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Global holomorphic 2-forms and pluricanonical systems on threefolds

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Abstract. The change of zero locus of a global holomorphic 2-form on a threefold under birational transformations is investigated. It is proved that existence of 2-forms with certain conditions on their zero loci on a threefold of nonnegative Kodaira dimension limits types of terminal singularities appearing on its minimal models. As a result of the restriction on the types of terminal singularities and Reid's Riemann-Roch formula, a universal bound N is found such that the linear system NK defines a birational map from a threefold of general type admitting those 2-forms, where K is the canonical bundle of the threefold.

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Contents

1 Introduction

Let X be a complex projective manifold of dimension n with some nontrivial global holomorphic forms, i.e; there is $0 \neq \omega \in H^0(X, \Omega_X^p)$ for some $p \geq 1$. It is known that the dimension of all global holomorphic forms $h^0(X, \Omega_X^p) = h^p(X, \mathcal{O}_X)$ is a birational invariant. Furthermore if X' is another complex projective manifold which is birational to X, an $\omega \in H^0(X, \Omega_X^p)$ corresponds to a unique $\omega' \in H^0(X', \Omega_{X'}^p)$. Let $Z(\omega)$ be the zero locus of ω. We are interested in the following general question:

Question 1.1. How does $Z(\omega)$ change under birational transformations?

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The question is trivial for $n = 1$. When $n = 2$, two birationally equivalent algebraic surfaces are connected by a sequence of blowing ups(downs) of points ((-1) curves). So it is sufficient to understand the change of $Z(\omega)$ after one such operation. Let $\pi : (X, E) \to (Y, P)$ be a blow up at a smooth point P and E the exceptional curve. Then locally $Z(\pi^*(\omega)) = E$ for a 2-form and $Z(\pi^*(\omega)) = E$ or $Z(\pi^*(\omega)) = Q \in E$ for a 1-form depending on whether $P \in Z(\omega)$.

The situation is more involved in dimension 3. It is easy to see that blowing up a point on X creates a two dimensional zero locus for the pull backs of the 2-forms while blowing up a smooth curve creates at least one dimensional zero locus. A some what interesting phenomenon is that one never creates isolated point as zero locus for pull backs of the 2-forms by simply blowing up along smooth centers.

Let us assume that the Kodaira dimension of X is nonnegative. It is now well known after the completion of Mori's program that to get to a minimal model of X one needs to perform a sequence of birational transformations which are called extremal contractions and flips. Two minimal models are connected by flops. So our question of understanding the changes of $Z(\omega)$ could be interpreted as that of understanding the changes of $Z(\omega)$ after those "elementary" ones. An unpleasant feature of doing extremal contractions and flips is that one gets into the category of singular varieties. More precisely we have to deal with threefolds with terminal singularities, which are isolated and are of quotient (or hyperquotient) of $\mathbb{C}^3$ (or some $V(F) \subset \mathbb{C}^4$) by a cyclic group G.

Our study is primarily motivated by the following open problem on pluricanonical systems in dimension bigger than 2:

Problem 1.2. Let X be a threefold of general type, i.e; a smooth projective variety of dimension 3 over $\mathbb{C}$ whose Kodaira dimension is also 3.

Find a universal N such that $|NK|$ defines a birational map from X.

In [L1] we have considered the subsheaf $\mathcal{E}$ generated by global 2-forms in Ω_X^2 and used it to construct maps from X. Using the properties of those maps, we are able to construct sections in pluricanonical systems on X, hence giving answer to the problem in the cases when the rank of $\mathcal{E}$ is 1 or 3.

We also investigated the case when the rank of $\mathcal{E}$ is 2 in [L2], where some partial results are obtained. However as pointed out there, it is not know whether 1.2 has an affirmative answer even in the case when $\mathcal{E}$ is a rank 2 vector bundle.

It should be pointed out that problem 1.2 has an affirmative answer for irregular threefolds of general type thanks to the work of [Ko1], for threefolds having minimal models over which the canonical sheaves are invertible [EL]. From another point of view for threefolds of nonnegative Kodaira dimension, one may ask whether there is a N such that the linear system $|NK|$ has a nontrivial member. The existence of such a bound is given for cases $\kappa(X) = 0, 1, 2$ by Kawamata, Mori, and Kollár respectively in [Ka1],[Mr3],and [Ko2].

Assume now that the rank of $\mathcal{E}$ is 2.

Definition 1.3. *The degenerating locus $D(\mathcal{E})$ of $\mathcal{E}$ is the closed subset of X such that $< \omega|_x, \omega \in H^0(X, \Omega_X^2) >$ is a vector space of dimension less than or equal to 1 for any $x \in D(\mathcal{E})$.*

Obviously $\dim D(\mathcal{E})$ could be 0, 1, 2 because of the rank 2 assumption.

Based on the understanding of changes of zero locus of a global 2-form after extremal divisorial contractions and flips, we are able to show

Theorem 6.1. *Let X be a smooth complex projective threefold of general type. Assume $h^2(X, \mathcal{O}_X) \geq 2$ and $\dim D(\mathcal{E}) \leq 1$. Then there is a universal N such that NK_X defines a birational map from X.*

Hence when X has enough global holomorphic 2-forms such that they generate a rank 2 vector bundle, problem 1.2 has an affirmative answer. We hope a generalization of our methods will allow us to treat the case when $\dim D(\mathcal{E}) = 2$ at least in terms of number of 2 dimensional irreducible components in $D(\mathcal{E})$ and vanishing orders of 2-forms on each component.

The proof of Theorem 6.1 also rests upon the following result obtained purely from analysis of contributions of singularities in Riemann-Roch formula.

Theorem 5.3. *Let X be a smooth threefold of general type whose minimal model has singularities of type $\frac{1}{r}(1, -1, 1)$ after $\mathbb{Q}$ smoothing as in [R] for $r \geq 2$. Then there is a universal N such that NK_X defines a birational map from X.*

Our study is also motivated by a result in an earlier version of a paper by F. Campana and T. Peternell [CP] which says that if a threefold has a global holomorphic 2-form with isolated points as its zero locus, then the canonical bundle K must be nef.

Indeed we have a counterexample to the above claim. However the following result is true and the example shows that the conditions given are sharp.

Theorem 2.2. *Let X be a smooth threefold of $\kappa(X) \geq 0$ with $h^0(X, \Omega_X^2) = h^2(X, \mathcal{O}_X) = l \geq 4$. Assume that there is subspace $V \subset H^0(X, \Omega_X^2)$ of dimension 4 and dimension of $Z(\omega)$ is bounded from above by 1 for $0 \neq \omega \in V$. Then X has a smooth minimal model.*

Our study is guided by the principle that the existence of nontrivial global holomorphic forms on a threefold should reflect certain properties on its minimal models. For example at the moment we have very little knowledge about what kind of combination of terminal singularities may appear on a minimal model, despite of quite extensive understanding of birational transformations in dimension 3. Indeed our study shows that the conditions put on the global 2-forms limit the types of singularities appearing in the process going to a minimal model. We hope that our study will shed some light on determining the “basket” of singularities on a minimal model via geometric global constraints.

There is also a fundamental question regarding the existence (and its structure) of zero locus of holomorphic forms on a threefold of general type. For example [Z] shows a global 1-form has nontrivial zero locus when the canonical bundle is ample. In [LZ] we will show that any global i-form, $1 \leq i \leq 3$, has nontrivial zero locus on a threefold of general type.

More precisely the paper is organized as following:

In Sect. 2 we discuss the change of zero locus of 2-forms on a smooth threefold under divisorial extremal contractions which result in a smooth minimal model. Theorem 2.2 is proved.

In Sect. 3 we identify global 2-forms on a threefold with terminal singularities as *locally invariant* 2-forms which come from the corresponding cyclic covers locally around the singularities and study the change of zero locus under special resolutions. Special (partial) resolutions are constructed for certain types of terminal singularities over which a 2-form nonzero at the singularity corresponds to a 2-form with only isolated points in its zero locus along the exceptional locus.

In Sect. 4 change of zero locus of locally invariant 2-forms under a flip is considered, with an assumption that the dimension of degenerating locus of $\mathcal{E}$ is bounded from above by 1. Indeed we predict the types of singularities in the flipped neighborhood when the types of singularities in the flipping neighborhood are restricted.

Reid's Riemann-Roch formula is investigated in Sect. 5 when the types of contributing singularities are of types $\frac{1}{r}(1, -1, 1)$. We obtain a universal N, independent of indices and number of singularities, such that the linear system $|NK|$ has at least two nontrivial elements. Theorem 5.4 is proved.

Finally combining results in sections 3, 4, and 5, Theorem 6.1 is proved in Sect. 6 based on the fact that our assumption on the global holomorphic 2-forms imply that types of terminal singularities on a minimal model are limited (even though the number of them and their indices are not limited). Examples are discussed.

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2 The smooth case

We begin with a lemma describing the zero locus of twisted 1-forms on $\mathbb{P}^2$.

Lemma 2.1. *The vector space* $H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(2))$ *has dimension 3.* $Z(\theta)$ *is a point for* $0 \neq \theta \in H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(2))$.

Proof. The result about the dimension is well known. Let U_0, U_1, U_2 be the standard covering of $\mathbb{P}^2$ with homogeneous coordinates $[z_0, z_1, z_2]$. It is easy to list three independent sections $\theta_1, \theta_2, \theta_3$ in $H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(2))$:

On U_0:

$$\theta_1 = z_0^2 d(\frac{z_1}{z_0}), \quad \theta_2 = -z_0 z_2 d(\frac{z_1}{z_0}) + z_0 z_1 d(\frac{z_2}{z_0}), \quad \theta_3 = z_0^2 d(\frac{z_2}{z_0}),$$

On U_1:

$$\theta_1 = -z_1^2 d(\frac{z_0}{z_1}), \quad \theta_2 = z_1^2 d(\frac{z_2}{z_1}), \quad \theta_3 = -z_1 z_2 d(\frac{z_0}{z_1}) + z_0 z_1 d(\frac{z_2}{z_1}),$$

On U_2:

$$\theta_1 = -z_1 z_2 d(\frac{z_0}{z_2}) + z_0 z_2 d(\frac{z_1}{z_2}), \quad \theta_2 = -z_2^2 d(\frac{z_1}{z_2}), \quad \theta_3 = -z_2^2 d(\frac{z_0}{z_2}).$$

The conclusion on $Z(\theta)$ is now clear. □

The following result tells us when one can expect smooth minimal model by looking at the zero locus of global 2-forms.

Theorem 2.2. *Let* X *be a smooth threefold of* $\kappa(X) \geq 0$. *Assume that* $h^0(X, \Omega_X^2) = h^2(X, \mathcal{O}_X) = l \geq 4$. *Assume that there is subspace* $V \subset H^0(X, \Omega_X^2)$ *of dimension 4 and dimension of* $Z(\omega)$ *is bounded from above by 1 for* $0 \neq \omega \in V$. *Then* X *has a smooth minimal model.*

Proof. If K_X is nef, there is nothing to prove.

Assume K_X is not nef. Let $f : X \to Y$ be an extremal contraction. Mori [Mr1] says that f is divisorial with the exceptional divisor $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)$ or $\mathcal{O}(-2)$, $\mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \otimes \mathcal{O}(-1)$, a cone E over rational normal curve of degree 2 with normal bundle $\mathcal{O}(-1)$, a $\mathbb{P}^1$-bundle over a smooth curve. The last case is the only one in which a divisor is contracted to a curve.

We will show that under the assumption of the theorem, divisor to curve is the only possible case that could happen. This is achieved through a case by case analysis.

Let E be the exceptional divisor of the contraction. One has

$$0 \to T_E \to T_X \to N_{E/X} \to 0,$$

whose dual is

$$0 \to N^*_{E/X} \to \Omega_X \to \Omega_E \to 0.$$

Taking wedge product one has

$$0 \to N^*_{E/X} \otimes \Omega_E \to \Omega^2_X \to \Omega^2_E \to 0.$$

It is left as an exercise to check that when $E \simeq \mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)$, $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \otimes \mathcal{O}(-1)$, and $E \simeq$ a singular quadric surface in $\mathbb{P}^3$ with normal bundle $\mathcal{O}_E \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$, one always have $E \subset Z(\omega)$ for every $\omega \in H^0(X, \Omega^2_X)$. This is impossible because of the existence of V.

When $E \simeq \mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$, one has

$$h^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(2)) = 3, h^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}) = 0.$$

Hence $H^0(\mathbb{P}^2, \Omega^2_X)$ is a three dimensional vector space, identified with $H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(2))$ when restricting on E by Lemma 2.1.

Considering the restriction map $\rho : H^0(X, \Omega^2_X) \to H^0(\mathbb{P}^2, \Omega^2_X)$ on the subspace V, there is a $0 \neq \omega \in H^0(X, \Omega^2_X)$ such that $E \subset Z(\omega)$, which is impossible by our assumption.

When f contracts a divisor to a curve, Y is again a smooth threefold of non-negative Kodaira dimension. In this case, the 2-forms on X are the pullbacks of those on Y. So we have on Y a subspace $V' \subset H^0(Y, \Omega^2_Y)$ such that the dimension of $Z(\omega')$ is bounded from above by 1 for every nonzero $\omega' \in H^0(Y, \Omega^2_Y)$. We may repeat the process when K_Y is not nef. Since divisorial contraction reduces Picard number, after finitely many steps a smooth threefold of the same Kodaira dimension with nef canonical bundle is reached. Thus X has a smooth minimal model. □

Remark. The proof of Theorem 2.2 follows closely to that of a statement by Campana and Peternell (an earlier version of Theorem 3.3 in [CP]) which says

Claim 2.3. Let X be a projective threefold and ω a 2-form on X with finite zero set $Z(\omega)$.

(1) Assume $\kappa(X) = -\infty$. Then X is a $\mathbb{P}^1$-bundle over a K3-surface or a torus and ω is a pull back.
(2) If $\kappa(X) \geq 0$, then K_X is nef.

The following example was used in [L1] for a different purpose. It actually provide us a counterexample to the claim. The example also demonstrates that the conditions in the previous theorem are effectively sharp.

Example 2.4. Let C be an elliptic curve with an involution τ. Let $Y = C \times C \times C/\tau$, where the action is defined as $\tau(x, y, z) = (\tau x, \tau y, \tau z)$. Y is a threefold with index 2 terminal singularities (4^3 of them). Let X be the blowup of all the singular points. X is of Kodaira dimension 0 and not minimal. Yet X has three linearly (actually polynomially) independent 2-forms with isolated points (on the exceptional divisors) as their zero locus by Lemma 2.1.

The claim on the zero locus can also be checked by a local computation. We work this out in detail since it serves as the starting point of what we do next in Sect. 3.

Locally around each singularity Q of Y, the action of τ is simply

$$(\tau x, \tau y, \tau z) = (-x, -y, -z),$$

where Q is the image of (0, 0, 0). The affine coordinate ring is $\mathbb{C}[u_0, u_1, u_2, v_0, v_1, v_2]/I$, where I is generated by $u_0u_1 = v_0^2, u_1u_2 = v_1^2, u_0u_2 = v_2^2$. The blow up of the origin at $\mathbb{C}^6$ resolve the singularity with the exceptional locus $\mathbb{P}^2 \subset W = Bl_O(V(I)) \subset \mathbb{C}^6 \times \mathbb{P}^5$. The invariant 2-form $dy \wedge dz$ is written as $\dfrac{du_1 \wedge du_2}{2v_1}$. On the open set $Z_0 \neq 0$,

$$\frac{du_1 \wedge du_2}{2v_1} = tdu_0 \wedge ds + sdt \wedge du_0 + u_0dt \wedge ds,$$

where $u_1 = u_0t^2, u_2 = u_0s^2$, which has one point as its zero locus while $dx \wedge dy$ and $dz \wedge dx$ have no zero locus on this open set. This way we see that the invariant 2-forms $dx \wedge dy$, $dy \wedge dz$, $dz \wedge dx$ correspond to three 2-forms on the resolution having exactly one point in the exceptional $\mathbb{P}^2$ as their zero locus.

Remark. If one takes a hyperelliptic curve C and does the same construction, there are two dimensional components in the zero loci of 2-forms considered above. However it is easy to check that a general 2-form has only isolated points in its zero locus.

Combining the proof of the above theorem and the rank 2 condition on the global 2-forms, we have

Corollary 2.5. *Let X be a threefold of $\kappa(X) \geq 0$ and $h^0(X, \Omega_X^2) \geq 4$. Assume that there is a subspace $V \subset H^0(X, \Omega_X^2)$ of dimension 4 such that it generates a rank 2 subsheaf of Ω_X^2 and any two linearly independent members of V do not share dimension 2 zero locus. Then X has a smooth minimal model.*

Proof. As in the proof of theorem 2.2 when K_X is not nef, we need to show contraction of $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$ is impossible.

Let $f : X \to Y$ be such a contraction and E be the exceptional divisor. It is shown that $H^0(E, \Omega_X^2) \simeq H^0(E, \Omega_E(2))$, a three dimensional vector space in which any non-zero member has one point as its zero locus in E according to Lemma 2.1. Now let

$$\rho : H^0(X, \Omega_X^2) \to H^0(E, \Omega_E(2))$$

be the restriction map composed with the above identification. Let ρ_V be the restriction of ρ on V. We see that the image of ρ_V has at most dimension 2 because the members of $H^0(X, \Omega_X^2)$ generate a rank 2 subsheaf. This implies that the kernel of ρ_V is at least of dimension 2. Let $\omega_1, \omega_2 \in \ker(\rho_V)$ and be linearly independent. Clearly $E \subset Z(\omega_1) \cap Z(\omega_2)$ which is a contradiction. □

Benveniste [B] showed that when X is a smooth threefold of general type with K_X nef, $8K_X$ defines a birational map. Hence our results imply

Corollary 2.6. *Let X be a threefold of general type with* $h^0(X, \Omega_X^2) \geq 4$. *Assume that there is subspace* $V \subset H^0(X, \Omega_X^2)$ *of dimension 4 such that one of the following is satisfied;*

(1) *Dimension of* $Z(\theta)$ *is bounded from above by 1 for* $0 \neq \theta \in V$.
(2) V *generates a rank 2 subsheaf of* Ω_X^2 *and any two linearly independent members of* V *do not share dimension 2 zero locus.*

Then $8K_X$ *defines a birational map.*

3 Zero locus under an extremal contraction

In this section we investigate the changes of zero locus of global 2-forms under certain extremal contractions. As mentioned before one encounters threefolds with terminal singularities after contractions. To keep track of changes of zero loci of global 2-forms it is convienient to identify global holomorphic forms on the smooth locus of a normal variety with (hyper)quotient singularities as those *global locally invariant* holomorphic forms because of the nature of the singularities involved. The cumbersome (and misleading as pointed out by the referee) name comes from the fact that if $f: Y \to X$ is a resolution of singularities, on a neighborhood U around each singularity P of X and for a global holomorphic form ω on Y, $\omega|_{f^{-1}U - f^{-1}P} = \omega|_{U-P}$, and the latter comes from an invariant holomorphic form on $\mathbb{C}^3$ (or on $\mathbb{C}^4$ restricting on a hypersurface). So one identifies a “global holomorphic" form on X with a holomorphic form on the non-singular locus of X patched together with those locally defined on the covering of each singularity which are invariant under the corresponding group action.

As explained in the introduction, the primary goal in this section is to identify those terminal singularities having the property of, under a particular resolution, admitting two independent 2-forms with isolated zeros on the exceptional divisors whose discrepancies are less than 1. This is a crucial step for the later development.

A three dimensional terminal singularity (Y, P) of index bigger than 1 has been classified by Mori (See [R] for details) as either a cyclic quotient of $\mathbb{C}^3$ of type $\frac{1}{r}(a, -a, 1)$ with $(r, a) = 1$ or one of the following hyperquotient singularities, one main series and 5 exceptional ones:

(1) $\frac{1}{r}(a, -a, 1, 0; 0) : xy + g(z^r, w)$, where $g \in m^2$, $(r, a) = 1$
(2) $\frac{1}{4}(1, 1, 3, 2; 2) : xy + z^2 + g(w)$ or $x^2 + z^2 + g(y, w)$, where $g \in m^4$
(3) $\frac{1}{2}(0, 1, 1, 1; 0) : xy + g(z, w)$, where $g \in m^4$
(4) $\frac{1}{3}(0, 2, 1, 1; 0) : x^2 + y^3 + g(z, t)$, where $g \in m^3$
(5) $\frac{1}{2}(1, 0, 1, 1; 0) : x^2 + g(y, z, w) + h(z, w)$, where $g \in m^3$, $h \in m^4$
(6) $\frac{1}{2}(1, 0, 1, 1; 0) : x^2 + y^3 + yg(z, w) + h(z, w)$, where $g, h \in m^4$

where $m = m_P$, the maximal ideal of P.

Indeed the functions g and h in (4)-(6) can be further specified. It is proved in [KSB] that every isolated singularity of type among (1)-(6) is actually terminal.

We first recall the main result in [L3], where the concept of index increasing divisorial contraction is defined and a classification is made.

Proposition 3.1. *Assume $\pi : (X, E) \to (Y, P)$ is an index increasing extremal divisorial contraction. Then P is a quotient singularity of type $\frac{1}{r}(a, -a, 1)$ or a hyperquotient singularity of type (1) with ord$(g) = 1$ and π is the weighted blow up which gives the coefficient $\frac{1}{r}$ for the exceptional divisor in K_X.*

We want to understand the change of zero locus when pulling back a locally invariant 2-form around P to X. First let us focus on the situation where P is a quotient singularity. In this case π is the weighted blow up with weight $\left(\frac{a}{r}, \frac{r-a}{r}, \frac{1}{r}\right)$ and X has two singularities of types $\frac{1}{a}(r, -r, 1)$ and $\frac{1}{r-a}(-r, r, 1)$, which are covered by three open sets U_0, U_1, U_2.

More precisely, let $\alpha : \mathbb{C}^3 \to (Y, P)$ be the quotient map. Let $\beta : A = \mathbb{C}^3 \to \mathbb{C}^3$ be the finite map which "homogenizes" the group action, defined by $\beta(x, y, z) = (x^a, y^{r-a}, z)$. The action of $G =< \sigma >$ is then lifted to an action on A by

$$\sigma(x, y, z) = (\xi_r x, \xi_r y, \xi_r z)$$

where ξ_r is a primitive r-th root of unity. Obviously $dx^a \wedge dy^{r-a}$, which is the pull back of $dx \wedge dy$ to A, is invariant under the group action. The weighted blow up is realized by blowing up the origin of A

$$A \times \mathbb{P}^2 \supset Bl_O(A) \overset{\text{blow up at O}}{\to} A = \mathbb{C}^3 \overset{\beta}{\to} \mathbb{C}^3 \overset{\alpha}{\to} (Y, P).$$

Let $[Z_0, Z_1, Z_2]$ be the coordinates of $\mathbb{P}^2$. On the open set where $Z_0 \neq 0$, The groups $G = \mathbb{Z}_r$, $G_1 = \mathbb{Z}_a =< \sigma_1 >$, and $G_2 = \mathbb{Z}_{r-a} =< \sigma_2 >$ act on

$\mathbb{C}^3 = (x, s, t)$, where $s = \dfrac{Z_1}{Z_0}$ and $t = \dfrac{Z_2}{Z_0}$, by

$$\sigma(x, s, t) = (\xi_r x, s, t), \sigma_2(x, s, t) = (x, \xi_{r-a} s, t).$$

So the quotient by $G \times G_2$ is a $\mathbb{C}^3$ with invariant coordinates (x^r, s^{r-a}, t).

U_0 is the quotient of $\mathbb{C}^3$ with coordinates (x^r, s^{r-a}, t) with $G_1 =< \sigma_1 >$ action

$$\sigma_1(x^r, s^{r-a}, t) = (\xi_a^r x^r, \xi_a^{-(r-a)} s^{r-a}, \xi_a^{-1} t).$$

Using s, t, $\omega = dx^a \wedge dy^{r-a}$ becomes $dx^a \wedge dx^{r-a} s^{r-a}$. If we view ω on U_0, it becomes $\omega = c dx^r \wedge ds^{r-a}$ where c is a nonzero constant. So it has no zero locus on U_0.

Similarly U_1 is the quotient of $\mathbb{C}^3$ (the quotient of $\mathbb{C}^3 = (s, y, t)$ under action of $G \times G_1$) with coordinates (s^a, y^r, t) with $G_2 =< \sigma_2 >$ action

$$\sigma_2(s^a, y^r, t) = (\xi_{r-a}^{-a} s^a, \xi_{r-a}^r y^r, \xi_{r-a}^{-1} t),$$

where $s = \dfrac{Z_0}{Z_1}$ and $t = \dfrac{Z_2}{Z_1}$. Using s, t, $\omega = dx^a \wedge dy^{r-a}$ becomes $\omega = d(y^a s^a) \wedge dy^{r-a}$. If we view it on U_1, it becomes $ds^a \wedge dy^r$. It has no zero locus on U_1.

U_2 is actually a $\mathbb{C}^3$ with coordinates (s^a, t^{r-a}, z^r). Using s, t, $\omega = dx^a \wedge dy^{r-a}$ becomes $\omega = d(z^a s^a) \wedge dz^{r-a} t^{r-a}$. If we view it on U_2, it becomes $z^r dz^a \wedge dt^{r-a} + s^a dz^r \wedge dt^{r-a} + t^{r-a} ds^a \wedge dz^r$. It has zero locus at $(0, 0, 0)$ on U_2.

If $a = 1$ or $r - 1$, there is another linearly independent 2-form $dy^{r-1} \wedge dz$ (or $dx^{r-1} \wedge dz$) which is not zero at P.

Summarizing the computation, we have

Lemma 3.2. *Let $\pi : (X, E) \to (Y, P)$ be an index increasing extremal contraction where P is a quotient terminal singularity. Let ω be a 2-form around P. Then $Z(\pi^*(\omega))$ contains E if $Z(\omega)$ contains P and $Z(\pi^*(\omega))$ contains exactly one isolated point Q around E which is smooth on X if ω does not vanish at P. There are at least two such 2-forms which are linearly independent over P if and only if $|a| = 1$ (i.e; $a = 1$ or $r - 1$).*

Remark. We have seen in Sect. 2 that when $r = 2$ there are three such 2-forms nonzero at P.

The hyperquotient case is treated similarly. We discuss type (1) singularity next. In the process we abuse the notation by identifying locally a 2-form ω on the hypersurface $V(F)$ with a restriction of a 2-form from the ambient space $\mathbb{C}^4$:

$$0 \to N^*_{V(F)/X} \otimes \Omega_V(F) \to \Omega^2_{\mathbb{C}^4} \to \Omega^2_{V(F)} \to 0,$$

away from the isolated singular point, where $N^*_{V(F)/X}$ is the dual of the normal bundle generated by dF.

Even though this identification is not unique, one checks that it has no effect on the statement regarding the nature of zero locus of ω.

Definition 3.3. *A singularity of type (1) in Mori's list is called type (1.1) if $a = 1$ or $r - 1$ and the order of g is 1.*

Let (Y, P) be a terminal singularity of type (1) with order of $g = 1$. The index increasing extremal contraction corresponds to the weighted blow up with weights $\left(\frac{a}{r}, \frac{r-a}{r}, \frac{1}{r}, 1\right)$, which is covered by four open sets U_0, U_1, U_2, U_3. It is realized by

$$A \times \mathbb{P}^3 \supset Bl_O(A) \overset{\text{blow up at O}}{\to} A = \mathbb{C}^4 \overset{\beta}{\to} B = \mathbb{C}^4 \supset V(F) \overset{\alpha}{\to} (Y, P) \subset \mathbb{C}^4/G.$$

α is the quotient map. β homogenizes the action of $G = \mathbb{Z}_r =< \sigma >$:

$$\beta(x, y, z, t) = (x^a, y^{r-a}, z, t^r), \sigma(x, y, z, t) = (\xi_r x, \xi_r y, \xi_r z, \xi_r t).$$

B is the quotient of A under group $G_1 =< \sigma_1 > \times G_2 =< \sigma_2 > \times G_4 =< \sigma_4 >$ action via

$$\sigma_1(x, y, z, t) = (\xi_a x, y, z, t), \sigma_2(x, y, z, t) = (x, \xi_{r-a} y, z, t),$$

$$\sigma_4(x, y, z, t) = (x, y, z, \xi_r t).$$

Let $[Z_0, Z_1, Z_2, Z_3]$ be the coordinates of $\mathbb{P}^3$. U_0 is the hyperquotient of $\mathbb{C}^4$ with coordinates (x^r, u^{r-a}, v, w^r) under G_1 action:

$$\sigma_1(x^r, u^{r-a}, v, w^r) = (\xi_a^r x^r, \xi_a^{a-r} u^{r-a}, \xi_a^{-1} v, \xi_a^{-r} w^r)$$

and the defining equation: $u^{r-a} + g(x^r v^r, x^r w^r)x^{-r}$, where $u = \frac{Z_1}{Z_0}$, $v = \frac{Z_2}{Z_0}$, and $w = \frac{Z_3}{Z_0}$. U_0 is a quotient of $\mathbb{C}^3$ of type $\frac{1}{a}(r, -1, -r)$.

U_1 is the hyperquotient of $\mathbb{C}^4$ with coordinates (u^a, y^r, v, w^r) under G_2 action:

$$\sigma_2(u^a, y^r, v, w^r) = (\xi_{r-a}^{-a} u^a, \xi_{r-a}^r y^r, \xi_{r-a}^{-1} v, \xi_{r-a}^{-r} w^r)$$

and the defining equation: $u^a + g(y^r v^r, y^r w^r)y^{-r}$, where $u = \frac{Z_0}{Z_1}$, $v = \frac{Z_2}{Z_1}$, and $w = \frac{Z_3}{Z_1}$. U_1 is a quotient of $\mathbb{C}^3$ of type $\frac{1}{r-a}(r, -1, -r)$.

U_2 is smooth.

U_3 is the hyperquotient of $\mathbb{C}^4$ with coordinates (u^a, v^{r-a}, w, t^r) under G_4 action:

$$\sigma_4(u^a, v^{r-a}, w, t^r) = (\xi_r^{-a}u^a, \xi_r^{a-r}v^{r-a}, \xi_r^{-1}w, \xi_r^r t^r)$$

and the defining equation: $u^a v^{r-a} + g(t^r w^r, t^r)t^{-r}$, where $u = \dfrac{Z_0}{Z_3}$, $v = \dfrac{Z_1}{Z_3}$, and $w = \dfrac{Z_2}{Z_3}$. U_3 is a hyperquotient of $\mathbb{C}^4$ of type $\dfrac{1}{r}(-a, a, -1, 0)$.

As before one checks that 2-form $dx^a \wedge dy^{r-a}$ does not have zero locus on U_0, U_1, U_2. It has one point as its zero locus on U_3 because

$$\begin{aligned} dx^a \wedge dy^{r-a} &= d(ut)^a \wedge d(vt)^{r-a} \\ &= t^r du^a \wedge dv^{r-a} + c_2 u^a dt^r \wedge dv^{r-a} + c_3 v^{r-a} du^a \wedge dt^r, \end{aligned}$$

where c_1, c_2 are nonzero scalars. The zero locus is the singular point on U_3.

Assume $a = 1$ (or $r - 1$). $dy^{r-1} \wedge dz$ (or $dx^{r-1} \wedge dz$) also has the singular point on U_3 as its zero locus.

The discussion leads to

Lemma 3.4. *Let $\pi : (X, E) \to (Y, P)$ be an index increasing extremal contraction where P is of type (1). Let ω be a 2-form around P. Then $Z(\pi^*(\omega))$ contains E if $Z(\omega)$ contains P and $Z(\pi^*(\omega))$ contains an isolated point Q on E which is a type (1) terminal singularity on X if ω does not vanish at P. There are at least two such 2-forms which are linearly independent over P if and only if P is of type (1.1).*

The above analysis shows

Corollary 3.5. *Let (Y, P) be a terminal singularity of quotient type with $|a| = 1$ or type (1.1) with $r = 2$. Then for any resolution X of P and E any exceptional divisor with discrepancy less than 1, the pull back of any nonzero locally invariant 2-form around P does not vanish on E. There are two linearly independent (over P) 2-forms nonzero at P whose pull backs on X are nonzero on E.*

Proof. One needs only the fact that there is a one-to-one birational correspondence between exceptional divisors of discrepancies less than 1 on two resolutions of P. □

For our purpose it is important to identify those terminal singularities with the similar property as the ones in 3.5.

Definition 3.6. *A singularity of type (4) in Mori's list is called of type (4.a) if the degree* 3 *homogeneous component of g is $z^3 + t^3$.*

Lemma 3.7. *Let (Y, P) be a singularity of type (4.a) and $\pi : (X, E) \to (Y, P)$ be the weighted blow up with weights $(1, \frac{2}{3}, \frac{1}{3}, \frac{1}{3})$. Then the pull backs of 2-forms represented by $dy \wedge dz$, $dy \wedge dt$ have an isolated point in their zero loci on E and the point is a quotient terminal singularity of index* 3.

Proof. The weighted blow up is covered by four open sets U_0, U_1, U_2, U_3 and is realized through

$$A \times \mathbb{P}^3 \supset Bl_O(A) \overset{\text{blow up at O}}{\to} A = \mathbb{C}^4 \overset{\beta}{\to} B = \mathbb{C}^4 \supset V(F) \overset{\alpha}{\to} (Y, P) \subset \mathbb{C}^4/G.$$

α is the quotient map. β homogenizes the action of $G = \mathbb{Z}_3 =< \sigma >$:

$$\beta(x, y, z, t) = (x^3, y^2, z, t), \sigma(x, y, z, t) = (\xi_3 x, \xi_3 y, \xi_3 z, \xi_3 t).$$

B is the quotient of A under group $G_1 =< \sigma_1 > \times G_2 =< \sigma_2 >$ action via

$$\sigma_1(x, y, z, t) = (\xi_3 x, y, z, t), \sigma_2(x, y, z, t) = (x, \xi_3 y, z, t).$$

Let $[Z_0, Z_1, Z_2, Z_3]$ be the coordinates of $\mathbb{P}^3$. On the open set $Z_0 \neq 0$, the action of $G \times G_2$ on (x, u, v, w), where $u = \frac{Z_1}{Z_0}$, $v = \frac{Z_2}{Z_0}$, and $w = \frac{Z_3}{Z_0}$, is

$$\sigma(x, u, v, w) = (\xi_3 x, u, v, w), \sigma_2(x, u, v, w) = (x, \xi_2 u, v, w).$$

The quotient is a $\mathbb{C}^4$ with invariant coordinates (x^3, u^2, v, w). The defining equation becomes

$$x^3 + x^3(u^2)^3 + v^3 + w^3 + ... = 0.$$

U_0 is the quotient of (x^3, u^2, v, w) under action of G_1:

$$\sigma_1(x^3, u^2, v, w) = (x^3, \xi_3^{-2}u^2, \xi_3^{-1}v, \xi_3^{-1}w).$$

So U_0 has the quotient singularity of type $\frac{1}{3}(1, -1, -1)$. The invariant 2-form $dy^2 \wedge dz$ on U_0 becomes

$$dx^2u^2 \wedge dxv = x^3 du^2 \wedge dv + \frac{1}{3}u^2 dx^3 \wedge dv + \frac{1}{3} v du^2 \wedge dx^3,$$

which has one point as its zero locus. Note that this point is the singular point of U_0.

On the open set $Z_1 \neq 0$, the action of $G \times G_1$ on (u, y, v, w), where $u = \frac{Z_0}{Z_1}$, $v = \frac{Z_2}{Z_1}$, and $w = \frac{Z_3}{Z_1}$, is

$$\sigma(u, y, v, w) = (u, \xi_3 y, v, w), \sigma_1(u, y, v, w) = (\xi_3 u, y, v, w).$$

The quotient is a $\mathbb{C}^4$ with invariant coordinates (u^3, y^3, v, w). The defining equation becomes

$$u^6 y^3 + 1 + v^3 + w^3 + ... = 0.$$

So U_1 has the quotient singularity of type $\frac{1}{2}(1,1,1)$. The invariant 2-form $dy^2 \wedge dz$ on U_0 becomes

$$dy^2 \wedge dyv = \frac{1}{3} dy^3 \wedge dv,$$

which has no zero locus.

One checks that $dy^2 \wedge dz$ does not have zero locus on U_2, U_3. The same process is applied to $dy^2 \wedge dt$ to show that it has one point as its zero locus, which is the singular point on U_0. □

The combination of 3.4 and 3.7 yields

Corollary 3.8. *If (Y, P) is of type (1.1) with $r > 2$, or type (4.a), then for any resolution X, there are exceptional divisors E_1 and E_2 with discrepancies less than 1 such that there are two 2-forms which are linearly independent over P and their pull backs on X do not contain E_1 as their zero loci. The pull back of any 2-form on X has E_2 in its zero locus.*

Proof. The first claim is clear from 3.4 and 3.7. For the second claim, we perform weighted blow up at the singular point with certain weights where the pull backs of the 2-forms vanish after the first weighted blow up. This produces an exceptional divisor E_2 with discrepancy $\frac{2}{r}$ $\left(\frac{2}{3}\right)$ for type (1.1) (type (4.a)). All pull back 2-forms vanish on E_2. □

As for singularities of other types, we have

Lemma 3.9. *Assume (Y, P) is not one of quotient type, type (1.1), type (4.a) . Then there exists a partial resolution $\pi : (X, E) \to (Y, P)$ with E having the minimal discrepancy such that there is at most one (up to linear dependency over P) 2-form ω, $\pi^*(\omega)_{|E} \neq 0$.*

Proof. This is a case by case checking using the explicit weighted blowups by Kawamata as described in [Ka2]. We work out only the case when P is of type (1) with $k =$ order of g bigger than 1. Kawamata's blow up uses weight $\left(\frac{a}{r}, \frac{kr-a}{r}, \frac{1}{r}, 1\right)$. The resulting X is covered by four open sets U_0, U_1, U_2, U_3. It is realized by

$$A \times \mathbb{P}^3 \supset Bl_O(A) \overset{\text{blow up at O}}{\to} A = \mathbb{C}^4 \overset{\beta}{\to} B = \mathbb{C}^4 \supset V(F) \overset{\alpha}{\to} (Y, P) \subset \mathbb{C}^4/G.$$

α is the quotient map. β homogenizes the action of $G = \mathbb{Z}_r =< \sigma >$:

$$\beta(x, y, z, t) = (x^a, y^{kr-a}, z, t^r), \sigma(x, y, z, t) = (\xi_r x, \xi_r y, \xi_r z, \xi_r t).$$

B is the quotient of A under group $G_1 =< \sigma_1 > \times G_2 =< \sigma_2 > \times G_4 =< \sigma_4 >$ action via

$$\sigma_1(x, y, z, t) = (\xi_a x, y, z, t), \sigma_2(x, y, z, t) = (x, \xi_{kr-a} y, z, t),$$

$$\sigma_4(x, y, z, t) = (x, y, z, \xi_r t).$$

Let $[Z_0, Z_1, Z_2, Z_3]$ be the coordinates of $\mathbb{P}^3$. U_0 is the hyperquotient of $\mathbb{C}^4$ with coordinates (x^r, u^{kr-a}, v, w^r) under G_1 action:

$$\sigma_1(x^r, u^{kr-a}, v, w^r) = (\xi_a^r x^r, \xi_a^{a-kr} u^{kr-a}, \xi_a^{-1} v, \xi_a^{-r} w^r)$$

and the defining equation: $u^{kr-a} + g(x^r v^r, x^r w^r)x^{-r}$, where $u = \dfrac{Z_1}{Z_0}$, $v = \dfrac{Z_2}{Z_0}$, and $w = \dfrac{Z_3}{Z_0}$. U_0 is a quotient of $\mathbb{C}^3$ of type $\dfrac{1}{a}(r, -1, -r)$. The nonzero 2-form represented by $dx \wedge dy$ ($dx^a \wedge dy^{kr-a}$ on A) on U_0 is

$$dx^a \wedge d(xu)^{kr-a} = (x^r)^{(k-1)} dx^r \wedge du^{kr-a}.$$

when $k > 1$, the 2-form vanishes on $x^r = 0$, which defines the exceptional locus in the corresponding open subset. □

So after all we have

Proposition 3.10. *Quotient singularities, type (1.1) with* $r = 2$ *are the only types of terminal singularities with indices bigger than* 1 *which admit a resolution such that there are two linearly independent 2-forms on each exceptional divisor* E *with discrepency less than* 1.

Remark. The above analysis should be compared with what may happen with index decreasing extremal contractions. The simplest example is to blow up a smooth point $P \in X$ to get a $E_1 \simeq \mathbb{P}^2 \subset X_1$ with normal bundle $\mathcal{O}(-1)$. Then we blow up a line in E_1 to get $F_1 \cup F_2 \subset X_2$ where $F_1 = \mathbb{P}^2$ is the proper transform of E_1 with normal bundle $\mathcal{O}(-2)$. Y is obtained from X_2 by blowing down F_1. Let ω be a 2-form non zero at P. The pullback of ω on X_2 vanishes on both F_1 and F_2, which corresponds to a 2-form on Y vanishing on the proper transform of F_2.

We end this section with the following observation. Given a smooth threefold X with nontrivial global holomorphic 2-forms. A smooth threefold X' birational equivalent to X is called “simpler” than X if

$$\text{Max}\{\text{number of isolated points in } Z(\theta), \theta \in H^0(X, \Omega_X^2\} <$$

$$\text{Max}\{\text{number of isolated points in } Z(\theta), \theta \in H^0(X', \Omega_{X'}^2\}.$$

An interesting question to be answered is that how one predicts the maximum of the number of isolated points in the zero locus for a global holomorphic 2-form on a "simplest" model and how two such models are related. For example if X and X' are birational equivalent and have corresponding 2-forms ω, ω' with the same number of isolated zeros (counting multiplicities), is it true that X and X' are isomorphic in codimension 1? Our calculation seemed to suggest that if X

admits a global 2-form with isolated (nondegenerate) zeros, then the number of singularities on a minimal model should be bounded from above by, in terms of Chern classes,

$$c_3 - c_1 c_2,$$

which is $c_3(\Omega^2_X)$. This kind of result is related to that in [Mi]. We plan to discuss it in the future.

4 Zero locus under a flip

In this section we study restriction of types of singularities which appear after a flip under the assumption that the degenerating locus of global holomorphic 2-forms is of dimension bounded from above by 1 on X.

Let $\phi : X \to Y$ be a small extremal contraction. Let $\phi : (X, C) \to (Y, P)$ be a corresponding extremal neighborhood. We plan to study the structure of a flip when the locally invariant 2-forms coming from those global forms around each singular point of X along C generate a rank 2 sheaf with no degenerating locus.

Let $\phi^+ : (X^+, C^+) \to (Y, P)$ be the flipped neighborhood. First we have

Lemma 4.1. *Assume X^+ has a terminal singularity Q^+ of index $r^+ > 1$ along C^+. Let ω be a locally invariant 2-form which is not zero at the singularities of X along C. Assume the singularities of X along C are of types $\frac{1}{r}(a, -a, 1)$, or type (1.1) with index 2, or index 1. Then ω^+ is not zero at Q^+.*

Proof. Let

$$\begin{array}{ccc} & \bar{X} & \\ f \swarrow & & \searrow g \\ (X, C) & \dashrightarrow & (X^+, C^+) \end{array}$$

be a common resolution of singularities of (X, C) and (X^+, C^+) which dominates the resolution of singularities of (X, C) constructed in 3.5.

The fact that K_{X^+} is ample along C^+ implies that on $\bar{X}$ we have

$$f^* K_X = g^* K_{X^+} + \sum_i a_i E_i$$

with $a_i \geq 0$ and E_i exceptional. $a_i > 0$ if $f(E_i) \subset C$.

Now there is a E on $\bar{X}$ such that $g(E) = Q^+$ and

$$K_{\bar{X}} = g^* K_{X^+} + aE + \text{others},$$

where $0 < a < 1$. This implies that

$$K_{\bar{X}} = f^* K_X + bE + \text{others},$$

with $0 < b \leq a < 1$. So $f(E) = Q$ where Q is a singular point of X along C of index larger than 1.

As in 3.5, $\bar{\omega}$, the pull back of ω on $\bar{X}$, is not zero on E since the discrepancy of E is less than 1. This says that ω^+ is not zero at Q^+. □

As a consequence of the previous result we have

Theorem 4.2. *Notations as before. Assume there are global locally invariant 2-forms which generate a rank two sheaf whose degenerating locus does not contain singularities of X along C. Suppose (X, C) has singularities along C of types $\frac{1}{r}(a, -a, 1)$ with $|a| = 1$, or type (1.1) with index 2, or index 1. Then (X^+, C^+) has singularities of types $\frac{1}{r}(a, -a, 1)$ with $|a| = 1$, or type (1.1) with index 2, or index 1 along C^+.*

Proof. Pick two locally invariant 2-forms ω_1, ω_2 which generate a rank 2 sheaf and are not zero at the singularities of X along C. Let Q^+ be a singularity of X^+ along C^+. Then the corresponding 2-forms ω_1^+, ω_2^+ on X^+ can not be zero at Q^+ if the index of Q^+ is bigger than 1 by lemma 4.1. Moreover in any resolution of Q^+, the pull backs of these 2-forms do not vanish on exceptional divisors whose discrepancies are less than 1 by the proof of 4.1. According to the classification done in 3.10, Q^+ must be of desired type. □

5 Riemann-Roch revisited

Let us first recall Reid's Riemann-Roch formula as described in [F] and [R]. Some of the intermediate formulas are needed later.

Let X be a threefold with terminal singularities of type $\frac{1}{r}(a, -a, 1)$ (same as that of $\frac{1}{r}(1, -1, b)$ for $ab \equiv 1 \text{mod r}$). Let ξ_r be a primitive n-th root of unity and define

$$\sigma(Q, n) = \sum_{k=1}^{r-1} \frac{\xi_r^{nk}}{(1 - \xi_r^k)(1 - \xi_r^{ak})(1 - \xi_r^{-ak})}$$

for the singularity Q. Then

$$\chi(\mathcal{O}_X(nK_X)) = \frac{(n-1)n(2n-1)}{12} K_X^3 + \chi(\mathcal{O}_X) + \frac{n\pi^* K_X \cdot c_2(Y)}{12}$$

(*) $$+ \sum_Q \frac{1}{r_Q}(\sigma(Q, n) - \sigma(Q, 0)),$$

where $\pi : Y \to X$ is a resolution of singularities.

Since

$$\sigma(Q, n) = \sum_{k=1}^{\bar{n}-1} \overline{b_Q k}(r_Q - \overline{b_Q k}) + \frac{r_Q^2 - 1}{24}(1 - 2\bar{n}),$$

here $\bar{?}$ means ? modulo r_Q, we have

$$\sigma(Q,n)-\sigma(Q,0)=\sum_{k=1}^{\bar{n}-1}\overline{kb_Q}(r_Q-\overline{kb_Q})-\frac{r_Q^2-1}{12}\bar{n}.$$

By a result of Kawamata, we know that

$$\pi^*K_X\cdot c_2(Y)=\sum_Q\frac{r_Q^2-1}{r_Q}-24\chi(\mathcal{O}_X).$$

We have also

$$\sum_{k=1}^{r_Q}\overline{kb_Q}(r_Q-\overline{kb_Q})=\sum_{k=1}^{r_Q-1}k(r_Q-k)=\frac{r_Q(r_Q^2-1)}{6}\ .$$

Now putting everything together, we have

$$\chi(\mathcal{O}(nK_X))=\frac{(n-1)n(2n-1)}{12}K_X^3+(1-2n)\chi(\mathcal{O}_X)\\+\sum_Q\sum_{k=1}^{n-1}\frac{\overline{kb_Q}(r_Q-\overline{kb_Q})}{2r_Q}\ .$$

If we further assume that X is minimal and of general type, K_X is both nef and big. As a consequence we have the vanishing of $H^i(\mathcal{O}(nK_X))$ for $i\geq 1$ and $n\geq 2$. So one can describe the plurigenus by the Riemann-Roch formula. That is for $n\geq 2$

$$P(n):=h^0(nK_X)\\=\frac{(n-1)n(2n-1)}{12}K_X^3+(1-2n)\chi(\mathcal{O}_X)+\sum_Q\sum_{k=1}^{n-1}\frac{\overline{kb_Q}(r_Q-\overline{kb_Q})}{2r_Q}.$$

If we define

$$S_k=\sum_Q\frac{\overline{kb_Q}(r_Q-\overline{kb_Q})}{2r_Q},$$

we have the following

Lemma 5.1. *Assume that $P(n)=0$, for $n=2,3,\ldots,N$. Then S_n can be written as*

(**) $$S_n=\frac{(n-2)\,(n+2)}{5}(S_3-S_2)+S_2$$

for $n=2,3,\ldots,N-1$.

Proof. Since $P(n) = 0$, for $n = 2, 3, \dots N$, letting Q_n be $P(n+1) - P(n)$, we get

$$Q_n = \frac{n^2}{2} K_X^3 - 2\chi(\mathcal{O}_X) + S_n = 0 \qquad \text{for} \qquad n = 2, \dots, N-1.$$

Repeat the step for Q_n, we have

$$Q_n - Q_{n-1} = \frac{2n-1}{2} K_X^3 + S_n - S_{n-1} = 0 \qquad \text{for} \qquad n = 3, \dots N-1.$$

Thus one gets

$$\frac{S_n - S_{n-1}}{2n-1} = \text{const} = \frac{S_3 - S_2}{5}$$

for $n = 3, \dots, N-1$.

The Lemma follows by taking sum of the above expressions from 3 to n. □

Now for our purpose we assume that each singularity Q is of type $\frac{1}{r_Q}(1, -1, 1)$, we define $\alpha_Q(k)$ such that $\alpha_Q(k)$ is the positive integer which satisfies

$$(\alpha_Q(k) + 1)r_Q > k \geq \alpha_Q(k) r_Q,$$

then

$$S_k = \sum_Q \frac{\overline{k}(r_Q - \overline{k})}{2r_Q} = \sum_Q \frac{(k - \alpha_Q(k)r_Q)(r_Q - k + \alpha_Q(k)r_Q)}{2r_Q},$$

and also

$$S_k = \frac{(k-2)(k+2)}{5}(S_3 - S_2) + S_2 = \frac{k^2-4}{20} N_2 + \sum_{r_Q \geq 3} \frac{k^2(r_Q - 5) + 6r_Q}{10 r_Q},$$

whose first few terms are

$$= \frac{k^2-4}{20} N_2 + \frac{18 - 2k^2}{30} N_3 + \frac{24 - k^2}{40} N_4 + \frac{3}{5} N_5 + \frac{k^2 + 36}{60} N_6 + \frac{2k^2 + 42}{70} N_7 + \dots,$$

where N_i is the number of singularities with $r = i$ in the Riemann-Roch formula.

Equating these two expressions of S_k we have, by letting $k = 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$, a sequence of 10 equations:

$$4N_3 - N_4 = 3N_2 + \sum_{i \geq 5} N_i$$

$$7N_3 + 2N_4 = 4N_2 + 3\sum_{i\geq 5} N_i$$

$$9N_3 + 4N_4 = 8N_2 + N_5 + 6\sum_{i\geq 6} N_i$$

$$3N_3 + N_4 = 2N_2 + N_6 + 2\sum_{i\geq 7} N_i$$

$$4N_3 + N_4 = 3N_2 + N_6 + 2N_7 + 3\sum_{i\geq 8} N_i$$

$$24N_3 + 9N_4 = 18N_2 + N_5 + 6N_6 + 11N_7 + 16N_8 + 21\sum_{i\geq 9} N_i$$

$$32N_3 + 12N_4 = 24N_2 + 3N_5 + 8N_6 + 13N_7 + 18N_8 + 23N_9 + 28\sum_{i\geq 10} N_i$$

$$39N_3 + 14N_4 = 28N_2 + N_5 + 11N_6 + 16N_7 + 21N_8 + 26N_9 + 31N_{10} + 36\sum_{i\geq 11} N_i$$

$$9N_3 + 3N_4 = 7N_2 + 3N_6 + 4N_7 + 5N_8 + 6N_9 + 7N_{10} + 8N_{11} + 9\sum_{i\geq 12} N_i$$

$$11N_3 + 4N_4 = 8N_2 + 2N_6 + 5N_7 + 6N_8 + 7N_9 + 8N_{10} + 9N_{11} + 10N_{12} + 11\sum_{i\geq 13} N_i$$

in which the coefficients of N_i repeat after certain i.

From the first 5 equations we obtain the following information:

$$N_3 = N_4 = N_5 = N_7$$

and

$$N_2 = N_6 = 0,$$

and

$$N_3 = \sum_{i\geq 8} N_i.$$

Jointly with next three equations, one gets

$$N_8 = N_9 = N_{10} = 0.$$

The 9-th equation provides

$$N_{11} = N_3, N_i = 0$$

for $i \geq 12$. The last equation says

$$N_3 = 10N_{12} + 11N_{13} + 11N_{14} + ... = 0.$$

Thus we have

Theorem 5.2. *Let X be a smooth threefold of general type whose minimal model has singularities of type* $\frac{1}{r}(1, -1, 1)$ *after* $\mathbb{Q}$ *smoothing as in [R] for* $r \geq 2$. *Then there is an* $N \leq 14$ *such that* $P(N) \geq 1$.

Proof. If there is a $k \leq 14$ such that $P(k) \geq 1$, we are done. Otherwise the above analysis implies that there are no singularities of indices bigger than 1. When that is so, K is invertible. $P(n)$ is a polynomial of degree 3 in n by formula (*). There is a $s \leq 5$ such that $P(s) \geq 1$. □

Indeed our numerical analysis provides us more information from Reid's Riemann-Roch formula.

Corollary 5.3. *Notations as in 5.2. There is an* N_1 *such that* $P(N_1) \geq 2$.

Proof. By assuming $P(n) = 0$ or 1 for $n = 2, 3, ..., 14$ and replacing in (**) S_k by $S'_k = S_k + \delta_k$ where $\delta_k = -1$, or 0, or 1, the above analysis shows that the number of singularities is bounded from above. The only thing one needs to be careful is that instead of obtaining equalities between N_is, one gets equalities involve certain constants which are bounded from above. The relation

$$N_3 = C_1 + 10N_{12} + 11N_{13} + 11N_{14} + ... = C_2,$$

where C_i are bounded, says the total number of singularities is bounded from above.

As done in Sect. 5 of [L1], by putting contributions from singularities of indices 2 to the right hand sides, we have from $P(2)$ and Q_2:

$$\frac{1}{2}K_X^3 - 3\chi(\mathcal{O}_X) - N_3\frac{1(3-1)}{6} - N_4\frac{1(4-1)}{8} - N_5\frac{1(5-1)}{10}.... = F_1,$$

$$2K_X^3 - 2\chi(\mathcal{O}_X) - N_3\frac{2(3-2)}{6} - N_4\frac{2(4-2)}{8} - N_5\frac{2(5-2)}{10}.... = F_2,$$

where $|F_i|$ are constants bounded from above. They imply

$$\frac{1}{2}K_X^3 - 3\chi(\mathcal{O}_X) - \frac{N_3}{6} - \frac{N_4}{8} - \frac{N_5}{10}.... = F'_1,$$

$$2K_X^3 - 2\chi(\mathcal{O}_X) - 4\frac{N_3}{6} - 4\frac{N_4}{8} - 4\frac{N_5}{10}.... = F'_2,$$

where $|F'_i|$ are bounded from above since the number of singularities is bounded. From these equations we obtain

$$10\chi(\mathcal{O}_X) = F'_2 - 4F'_1,$$

which implies that $\chi(\mathcal{O}_X)$ is bounded from above. Theorem 5.1 in [L1] then claims that there is an N_1 with $P(N_1) \geq 2$. □

An immediate consequence is

Theorem 5.4. *Let X be a smooth threefold of general type whose minimal model has singularities of type $\frac{1}{r}(1, -1, 1)$ after $\mathbb{Q}$ smoothing as in [R] for $r \geq 2$. Then there is a universal N such that NK_X defines a birational map from X.*

Proof. By Corollary 5.3 there is a universal N_1 such that $P(N_1) \geq 2$. A result of Kollár [Ko1] says if $N = 11N_1 + 5$, NK defines birational map. $\square$

6 An application to plurigenera

Let X be a smooth projective threefold and $\mathcal{E}$ the subsheaf of Ω_X^2 generated by global 2-forms . Our main theorem in this section is

Theorem 6.1. *Assume X is of general type, $h^2(\mathcal{O}) \geq 2$ and the degenerating locus of $\mathcal{E}$ is of dimension bounded from above by 1. Then there is a universal N such that NK defines a birational map from X.*

Proof. We run minimal model program on X. In the process of doing divisorial extremal contractions and flips, terminal singularities may appear. When a singularity is not of index 1, we argue that the singularities have to be among types $\frac{1}{r}(a, -a, 1)$ with $|a| = 1$, or type (1.1) with index 2 and there are two locally invariant 2-forms nonzero and linearly independent at each singularity. This is proved inductively with the first step being obvious.

By results in Sect. 3, when a singularity P is a result of a divisor to point extremal contraction $f_i : (X_i, E_i) \to (X_{i+1}, P)$, the bound on the dimension of degenerating locus assures that it is of type $\frac{1}{r}(a, -a, 1)$ with $|a| = 1$, type (1) with $|a| = 1$ or exceptional. We claim that the singularity must be of type $\frac{1}{r}(a, -a, 1)$ with $|a| = 1$, or type (1.1) with index 2. For otherwise we consider a common resolution:

$$\begin{array}{ccc} & \bar{X} & \\ \alpha \swarrow & & \searrow \beta \\ (X_i, E_i) & & (X', E') \\ f_i \searrow & & \swarrow g_i \\ & (X_{i+1}, P) & \end{array}$$

where g_i is the partial resolution constructed in 3.9. By induction X_i has only those specified singularities. This implies that for $\bar{E}$ on $\bar{X}$ exceptional over X_{i+1} with discrepancy less than 1, $(f_i \circ \alpha)^*(\omega_j)_{|\bar{E}} \neq 0$ for $j = 1, 2$, where ω_1, ω_2 are two nonzero 2-forms around P linearly independent over P by 3.7. But this is against the conclusion of 3.9 considering $(g_i \circ \beta)^*(\omega_j)$ when they are restricted on the proper transform of E' on $\bar{X}$.

The same argument is applied to the divisorial contraction situation f_i : $(X_i, E_i) \to (X_{i+1}, C)$ where a divisor is contracted to a curve C. Let $P \in C$ be a terminal singularity of index bigger than 1. X_i has to have a singularity Q on E_i of index bigger than 1. By induction Q is of the desired type and there are two nonzero locally invariant 2-forms linearly independent at Q. Now argue as in the previous case, we see that P must be of desired type.

Now let $\phi_i : (X_i, C_i) \dashrightarrow (X_i^+, C_i^+)$ be a flip. 4.2 says that a singularity (of index bigger than 1) of X_i^+ along C_i^+ must be of the desired type.

As a conclusion we see that singularities occur on a minimal model X_{min} are either of index 1, or $\dfrac{1}{r}(a, -a, 1)$ with $|a| = 1$, or type (1.1) with index 2, or index 1. Using notations from Sect. 5, we may assume the singularities contributing to the plurigenus formula are all of type $\frac{1}{r}(1, -1, 1)$. Theorem 5.4 gives us the desired conclusion. □

We construct an example of a (regular) threefold with the required conditions:

Example 6.2. Let S be a smooth (regular) projective surface with $h^0(\Omega_S^2) \geq 2$. Let $X \subset S \times S$ be a general hypersurface section. Using a result of Lefschetz we see that X is (regular) and there is an isomorphism between $H^0(X, \Omega_X^2)$ and $H^0(S \times S, \Omega_{S\times S}^2)$, which is generated by the pull backs of 2-forms on S through the two projections. So the 2-forms on X generate a sheaf $\mathcal{E}$ of rank 2. Moreover the degenerating locus of $\mathcal{E}$ has dimension bounded from above by 1 if the canonical system on S has only isolated base points.

Examples of a (regular) X with $h^2(\mathcal{O}) \geq 2$ and type (1) singularities with arbitrarily high indices may be obtained by modifying an example of Reid in [R]:

Example 6.3. Let V be a smooth (regular) threefold with $h^2(\mathcal{O}) \geq 2$. Consider the product $V \times \mathbb{P}^3$ with an order r cyclic group G action

$$g(x, [z_0, z_1, z_2, z_3]) = (x, [z_0, \xi_r z_1, \xi_r^{r-1} z_2, \xi_r z_3]),$$

where ξ is a r-th primitive root of 1. Let $Y = V \times \mathbb{P}^3/G$. Let X be an intersection of three general hypersurface sections. X is a (regular) threefold of general type with quotient terminal singularities of the same type $\frac{1}{r}(1, -1, 1)$. The global 2-forms on V are invariant under the group action, hence can be view as global locally invariant 2-forms on X. Let X' be the economical resolution of those quotient singularities on X. X' is a smooth threefold of general type with those singularities on its minimal models. Moreover, if the 2-forms on V generate a rank two vector bundle, the 2-forms on X' generate a rank two sheaf whose degenerating locus is of dimension bounded from above by 0.

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