Math. Ann. 316, 103–137 (2000)

Mathematische Annalen

A Liouville theorem for vector-valued nonlinear heat equations and applications

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Received: 31 July 1998 / in final form: 13 May 1999

Mathematics Subject Classification (1991): 35K40, 35K55, 35A20

1 Introduction

We are concerned in this paper with blow-up solutions of

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + F(|u|)u & \text{in} \quad \Omega \times [0,T) \\ u = 0 & \text{on} \quad \partial\Omega \times [0,T) \\ u(.,0) = u_0 & \text{in} \quad \Omega \end{cases} \tag{1}$$

where
$u : (x,t) \in \Omega \times [0,T) \to \mathbb{R}^M$, $u_0 : \Omega \to \mathbb{R}^M$,
Ω is a bounded convex regular open set of $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$, $T > 0$,
$(\Delta u)_i = \Delta u_i$, $|u|$ is the euclidian norm of u in $\mathbb{R}^M$
$F : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying

$$F(|u|) \sim |u|^{p-1} \text{ as } |u| \to +\infty$$

(in a suitable norm) with

$$p > 1 \text{ and } (3N-4)p < 3N+8. \tag{2}$$

We also consider the following condition on p valid for scalar equations ($M = 1$) with nonnegative initial data:

$$u_0 \geq 0 \text{ and } (N-2)p < N+2. \tag{3}$$

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The Cauchy problem for system (1) can be solved (for example) in $L^\infty(\mathbb{R}^N, \mathbb{R}^M)$. If the maximal solution $u(t)$ is defined on $[0, T)$ with $T < +\infty$, then

$$\lim_{t\to T} \|u(t)\|_{L^\infty} = +\infty.$$

We say that $u(t)$ blows-up at time T. If $a \in \Omega$ satisfies $|u(x_n, t_n)| \to +\infty$ as $n \to +\infty$ for some sequence $(x_n, t_n) \to (a, T)$, then a is called a blow-up point of u. The set of all blow-up points of $u(t)$ is called the blow-up set of $u(t)$ and will be denoted by S.

The existence of blow-up solutions for systems of the type (1) has been proved by several authors (Friedman [Fri65], Fujita [Fuj66], Levine [Lev73], Ball [Bal77],..). Many authors have been concerned by the asymptotic behavior of $u(t)$ at blow-up time, near blow-up points. Let us point out that a great deal of the known results are valid only for scalar equations with nonnegative initial data (case (3)), typically for the equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p,\ \ p > 1,\ \ \left(N \le 2 \text{ or } p < \frac{N+2}{N-2}\right); \tag{4}$$

indeed, in the case (3), one can use the maximum principle which does not hold in general in the case (2). On the contrary, the results in the vectorial case or even in the scalar case with no positivity condition remain very poor.

Let us give a sketch of the known results both in cases (2) and (3). For simplicity in the notations, we assume that

$$F(|u|) = |u|^{p-1}$$

and consider the equation

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u. \tag{5}$$

Consider $u(t)$ a solution of (5) which blows-up at time T at a point $a \in \Omega$. The study of the behavior of $u(t)$ near (a, T) has been done through the introduction of the following similarity variables:

$$y = \frac{x-a}{\sqrt{T-t}},\ \ s = -\log(T-t),\ \ w_a(y, s) = (T-t)^{\frac{1}{p-1}} u(x, t). \tag{6}$$

It is readily seen from (5) that w_a (or simply w) satisfies the following equation: $\forall s \ge -\log T$, $\forall y \in W_{a,s} \equiv e^{\frac{s}{2}}(\Omega - a)$,

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1}w. \tag{7}$$

The following Lyapunov functional is associated with (7):

$$E(w) = \int_{W_{a,s}} \left(\frac{1}{2}|\nabla w|^2 + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1} \right) \rho(y) dy \tag{8}$$

where

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}. \tag{9}$$

In the case (3) (equation (4)), Giga and Kohn showed in [GK85], [GK87] and [GK89] that

$$\forall x \in \Omega,\ \forall t \in [0, T),\ |u(x,t)| \leq C(T-t)^{-\frac{1}{p-1}} \tag{10}$$

for some constant $C > 0$. They also showed that

$$w_a(y,s) \to \kappa \equiv (p-1)^{-\frac{1}{p-1}} \text{ as } s \to +\infty, \tag{11}$$

uniformly on compact sets. This estimate has been refined until the higher order by Filippas, Kohn and Liu [FK92], [FL93], Herrero and Velázquez [HV93], [HV92a], [HV92b], [Vel93a]. A notion of limiting blow-up profile has been developed both in variables (x, t) and (y, s) by Bricmont and Kupiainen [BK94], Merle and Zaag [MZ97], Zaag [Zaa98], Herrero and Velázquez.

In [MZ98a], a further step was accomplished in the understanding of the behavior of nonnegative scalar solutions of (1). We proved there the following Liouville Theorem for equation (7):

Let w be a nonnegative solution of (7) defined for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$ such that $w \in L^\infty(\mathbb{R}^N \times \mathbb{R})$. Then, necessarily one of the following cases occurs:

$$w \equiv 0 \text{ or } w \equiv \kappa \text{ or } \exists s_0 \in \mathbb{R} \text{ such that } w(y,s) = \varphi(s - s_0) \tag{12}$$

$$\text{where } \varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}} \text{ and } \kappa = (p-1)^{-\frac{1}{p-1}}.$$

From this Theorem we derived in [MZ98a] the following localization theorem:

$\forall \epsilon > 0$, $\exists C_\epsilon > 0$ *such that* $\forall t \in [\frac{T}{2}, T)$, $\forall x \in \mathbb{R}^N$,

$$\left| \frac{\partial u}{\partial t} - u^p \right| \leq \epsilon u^p + C_\epsilon. \tag{13}$$

We also derived in [MZ98b] the following uniform estimates of order one (in the case $\Omega = \mathbb{R}^N$):

$\exists C_i > 0$, $i = 1, 2, 3$ such that $\forall \epsilon > 0$, $\exists s_0(\epsilon) \geq -\log T$ such that $\forall s \geq s_0$, $\forall a \in \mathbb{R}^N$,

$$\kappa \leq \|w_a(s)\|_{L^\infty} \leq \kappa + \left(\frac{N\kappa}{2p} + \epsilon\right)\frac{1}{s}, \ \|\nabla^i w_a(s)\|_{L^\infty} \leq \frac{C_i}{s^{i/2}} \text{ for } i = 1, 2, 3, \tag{14}$$

where w_a is defined in (6) and $\nabla^i w$ stands for the differential of w of order i.

The results (14) and (13) are direct consequences of the Liouville Theorem (12) which is valid only for positive scalar solutions of (1).

As to the case (2), the starting point was the proof by Giga and Kohn [GK87] of the validity of the global estimate (10). In [FM95], Filippas and Merle showed that

$$w_a(y, s) \to \kappa\omega_a \text{ as } s \to +\infty$$

uniformly on compact sets, for some $\omega_a \in S^{M-1}$. No other results were known.

In this paper, we extend the validity of the Liouville Theorem (12) to the vectorial case and obtain the following theorem which classifies all connections in L^∞_{loc} between critical points of (7) (which are according to [GK85]: 0 and $\kappa\omega$ for all $\omega \in S^{M-1}$). This Theorem is in some sense a classification of "critical points at infinity" (in a parabolic sense) for equation (7).
Note that this Theorem is valid not only for p satisfying (2) but for all subcritical p, that is under the condition

$$p > 1 \text{ and } (N-2)p < N + 2. \tag{15}$$

Theorem 1 (Liouville Theorem for equation (7)) *Assume (15) and consider w a solution of (7) defined for all $(y, s) \in \mathbb{R}^N \times \mathbb{R}$ such that $w \in L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^M)$. Then necessarily one of the following cases occurs:*
i) $w \equiv 0$,
ii) $\exists \omega_0 \in S^{M-1}$ such that $w \equiv \kappa\omega_0$,
iii) $\exists s_0 \in \mathbb{R}$, $\exists \omega_0 \in S^{M-1}$ such that $w(y, s) = \varphi(s - s_0)\omega_0$ where

$$\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}.$$

Remark. In [GK85], Giga and Kohn assumed in addition to the hypotheses of Theorem 1 that

$$\limsup_{s\to+\infty} |w(0, s)| > 0, \tag{16}$$

and proved that in this case, ii) occurs (Theorem 2 page 310). Indeed, under assumption (16), it follows directly from energy arguments that w is a stationary solution of Equation (7). We concentrate in our proof on the classification

of non stationary solutions. This will need introduction of new tools such as a combination of the linearization of the equation as s goes to $-\infty$, the use of a geometric invariance of equation (4) and a blow-up criterion for equation (7), sharp for data close to stationary solutions.
This Theorem has an equivalent formulation for solutions of (5) via the transformation (6).

Corollary 1 (A Liouville Theorem for equation (5)) *Assume that (15) holds and that u is a solution in L^∞ of (5) defined for $(x,t) \in \mathbb{R}^N \times (-\infty, T)$. Assume in addition that $|u(x,t)| \le C(T-t)^{-\frac{1}{p-1}}$. Then $u \equiv 0$ or there exist $T_0 \ge T$ and $\omega_0 \in S^{M-1}$ such that $\forall (x,t) \in \mathbb{R}^N \times (-\infty, T)$, $u(x,t) = \kappa(T_0 - t)^{-\frac{1}{p-1}}\omega_0$.*

Our second contribution in this paper is to show that the global estimate (10) of Giga and Kohn which is valid in the cases (2) and (3) is in fact uniform with respect to u_0.

Theorem 2 (Uniform estimates with respect to u_0) *Assume condition (2) holds and consider u a solution of (5) which blows-up at time $T < T_0$ and satisfies $\|u(0)\|_{C^2(\Omega)} \le C_0$. Then, there exists $C(C_0, T_0)$ such that $\forall t \in [0, T)$, $\|u(t)\|_{L^\infty(\Omega)} \le Cv(t)$ where $v(t) = \kappa(T-t)^{-\frac{1}{p-1}}$ is the solution of*

$$v' = v^p \text{ and } v(T) = +\infty.$$

Remark. We suspect that this result is true with no condition on T.
Let us remark that we suspect this Theorem to be valid in the case (15).

Theorems 1 and 2 have important consequences in the understanding of the blow-up behavior for equation (5) in the case (2). We have the following localization result which compares (5) with the associated ODE

$$u' = u^p.$$

Theorem 3 (Uniform ODE Behavior) *Assume that (2) holds and consider $T \le T_0$ and $\|u_0\|_{C^2(\Omega)} \le C_0$. Then, $\forall \epsilon > 0$, there is $C(\epsilon, C_0, T_0)$ such that $\forall x \in \Omega$, $\forall t \in [0, T)$,*

$$\left| \frac{\partial u}{\partial t}(x,t) - |u|^{p-1}u(x,t) \right| \le \epsilon |u(x,t)|^p + C.$$

Remark. Note that the condition $u(0) \in C^2$ in Theorems 2 and 3 is not restrictive, because of the regularizing effect of the heat equation.

As direct consequences of Theorem 3, we have the following striking corollary:

Corollary 2 *Assume that (2) holds and consider $u(t)$ a solution of (5). Let $a \in \Omega$ be a blow-up point of $u(t)$. Then,*
i) $|u(x,t)| \to +\infty$ as $(x,t) \to (a,T)$,
*ii) (***Approximate scalar behavior of** $|u|$*) $\exists \delta > 0$ such that $\forall x \in B(a,\delta)$, $\forall t \in [T-\delta, T)$,*

$$\frac{\partial |u|}{\partial t}(x,t) > 0 \text{ and } |u(x,t)| > 0.$$

iii) If $M = 1$ and $u(a,t) \sim \epsilon\kappa(T-t)^{-\frac{1}{p-1}}$ where $\epsilon \in \{-1,1\}$, then $\exists \delta > 0$ such that $\forall x \in B(a,\delta)$, $\forall t \in [T-\delta, T)$,

$$\epsilon u(x,t) > 0 \text{ and } \epsilon \frac{\partial u}{\partial t}(x,t) > 0.$$

We now set in the case (2) some results which were known before only in the scalar case with nonnegative initial data. These results follow from Theorems 1 and 2 and the proofs of the positive case.

Theorems 1 and 2 yield the following uniform estimates of order 1 for solution of (5):

Theorem 4 **(L^∞ refined estimates for $w(s)$ and $u(t)$ at blow-up)** *Assume that (2) holds. Then, there exist positive constants C_i for $i = 1, 2, 3$ such that if u is a solution of (5) which blows-up at time T and satisfies $u(0) \in C^3(\mathbb{R}^N)$, then $\forall \epsilon > 0$, there exists $s_1(\epsilon) \geq -\log T$ such that*

i) $\forall s \geq s_1$, $\forall a \in \mathbb{R}^N$,

$$\begin{array}{ll} \|w_a(s)\|_{L^\infty} \leq \kappa + (\frac{N\kappa}{2p} + \epsilon)\frac{1}{s}, & \|\nabla w_a(s)\|_{L^\infty} \leq \frac{C_1}{\sqrt{s}}, \\ \|\nabla^2 w_a(s)\|_{L^\infty} \leq \frac{C_2}{s}, & \|\nabla^3 w_a(s)\|_{L^\infty} \leq \frac{C_3}{s^{3/2}}, \end{array}$$

where $\kappa = (p-1)^{-\frac{1}{p-1}}$,

ii) $\forall t \geq T - e^{-s_1}$,

$$\begin{array}{l} \|u(t)\|_{L^\infty} \leq \left(\kappa + (\frac{N\kappa}{2p} + \epsilon)\frac{1}{|\log(T-t)|}\right)(T-t)^{-\frac{1}{p-1}}, \\ \|\nabla^i u(t)\|_{L^\infty} \leq C_i \frac{(T-t)^{-(\frac{1}{p-1}+\frac{i}{2})}}{|\log(T-t)|^{i/2}} \end{array}$$

for $i = 1, 2, 3$.

Remark. Note that these estimates are sharp (see for example [MZ97]). If $v : \mathbb{R}^N \to \mathbb{R}$ is regular, $\nabla^i v$ stands for the differential of order i of v. For all $y \in \mathbb{R}^N$, we define $|\nabla v(y)|^2 = \sum_{j=1}^N \left(\partial_j v(y)\right)^2$, $|\nabla^2 v(y)| = \sup_{z \in \mathbb{R}^N} \frac{\left|z^T \nabla^2 v(y) z\right|}{|z|^2}$

and $|\nabla^3 v(y)| = \sup_{\alpha,\beta,\gamma\in\mathbb{R}^N} \left| \sum_{i,j,k} \frac{\alpha_i}{|\alpha|}\frac{\beta_j}{|\beta|}\frac{\gamma_k}{|\gamma|}\partial^3_{i,j,k} v(y)\right|$.
In addition, $\|v\|_{L^\infty} = \sup_{y\in\mathbb{R}^N} |v(y)|$ and $\|\nabla^i v\|_{L^\infty} = \sup_{y\in\mathbb{R}^N} |\nabla^i v(y)|$.

We also obtain information on the limiting blow-up profile for equation (7):

Proposition 1 (Existence of a blow-up profile for equation (5)) *Assume (2) holds and consider $u(t)$ a solution of (5) which satisfies $u(0) \in H^1(\mathbb{R}^N)$ and blows-up at (a, T). Then, there exist $\omega_a \in S^{M-1}$, Q a $N \times N$ orthonormal matrix and $l \in \{0, ..., N\}$ such that $\forall K > 0$*

$$\sup_{|y|\le K\sqrt{s}} \left| w_a(Qy, s) - \left(p - 1 + \frac{(p-1)^2}{4p}\sum_{i=1}^{l}\frac{y_i^2}{s}\right)^{-\frac{1}{p-1}} \omega_a \right| \to 0 \text{ as } s \to +\infty. \tag{17}$$

Remark. In the case $l = 0$, Proposition 1 yields $(p-1)^{-\frac{1}{p-1}} = \kappa$ as asymptotic behavior for w_a. This corresponds to a degenerate blow-up rate, and one can find an other blow-up profile in the scale $y \sim \exp\left(+\frac{s}{2}\left(1 - \frac{1}{k}\right)\right)$ for some $k \in \mathbb{N}\backslash\{0, 1\}$.
Remark. In the case of single point blow-up with $l = N$ and $M = 1$, we use the Liouville Theorem and show with Fermanian-Kammerer in [FKMZ] that the behavior (17) is stable under perturbations of initial data. Moreover, the convergence is uniform in a neighborhood of a given initial data. In other words, if $\hat{u}(t)$ is a solution of (4) which blows-up at time $\hat{T}$ only at one point $\hat{a}$ with the behavior (17) (with $l = N$ and $\omega_a = 1$), then, there exists a neighborhood $\mathcal{V}_0$ of $\hat{u}(0)$ such that for all $u_0 \in \mathcal{V}_0$, the solution $u(t)$ of (4) with initial data u_0 blows-up in finite time $T(u_0)$ at only one blow-up point and for all $K > 0$,

$$\sup_{u_0\in\mathcal{V}_0,\ |y|\le K\sqrt{s}} \left| w_{a,T}(y, s) - \left(p - 1 + \frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{1}{p-1}} \right| \to 0 \text{ as } s \to +\infty$$

where $w_{a,T} = w_{a(u_0),T(u_0)}$ is defined from $u(t)$ by (6). Moreover, $a(u_0) \to \hat{a}$ and $T(u_0) \to \hat{T}$ as $u_0 \to \hat{u}(0)$.

Theorem 3 shows that the blow-up phenomenon is continuous with respect to initial data. In [Mer92], Merle shows that the blow-up time is continuous with respect to initial data in $L^\infty \cap H^1(\Omega)$. If S is the blow-up set of $u(t)$, we know from standard parabolic estimates that we can define the blow-up profile $u^* \in C(\Omega\backslash S)$ outside the singular set by

$$\forall x \in \Omega\backslash S,\ u^*(x) = \lim_{t\to T} u(x, t),$$

and that the convergence is uniform on every compact set of $\Omega\backslash S$. In the following Proposition, we show that the blow-up profile is continuous with respect to initial data.

Proposition 2 (Continuity of the blow-up profile with respect to initial data) *Assume that condition (2) holds. Let $u_{0n} \to u_0$ in $L^\infty \cap H^1(\Omega)$ and denote by $u_n(x,t)$ the solution of (5) with initial data u_{0n}. Denote by T_n and u_n^* the blow-up time and profile of $u_n(t)$.*
(A) Continuity at the regular points of $u(t)$.
i) $u_n^* \to u^*$ as $n \to +\infty$ uniformly on compact sets of $\Omega\backslash S$.
ii) If $t_n \to T$, then $u_n(x,t_n) \to u^*(x)$ uniformly on compact sets of $\Omega\backslash S$.
(B) Continuity at the blow-up points of $u(t)$.
$\forall A > 0$, $\exists \epsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\exists t_0 < T$ such that $\forall n \geq n_0$, $\forall x \in \Omega$ such that $d(x,S) \leq \epsilon$, $\forall t \in [t_0, T_n)$, $|u_n(x,t)| \geq A$.

Remark. (A) was proved in [Mer92]. In the contrary, only a local version (localized near a blow-up point of $u(t)$ of a particular type) was proved in [Mer92].

By the same techniques as in [MZ98b], we have the following equivalence result of several notions of blow-up profiles for equation (5):

Proposition 3 (Equivalence of different notions of blow-up profiles at a singular point) *Assume that condition (2) holds. Let $x_0 \in \mathbb{R}^N$ be an isolated blow-up point of $u(t)$ solution of (4) such that $u_0 \in H^1(\mathbb{R}^N)$ and $\omega \in S^{M-1}$. The following blow-up behaviors of $u(t)$ near x_0 or $w(s) = w_{x_0}(s)$ (defined in (6)) are equivalent:*

(A) $\forall R > 0$, $\sup_{|y|\leq R} \left| w(y,s) - \left[\kappa + \frac{\kappa}{2ps}(N - \frac{1}{2}|y|^2)\right]\omega \right| = o\left(\frac{1}{s}\right)$ as $s \to +\infty$ where $\kappa = (p-1)^{-\frac{1}{p-1}}$,

(B) $\exists \epsilon_0 > 0$ such that $\left\| w(y,s) - f_0(\frac{y}{\sqrt{s}})\omega \right\|_{L^\infty(|y|\leq \epsilon_0 e^{s/2})} \to 0$ as $s \to +\infty$ with $f_0(z) = (p - 1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}$,

(C) $\exists \epsilon_0 > 0$ such that if $|x - x_0| < \epsilon_0$, then $u(x,t) \to u^*(x)$ as $t \to T$ and $u^*(x) \sim \left[\frac{8p|\log|x-x_0||}{(p-1)^2|x-x_0|^2}\right]^{\frac{1}{p-1}} \omega$ as $x \to x_0$.

One further result concerns the size of the blow-up set:

Proposition 4 (Size of the blow-up set) *Assume that (2) holds and $M = 1$. Consider $u(t) \in H^1 \cap L^\infty(\Omega)$ a solution of (5) that blows-up at time T. Let S be its blow-up set. Then S is compact and the $(N-1)-$Hausdorff measure of S is finite.*

We now present in section 2 the proof of the Liouville Theorem 1 in the scalar case. Section 3 is devoted to the control of $\|u(t)\|_{L^\infty}$ (Theorem 2) and the ODE behavior (Theorem 3) uniformly with respect to initial data. In section 4, we use modulation theory to adapt to the vectorial case the proof of the Liouville Theorem 1.

2 Liouville Theorem for equation (7)

In this section, we prove Theorem 1 in the case $M = 1$. Similar ideas with the use of the modulation theory yield the result for general M (see section 4 for the case $M \geq 2$).
Note that for the Liouville Theorem, we assume that p satisfies the more general condition (15) and not only the condition (2).

The proof follows the same pattern as the analogous one presented in [MZ98a] in the case of nonnegative data. Indeed, all the arguments presented in [MZ98a] remain valid for solutions with no sign, except the following blow-up criterion for equation (7) which is specific for nonnegative data:

Let w be a nonnegative solution of (7) and assume that $\int_{\mathbb{R}^N} w(y, s_0)\rho(y)dy > \kappa \int_{\mathbb{R}^N} \rho(y)dy$ for some $s_0 \in \mathbb{R}$. Then, w blows-up at some time $S > s_0$.

Note that the criterion breaks even in the case $M = 1$ if there is no sign condition. Therefore, it is enough to replace this criterion by another suitable one, valid for solutions with no sign, so that the proof of [MZ98a] can be adapted in the current case (and in the vectorial case).

Let us first introduce the following functional defined for all $W \in H^1_\rho(\mathbb{R}^N)$

$$I(W) = -2E(W) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^N} |W(y)|^2 \rho(y)dy\right)^{\frac{p+1}{2}} \tag{18}$$

where E is defined in (8), and the following blow-up criterion valid for vectorial solutions of (7):

Proposition 2.1 (Blow-up criterion for vectorial solutions of (7))
Let w be a solution of (7) which satisfies

$$I(w(s_0)) > 0 \tag{19}$$

for some $s_0 \in \mathbb{R}$. Then, w blows-up at some time $S > s_0$.

Remark. This Proposition and the fact that $I(\kappa) = 0$ yield informations on the solutions of (7) close to κ in the energy space.
In the following, we will prove Proposition 2.1 and then give a sketch of the arguments of the proof of the Liouville Theorem, since they are the same as

those in [MZ98a]. Only the arguments related to the new blow-up criterion will be expanded.

Proof of Proposition 2.1. We proceed by contradiction and suppose that w is defined for all $s \in [s_0, +\infty)$. According to (7) and (8), we have $\forall s \geq s_0$,

$$\begin{aligned}\frac{d}{ds}\int |w(y,s)|^2\rho dy &= 2\int \left(-|\nabla w(y,s)|^2 - \frac{|w(y,s)|^2}{p-1} + |w(y,s)|^{p+1}\right)\rho dy\\ &= -4E(w(s)) + \frac{2(p-1)}{p+1}\int |w|^{p+1}\rho dy\\ &\geq -4E(w(s_0)) + \frac{2(p-1)}{p+1}\left(\int |w|^2\rho dy\right)^{\frac{p+1}{2}}\end{aligned}$$

where we used Jensen's inequality ($\int \rho dy = 1$) and the fact that E is decreasing in time.

If we set

$$z(s) = \int |w(y,s)|^2\rho dy,\ \alpha = -4E(w(s_0)) \text{ and } \beta = \frac{2(p-1)}{p+1}, \tag{20}$$

then this reads:

$$\forall s \geq s_0,\ z'(s) \geq \alpha + \beta z(s)^{\frac{p+1}{2}}. \tag{21}$$

With (20) and (18), the condition (19) reads: $\alpha + \beta z(s_0)^{\frac{p+1}{2}} > 0$. By a classical argument, we have from this and from (21)

$$\forall s \geq s_0,\ z'(s) > 0 \text{ and } \alpha + \beta z(s)^{\frac{p+1}{2}} > 0.$$

Using a direct integration, we obtain:

$$\forall s \geq s_0,\ s - s_0 \leq \int_{z(s_0)}^{z(s)} \frac{dx}{\alpha + \beta x^{\frac{p+1}{2}}} \leq \int_{z(s_0)}^{+\infty} \frac{dx}{\alpha + \beta x^{\frac{p+1}{2}}} = C(z(s_0)) < +\infty$$

since $p > 1$. Thus, a contradiction follows and Proposition 2.1 is proved. $\square$

Proof of Theorem 1 in the scalar case. We assume $p > 1$ and $p < \frac{N+2}{N-2}$ if $N \geq 3$, and consider $w \in L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ a solution of (7). We proceed in two parts in order to show that w depends only on s:
- In Part I, we show from the dissipative character of the equation that w has a limit $w_{\pm\infty}$ as $s \to \pm\infty$ with $w_{\pm\infty}$ a critical point of (7), that is $w_{\pm\infty} \equiv 0$, κ or $-\kappa$. We then focus on the nontrivial case $(w_{-\infty}, w_{+\infty}) = (\kappa, 0)$ and show from a linear study of the equation around κ that w goes to κ as $s \to -\infty$ in three possible ways.
- In Part II, we show that one of these three ways corresponds to $w(y,s) =$

$\varphi(s-s_0)$ for some $s_0 \in \mathbb{R}$ where $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$. In the two other cases, we find a contradiction from nonlinear informations:

- the blow-up criterion of Proposition 2.1 (for w close to κ),
- the following geometrical transformation:

$$a \in \mathbb{R}^N \to w_a \text{ defined by } w_a(y,s) = w(y + ae^{\frac{s}{2}}, s) \tag{22}$$

which keeps (7) invariant (thanks to the translation invariance of equation (5)).

Part I: Possible behaviors of w as $s \to \pm\infty$

We proceed in two steps: First, we find limits $w_{\pm\infty}$ for w as $s \to \pm\infty$. In a second step, we focus on the linear behavior of w as $s \to -\infty$, in the case $w_{-\infty} = \kappa$.

Step 1: Limits of w as $s \to \pm\infty$

Proposition 2.2 (Limits of w as $s \to \pm\infty$) $w_{+\infty}(y) = \lim_{s\to+\infty} w(y,s)$ *exists and is a critical point of (7). The convergence holds in L^2_ρ, the L^2 space associated to the Gaussian measure $\rho(y)dy$ where ρ is defined in (9), and uniformly on each compact subset of $\mathbb{R}^N$. The same statement holds for $w_{-\infty}(y) = \lim_{s\to-\infty} w(y,s)$.*

Proof. See Step 1 in section 3 in [MZ98a]. □

Proposition 2.3 (Stationary problem for (7)) *The only nonnegative bounded global solutions in $\mathbb{R}^N$ of*

$$0 = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1}w \tag{23}$$

are the constant ones: $w \equiv 0$, $w \equiv -\kappa$ and $w \equiv \kappa$.

Proof. One can derive the following Pohozaev identity for each bounded solution of equation (7) in $\mathbb{R}^N$ (see Proposition 2 in [GK85]):

$$(N+2-p(N-2)) \int |\nabla w|^2 \rho dy + \frac{p-1}{2} \int |y|^2 |\nabla w|^2 \rho dy = 0. \tag{24}$$

Hence, for $(N-2)p \le N+2$, w is constant. Thus, $w \equiv 0$ or $w \equiv \kappa$ or $w \equiv -\kappa$. □

From Propositions 2.2 and 2.3, we have $w_{\pm\infty} \equiv 0$ or $w_{\pm\infty} \equiv \kappa$ or $w_{\pm\infty} \equiv -\kappa$. Since E is a Lyapunov functional for w, one gets from (8) and (7):

$$\int_{-\infty}^{+\infty} ds \int_{\mathbb{R}^N} \left| \frac{\partial w}{\partial s}(y,s) \right|^2 \rho dy = E(w_{-\infty}) - E(w_{+\infty}). \tag{25}$$

Therefore, since $E(\kappa) = E(-\kappa) > 0 = E(0)$, there are only two cases:
1 - $E(w_{-\infty}) - E(w_{+\infty}) = 0$. This implies by (25) that $\frac{\partial w}{\partial s} \equiv 0$, hence w is a

stationary solution of (7) and $w \equiv 0$ or $w \equiv \kappa$ or $w \equiv -\kappa$ by Proposition 2.3.
2 - $E(w_{-\infty}) - E(w_{+\infty}) > 0$. This occurs only if $w_{+\infty} \equiv 0$ and $w_{-\infty} \equiv \kappa$ or $-\kappa$. It remains to treat this case. Since (7) is invariant under the transformation $w \to -w$, it is enough to focus on the case:

$$(w_{-\infty}, w_{+\infty}) \equiv (\kappa, 0). \tag{26}$$

Remark. The case 1 contains the case studied in [GK85]. Indeed, the authors had there $E(w_{-\infty}) = E(w_{+\infty})$ and $w_{+\infty} > 0$ (assuming (16)). Therefore w is a stationary solution of (7).

Step 2: Linear behavior of w near κ as $s \to -\infty$

Let us introduce $v = w - \kappa$. From (7), v satisfies the following equation: $\forall (y, s) \in \mathbb{R}^{N+1}$,

$$\frac{\partial v}{\partial s} = \mathcal{L}v + f(v) \tag{27}$$

$$\text{where } \mathcal{L}v = \Delta v - \frac{1}{2} y.\nabla v + v \text{ and}$$
$$f(v) = |v+\kappa|^{p-1}(v+\kappa) - \kappa^p - p\kappa^{p-1} v. \tag{28}$$

Since w is bounded in L^∞, we assume $|v(y,s)| \le C$ and $|f(v)| \le C|v|^2$.
$\mathcal{L}$ is self-adjoint on $\mathcal{D}(\mathcal{L}) \subset L^2_\rho$. Its spectrum is

$$\operatorname{spec}(\mathcal{L}) = \{1 - \frac{m}{2} \mid m \in \mathbb{N}\}, \tag{29}$$

and it consists of eigenvalues. The eigenfunctions of $\mathcal{L}$ are derived from Hermite polynomials:

– $N = 1$:
All the eigenvalues of $\mathcal{L}$ are simple. For $1 - \frac{m}{2}$ corresponds the eigenfunction

$$h_m(y) = \sum_{n=0}^{[\frac{m}{2}]} \frac{m!}{n!(m-2n)!} (-1)^n y^{m-2n}. \tag{30}$$

– $N \ge 2$:
We write the spectrum of $\mathcal{L}$ as

$$\operatorname{spec}(\mathcal{L}) = \{1 - \frac{m_1 + \ldots + m_N}{2} | m_1, \ldots, m_N \in \mathbb{N}\}.$$

For $(m_1, \ldots, m_N) \in \mathbb{N}^N$, the eigenfunction corresponding to $1 - \frac{m_1+\ldots+m_N}{2}$ is

$$h_{(m_1,\ldots,m_N)} : y \longrightarrow h_{m_1}(y_1) \ldots h_{m_N}(y_N), \tag{31}$$

where h_m is defined in (30). In particular,
*1 is an eigenvalue of multiplicity 1, and the corresponding eigenfunction is

$$H_0(y) = 1, \tag{32}$$

*$\frac{1}{2}$ is of multiplicity N, and its eigenspace is generated by the orthogonal basis $\{H_{1,i}(y) | i = 1, ..., N\}$, with $H_{1,i}(y) = h_1(y_i)$; we note

$$H_1(y) = (H_{1,1}(y), ..., H_{1,N}(y)), \tag{33}$$

*0 is of multiplicity $\frac{N(N+1)}{2}$, and its eigenspace is generated by the orthogonal basis $\{H_{2,ij}(y) | i, j = 1, ..., N, i \le j\}$, with $H_{2,ii}(y) = h_2(y_i)$, and for $i < j$, $H_{2,ij}(y) = h_1(y_i)h_1(y_j)$; we note

$$H_2(y) = (H_{2,ij}(y), i \le j). \tag{34}$$

Since the eigenfunctions of $\mathcal{L}$ constitute a total orthonormal family of L^2_ρ, we expand v as follows:

$$v(y,s) = \sum_{m=0}^{2} v_m(s).H_m(y) + v_-(y,s) \tag{35}$$

where
$v_0(s)$ is the projection of v on H_0,
$v_{1,i}(s)$ is the projection of v on $H_{1,i}$, $v_1(s) = (v_{1,i}(s), ..., v_{1,N}(s))$, $H_1(y)$ is given by (33),
$v_{2,ij}(s)$ is the projection of v on $H_{2,ij}$, $i \le j$, $v_2(s) = (v_{2,ij}(s), i \le j)$, $H_2(y)$ is given by (34),
$v_-(y,s) = P_-(v)$ and P_- is the projector on the negative subspace of $\mathcal{L}$.

With respect to the positive, null and negative subspaces of $\mathcal{L}$, we write

$$v(y,s) = v_+(y,s) + v_{null}(y,s) + v_-(y,s) \tag{36}$$

where $v_+(y,s) = P_+(v) = \sum_{m=0}^{1} v_m(s).H_m(y)$,
$v_{null}(y,s) = P_{null}(v) = v_2(s).H_2(y)$, P_+ and P_{null} are the L^2_ρ projectors respectively on the positive subspace and the null subspace of $\mathcal{L}$.

Now, we show that as $s \to -\infty$, either $v_0(s)$, $v_1(s)$ or $v_2(s)$ is predominant with respect to the expansion (35) of v in L^2_ρ. At this level, we are not able to use a center manifold theory to get the result (see [FK92] page 834-835 for more details). In some sense, we are not able to say that the nonlinear terms in the function of space are small enough. However, using similar techniques as in [FK92], we are able to prove the result. We have the following:

Proposition 2.4 (Linear classification of the behaviors of w as $s \to -\infty$) *As $s \to -\infty$, one of the following cases occurs:*
i) $|v_1(s)| + \|v_{null}(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(v_0(s))$,

$$\forall s \le s_0,\ v_0'(s) = v_0(s) + O\left(v_0(s)^2\right) \tag{37}$$

and there exists $C_0 \in \mathbb{R}$ such that

$$\|v(y,s) - C_0 e^s\|_{H^1_\rho} = o(e^s), \tag{38}$$

and $\forall \epsilon > 0$,

$$v_0(s) = C_0 e^s + O(e^{(2-\epsilon)s}) \text{ and } v_1(s) = O(e^{(2-\epsilon)s}). \tag{39}$$

ii) $|v_0(s)| + \|v_{null}(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(v_1(s))$ *and* $\exists C_1 \in \mathbb{R}^N \backslash \{0\}$ *such that* $\|v(y,s) - e^{\frac{s}{2}} C_1.y\|_{H^1_\rho} = o(e^{\frac{s}{2}})$, $v_1(s) \sim C_1 e^{s/2}$ *and* $v_0(s) \sim \frac{p}{\kappa}|C_1|^2 e^s$,
iii) $\|v_+(y,s)\|_{L^2_\rho} + \|v_-(y,s)\|_{L^2_\rho} = o(\|v_{null}(y,s)\|_{L^2_\rho})$ *and there exists* $l \in \{1,...,N\}$ *and Q an orthonormal $N \times N$ matrix such that*
$\left\|v(Qy,s) - \frac{\kappa}{4ps}\left(2l - \sum_{i=1}^{l} y_i^2\right)\right\|_{H^1_\rho} = o(\frac{1}{s})$,
$v_{null}(Qy,s) = \frac{\kappa}{4ps}\left(2l - \sum_{i=1}^{l} y_i^2\right) + O\left(\frac{1}{s^{1+\delta}}\right)$, $v_1(s) = O\left(\frac{1}{s^2}\right)$ *and* $v_0(s) = O\left(\frac{1}{s^2}\right)$ *for some* $\delta > 0$.

Proof. See Propositions 3.5, 3.6, 3.9 and 3.10 in [MZ98a]. Although only L^2_ρ norms appear in those Propositions, one can see that the proof of Proposition 3.5 in [MZ98a] can be adapted without difficulties to yield H^1_ρ estimates (see section 6 in [FK92] for a similar adaptation). $\square$

Part II: Conclusion of the proof

The crucial point is to note that $I(\kappa) = 0$ where I is defined in (18). Thus, the use of the geometrical transformation $w \to w_a$ (see (22)) and the blow-up argument of Proposition 2.1 applied to $w_a(s)$ will introduce some rigidity on the behavior of $w(s)$ as $s \to -\infty$.
We proceed in two steps:
- In Step 1, we show that if the case $i)$ of Proposition 2.4 occurs, then $w(y,s) = \varphi(s - s_0)$ for some $s_0 \in \mathbb{R}$.
- In Step 2, we show by means of Proposition 2.1 and the transformation (22) that cases $ii)$ and $iii)$ of Proposition 2.4 yield a contradiction.

Step 1: Case $i)$ of Proposition 2.4: the relevant case

Proposition 2.5 *Assume that case $i)$ of Proposition 2.4 occurs, then:*
i) $C_0 < 0$,
ii) $\forall y \in \mathbb{R}^N$, $\forall s \in \mathbb{R}$, $w(y,s) = \varphi(s - s_0)$ *where* $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$ *and* $s_0 = -\log\left(-\frac{(p-1)C_0}{\kappa}\right)$.

Proof.
i) We proceed by contradiction in order to eliminate successively the cases $C_0 = 0$ and $C_0 > 0$.
- Suppose $C_0 = 0$, then one can see from (37) and (39) that $\forall s \le s_1$, $v_0(s) = 0$ for some $s_1 \in \mathbb{R}$. Since $\|v(s)\|_{L^2_\rho} \sim v_0(s)$ as $s \to -\infty$, we have $\forall s \le s_2$, $\forall y \in \mathbb{R}^N$, $v(y,s) = 0$ and $w(y,s) = \kappa$ for some $s_2 \in \mathbb{R}$. From the uniqueness of the solution of the Cauchy problem for equation (7), we have $w \equiv \kappa$ in all $\mathbb{R}^N \times \mathbb{R}$, which contradicts the fact that $w \to 0$ as $s \to +\infty$ (see (26)). Hence, $C_0 \neq 0$.
- Suppose now that $C_0 > 0$. We will prove that

$$I(w(s)) = -2E(w(s)) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^N} |w(y,s)|^2 \rho(y)dy\right)^{\frac{p+1}{2}} > 0 \qquad (40)$$

for some $s \in \mathbb{R}$, which is the blow-up condition of Proposition 2.1, in contradiction with the global boundedness of w.

Since $w = \kappa + v$ and κ is a critical point of $E : H^1_\rho(\mathbb{R}^N) \to \mathbb{R}$ (see Proposition 2.3), we have

$$E(w(s)) = E(\kappa) + O\left(\|v(s)\|^2_{H^1_\rho}\right) = \frac{\kappa^2}{2(p+1)} + O\left(\|v(s)\|^2_{H^1_\rho}\right). \qquad (41)$$

For the second term in (40), we use $w = \kappa + v$ and write
$\int |w(y,s)|^2 \rho dy = \kappa^2 + 2\kappa \int v(y,s)\rho dy + \int |v(y,s)|^2 \rho dy$
$= \kappa^2 + 2\kappa v_0(s) + \int |v(y,s)|^2 \rho dy$. Therefore,
$\frac{p-1}{p+1}\left(\int |w(y,s)|^2 \rho dy\right)^{\frac{p+1}{2}} = \frac{\kappa^2}{p+1} + \kappa v_0(s) + O(\|v(s)\|^2_{L^2_\rho})$. Combining this with (41) and using (39) and (38), we end up with

$$I(w(s)) \sim \kappa v_0(s) \sim \kappa C_0 e^s > 0 \text{ as } s \to -\infty$$

which is the blow-up condition of Proposition 2.1. Contradiction. Thus, $C_0 < 0$.

ii) Let us introduce $V(y,s) = w(y,s) - \varphi(s - s_0)$ where $\varphi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$ and $s_0 = -\log\left(-\frac{(p-1)C_0}{\kappa}\right)$. Since φ is a solution of

$$\varphi'(s) = -\frac{\varphi(s)}{p-1} + \varphi(s)^p,$$

we see from (7) that V satisfies the following equation:

$$\frac{\partial V}{\partial s} = (\mathcal{L} + l(s))V + F(V) \tag{42}$$

where $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$, $l(s) = -\frac{pe^{s-s_0}}{(p-1)(1+e^{s-s_0})}$ and $F(V) = |\varphi+V|^{p-1}(\varphi+V)-\varphi^p - p\varphi^{p-1}V$. Note that $\forall s \le 0$, $|F(V)| \le C|V|^2$.

Besides, we have from $i)$ of Proposition 2.4 and the choice of s_0 that

$$|V_0(s)| + |V_1(s)| = O(e^{(2-\epsilon)s}) \text{ and } \|V_{null}(s)\|_{L^2_\rho} + \|V_-(s)\|_{L^2_\rho} = o(e^s) \tag{43}$$

as $s \to -\infty$. Using the linear classification at infinity of solutions of equation (42) under the conditions (43) (see Proposition 3.7 in [MZ98a]), we get $V \equiv 0$ on $\mathbb{R}^N \times \mathbb{R}$. Thus, $\forall y \in \mathbb{R}^N$, $\forall s \in \mathbb{R}$,

$$w(y,s) = \varphi(s - s_0).$$

□

Step 2: Cases $ii)$ and $iii)$ of Proposition 2.4: blow-up cases In both cases $ii)$ and $iii)$ of Proposition 2.4, we will find $s_0 \in \mathbb{R}$ and $|a_0| \le e^{-\frac{s_0}{2}}$ such that $I(w_{a_0}(s_0)) > 0$ where I is defined in (18), which implies by Proposition 2.1 that w_{a_0} blows-up in finite time $S > s_0$, in contradiction with $\|w_{a_0}\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} = \|w\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} < +\infty$. We give in the following lemma an expansion of $I(w_a(s))$ as $s \to -\infty$ and $ae^{s/2} \to 0$, which will allow us to conclude:

Lemma 2.6
a - Assume that case $ii)$ or $iii)$ of Proposition 2.4 holds, then

$$I(w_a(s)) = \kappa \int v(y,s)\rho(y - ae^{s/2})dy + O\left(\|v(s)\|^2_{H^1_\rho}\right)$$

as $s \to -\infty$ and $ae^{s/2} \to 0$. Moreover,
b - In case $ii)$: $\int v(y,s)\rho(y-ae^{s/2})dy = a.C_1e^s + o(|a|e^s) + O(se^s)$,
c - In case $iii)$: $\int v(y,s)\rho(y-ae^{s/2})dy =$
$$\frac{\kappa}{4p|s|}\sum_{i=1}^{l}\int (z_i^2-2)(Qae^{s/2}.z)^2\rho(z)dz + O\left(\frac{1}{s^2}\right) + O\left(\frac{|a|^2e^s}{|s|^{1+\delta}}\right) + O\left(\frac{|a|^3e^{\frac{3s}{2}}}{|s|}\right).$$

Proof. see Appendix A. □

This lemma allows us to conclude. Indeed,
– if case $ii)$ of Proposition 2.4 holds, then
$I(w_a(s)) = \kappa a.C_1e^s + o(|a|e^s) + O(se^s)$. We fix s_0 negative enough and $a_0 = \frac{1}{|s_0|}\frac{C_1}{|C_1|}e^{-s_0/2}$ to get

$$I(w_{a_0}(s_0)) \ge \frac{1}{2}\kappa a_0.C_1e^{s_0} = \kappa\frac{e^{s_0/2}}{2|s_0|}|C_1| > 0.$$

This implies by Proposition 2.1 that w_{a_0} blows-up at time $S > s_0$. Contradiction.
– If case $iii)$ of Proposition 2.4 holds, then
$I(w_a(s)) = \frac{\kappa^2}{4p|s|}\sum_{i=1}^{l}\int (z_i^2-2)(Qae^{s/2}.z)^2\rho(z)dz + O\left(\frac{1}{s^2}\right) + O\left(\frac{|a|^2e^s}{|s|^{1+\delta}}\right) + O\left(\frac{|a|^3e^{\frac{3s}{2}}}{|s|}\right)$. We fix s_0 negative enough and $a_0 = \frac{e^{-s_0/2}}{|s_0|^{1/4}}Q^{-1}e_1$ where $e_1 = (1, 0, ..., 0)$ so that we get

$$I(w_{a_0}(s_0)) \geq \frac{1}{2}\frac{\kappa^2}{4p|s_0|}\sum_{i=1}^{l}\int (z_i^2-2)(\frac{e_1}{|s_0|^{1/4}}.z)^2\rho(z)dz = \frac{\kappa^2}{p|s_0|^{3/2}} > 0$$

by (9). This implies by Proposition 2.1 that w_{a_0} blows-up at time $S > s_0$. Contradiction.

This concludes the proof of Theorem 1 in the scalar case. □

3 Uniform estimates for nonlinear heat equations

In this section, we prove uniform bounds on solutions of (5) (Theorem 2) and deduce several applications of Theorems 1 and 2 for nonlinear heat equations. In particular, we prove uniform bounds and the ODE like behavior of the solution (Theorems 3 and 4 and Corollary 2). We treat only the case $\Omega = \mathbb{R}^N$. The case where Ω is a convex bounded $C^{2,\alpha}$ domain can be treated in the same way, by using regularity results near the boundary (see [GK87], lemma 3.4).
In the end of the section, we give a sketch of the proof of various consequences of Theorems 3 and 4 presented in the introduction.

Proof of Theorem 2: Uniform L^∞ bounds on the solution.

Consider $u_0 \in C^2$ such that $\|u_0\|_{C^2} \leq C_0$ and $u(t)$ solution of (5) with initial data u_0 blows-up at T with $T < T_0$. We claim that there is $C = C(C_0, T_0)$ such that $\|u(t)\|_{L^\infty}$ is controlled by $Cv(t)$ where v is the solution of the ODE $v' = v^p$ which blows-up at the same time T as $u(t)$. The result mainly follows from blow-up argument giving local energy estimates and the fact that these estimates yield L^∞ estimates (from Giga-Kohn [GK87]).

Step 1: Estimates on $u(t)$ for small time

Lemma 3.1 (C^2 bounds for small time) *There is $t_0 = t_0(C_0) > 0$ such that:*
i) for all $t \in [0, t_0]$, $\|u(t)\|_{L^\infty} \leq 2C_0$,
ii) for all $t \in [0, t_0]$, $\|u(t)\|_{C^2} \leq 2C_0$,
iii) for all $\alpha \in (0, 1)$, $\|\Delta u\|_{C^\alpha(D)} \leq C_1(\alpha, C_0)$where

$$\|a\|_\alpha = \sup_{(x,t)\neq(x',t')\in D}\frac{\left|a(x,t)-a(x',t')\right|}{\left(|x-x'|+|t-t'|^{1/2}\right)^\alpha}$$

where $D = \mathbb{R}^N \times [\frac{t_0}{2}, t_0]$.

Proof: We start with $i)$ and $ii)$. Since u satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t-s)|u(s)|^{p-1}u(s)ds,$$

we have

$$\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \int_0^t \|u(s)\|_{L^\infty}^p ds.$$

Thus, by a priori estimates, we have $\forall t \in [0, t_0]$, $\|u(t)\|_{L^\infty} \leq 2C_0$ where $t_0 = 2^{-p}C_0^{1-p}$.
Similarly, we obtain $\forall t \in [0, t_0]$, $\|u(t)\|_{C^2} \leq 2C_0$ where $t_0 = t_0(C_0)$.

$iii)$ We use the following lemma:

Lemma 3.2 *Assume that* h *solves*

$$\frac{\partial h}{\partial \tau} = \Delta h + a(\xi, \tau)h$$

for $(\xi, \tau) \in D$ *where* $D = B(0,3) \times [0, t_0]$ *and* $t_0 \leq T_0$. *Assume in addition that* $\|a\|_{L^\infty} + |a|_{\alpha,D}$ *is finite, where*

$$|a|_{\alpha,D} = \sup_{(\xi,\tau),(\xi',\tau')\in D} \frac{\left|a(\xi,\tau) - a(\xi',\tau')\right|}{\left(|\xi - \xi'| + |\tau - \tau'|^{1/2}\right)^\alpha} \tag{44}$$

and $\alpha \in (0,1)$. *Then,*

$$\|h\|_{C^2(D')} + |\nabla^2 h|_{\alpha,D'} \leq K\|h\|_{L^\infty(D)}$$

where $K = K\left(\|a\|_{L^\infty(D)} + |a|_{\alpha,D}\right)$ *and* $D' = B(0,1) \times [\frac{t_0}{2}, t_0]$.

Proof: see Lemma 2.10 in [MZ98b]. □

Step 2: Energy bounds in similarity variables

From the blow-up argument for equation (7) (Proposition 2.1) and the monotonicity of the energy E, we have:

Lemma 3.3 *There is* $C_1 = C_1(C_0, T_0)$ *such that* $\forall s \geq s_0 = -\log T$, $\forall a \in \mathbb{R}^N$,
i) $|E(w_a(s))| \leq C_1$ *and* $\int |w_a(y,s)|^2 \rho(y)dy \leq C_1$,
ii) $\int_s^{s+1} \int \left(|w_a(y,s)|^{p+1} + |\nabla w_a(y,s)|^2 + \left|\frac{\partial w_a}{\partial s}(y,s)\right|^2\right) \rho(y)dyds \leq C_1$,
iii) $\int_s^{s+1} \left(\int |w_a(y,s)|^{p+1}\rho(y)dy\right)^2 ds \leq C_1$ *where* w_a *and* E *are defined respectively in (6) and (8).*

Proof. Following [GK87], we note $w = w_a$.

i) First we have that $\forall s \in [s_0, +\infty)$, $\frac{d}{ds}E(w_a(s)) \le 0$, $E(w_a(s)) \le E(w_a(s_0)) \le C(C_0, T_0)$. Let us note from the blow-up result of Proposition 2.1 that $\forall s \in [s_0, +\infty)$,

$$I(w(s)) = -2E(w(s)) + \frac{p-1}{p+1}\left(\int |w(y,s)|^2\rho(y)dy\right)^{\frac{p+1}{2}} \le 0.$$

Thus, $\left(\int |w(y,s)|^2\rho(y)dy\right)^{\frac{p+1}{2}} \le \frac{2(p+1)}{p-1}E(w(s)) \le C(C_0, T_0)$ and we have i).

ii) We have

$$\frac{d}{ds}\int |w(y,s)|^2\rho(y)dy = -2E(w(s)) + \frac{p-1}{p+1}\int |w(y,s)|^{p+1}\rho(y)dy.$$

Therefore, by integration and i), $\int_s^{s+1}\int |w(y,s)|^{p+1}\rho(y)dyds \le C_1$.
From the bound on $\int |w(y,s)|^2\rho(y)dy$, $E(w(s))$ and
$\int_s^{s+1}\int |w(y,s)|^{p+1}\rho(y)dyds$, we obtain the bound on
$\int_s^{s+1}\int |\nabla w(y,s)|^2\rho(y)dyds$, and from the variation of the energy,
$\left|\int_s^{s+1}\int \left|\frac{\partial w}{\partial s}(y,s)\right|^2\rho(y)dyds\right| \le |E(w(s))| + |E(w(s+1))| \le 2C_1$.

iii) We write
$-\int |\nabla w(y,s)|^2\rho(y)dy + \int |w(y,s)|^{p+1}\rho(y)dy$
$= \int \frac{\partial w}{\partial s}(y,s)w(y,s)\rho(y)dy + \frac{1}{p-1}\int |w(y,s)|^2\rho(y)dy.$
Since $\left|\int |\nabla w(y,s)|^2\rho(y)dy - \frac{2}{p+1}\int |w(y,s)|^{p+1}\rho(y)dy\right| \le C_1$, we have
$\int |w(y,s)|^{p+1}\rho(y)dy \le C_1\left(\int \left|\frac{\partial w}{\partial s}(y,s)\right|^2\rho dy\right)^{\frac{1}{2}}\left(\int |w(y,s)|^2\rho(y)dy\right)^{\frac{1}{2}} + C_1$,
then,
$\left(\int |w(y,s)|^{p+1}\rho(y)dy\right)^2 \le C_1\left(1 + \int \left|\frac{\partial w}{\partial s}(y,s)\right|^2\rho(y)dy\right)$.
Thus, by integration we have the conclusion. □

Step 3: L^∞ bound in similarity variables

We have the following proposition, where L^∞ bound can be derived from energy bounds:

Proposition 3.4 (Giga-Kohn, L^∞ bound on w) *Assume that we have the bounds of lemma 3.3 on w in the interval $[s, s+1]$ for a given C_1, then for all $\delta \in (0,1)$, there exists $C_2(C_1, \delta)$ such that $|w_a(0, s+\delta)| \le C_2$.*

Proof. See lemma 3.2 in [GK87]. □

Step 4: Conclusion of the proof: L^∞ bounds with respect to C_0 and T_0

We can see that these arguments yield uniform bounds on the solution.
- On one hand, we have from Step 1,

$$\forall t \in [0, t_0(C_0)],\ \|u(t)\|_{L^\infty} \le 2C_0. \tag{45}$$

- On the other hand, we have from Proposition 3.4 and Step 2, for all $\delta_0 \in (0,1)$, $\forall s \in [s_0 + \delta_0, +\infty)$, $\|w(s)\|_{L^\infty} \le C_2(C_1, \delta_0)$, therefore

$$\forall t \in [T(1-e^{-\delta_0}), T),\ \|u(t)\|_{L^\infty} \le \frac{C_2}{(T-t)^{\frac{1}{p-1}}}. \tag{46}$$

Taking $\delta_0 = \delta_0(T_0, t_0)$ such that $T_0(1-e^{-\delta_0}) \le \frac{t_0}{2}$, and using (45) and (46) we obtain $\forall t \in [0,T)$, $\|u(t)\|_{L^\infty} \le \frac{C_3}{(T-t)^{\frac{1}{p-1}}}$ where

$C_3(C_0, T_0) = \max(C_2(C_1, \delta_0), 2C_0 T_0^{\frac{1}{p-1}})$.

This concludes the proof of Theorem 2. □

Let us prove now the uniform pointwise control of the diffusion term by the nonlinear term, which asserts that the solution $u(t)$ behaves everywhere like the ODE $v' = v^p$.

Proof of Theorem 3 (Uniform ODE behavior). The main ideas are the same as in [MZ98a] where the proof was presented for a given positive solution. But we will present the proof in a different way which allows us to obtain a constant uniform with respect to initial data.

We argue by contradiction. Let us consider u_n solution of (5) with initial data u_{0n} such that $\|u_{0n}(t)\|_{C^2} \le C_0$, $u_n(t)$ blows-up at time $T_n < T_0$ and for some $\epsilon_0 > 0$, the statement

$$|\Delta u| \le \epsilon_0 |u|^p + n \text{ on } \mathbb{R}^N \times [0, T_n) \tag{47}$$

is not valid. Therefore, there is $(x_n, t_n) \in \mathbb{R}^N \times [0, T_n)$ such that

$$|\Delta u_n(x_n, t_n)| \ge \epsilon_0 |u_n(x_n, t_n)|^p + n. \tag{48}$$

Considering $\tilde{u}_n(x,t) = u_n(x_n + x, t)$, we can assume

$$x_n = 0.$$

From the uniform estimates and the parabolic regularity, we have

$$T_n - t_n \to 0 \text{ as } n \to +\infty.$$

Indeed, from Theorem 2, $\exists C_2(C_0, T_0) > 0$ such that $\forall t \in [0, T_n)$, $\|u_n(t)\|_{L^\infty} \le \frac{C_2}{(T_n - t)^{\frac{1}{p-1}}}$.

Introducing $w_n(y,s)$ for all $y \in \mathbb{R}^N$ and $s \ge s_{0n} = -\log T_n$ by

$$y = \frac{x-a}{\sqrt{T_n - t}},\ s = -\log(T_n - t),\ w_n(y,s) = (T_n - t)^{\frac{1}{p-1}} u_n(x,t),$$

we have $\forall s \in [s_{0n}, +\infty)$, $\|w_n(s)\|_{L^\infty} \le C_2$, where $s_{0n} = -\log T_n$. From parabolic regularity applied to equations (5) and (7), there is C' such that $\forall s \in [s_0, +\infty)$, $\|\Delta w_n(s)\|_{L^\infty} \le C'$.
Thus, $\forall t \in [0, T_n)$, $\|\Delta u_n(t)\|_{L^\infty} \le \frac{C'}{(T_n-t)^{\frac{p}{p-1}}}$.
From (48), we have

$$\frac{C'}{(T_n - t_n)^{\frac{p}{p-1}}} \ge \|\Delta u_n(t_n)\|_{L^\infty} \ge |\Delta u_n(x_n, t_n)| \ge n$$

and $T_n - t_n \to 0$ as $n \to +\infty$.

Let us now consider two cases.
In the region where the solution $u_n(t)$ is of the same order as the solution of the ODE blowing-up at T_n (called the very singular region), the Liouville Theorem 1 in similarity variables yields a contradiction.
For the other regions, we can control the nonlinear term by using in some sense wellposedness for small data in some localized energy space (subcritical behavior). This allows us to transport the information from the very singular region everywhere.

i) Estimates in the very singular region. $|u_n(0, t_n)|(T_n - t_n)^{\frac{1}{p-1}} \to \delta_0 \ne 0$ *as* $n \to +\infty$.

A compactness procedure and the Liouville Theorem yield a contradiction.
We now consider $\tilde{w}_n(y, s) = w_n(s_n + s, y)$ where $s_n = -\log(T_n - t_n) \to +\infty$ as $n \to +\infty$.
$\tilde{w}_n$ is a solution of (7) for $(y, s) \in \mathbb{R}^N \times [s_{0n} - s_n, +\infty)$ such that $\forall s \ge s_{0n} - s_n + 1$, $\|\tilde{w}_n(s)\|_{L^\infty(\mathbb{R}^N)} \le C$, $\forall R > 0$, $\|\tilde{w}_n\|_{C^{2,1}_\alpha(B(0,R)\times[-R,R])} \le C'(R)$, and $|\Delta \tilde{w}_n(0, 0)| \ge \epsilon_0 |\tilde{w}_n(0, 0)|^p \ge \epsilon_0 \frac{\delta_0^p}{2} \ge \delta_0' > 0$, where for all $D \subset \mathbb{R}^N \times \mathbb{R}$,

$$\begin{aligned}\|w\|_{C^{2,1}_\alpha(D)} &= \|w\|_{L^\infty(D)} + \|\nabla w\|_{L^\infty(D)} + \|\nabla^2 w\|_{L^\infty(D)} + \|\nabla^2 w\|_{\alpha, D} \\ &\quad + \|\frac{\partial w}{\partial t}\|_{L^\infty(D)} + \|\frac{\partial w}{\partial s}\|_{\frac{\alpha}{2}, D}\end{aligned}$$

and $\|u\|_{\alpha,D}$ is defined in (44). Note that $s_n \to +\infty$ and $s_{0n} = -\log T_n \le -\log t_0(C_0)$ by lemma 3.1. Therefore, $s_{0n} - s_n \to -\infty$. By compactness procedure, $\tilde{w}_n \to w$ as $n \to +\infty$ on compact sets of $\mathbb{R}^N \times \mathbb{R}$ where w is solution of (7) for $(y, s) \in \mathbb{R}^N \times \mathbb{R}$ such that

$$\forall s \in \mathbb{R},\ \|w(s)\|_{L^\infty} \le C \text{ and } |\Delta w(0, 0)| \ge \delta_0' > 0.$$

From Theorem 1, we have a contradiction, since all the globally bounded solutions w of (7) defined on $\mathbb{R}^N \times \mathbb{R}$ satisfy $w(y, s) = w(s)$ and $\Delta w(y, s) = 0$.

ii) Estimates in the singular region: $u_n(0, t_n)(T_n - t_n)^{\frac{1}{p-1}} \to 0$.

We now consider the case where

$$u(0,t_n)(T_n-t_n)^{\frac{1}{p-1}} \to 0 \text{ as } n \to +\infty. \tag{49}$$

Again, by the Liouville Theorem and the local energy estimates (which allow us to control the nonlinear term), we transport the information obtained in the very singular region to obtain a contradiction in this case.

Step 1: Compactness procedure outside the singular region

We have from Theorem 2 and its proof

$$\forall t \in [0,T_n),\ \forall n,\ \|u_n(t)\|_{L^\infty} \le \frac{C}{(T_n-t)^{\frac{1}{p-1}}} \text{ and } \|u_n(t)\|_{C^2} \le \frac{C}{(T_n-t)^{\frac{p}{p-1}}}.$$

By a compactness procedure, we can assume that $T_n \to T^*$ where $t_0(C_0) < T^* \le T_0$ and $u_n(x,t) \to u(x,t)$ in $C^{2,1}_{loc}(\mathbb{R}^N \times [0,T^*))$ where $\forall t \in [0,T^*)$, $\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u$,

$$\|u(t)\|_{L^\infty} \le \frac{C_1}{(T^*-t)^{\frac{1}{p-1}}} \text{ and } \|u(t)\|_{C^2} \le \frac{C_1}{(T^*-t)^{\frac{p}{p-1}}},$$

and for all $D \subset \mathbb{R}^N \times \mathbb{R}$,

$$\|u\|_{C^{2,1}(D)} = \|u\|_{L^\infty(D)} + \|\nabla u\|_{L^\infty(D)} + \|\nabla^2 u\|_{L^\infty(D)} + \|\frac{\partial u}{\partial t}\|_{L^\infty(D)}.$$

We claim:

Lemma 3.5 *$u(t)$ blows-up at T^* and 0 is a blow-up point of $u(t)$.*

Let us recall the following result which asserts that the smallness of the following weighted energy (related to the energy $E(w_a)$ defined in (8)):

$$\mathcal{E}_{a,t}(u) = t^{\frac{2}{p-1}-\frac{N}{2}+1}\int \left[\frac{1}{2}|\nabla u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right]\rho(\frac{x-a}{\sqrt{t}})dx$$
$$+\frac{1}{2(p-1)}t^{\frac{2}{p-1}-\frac{N}{2}}\int |u(x)|^2\rho(\frac{x-a}{\sqrt{t}})dx$$

implies an L^∞ bound on $u(x,t)$ locally in space-time.

Proposition 3.6 (Local energy smallness result) *There exists $\sigma_0 > 0$ such that for all $\delta' > 0$ and $\theta' > 0$, $\forall t' \in [0, T_n - \theta']$, if $\forall x \in B(0,\delta')$, $\mathcal{E}_{x,T_n-t'}(u_n) \le \sigma_0$, then*
- $\forall |x| \le \delta'$, $\forall t \in [\frac{t'+T_n}{2}, T_n)$, $|u_n(x,t)| \le \frac{C\sigma_0^\theta}{(T_n-t)^{\frac{1}{p-1}}}$
- *Moreover, if* $\forall |x| \le \delta'$, $|u_n(x,\frac{t'+T_n}{2})| \le M'$ *then* $\forall |x| \le \frac{\delta'}{2}$, $\forall t \in [\frac{t'+T_n}{2}, T_n)$, $|u_n(x,t)| \le M^*$ *where* $M^* = M^*(M',\delta',\theta')$.

Proof. See [GK89] and [Mer92] (Proposition 2.5). □

Proof of lemma 3.5. By contradiction, there is $M, \delta > 0$ such that

$$\forall |x| \leq 4\delta,\ \forall t \in [0, T^*),\ |u(x,t)| \leq M. \tag{50}$$

From a stability result with respect to the initial data of this property, we obtain a contradiction.
Indeed, from (50) and direct calculations, there is then t^* such that $\forall |x| \leq \delta$, $\mathcal{E}_{x,T^*-t^*}(u(t^*)) \leq \frac{\sigma_0}{2}$. We now fix t^*. Then, for n large, $\forall |x| \leq \delta$, $\mathcal{E}_{x,T_n-t^*}(u_n)(t^*) \leq \sigma_0$, and $\forall |x| \leq \delta, \forall t \in [0, \frac{t^*+T_n}{2}], |u_n(x,t)| \leq 2M$. Therefore, form Proposition 3.6, $\forall |x| \leq \frac{\delta}{2}, \forall t \in [\frac{t^*+T_n}{2}, T_n), |u_n(x,t)| \leq M^*$.

By a classical regularity argument, we have $\forall |x| \leq \frac{\delta}{4}$, $\forall t \in [\frac{3T_n}{4}, T_n)$, $|\Delta u_n(0,t_n)| \leq M^{**}(M^*, M)$ which is a contradiction with the fact that $|\Delta u_n(0,t_n)| \to +\infty$ as $n \to +\infty$ and the fact that $T_n - t_n \to 0$. This concludes the proof of Lemma 3.5. □

Step 2: Choice of the scaling parameter

From the fact that 0 is a blow-up point of u, we are able to choose a suitable scaling parameter connecting $(0, t_n)$ and the "very singular region" of u_n. We are now reduced to the same proof as in [MZ98a]. Consider $\kappa_0 \in (0, \kappa)$ a constant such that $\mathcal{E}_{0,1}(\kappa_0) \leq \frac{\sigma_0}{2}$ ($\mathcal{E}_{0,1}(0) = 0$ yields the existence of such a κ_0).
Since 0 is a blow-up point of u,

$$u(0,t)(T^*-t)^{\frac{1}{p-1}} \to \kappa\omega.$$

where $\omega \in S^{M-1}$. (Note that this follows from the results of Giga and Kohn [GK89] and Filippas and Merle [FM95]. If $M = 1$, then $\omega = \pm 1$).
In particular, there is $t_0 \geq 0$ such that $\forall t \in [t_0, T^*)$, $|u(0,t)|(T^*-t)^{\frac{1}{p-1}} \geq \frac{3\kappa+\kappa_0}{4}$.
Therefore, by continuity arguments, for all $t \in [t_0, T^*)$, there is a $n(t)$ such that

$$\forall n \geq n(t),\ |u_n(0,t)|(T_n-t)^{\frac{1}{p-1}} \geq \frac{\kappa+\kappa_0}{2}. \tag{51}$$

From (49) and (51), we have the existence of $\tilde{t}_n \in [0, t_n]$ such that
$|u_n(0,\tilde{t}_n)|(T_n-\tilde{t}_n)^{\frac{1}{p-1}} = \kappa_0$ and $\forall t \in (\tilde{t}_n, t_n]$, $|u_n(0,t)|(T_n-t)^{\frac{1}{p-1}} < \kappa_0$.
We will see in Step 3 that $u(0,\tilde{t}_n) \sim \frac{C}{(T_n-\tilde{t}_n)^{\frac{1}{p-1}}}$.
We have $\tilde{t}_n \to T^*$ from (51).

Let us now consider

$$v_n(\xi,\tau) = (T_n-\tilde{t}_n)^{\frac{1}{p-1}} u_n(\xi\sqrt{T_n-\tilde{t}_n}, \tilde{t}_n + \tau(T_n-\tilde{t}_n)).$$

Step 3: Conclusion of the proof

From the Liouville Theorem stated for equation (5) (Corollary 1) and energy estimates, we show that the nonlinear term is "subcritical" on compact sets of $\mathbb{R}^N \times (-\infty, 1]$. In particular, we have $v_n(\xi,\tau) \to v(\tau)\omega_0$ where $\omega_0 \in S^{M-1}$, $v' = v^p$ and $v(0) = \kappa_0$ uniformly on compact sets of $\mathbb{R}^N \times (-\infty, 1]$ (Note that $v(\tau) = \kappa\left(\left(\frac{\kappa}{\kappa_0}\right)^{p-1} - \tau\right)^{-\frac{1}{p-1}}$ and $v(1) < +\infty$).

We have from the definition of v_n that

- v_n is defined for all $\tau \in [\tau_n, 1)$ where $\tau_n \to -\infty$ (since $T_n - \tilde{t}_n \to 0$) and satisfies

$$\frac{\partial v_n}{\partial \tau} = \Delta v_n + |v_n|^{p-1} v_n.$$

- $\|v_n(\tau)\|_{L^\infty} \le C \frac{(T_n-\tilde{t}_n)^{\frac{1}{p-1}}}{[(1-\tau)(T_n-\tilde{t}_n)]^{\frac{1}{p-1}}} \le \frac{C}{(1-\tau)^{\frac{1}{p-1}}}$, $\|v_n(\tau)\|_{C^2} \le \frac{C'}{(1-\tau)^{\frac{p}{p-1}}}$ and $|v_n(0,0)| = \kappa_0$.

We can assume $v_n \to v$ in $C^{2,1}_{loc}(\mathbb{R}^N \times (-\infty, 1))$ where

$$\frac{\partial v}{\partial \tau} = \Delta v + |v|^{p-1} v$$

$$|v(0,0)| = \kappa_0 \text{ and } \|v(\tau)\|_{L^\infty} \le \frac{C'}{(1-\tau)^{\frac{1}{p-1}}}.$$

From Corollary 1, (that is using in some sense the Liouville Theorem in the very singular region), we have $v(\xi,\tau) = v(\tau)\omega_0$ for some $\omega_0 \in S^{M-1}$. Thanks to this result, we have uniformly with respect to $|\xi| \le 2$,

$$\mathcal{E}_{\xi,1}(v_n(0)) \to \mathcal{E}_{\xi,1}(v(0)) = \mathcal{E}_{\xi,1}(\kappa_0) \le \frac{\sigma_0}{2}.$$

Thus, for n large, $\forall |\xi| \le 2$, $\mathcal{E}_{\xi,1}(v_n(0)) \le \sigma_0$, $|v_n(\xi, \frac{1}{2})| \le 2v(\frac{1}{2})$, and by Proposition 3.6, $\forall |\xi| \le \frac{1}{2}$, $\forall \tau \in [\frac{1}{2}, 1)$, $|v_n(\xi,\tau)| \le M^*$.

By lemma 3.2, there is M^* such that $\forall |\xi| \le \frac{1}{4}$, $\forall \tau \in [\frac{3}{4}, 1]$,
$\left|\frac{\partial v_n}{\partial t}\right|_{\frac{1}{2},[-\frac{1}{4},\frac{1}{4}]^N \times [\frac{3}{4},1]} + |\Delta v_n|_{\frac{1}{2},[-\frac{1}{4},\frac{1}{4}]^N \times [\frac{3}{4},1]} \le M^{**}$ where $|a|_{\alpha,D}$ is defined in (44).
In particular, $|\Delta v_n|$ and $\left|\frac{\partial v_n}{\partial t}\right|$ are uniformly continuous on $(\xi,\tau) \in B_{1/4} \times [\frac{3}{4}, 1]$ (with a constant independent from n). Thus, $v_n(0,\tau) \to v(\tau)\omega_0$ and $\Delta v_n(0,\tau) \to \Delta v(0,\tau)\omega_0 = 0$ uniformly for $\tau \in [0,1]$ as $n \to +\infty$.

For $\tau_n = \frac{t_n - \tilde{t}_n}{T_n - \tilde{t}_n} \in [0,1]$, we have from (47)
$|\Delta v_n(\tau_n, 0)| = (T_n - \tilde{t}_n)^{\frac{p}{p-1}} |\Delta u_n(0,t_n)| \ge \frac{\epsilon_0}{2} |u_n(0,t_n)|^p (T_n - \tilde{t}_n)^{\frac{p}{p-1}}$
$\ge \frac{\epsilon_0}{2} |v_n(0,\tau_n)|^p$. Let $n \to +\infty$, we obtain

$$0 \ge \frac{\epsilon_0}{2}\left(\min_{\tau \in [0,1]} v(\tau)\right)^p \ge \frac{\epsilon_0}{2}\kappa_0^p$$

which is a contradiction. This concludes the proof of Theorem 3. □

Let us sketch some consequences of these Theorems.

Corollary 2. It is obvious that $iii)$ is an immediate consequence of $ii)$. For $i)$ and $ii)$, see section 2.2 in [MZ98a] and work with $|u|$ instead of u. □

Theorem 4. The proof is divided in two parts. In a first part, by a contradiction argument, we prove that $\forall a \in \mathbb{R}^N$, $\|w_a(s)\|_{L^\infty} \to \kappa$ and $\|\nabla^i w_a(s)\|_{L^\infty} \to 0$ as $s \to +\infty$. The proof of Theorem 1.1 in [MZ98a] is valid in this case.

In a second part, by slightly adapting the proof presented in [MZ98b], we use a priori estimates and a contradiction argument to get the conclusion. More precisely, one should use the new blow-up criterion of equation (7) of Proposition 2.1, rather than the one specific for nonnegative data in the scalar case. □

Proposition 1. The proof of Theorem 2 in [MZ98b] is valid in this case, with obvious changes. □

Proposition 2. For (A), see Proposition 2.3 in [Mer92].

(B) is a direct consequence of continuity arguments and the uniform ODE behavior of Theorem 3. □

Proposition 3. The proof of Theorem 3 in [MZ98b] is valid in this case. □

Proposition 4. Thanks to the results of Giga and Kohn in [GK89], S is compact.

Using $iii)$ of Corollary 2, we find for each $a \in S$, $\epsilon_a > 0$ and $t_a < T$ such that $u(x,t)$ has a constant sign on $B(a,\epsilon_a) \times [t_a, T)$. Since S is compact, we can extract a finite collection a_1, ..., a_l such that

$$S \subset \cup_{i=1}^{l} B(a_i, \frac{\epsilon_{a_i}}{2}). \tag{52}$$

Since u has a constant sign on $B(a,\epsilon_{a_i}) \times [t_{a_i}, T)$, we can define $u_i \in C(\mathbb{R}^N \times [0,T), \mathbb{R})$ such that:

i) supp $u_i \subset B(a_i,\epsilon_{a_i}) \times [t_{a_i}, T)$,

ii) $\exists \eta_i \in \{-1,1\}$ such that $\forall (x,t) \in B(a_i, \frac{\epsilon_{a_i}}{2}) \times [\frac{t_{a_i}+T}{2}, T)$, $u_i(x,t) = \eta_i u(x,t)$,

iii) $\forall (x,t) \in \mathbb{R}^N \times [0,T)$, $u_i(x,t) \geq 0$ and

$$\frac{\partial u_i}{\partial t} = \Delta u_i + u_i^p + g_i(x,t), \tag{53}$$

with supp $g_i \subset \{\frac{\epsilon_{a_i}}{2} \leq |x| \leq \epsilon_{a_i}\}$.

iv) u_i blows-up at time T, on a blow-up set S_i containing $S \cap B(0, \frac{\epsilon_{a_i}}{2})$ (use ii)).

We claim that the results of Velázquez in [Vel93a], [Vel92] and [Vel93b] are valid for equation (53), therefore, the $(N-1)$ dimensional Hausdorff measure of S_i is finite.

Using iv) and (52), we get the conclusion. □

4 Generalization to the vectorial case

We prove Theorem 1 in the vectorial case in this section. The proof follows the same structure as the scalar case presented in section 2. Therefore, we will summarize the similar arguments and focus on those which are particular to the vectorial structure.

We recall that we consider all subcritical values of p (condition (15)) and not only the condition (2).

Part I: Possible behaviors of w as $s \to \pm\infty$

Step 1: Limits of w as $s \to \pm\infty$

The knowledge of the stationary solutions associated to (7) is crucial. The Pohozaev equality (24) is still valid, therefore, the stationary solutions are formed by the isolated point 0 and the continuum $\kappa\omega$ where $\omega \in S^{M-1}$, and this is the main difficulty in handling the vectorial case. Indeed, if all the possible limits were isolated points, no real difficulty would be encountered. Nevertheless, by using the compactness procedure as in the scalar case, one can show that:

- either $\|w(s)\|_{L^2_\rho} \to 0$ as $s \to +\infty$,
- or $\min_{\omega \in S^{M-1}} \|w(s) - \kappa\omega\|_{L^2_\rho} \to 0$ as $s \to +\infty$.

In this latter case, using a modulation theory, Filippas and Merle in [FM95], prove that w actually approaches a particular stationary solution $\kappa\omega_{+\infty}$ in the continuum κS^{M-1} as $s \to +\infty$.

In conclusion, we have $w(y,s) \to w_{+\infty}$ in L^2_ρ as $s \to +\infty$, where $w_{+\infty} \in \{0\} \cup \kappa S^{M-1}$.

Symmetrically, using similarly a modulation theory as in [FM95], we also have $w(y,s) \to w_{-\infty}$ as $s \to -\infty$, where $w_{-\infty} \in \{0\} \cup \kappa S^{M-1}$. The convergence holds also uniformly on compact sets of $\mathbb{R}^N$.

Using the energy estimate (25) and the fact that $\forall \omega, \omega' \in S^{M-1}$, $E(\kappa\omega) = E(\kappa\omega') > 0$ and $E(0) = 0$, we see that unless $w \equiv 0$ or $w \equiv \kappa\omega$ for some $\omega \in S^{M-1}$, there is only one non trivial case to consider:

$$(w_{-\infty}, w_{+\infty}) = (\kappa\omega_{-\infty}, 0) \tag{54}$$

where $\omega_{-\infty} \in S^{M-1}$.

From the rotation invariance of (7), we can assume that $\omega_{-\infty} = \epsilon_1$, the first element of the canonical base of $\mathbb{R}^M$. Let us remark that the modulation theory method presented in [FM95] yields also

$$\forall s \le -1,\ \|w(s) - \kappa\epsilon_1\|_{L^2_\rho} \le \frac{C}{s}. \tag{55}$$

In the following, we will find $s_0 \in \mathbb{R}$ such that $\forall (y,s) \in \mathbb{R}^N \times \mathbb{R}$, $w(y,s) = \kappa(1+e^{s-s_0})^{-\frac{1}{p-1}}\epsilon_1$, which will conclude the proof of the Theorem.

Step 2: Linear behavior of w near $\kappa\epsilon_1$ as $s \to -\infty$

Let $v = w - \kappa\epsilon_1$. We expand $v(y,s) = \sum_{i=1}^{M} v_i(y,s)\epsilon_i$ with respect to the canonical base of $\mathbb{R}^M$, where $v_i : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$. From (7), we see that v satisfies the following equation: $\forall (y,s) \in \mathbb{R}^{N+1}$,

$$\frac{\partial v}{\partial s} = \mathcal{L}_M v + f(v) \tag{56}$$

where $\mathcal{L}_M$ is the self-adjoint diagonal operator $(\mathcal{D}(\mathcal{L}))^M \to \left(L^2_\rho(\mathbb{R}^N,\mathbb{R})\right)^M$ given by

$$\mathcal{L}_M = \begin{pmatrix} \mathcal{L} & 0 & .. & 0 \\ 0 & \mathcal{L}-1 & .. & 0 \\ .. & .. & .. & 0 \\ 0 & 0 & .. & \mathcal{L}-1 \end{pmatrix} \tag{57}$$

and defined by $\mathcal{L}_M(v) = (\mathcal{L}v_1, (\mathcal{L}-1)v_2, ..., (\mathcal{L}-1)v_M)$, $\mathcal{L} = \Delta - \frac{1}{2}y.\nabla + 1$ and $f(v) = |\kappa\epsilon_1 + v|^{p-1}(\kappa\epsilon_1 + v) - \frac{\kappa}{p-1}\epsilon_1 - \frac{v}{p-1} - v_1\epsilon_1$.
From (29), the spectrum of $\mathcal{L}_M$ is

$$\text{spec}(\mathcal{L}_M) = \{1 - \frac{m}{2} \mid m \in \mathbb{N}\}.$$

The set of all eigenfunctions of $\mathcal{L}_M$ is

$$\{h_{(m_1,..,m_N)}\epsilon_i \mid (m_1,...,m_N) \in \mathbb{N}^N,\ 1 \le i \le M\}$$

where $h_{(m_1,..,m_N)}$ is defined in (31) and satisfies

$$\mathcal{L}_M\left(h_{(m_1,...,m_N)}\epsilon_1\right) = \left(1 - \frac{m_1 + .. + m_N}{2}\right) h_{(m_1,...,m_N)}\epsilon_1,$$
$$\forall i \ge 2,\ \mathcal{L}_M\left(h_{(m_1,...,m_N)}\epsilon_i\right) = -\frac{m_1 + .. + m_N}{2} h_{(m_1,...,m_N)}\epsilon_i.$$

Let P_n be the $L^2_\rho(\mathbb{R}^N,\mathbb{R})$ projector on

$$\{h_{(m_1,...,m_N)} \mid m_1 + ... + m_N = n\}. \tag{58}$$

We expand each coordinate v_i of v and then v as follows

$$v_i(y,s) = \sum_{n\in\mathbb{N}} P_n(v_i(s))$$
$$v(y,s) = \sum_{n\in\mathbb{N}} \sum_{i=1}^{M} P_n(v_i)\epsilon_i$$

Let us use this notation and give the projection of v on the eigenspace of $\mathcal{L}_M$ corresponding to the eigenvalue λ, in the case $\lambda = 1$, $\frac{1}{2}$ or 0:
$\lambda = 1$: the projection is $P_0(v_1)$,
$\lambda = \frac{1}{2}$: the projection is $P_1(v_1)$,
$\lambda = 0$: the projection is $P_2(v_1) + \sum_{i=2}^{M} P_0(v_i)$.

The following Proposition (analogous to Proposition 2.4) asserts that when $s \to -\infty$, the projection of v on the eigenspace of $\mathcal{L}_M$ corresponding to 1, $\frac{1}{2}$ or 0 dominates the others.

Proposition 4.1 (Linear estimates) *One of the following cases occurs as $s \to -\infty$:*
i) (eigenspace of $\lambda = 1$): $\|v - P_0(v_1)\|_{L^2_\rho} = o\left(\|P_0(v_1)\|_{L^2_\rho}\right)$,
ii) (eigenspace of $\lambda = \frac{1}{2}$): $\|v - P_1(v_1)\|_{L^2_\rho} = o\left(\|P_1(v_1)\|_{L^2_\rho}\right)$,
iii) (eigenspace of $\lambda = 0$):

$$\left\|v - \left(P_2(v_1) + \sum_{i=2}^{M} P_0(v_i)\right)\right\|_{L^2_\rho} = o\left(\left\|P_2(v_1) + \sum_{i=2}^{M} P_0(v_i)\right\|_{L^2_\rho}\right).$$

Proof. The proof of Proposition 3.5 in [MZ98a] is valid in this case with obvious adaptations. □

Part II: Conclusion of the proof

We handle in this Part the three cases of Proposition 4.1 to show that the first case corresponds to the solution $w(y,s) = \varphi(s - s_0)\epsilon_1$ where $\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}$ for some $s_0 \in \mathbb{R}$, whereas the two others yield a contradiction.

The proof is the same as in the scalar case thanks to the following facts:

- *Nonlinear estimate*: The blow-up criterion and its proof hold without any adaptations in the vectorial case.

- *Linear estimate*: Considering v_1, we reduce the study to the scalar case. Indeed, from (56), v_1 satisfies the following equation:

$$\frac{\partial v_1}{\partial s} = \mathcal{L} v_1 + f_1(v) \tag{59}$$

where $f_1(v) = |\kappa\epsilon_1 + v|^{p-1}(\kappa + v_1) - \frac{\kappa}{p-1} - \frac{p}{p-1} v_1$, which is almost the same as the equation (27) satisfied by v in the scalar case. We have in fact the following Proposition:

Proposition 4.2 *In all cases, $i)$, $ii)$ and $iii)$ of Proposition 4.1, $v(s) \sim v_1(s)$ in the L^2_ρ norm.*

Proof. See Appendix B. □

We now reduce the problem to the study of v_1, so that all the asymptotic computations performed on v in the scalar case remain valid for v_1 in the vectorial case. Therefore, we conclude as follows:

Assume that case $i)$ of Proposition 4.1 holds. Then, $w(y,s)=\varphi(s-s_0)\epsilon_1$ *where* $\varphi(s)=\kappa(1+e^s)^{-\frac{1}{p-1}}$ *for some* $s_0\in\mathbb{R}$.

Assume that case $ii)$ or $iii)$ of Proposition 4.1 holds. Then, there exists $a_0\in\mathbb{R}^N$ *such that* w_{a_0} *defined in (22) blows-up in finite time* $S>s_0$. *Contradiction.*

This concludes the proof of Theorem 1 in the vectorial case. $\square$

A Proof of Lemma 2.6

Proof of a -: Since $w=\kappa+v$, we write from (18), (8) and (22): $\forall a\in\mathbb{R}^N$, $\forall s\in\mathbb{R}$, $I(w_a(s))=I_1+I_2+I_3$ where
$I_1=-2\int|\nabla v(y,s)|^2\rho(y-\alpha)dy$,
$I_2=-2\int G(v(y,s))\rho(y-\alpha)dy$,
$I_3=\frac{p-1}{p+1}\left(\int_{\mathbb{R}^N}|\kappa+v(y,s)|^2\rho(y-\alpha)dy\right)^{\frac{p+1}{2}}$,
$\alpha=ae^{s/2}$ and

$$G(v)=\frac{|\kappa+v|^2}{2(p-1)}-\frac{|\kappa+v|^{p+1}}{p+1}. \tag{60}$$

Let us expand in the following I_1, I_2 and I_3 as $s\to-\infty$ and $\alpha=ae^{s/2}\to 0$.

For I_1, we write from (9): $\rho(y-\alpha)=\sqrt{\rho(y)}\sqrt{\rho(y)}e^{-\frac{|\alpha|^2}{4}}e^{\frac{\alpha.y}{2}}$. By Cauchy-Schwartz's inequality, we deduce
$|I_1|\le C\left(\int|\nabla v(y,s)|^4\rho(y)dy\right)^{1/2}\left(\int e^{\alpha.y}\rho(y)dy\right)^{1/2}$. Since $|\alpha|\le 1$, we have $e^{\alpha.y}\le\exp\left(9|\alpha|^2+\frac{|y|^2}{9}\right)$, therefore, $\int e^{\alpha.y}\rho(y)dy\le C$. Hence,

$$|I_1|\le C\left(\int|\nabla v(y,s)|^4\rho(y)dy\right)^{1/2}. \tag{61}$$

The following lemma asserts that $|\nabla v|^2$ is in fact quadratic in the L^2_ρ norm, both in cases $ii)$ and $iii)$ of Proposition 2.4.

Lemma A.1 (v^2 and $|\nabla v|^2$ are quadratic in L^2_ρ) *Assume that case $ii)$ or $iii)$ of Proposition 2.4 holds, then,* $\forall s\le s_0$
$\left(\int|v(y,s)|^4\rho(y)dy\right)^{1/4}\le C\left(\int|v(y,s)|^2\rho(y)dy\right)^{1/2}$ *and*
$\left(\int|\nabla v(y,s)|^4\rho(y)dy\right)^{1/4}\le C\left(\int|\nabla v(y,s)|^2\rho(y)dy\right)^{1/2}$.

This property has been noticed by Filippas and Liu [FL93] who used a result by Herrero and Velázquez [HV93] that asserts that all L^q norms of v and ∇v with respect to the measure ρdy are equivalent, with a controlled delay in time. For more details, see the proof of lemma A.1 below.

With this lemma and (61), we get $\forall s \le s_0$, $\forall |\alpha| \le 1$,

$$|I_1| \le C \|v(s)\|^2_{H^1_\rho}. \tag{62}$$

We focus now on I_2. We get from (60) $G(0) = \frac{\kappa^2}{2(p+1)}$ and $\nabla G(0) = 0$. Since v is globally bounded, we deduce that $\left| G(v) - \frac{\kappa^2}{2(p+1)} \right| \le C|v|^2$. Therefore, $I_2 = -\frac{\kappa^2}{p+1} + O\left(\int |v(y,s)|^2 \rho(y-\alpha) dy\right)$.
As we did for I_1, we can use Cauchy-Schwartz's inequality and lemma A.1 to get

$$\int |v(y,s)|^2 \rho(y-\alpha) dy = O\left(\int |v(y,s)|^2 \rho(y) dy\right). \tag{63}$$

Therefore,

$$I_2 = -\frac{\kappa^2}{p+1} + O\left(\|v(s)\|^2_{H^1_\rho}\right). \tag{64}$$

For I_3, we write
$I_3 = \frac{p-1}{p+1}\left(\int_{\mathbb{R}^N} |\kappa + v(y,s)|^2 \rho(y-\alpha) dy\right)^{\frac{p+1}{2}}$
$= \frac{p-1}{p+1}\left(\kappa^2 + 2\kappa \int v(y,s)\rho(y-\alpha)dy + \int v^2 \rho(y-\alpha)dy\right)^{\frac{p+1}{2}}$
$= \frac{\kappa^2}{p+1}\left(1 + \frac{2}{\kappa}\int v(y,s)\rho(y-\alpha)dy + O\left(\|v(s)\|^2_{H^1_\rho}\right)\right)^{\frac{p+1}{2}}$ according to (63).
By Cauchy-Schwartz's inequality and (63), we have:
$\left(\int v(y,s)\rho(y-\alpha)dy\right)^2 \le \int v(y,s)^2 \rho(y-\alpha) dy \le C\|v(s)\|^2_{L^2_\rho}$.
Since $\|v(s)\|_{H^1_\rho} \to 0$ as $s \to -\infty$, we end up with

$$I_3 = \frac{\kappa^2}{p+1} + \kappa \int v(y,s)\rho(y-\alpha)dy + O\left(\|v(s)\|^2_{H^1_\rho}\right). \tag{65}$$

Gathering (62), (64) and (65), we get

$$I(w_a(s)) = \kappa \int v(y,s)\rho(y-\alpha)dy + O\left(\|v(s)\|^2_{H^1_\rho}\right)$$

as $s \to -\infty$ and $\alpha = ae^{s/2} \to 0$.

It remains then to prove lemma A.1 in order to conclude the proof of lemma 2.6 a -.

Proof of lemma A.1. The main feature in the proof of this lemma is an a priori estimate on bounded solutions of

$$\psi_s \leq (\mathcal{L} + C)\psi \tag{66}$$

due to Herrero and Velázquez. Their result asserts that all L^q norms with respect to ρdy are equivalent up to a controlled delay in time.

Lemma A.2 (Herrero-Velázquez) *Assume that ψ solves (66) and $|\psi| \leq B < \infty$. Then for any $r > 1$, $q > 1$ and $L > 0$, there exist $s_0^* = s_0^*(q, r)$ and $C = C(r, q, L) > 0$ such that*

$$\left(\int |\psi(y, s + s^*)|^r \rho dy\right)^{1/r} \leq C \left(\int |\psi(y, s)|^q \rho dy\right)^{1/q}$$

for any $s \in \mathbb{R}$ and any $s^* \in [s_0^*, s_0^* + L]$.

Proof. See lemma 2.3 in [HV93]. □

According to (7) and (27), v and ∇v satisfy

$$\frac{\partial v}{\partial s} = \mathcal{L}v + f(v),$$

$$\frac{\partial \nabla v}{\partial s} = \mathcal{L}\nabla v - \left(\frac{1}{p-1} + \frac{1}{2}\right)\nabla v + p|v|^{p-1}\nabla v$$

with $|f(v)| \leq C|v|^2$.
Since v is bounded, ∇v is also globally bounded by the parabolic regularity, and we deduce that $|v|$ and $|\nabla v|$ satisfy (66). Therefore, lemma A.2 is valid for $|v|$ and $|\nabla v|$.
We prove the estimate of lemma A.1 only for ∇v in the case where $ii)$ of Proposition 2.4 holds. The three other cases follow in the same way.
Notice that in this case $\|\nabla v(s)\|_{L^2_\rho} \sim C_0 e^{s/2}$ as $s \to -\infty$ for some $C_0 > 0$. Therefore, $\forall s \leq s_0$,

$$\frac{C_0}{2} e^{s/2} \leq \left(\int |\nabla v(y, s)|^2 \rho dy\right)^{1/2} \leq 2C_0 e^{s/2}. \tag{67}$$

Set $s^* = s_0(2, 4)$ and $C^* = C(4, 2, 1)$. Then, according to lemma A.2 and (67): $\forall s \leq s_0$,
$\left(\int |\nabla v(y, s)|^4 \rho dy\right)^{1/4} \leq C^* \left(\int |\nabla v(y, s - s^*)|^2 \rho dy\right)^{1/2} \leq C^* \times 2C_0 e^{\frac{s-s^*}{2}} \leq 2C^* e^{-s^*/2} \times 2 \left(\int |\nabla v(y, s)|^2 \rho dy\right)^{1/2}$ which is the desired estimate.

Proof of b -. Use $ii)$ of Proposition 2.4 and see the proof of $ii)$ of Proposition 3.9 in [MZ98a].

Proof of c -. Use $iii)$ of Proposition 2.4 and see the proof of $ii)$ of Proposition 3.10 in [MZ98a]. □

B Proof of Proposition 4.2

The result is obvious from Proposition 4.1 if case $i)$ or $ii)$ holds. Thus, we assume that case $iii)$ of Proposition 4.1 holds.

We claim the following lemma

Lemma B.1 *Assume that Case $iii)$ of Proposition 4.1 holds. Then*

$$\left\| \sum_{i=2}^{M} P_0(v_i) \right\|_{L^2_\rho} = o\left(\|P_2(v_1)\|_{L^2_\rho} \right).$$

With this lemma, $iii)$ of Proposition 4.1 yields

$$\|v - P_2(v_1)\|_{L^2_\rho} = o(\|P_2(v_1)\|_{L^2_\rho}) \text{ as } s \to -\infty.$$

Therefore, v_1 dominates all v_i for $i \geq 2$, and Proposition 4.2 follows. It remains for us then to prove lemma B.1.

Proof of lemma B.1. We proceed in 3 steps. In Steps 1 and 2, we find equations satisfied by $P_2(v_1)$ and $P_0(v_i)$ for $i \geq 2$. In Step 3, we use these equations to compare them as $s \to -\infty$.

Step 1: Equation satisfied by $P_2(v_1)$

Arguing as in Proposition C.1 in [MZ98a], we can write from (58) and (31):

$$P_2(v_1)(y,s) = y^T A(s) y - 2 tr A(s)$$

where $A(s)$ is a C^1 symmetric $N \times N$ matrix, and deduce form (59) the equation satisfied by $A(s)$:

$$A'(s) = \frac{4p}{\kappa} A(s)^2 + o\left(\|v(s)\|^2_{L^2_\rho(\mathbb{R}^N, \mathbb{R}^M)} \right). \tag{68}$$

We can also introduce N C^1 eigenvalues of $A(s)$, $(\lambda_k(s))_{k=1,..,N}$ which satisfy by (68):

$$\forall k \in \{1, .., N\},\ \lambda'_k(s) = \frac{4p}{\kappa} \lambda_k(s)^2 + o\left(\|v(s)\|^2_{L^2_\rho(\mathbb{R}^N, \mathbb{R}^M)} \right) \tag{69}$$

$$\text{and} \qquad \frac{1}{c} \sum_{k=1}^{N} \lambda_k(s)^2 \leq \|P_2(v_1)\|^2_{L^2_\rho} \leq c \sum_{k=1}^{N} \lambda_k(s)^2 \tag{70}$$

for some $c > 0$ (see lemmas C.4 and C.5 in [MZ98a]).

Step 2: Equation satisfied by $P_0(v_i)$, $i \geq 2$

We have the following lemma:

Lemma B.2 $\forall i \geq 2$,

$$\frac{dP_0(v_i(s))}{ds} = o\left(\|v(s)\|^2_{L^2_\rho(\mathbb{R}^N,\mathbb{R}^M)}\right) \text{ as } s \to -\infty. \tag{71}$$

Proof. According to (56), $\forall i \geq 2$, v_i satisfies the following equation:

$$\frac{\partial v_i}{\partial s} = (\mathcal{L} - 1)v_i + f_i(v) \tag{72}$$

where $f_i(v) = |\kappa\epsilon_1 + v|^{p-1}v_i - \frac{v_i}{p-1}$. Since $P_0(v_i) = \int v_i(y,s)\rho(y)dy$ and $\int(\mathcal{L}-1)v_i\rho dy = 0$ (see (30) with $m = 0$), equation (72) gives

$$\frac{dP_0(v_i(s))}{ds} = \int_{\mathbb{R}^N} f_i(v)\rho(y)dy. \tag{73}$$

Since $|v(y,s)| \leq C_0 < +\infty$, we expand $f_i(v)$ until the third order as follows: $\left|f_i(v) - \frac{v_i v_1}{\kappa}\right| \leq C|v|^3$. Therefore,

$$\left|\int_{\mathbb{R}^N} f_i(v)\rho dy - I\right| \leq CII \tag{74}$$

where $I = \frac{1}{\kappa}\int v_i(y,s)v_1(y,s)\rho(y)dy$ and $II = \int |v(y,s)|^3\rho(y)dy$.
Let us estimate I first:
$I = \frac{1}{\kappa}P_0(v_i)\int v_1(y,s)\rho dy + \frac{1}{\kappa}\int(v_i - P_0(v_i))v_1(y,s)\rho dy$
$= \frac{1}{\kappa}P_0(v_i)P_0(v_1) + \frac{1}{\kappa}\int(v_i - P_0(v_i))v_1(y,s)\rho dy$. Hence,
$|I| \leq C|P_0(v_i)||P_0(v_1)| + \|v_i - P_0(v_i)\|_{L^2_\rho}\|v_1(s)\|_{L^2_\rho}$. Since Case $iii)$ of Proposition 4.1 holds, we have $|P_0(v_1)| + \|v_i - P_0(v_i)\|_{L^2_\rho} = o(\|v(s)\|_{L^2_\rho})$. Thus,

$$|I| = o(\|v(s)\|^2_{L^2_\rho}). \tag{75}$$

We use the following lemma to estimate II:

Lemma B.3 *There exists $\delta_0 > 0$ and an integer $k > 4$ such that for all $\delta \in (0, \delta_0)$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \leq s_0$,*

$$\int |v|^2|y|^k\rho dy \leq c_0(k)\delta^{4-k}\int\left[P_2(v_1)^2 + \sum_{i=2}^M P_0(v_i)^2\right]\rho dy.$$

Proof. The proof is in all points similar to the proof of lemma C.2 in [MZ98a]. □

Using the same techniques as in the proof of Proposition C.1 in [MZ98a], one can easily show that

$$II = o\left(\|v(s)\|^2_{L^2_\rho}\right). \tag{76}$$

Combining (73), (74) and (76) concludes the proof of lemma B.2. □

Step 3: Comparison of $P_2(v_1)$ and $\sum_{i=2}^{M} P_0(v_i)$

Let

$$X(s)^2 = \sum_{i=2}^{M} P_0(v_i(s))^2 \text{ and } Z(s)^2 = X(s)^2 + \sum_{k=1}^{N} \lambda_k(s)^2. \tag{77}$$

According to (70), it is enough to prove that

$$X(s) = o\left(\sqrt{\sum_{k=1}^{N} \lambda_k(s)^2}\right) \text{ as } s \to -\infty. \tag{78}$$

Since $\|v(s)\|_{L^2_\rho} \sim \left\| P_2(v_1) + \sum_{i=2}^{M} P_0(v_i) \right\|_{L^2_\rho}$, we have from (69), (71), (77) and (70):

$$\begin{cases} \lambda_k' = \frac{4p}{\kappa}\lambda_k^2 + o(Z(s)^2) \text{ for } k = 1, .., N \\ X' = o(Z(s)^2) \end{cases} \tag{79}$$

as $s \to -\infty$, and from (55), $Z(s)^2 = O\left(\frac{1}{s^2}\right)$. This gives by (79)

$$X(s) = o\left(\frac{1}{s}\right). \tag{80}$$

From (77) and (79), we have by simple calculations:

$$Z'(s) \le CZ(s)^2 \tag{81}$$

for some $C > 0$. $Z(s)$ can never be zero. Indeed, if $Z(s_0) = 0$ for some $s_0 \in \mathbb{R}$, then $\|v(s_0)\|_{L^2_\rho} = 0$, and $v \equiv 0$ on $\mathbb{R}^N \times [s_0, +\infty)$ by the uniqueness of the solution to the Cauchy problem of (27). This contradicts the fact that $v \to -\kappa\epsilon_1$ as $s \to +\infty$ (see (54)). Therefore, (81) yields: $\forall s \le s_1$,

$$Z(s) \ge \frac{C'}{|s|} \tag{82}$$

for some $s_1 \in \mathbb{R}$ and $C' > 0$. Combining (80), (82) and (77) gives the conclusion (78) and concludes the proofs of lemma B.1 and Proposition 4.2 too. $\square$

References

[Bal77] J. M. Ball. Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. Quart. J. Math. Oxford Ser. (2), 28(112):473–486, 1977.

[BK94] J. Bricmont and A. Kupiainen. Universality in blow-up for nonlinear heat equations. Nonlinearity, 7(2):539–575, 1994.

[FK92] S. Filippas and R. V. Kohn. Refined asymptotics for the blowup of $u_t - \Delta u = u^p$. Comm. Pure Appl. Math., 45(7):821–869, 1992.

[FKMZ] C. Fermanian Kammerer, F. Merle, and H. Zaag. Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view. preprint.

[FL93] S. Filippas and W. X. Liu. On the blowup of multidimensional semilinear heat equations. Ann. Inst. H. Poincaré Anal. Non Linéaire, 10(3):313–344, 1993.

[FM95] S. Filippas and F. Merle. Modulation theory for the blowup of vector-valued nonlinear heat equations. J. Differential Equations, 116(1):119–148, 1995.

[Fri65] A. Friedman. Remarks on nonlinear parabolic equations. In Proc. Sympos. Appl. Math., Vol. XVII, pages 3–23. Amer. Math. Soc., Providence, R.I., 1965.

[Fuj66] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sect. I, 13:109–124, 1966.

[GK85] Y. Giga and R. V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. Comm. Pure Appl. Math., 38(3):297–319, 1985.

[GK87] Y. Giga and R. V. Kohn. Characterizing blowup using similarity variables. Indiana Univ. Math. J., 36(1):1–40, 1987.

[GK89] Y. Giga and R. V. Kohn. Nondegeneracy of blowup for semilinear heat equations. Comm. Pure Appl. Math., 42(6):845–884, 1989.

[HV92a] M. A. Herrero and J. J. L. Velázquez. Blow-up profiles in one-dimensional, semilinear parabolic problems. Comm. Partial Differential Equations, 17(1-2):205–219, 1992.

[HV92b] M. A. Herrero and J. J. L. Velázquez. Flat blow-up in one-dimensional semilinear heat equations. Differential Integral Equations, 5(5):973–997, 1992.

[HV93] M. A. Herrero and J. J. L. Velázquez. Blow-up behaviour of one-dimensional semilinear parabolic equations. Ann. Inst. H. Poincaré Anal. Non Linéaire, 10(2):131–189, 1993.

[Lev73] H. A. Levine. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. Arch. Rational Mech. Anal., 51:371–386, 1973.

[Mer92] F. Merle. Solution of a nonlinear heat equation with arbitrarily given blow-up points. Comm. Pure Appl. Math., 45(3):263–300, 1992.

[MZ97] F. Merle and H. Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. Duke Math. J., 86(1):143–195, 1997.

[MZ98a] F. Merle and H. Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. Comm. Pure Appl. Math., 51(2):139–196, 1998.

[MZ98b] F. Merle and H. Zaag. Refined uniform estimates at blow-up and applications for nonlinear heat equations. Geom. Funct. Anal., 8(6):1043–1085, 1998.

[Vel92] J. J. L. Velázquez. Higher-dimensional blow up for semilinear parabolic equations. Comm. Partial Differential Equations, 17(9-10):1567–1596, 1992.

[Vel93a] J. J. L. Velázquez. Classification of singularities for blowing up solutions in higher dimensions. Trans. Amer. Math. Soc., 338(1):441–464, 1993.

[Vel93b] J. J. L. Velázquez. Estimates on the $(n-1)$-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation. Indiana Univ. Math. J., 42(2):445–476, 1993.

[Zaa98] H. Zaag. Blow-up results for vector-valued nonlinear heat equations with no gradient structure. Ann. Inst. H. Poincaré Anal. Non Linéaire, 15(5):581–622, 1998.