_1 such that $\Sigma_1^g - C_1 = \Sigma_1^{g-1} \sqcup \Sigma_2^1$; in other words C_1 separates Σ_1^g into two components, a 1-punctured surface of genus $g-1$ and a 2-punctured torus (see Fig. 1).

Then the polyhedron $\mu(\bar{\mathscr{S}}_g(t))$ decomposes into a union of slices according to the value of μ_1, where μ_1 is the component of μ associated to C_1:

$$\mu(\bar{\mathscr{S}}_g(t)) = \bigcup_{s\in[0,1]} \mu(\bar{\mathscr{S}}_{g-1}(s)) \times \mu(\bar{\mathscr{S}}_1(s,t)) . \tag{13}$$

Introducing

$$\Phi(s,t) = \mathrm{vol}_{\mathrm{eucl}}(\mu(\bar{\mathscr{S}}_1(s,t))) ,$$

we see the following (which is an immediate consequence of (13)):

Theorem 7. *We have for $g \geqq 2$ that*

$$\mathrm{vol}_{\mathrm{eucl}}(\mu(\bar{\mathscr{S}}_g(t))) = \int_{s=0}^{1} ds \Phi(s,t) \mathrm{vol}_{\mathrm{eucl}}(\mu(\bar{\mathscr{S}}_{g-1}(s))) . \tag{14}$$

Theorem 7 suffices to determine $S_g(t)$ provided one knows $S_1(t)$.

We must now study the relation between the Euclidean volume $\mathrm{vol}_{\mathrm{eucl}}$ and the volume vol_μ. The correct normalization of the measure vol_μ is that which assigns unit volume to the fundamental domain determined by the lattice Λ, where the torus $\mathfrak{t}/\Lambda$ acts *effectively* on M. In the case of $\bar{\mathscr{S}}_g$ (corresponding to a closed surface) it was shown in [JW3] (Proposition 3.10) that the lattice Λ is spanned by the standard unit basis vectors e_j $(j = 1, \ldots, 3g-3)$ of $\mathbb{R}^{3g-3}$ and the additional vectors $\tilde{e}_\gamma (\gamma = 1, \ldots, 2g-2)$ where

$$\tilde{e}_\gamma = \tfrac{1}{2}(e_{j_1(\gamma)} + e_{j_2(\gamma)} + e_{j_3(\gamma)})$$

if $C_{j_1(\gamma)}$, $C_{j_2(\gamma)}$, $C_{j_3(\gamma)}$ are the three boundary circles of the pair of pants P_γ.

When $\Sigma = \Sigma_r^g$ has r boundary components, we have the following generalization ([JW3], Proposition 3.13):

Proposition 8. *The pants decomposition of Σ_r^g determines a torus $K = \mathbb{R}^{3g-3+d}/\Lambda$ which acts effectively on an open dense set in $\bar{\mathscr{S}}_g(\mathbf{t})$ preserving the symplectic structure. The lattice Λ is spanned by the standard unit basis vectors $e_j (j = 1, \ldots, 3g-3)$ of $\mathbb{R}^{3g-3+d}$ and the additional vectors $\tilde{e}_\gamma (\gamma = 1, \ldots, 2g-2)$ where*

$$\tilde{e}_\gamma = \frac{1}{2} \sum_{j(\gamma)} e_{j(\gamma)}$$

and we sum over $j(\gamma)$ corresponding to those boundary circles $C_{j(\gamma)}$ of the pair of pants P_γ which are not components of the boundary $\partial \Sigma_r^g$ of Σ_r^g.

The result of Proposition 8 is that the correctly normalized volume vol_μ of $\mu(U)$ (for which Theorem 2 holds) is not the Euclidean volume $\mathrm{vol}_{\mathrm{eucl}}$ but rather

$$\mathrm{vol}_\mu = \frac{\mathrm{vol}_{\mathrm{eucl}}}{2^{2g-3+d}} \,. \tag{15}$$

We must now study the function $\Phi(s,t)$ specifying the volume of the family of moduli spaces associated to the 2-punctured torus. We may form a 2-punctured torus by gluing together two 3-punctured spheres along two of their boundary components (see Fig. 2).

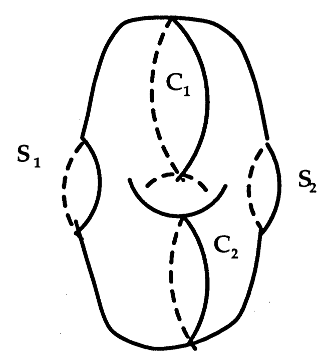

Fig. 2. A pants decomposition of a 2-punctured torus

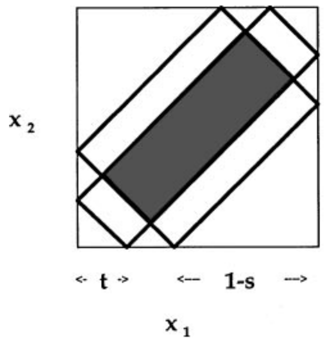

Fig. 3. The image of the moment map for the moduli space $\bar{\mathscr{S}}_1(s,t)$ associated to a 2-punctured torus, where the conjugacy classes of the holonomies around the two boundary components C_1, C_2 are constrained by the parameters s and t. The moment maps for the S^1 actions corresponding to the curves C_1, C_2 take the values x_1, x_2

Let the glued boundary components then be C_1, C_2 and the remaining boundary components S_1 and S_2. Then an application of Theorem 6 shows that $\mu(\bar{\mathscr{S}}_1(s,t))$ is the intersection of two rectangles $\square_s \cap \square_t$ in $[0,1]^2$, where

$$\square_s = \{(x_1, x_2) \in [0,1]^2 : \ |x_1 - x_2| \leqq s, |x_1 - (1 - x_2)| \leqq 1 - s\} .$$

If $s < t$ then $\square_s \cap \square_t$ is just a rectangle of side lengths $\sqrt{2}s$ and $\sqrt{2}(1-t)$ (see Fig. 3).

Furthermore, if $s \neq t$ then $\square_s \cap \square_t \subset (0,1)^2$: but $\mu^{-1}((0,1)^2)$ is the open set $U \subset \bar{\mathscr{S}}_1(s,t)$ where the action of $(S^1)^2$ is defined. So if $s \neq t$ then $\bar{\mathscr{S}}_1(s,t)$ has a *global* action of $(S^1)^2$ such that the image of the moment map is the rectangle $\square_s \cap \square_t$. Applying Delzant's theorem [Del] we see that $\bar{\mathscr{S}}_1(s,t)$ is symplectomorphic to $S^2 \times S^2$, where if $s < t$ the two copies of S^2 have symplectic volumes $\sqrt{2}s$ and $\sqrt{2}(1-t)$.

We then have the following formula (which follows immediately from Fig. 3):

Proposition 9. *The Euclidean volume* $\Phi(s,t)$ *is given by*

$$\Phi(s,t) = 2s(1-t)H(t-s) + 2t(1-s)H(s-t) , \tag{16}$$

where

$$H(x) = \begin{cases} 0, & x < 0 ; \\ 1, & x \geqq 0 . \end{cases}$$

Differentiating the expression in (16) we find a formula for $(d/dt)^2\Phi(s,t)$ in terms of delta functions $\delta(t-s)$ and their derivatives. In fact, rewriting (16) as

$$\Phi(s,t) = 2\{(s-t)H(t-s) + t - ts\} \tag{17}$$

we find

$$\frac{d}{dt}\Phi(s,t) = 2\{-H(t-s) + (s-t)\delta(t-s) + 1 - s\} , \tag{18}$$

so that

$$\left(\frac{d}{dt}\right)^2 \Phi(s,t) = 2\left(-2\delta(s-t) + (s-t)\frac{d}{dt}\delta(t-s)\right) . \tag{19}$$

Define

$$S_g^e(t) = \mathrm{vol}_{\mathrm{eucl}}\, \mu(\bar{\mathscr{S}}_g(t)) = \int \Phi(s,t) S_{g-1}^e(s) ds . \tag{20}$$

Then we find from (19) and (20) that

$$\begin{aligned}\frac{d^2}{dt^2} S_g^e(t) &= 2\int \left\{-2\delta(t-s) + (s-t)\frac{d}{dt}\delta(t-s)\right\} S_{g-1}^e(s) ds \\ &= -4S_{g-1}^e(t) + 2\int (s-t)\left(-\frac{d}{ds}\delta(t-s)\right) S_{g-1}^e(s) ds . \end{aligned} \tag{21}$$

Integrating by parts, we have that

$$\begin{aligned}\int (s-t)\left\{\frac{d}{ds}\delta(t-s)\right\} S_{g-1}^e(s) ds &= -S_{g-1}^e(t) - \int \delta(t-s)(s-t)\frac{d}{ds} S_{g-1}^e(s) ds \\ &= -S_{g-1}^e(t) \end{aligned} \tag{22}$$

(the second term vanishes because $\int x\delta(x) f(x) = 0$ for any f). Hence we find

$$\frac{d^2}{dt^2} S_g^e(t) = -2S_{g-1}^e(t) . \tag{23}$$

Using (15) we see that

$$S_g(t) = \frac{S_g^e(t)}{2^{2g-2}} .$$

This transforms (23) as follows:

Proposition 10. *The volumes $S_g(t)$ of the moduli space associated to a 2-manifold of genus g satisfy the following recurrence relation*:

$$\frac{d^2}{dt^2} S_g(t) = -\tfrac{1}{2} S_{g-1}(t) . \tag{24}$$

To use the recursion relation (24) to determine the functions $S_g(t)$, we recall that the Bernoulli polynomials $P_m(x)$ (for $x \in [0,1]$) may be characterized by the following three properties:

$$\begin{aligned} &(a)\ P_0(x) &&= 1 \\ &(b)\ \frac{d}{dx} P_m(x) &&= P_{m-1}(x) \\ &(c)\ \int P_m(x) dx = 0 . && \end{aligned} \tag{25}$$

The following equations thus suffice to characterize the *odd degree* Bernoulli polynomials P_{2m-1}:

$$\begin{aligned} &(a)\ P_1(x) &&= x - \tfrac{1}{2} \\ &(b)\ \frac{d^2}{dx^2} P_{2m+1}(x) &&= P_{2m-1}(x) \\ &(c)\ \int P_{2m-1}(x)dx &&= \int \frac{dP_{2m-1}}{dx} dx = 0. \end{aligned} \tag{26}$$

Equation (26c) may be replaced by

$$\int P_{2m-1}(x)dx = 0, \qquad P_{2m-1}(0) = P_{2m-1}(1)\,. \tag{26c$'$}$$

Notice that

$$P_{2m-1}(x) = -P_{2m-1}(1-x)$$

(since the function $P'_{2m-1}(x) = -P_{2m-1}(1-x)$ satisfies the equations (26)).

We shall prove the following Proposition:

Proposition 11. *For* $0 < t < 1$ *we have*

$$S_g(1-t) = (-1)^g 2^g P_{2g-1}\left(\frac{1-t}{2}\right)\,. \tag{27}$$

This was originally proved by Donaldson ([Don], (22)); we shall give a simple proof based on our decomposition theorem (Theorem 6). We extend the definition of $S_g(t)$ to $t \in [-1, 1]$ by setting $S_g(-t) = -S_g(t)$ for $t \in [0, 1]$.

If we define $\tilde{P}_{2g-1}(x)$ for $x \in [0, 1]$ by

$$S_g(t) = (-1)^g 2^g \tilde{P}_{2g-1}\left(\frac{1-t}{2}\right)\,, \tag{28}$$

then $\tilde{P}_{2g-1}(x)$ is defined for $x \in [0, 1]$ and satisfies

$$\tilde{P}_{2g-1}(1-x) = -\tilde{P}_{2g-1}(x)\,. \tag{29}$$

We find then that $\tilde{P}_{2g-1}$ satisfies (26) (b) and (c$'$): (b) follows immediately from (24) while (c$'$) follows from (29) and the fact that $\tilde{P}_{2g-1}(1) = (-1)^g 2^g S_g(0) = 0$. The latter equation is true because

$$S_g(0) = \int_0^1 ds \Phi(s, 0) S_{g-1}(s) \tag{30}$$

and $\Phi(s, 0) = 2sH(-s) = 0$ for $0 \leqq s \leqq 1$.

To prove that $\tilde{P}_{2g-1}$ satisfies (26) (a) we need only consider the function $S_1(1-t)$. Since a one-punctured torus can be obtained by gluing together two boundary components of a three-punctured sphere, we see from

(6) that $\mu(\mathscr{S}_1(1-t))$ is the line $\{x\colon |2x-1| \leqq t\}$ which has length t; hence $t = S_1(1-t) = -2\tilde{P}_1\left(\frac{1-t}{2}\right)$, which gives $\tilde{P}_1\left(\frac{1-t}{2}\right) = -t/2$ or

$$\tilde{P}_1(x) = \tfrac{1}{2} - x\ .$$

This completes the verification of the formula (24): we have recovered the inductive characterization of the odd degree Bernoulli polynomials $P_{2g-1}(t)$.

Donaldson introduces the related moduli space $\mathscr{Q}_g(t)$ which is the moduli space of flat connections on a surface Σ_2^g with holonomies t and $1-t$ around the two boundary components. One may define

$$Q_g(t) = \mathrm{vol}_\omega \mathscr{Q}_g(t) \tag{31}$$

and

$$Q_g^e(t) = \mathrm{vol}_{\mathrm{eucl}}\, \mu(\mathscr{Q}_g(t))\ . \tag{32}$$

We then have the following Proposition, which is due originally to Donaldson ([Don], (23)):

Proposition 12. *The function Q_g is given by*

$$Q_g(t) = (-1)^{g+1}2^g\left(P_{2g}\left(t+\tfrac{1}{2}\right) - P_g\left(\tfrac{1}{2}\right)\right) \tag{33}$$

when $0 < t < \frac{1}{2}$.

Our methods also permit us to give a simple proof of this Proposition. Using the fact that Σ_2^g may be obtained from Σ_1^g by attaching one boundary component of a pair of pants to the boundary of Σ_1^g, we find that

$$Q_g^e(t) = \int ds V(s,t) S_g(s) \tag{34}$$

where

$$V(s,t) = S_0(s,t,1-t) = H\left(\frac{s+1}{2} - t\right) H\left(\frac{s-1}{2} + t\right)$$

(this formula follows from (6)). Thus we have

$$\frac{d}{dt}V(s,t) = -\,\delta\left(\frac{s}{2} + \frac{1}{2} - t\right) H\left(\frac{s}{2} - \frac{1}{2} + t\right) + \delta\left(\frac{s}{2} - \frac{1}{2} + t\right) H\left(\frac{s}{2} + \frac{1}{2} - t\right)\ , \tag{35}$$

so that

$$\begin{aligned}\frac{d}{dt}Q_g^e(t) &= \int ds S_g(s)\left\{-\delta\left(\frac{s}{2} + \frac{1}{2} - t\right) H\left(\frac{s}{2} - \frac{1}{2} + t\right) + \delta\left(\frac{s}{2} - \frac{1}{2} + t\right) H\left(\frac{s}{2} + \frac{1}{2} - t\right)\right\} \\ &= 2H(1-2t)S_g(1-2t) - 2H(2t-1)S_g(2t-1) \\ &= 2S_g(1-2t), \quad t < \frac{1}{2}\ .\end{aligned} \tag{36}$$

Because $Q_g = Q_g^e/2^{2g-1}$, while $S_g = S_g^e/2^{2g-2}$ (by Proposition 8), we obtain

$$\frac{d}{dt} Q_g(t) = S_g(1-2t) \quad \text{when } t \leqq \tfrac{1}{2} \, . \tag{37}$$

Using the formula (27) for $S_g(t)$, we now find that

$$S_g(1-2t) = (-1)^g 2^g P_{2g-1}\left(\tfrac{1}{2} - t\right) = (-1)^{g+1} 2^g P_{2g-1}\left(\tfrac{1}{2} + t\right) \, ,$$

so that

$$\frac{d}{dt} Q_g(t) = (-1)^{g+1} 2^g P_{2g-1}\left(\tfrac{1}{2} + t\right) \, . \tag{38}$$

It follows from this and (24) that

$$Q_g(t) = (-1)^{g+1} 2^g (P_{2g}\left(\tfrac{1}{2} + t\right) + c) \tag{39}$$

for some constant c; to verify that $c = -P_{2g}(\frac{1}{2})$, it suffices to check that $Q_g(0) = 0$. This however is obvious from (34) since

$$Q_g(0) = \int ds V(s,0) S_g(s)$$

and $V(s,0) = H(s+1)H(s-1) = 0$ for $s < 1$. This completes the proof of Proposition 12.

A generalization of the equation (24) which applies to symplectic volumes of moduli spaces of flat connections on a general compact Lie group G is given in [JW5]: it is

$$\prod_{\gamma > 0} \gamma(\partial/\partial t)^2 S_g^G(t) = \kappa S_{g-1}^G(t) \, , \tag{40}$$

where $S_g^G(t)$ is the symplectic volume of the moduli space of flat G connections on a surface of genus g with one boundary component for which the holonomy around the boundary is given by $t \in \mathrm{Lie}(T)$.

References

[A] M.F. Atiyah: Convexity and Commuting Hamiltonians. Bull. Lond. Math. Soc. **14**, (1981) 1

[AB] M.F. Atiyah, R. Bott: The Yang Mills Equations over Riemann Surfaces. Phil. Trans. Roy. Soc. London **A308** (1982) 523

[BSz] A. Bertram, A. Szenes: Hilbert Polynomials of Moduli Spaces of Rank 2 Vector Bundles II. Topology **32**, (1993) 599

[Del] T. Delzant: Hamiltoniens Periodiques et Images Convexes de l'Application Moment. Bull. Soc. Math. France **116**, (1988) 315

[Don] S.K. Donaldson: Gluing Techniques in the Cohomology of Moduli Spaces. In Topological Methods in Modern Mathematics (volume in honor of the sixtieth birthday of John Milnor), Publish or Perish (1993) 137

[G1] W. Goldman: The Symplectic Nature of Fundamental Groups of Surfaces. Adv. Math. **54**, (1984) 200

[G2] W. Goldman: Invariant Functions on Lie Groups and Hamiltonian Flows of Surface Group Representations. Inv. Math. **85**, (1986) 263

[Gr] A. Grothendieck: Esquisse d'un Programme. Unpublished manuscript

[GLS] V. Guillemin, E. Lerman, S. Sternberg: Symplectic Fibrations and Multiplicity Diagrams. Cambridge University Press (to appear)
[GS1] V. Guillemin, S. Sternberg: Convexity Properties of the Moment Mapping. Inv. Math. **67**, (1982) 491
[GS2] V. Guillemin, S. Sternberg: Symplectic Techniques in Physics, Cambridge University Press, 1984
[J] L.C. Jeffrey: Extended moduli spaces of flat connections on Riemann surfaces. Math. Annalen **298**, (1994) 667
[JW1] L.C. Jeffrey, J. Weitsman: Bohr–Sommerfield Orbits in the Moduli Space of Flat Connections and the Verlinde Dimension Formula. Commun. Math. Phys., **150**, (1992) 593
[JW2] L.C. Jeffrey, J. Weitsman: Half Density Quantization of the Moduli Space of Flat Connections and Witten's Semiclassical Manifold Invariants. Topology, **32**, (1993) 509
[JW3] L.C. Jeffrey, J. Weitsman: Toric Structures on the Moduli Space of Flat Connections on a Riemann Surface: Volumes and the Moment Map. Institute for Advanced Study Preprint IASSNS-HEP-92/25; Adv. Math. **106**, (1994) 151
[JW4] L.C. Jeffrey, J. Weitsman: Torus Actions, Moment Maps, and the Symplectic Geometry of the Moduli Space of Flat Connections on a Two-Manifold. To appear in Proceedings of the AMS Summer Conference on Conformal Field Theory, Topological Field Theory, and Quantum Groups (Mt. Holyoke College, June, 1992) (ed. P. Sally) Contemporary Mathematics **175**, (1995)
[JW5] L.C. Jeffrey, J. Weitsman: In preparation
[JW6] L.C. Jeffrey, J. Weitsman: Torus actions and the topology and symplectic geometry of flat connections on 2-manifolds. In Topology, Geometry and Field Theory (eds. K. Fukaya, M. Furuta, T. Kohno and D. Kotschick), Proceedings of the Taniguchi Foundation Symposium on Low Dimensional Topology and Topological Field Theory, Kyoto, World Scientific, (1994) 49
[Ki] F. Kirwan: The Cohomology Rings of Moduli Spaces of Vector Bundles over Riemann Surfaces. J. Amer. Math. Soc., **5**, (1992) 853
[MS] V. Mehta, C.S. Seshadri: Moduli of Vector Bundles on Curves with Parabolic Structures, Math. Ann. **248**, (1980) 205
[NR] M.S. Narasimhan, T.R. Ramadas: Factorization of Generalized Theta Functions. I. Invent. Math. **114**, (1993) 565
[NS] M.S. Narasimhan, C.S. Seshadri: Stable and Unitary Vector Bundles on a Compact Riemann Surface. Annals of Math. **82**, (1965) 540
[Ne] P.E. Newstead: Characteristic Classes of Stable Bundles on an Algebraic Curve. Trans. Amer. Math. Soc. **69**, (1972) 337
[Sz] A. Szenes: Hilbert Polynomials of Moduli Spaces of Rank 2 Vector Bundles I. Topology **32**, (1993) 587
[T] M. Thaddeus: Conformal Field Theory and the Cohomology of the Moduli Space of Stable Bundles. J. Diff. Geom. **35**, (1992) 131
[V] E. Verlinde: Fusion Rules and Modular Transformations in 2d Conformal Field Theory. Nucl. Phys. **B300**, (1988) 351
[We1] J. Weitsman: Real Polarization of the Moduli Space of Flat Connections on a Riemann Surface. Commun. Math. Phys. **145**, (1992) 425
[We2] J. Weitsman: Geometry of the Intersection Ring of the Moduli Space of Flat Connections and the Conjectures of Newstead and Witten. Preprint (1993)
[Wit] E. Witten: On Quantum Gauge Theories in Two Dimensions. Commun. Math. Phys. **140**, (1991) 153