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Normalized solutions for Kirchhoff equations with Sobolev critical exponent and mixed nonlinearities

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Abstract

This paper focuses on the existence of normalized solutions for the following Kirchhoff equation:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+\lambda u=u^5+\mu|u|^{q-2}u,\ x\in\mathbb{R}^3,\\ \int_{\mathbb{R}^3}u^2\mathrm{d}x=c,\end{cases}$$

where $a, b, c > 0$, $\mu \in \mathbb{R}$ and $2 < q < 6$, $\lambda \in \mathbb{R}$ will arise as a Lagrange multiplier that is not a priori given. By using new analytical techniques, the paper establishes several existence results for the case $\mu > 0$:

(1) The existence of two solutions, one being a local minimizer and the other of mountain-pass type, under explicit conditions on c when $2 < q < \frac{10}{3}$.
(2) The existence of a mountain-pass type solution under explicit conditions on c when $\frac{10}{3} \le q < \frac{14}{3}$.
(3) The existence of a ground state solution for all $c > 0$ when $\frac{14}{3} \le q < 6$.

Furthermore, the paper presents the first non-existence result for the case $\mu \le 0$ and $2 < q < 6$. In particular, refined estimates of energy levels are proposed, suggesting a new threshold of compactness in the L^2-constraint. This study addresses an open problem for $2 < q < \frac{10}{3}$ and fills a gap in the case $\frac{10}{3} \le q < \frac{14}{3}$. We believe that our approach can be applied to a broader range of nonlinear terms with Sobolev critical growth, and the underlying ideas have potential for future development and applicability.

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1 Introduction

In this paper, we study the existence of normalized solutions for the following Kirchhoff equation:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+\lambda u=u^5+\mu|u|^{q-2}u,\ x\in\mathbb{R}^3, \\ \int_{\mathbb{R}^3}u^2\mathrm{d}x=c, \end{cases} \tag{1.1}$$

where $a, b > 0$ and $c > 0$ are given constants, $\lambda \in \mathbb{R}$ will arise as a Lagrange multiplier and is not a priori given, $\mu \in \mathbb{R}$ and $2 < q < 6$. Here 6 is the Sobolev critical exponent. Normalized solutions to (1.1) can be obtained as critical points of the energy functional $\Phi : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\Phi(u)=\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^2-\frac{1}{6}\int_{\mathbb{R}^3}u^6\mathrm{d}x-\frac{\mu}{q}\int_{\mathbb{R}^3}|u|^q\mathrm{d}x \tag{1.2}$$

restricted on

$$\mathcal{S}_c=\left\{u\in H^1(\mathbb{R}^3):\|u\|_2^2=c\right\}. \tag{1.3}$$

The first equation of (1.1) is a special form of the Kirchhoff type equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u+\lambda u=f(u), \tag{1.4}$$

where $N \geq 1$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, which was proposed by Kirchhoff as an extension of the classical D'Alembert's wave equations, describing free vibrations of elastic strings. Mathematically, (1.4) is often referred to be nonlocal as the appearance of the term $\left(\int_{\mathbb{R}^N}\nabla u|^2dx\right)\Delta u$ implies that (1.4) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties, which make the study of (1.4) particularly interesting. After the pioneering work of Lions [13], where a functional analysis approach was proposed, the Kirchhoff type equations began to call attention of researchers.

For the study of (1.4), there exist two distinct options regarding the frequency parameter λ, leading to two different research fields. A possible choice is fixing $\lambda \in \mathbb{R}$, or even with an additional external and fixed potential $V(x)$. This direction has been extensively studied in the last ten years, there are numerous relevant literature sources, and we will not list them here.

Alternatively, it is of great interest to investigate solutions to (1.4) that possess a prescribed L^2-norm, which are commonly referred to as normalized solutions. In this situation, the frequency $\lambda \in \mathbb{R}$ is an unknown parameter and acts as a Lagrange multiplier with respect to the constraint $\mathcal{S}_N(c) = \left\{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = c\right\}$. Normalized solutions to (1.4) can be obtained as critical points of the energy functional

$\Phi_N : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\Phi_N(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^N} F(u)\mathrm{d}x \tag{1.5}$$

on the constraint $\mathcal{S}_N(c)$, where $F(u) := \int_0^u f(t)\mathrm{d}t$. From the physical point of view, finding normalized solutions seems to be particularly meaningful because the L^2-norm of such solutions is a preserved quantity of the evolution and their variational characterization can help to analyze the orbital stability or instability, see, for example, [2, 14]. Despite its physical relevance, there have been few works available on this topic. In particular, when considering the critical growth case, we are only aware of the papers [11, 12, 21]. Before delving into the results motivate our research, let us highlight some novel aspects in the study of (1.4) with an L^2-constraint in the next subsection.

1.1 Previous developments and some perspectives

From a variational point of view, besides the Sobolev critical exponent $2^* := \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = \infty$ for $N = 1, 2$, a new L^2-critical exponent $q_N := 2 + \frac{8}{N}$ arises that plays a pivotal role in the study of normalized solutions to (1.4). This threshold determines whether the constrained functional Φ_N remains bounded from below on $\mathcal{S}_N(c)$ and consequently influences our choice of approaches when searching for constrained critical points. As far as we know, in this regard, the first results for (1.4) with $f(u) = |u|^{q-2}u$ can be attributed to the work by Ye in a sequence of papers [18–20]. These results are summarized in Table 1.

In particular, for $q > 2 + \frac{8}{N}$, Φ_N is always unbounded from below on $\mathcal{S}_N(c)$ since it can be easily derived that $\Phi_N\left(t^{N/2}u_t(x)\right) \to -\infty$ as $t \to \infty$, where

$$u_t(x) := u(tx), \quad \forall\, x \in \mathbb{R}^N, \ t > 0 \tag{1.6}$$

is a dilation preserving the L^2-norm, that is $\|t^{N/2}u_t\|_2 = \|u\|_2$ for $t > 0$, and this situation corresponds to what is termed as an L^2-supercritical case. In this case, more efforts are always needed since one cannot search for a global minimum of Φ_N restricted

Table 1 Results on (1.4) with $f(u) = |u|^{q-2}u$

q	c	Type of solutions
$2 < q < 2 + \frac{4}{N}$	$c > 0$	A global minimizer
$2 + \frac{4}{N} \leq q < 2 + \frac{8}{N}$	$c > c_q$	
$q = 2 + \frac{8}{N}$	$0 < c < c^*$	No solution
$2 + \frac{8}{N} < q < 2^*$	$c > 0$	A mountain pass type solution

on $\mathcal{S}_N(c)$ and only identify a suspected critical level. Later, the results in the case $2 + \frac{8}{N} < q < 2^*$ of Table 1 were further extended in [6] to the more general L^2-supercritical case with Sobolev subcritical growth, where $f(u) \sim \sum_{i=1}^{m} |u|^{q_i-2}u$ ($2 + \frac{8}{N} < q_i < 2^*$ and $m \geq 2$). Furthermore, as observed from Table 1, when $p \in (2, 2 + \frac{8}{N})$, the exponent $2 + \frac{4}{N}$ also plays an important role in the investigation of normalized solutions. In fact, it corresponds to the L^2-critical exponent in the study of normalized solutions to the Schrödinger equation, specifically (1.4) with $b = 0$. It is worth emphasizing that for the Schrödinger equation (1.4) with $b = 0$, the L^2-critical exponent is always strictly smaller than the Sobolev critical exponent, specifically $2 + \frac{4}{N} < \frac{2N}{N-2}$. However, for the Kirchhoff equation (1.4) with $b > 0$, the L^2-critical exponent is strictly smaller than the Sobolev critical exponent only when $N \leq 3$, that is, $q_N = 2 + \frac{8}{N} < 2^* = \frac{2N}{N-2}$ if and only if $N \leq 3$. This explains why the research on normalized solutions for the Kirchhoff equation is predominantly focused on the case of $N \leq 3$, and when the nonlinearity exhibits Sobolev critical growth, it suffices to consider the case of $N = 3$.

It is well-known that compared to the subcritical growth case, the Sobolev critical growth case of (1.4) presents additional challenges in terms of the compactness analysis, especially when considering the L^2-constraint. To the best of our knowledge, the first work on the Sobolev critical growth case is due to Zhang–Han [21]. They established the existence of normalized solutions to (1.1) when $\mu = 1$ and $\frac{14}{3} \leq q < 6$ by calculating the threshold of the mountain pass level. Subsequently, Li–Nie–Zhang [12] obtained similar results in the L^2-supercritical case $\frac{14}{3} < q < 6$ using a different method that relies on the Sobolev subcritical approximation. However, their results require $\mu > 0$ to be large enough in (1.1). More recently, Li–Luo–Yang [11] further extended these results on (1.1). However, their work is restricted to the power ranges: $2 < q < \frac{10}{3}$ or $\frac{14}{3} \leq q < 6$, and leaves a gap: $\frac{10}{3} \leq q < \frac{14}{3}$. The significant findings from their research are summarized in Table 2.

In Table 2, despite explicitly identifying the range of existence for local minima with respect to μ for $2 < q < \frac{10}{3}$, the expression for the upper bound $\mu_*(c, q)$ is excessively convoluted. Moreover, two open problems, labeled as (Q1) and (Q2), remain unaddressed. It is noteworthy that $\frac{10}{3}$ and $\frac{14}{3}$ are the L^2-critical exponents in the case of $N = 3$ to (1.4) with $b = 0$ and (1.4) with $b > 0$, respectively. When $b = 0$, Eq. (1.1) reduces to the three-dimensional scenario ($N = 3$) of the Schrödinger equation

Table 2 Existence results on (1.1) in [11]

q	μ $(c > 0)$	Type of solutions	Open problem
$2 < q < \frac{10}{3}$	$\mu < \mu_*(c, q)$ small enough	A local minimizer	**(Q1):** *Is there a second solution?*
$\frac{10}{3} \leq q < \frac{14}{3}$	?	?	**(Q2):** *What happens?*
$\frac{14}{3} \leq q < 6$	$\mu > 0$	A mountain pass type solution	

Table 3 Existing results on (1.7)

q	Type of solutions	Energy level	References
$2 < q < 2 + \frac{4}{N}$	One local minimizer	$:= m_c < 0$	[5, 8, 15]
	A second solution	$< m_c + \frac{1}{N}\mathcal{S}^{\frac{N}{2}}$	[9] for $N \geq 4$
			[5, 16] for $N \geq 3$
$2 + \frac{4}{N} \leq q < 2^*$	A mountain pass type solution	$< \frac{1}{N}\mathcal{S}^{\frac{N}{2}}$	[5, 10, 15, 16]

with Sobolev critical growth:

$$\begin{cases} -\Delta u + \lambda u = \mu|u|^{q-2}u + |u|^{2^*-2}u, \ x \in \mathbb{R}^N, \ \ N \geq 3, \\ \int_{\mathbb{R}^N} u^2 \mathrm{d}x = c, \end{cases} \tag{1.7}$$

which can be viewed as a counterpart of the classical Brezis–Nirenberg problem in the context of L^2-constraint. In addition to the Sobolev critical growth, an important feature of this kind of problem lies in the fact that the presence of multiple powers destroys the scale invariance of the homogeneous equation, and thus it is called a mixed problem. Such a problem has become an active research topic, as seen in references such as [5, 8–10, 15, 16]. In these references, some existence results were established for certain small values of $c > 0$, some of which are summarized in Table 3.

Here and in the rest of the paper, $\mathcal{S}$ denotes the best constant for the Sobolev inequality, i.e., for any $N \geq 3$ there exists an optimal constant $\mathcal{S} > 0$ depending only on N, such that

$$\mathcal{S}\|u\|_{2^*}^2 \leq \|\nabla u\|_2^2, \ \ \forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^N). \ \text{(Sobolev inequality)} \tag{1.8}$$

Remark that the existence of a second solution to (1.7) when $2 < q < 2 + \frac{4}{N}$ had been raised as an open problem in [15], subsequently, it was addressed, as presented in Table 3.

Compared to the case $b = 0$, the study of (1.1) with $b > 0$ is much more challenging, due to the additional difficulties caused by the combined effect of the nonlocal term of $(\|\nabla u\|_2^2)\Delta u$ and multiple powers. For example,

(i) The functional Φ is comprised of four distinct terms that exhibit varying scaling behavior with respect to the dilation $t^{3/2}u(t\cdot)$. The intricate interplay among these terms makes it more difficult to ascertain the types of critical points for Φ on $\mathcal{S}_c$.

(ii) It is widely recognized that establishing the compactness in critical growth problems hinges on obtaining rigorous upper bound estimates for the minimax levels. This has only been achieved when $b = 0$, specifically:

$$M(c) < \begin{cases} m_c + \frac{1}{3}\mathcal{S}^{\frac{3}{2}}, & \text{if } 2 < q < \frac{10}{3}, \ \text{where } m_c \text{ is defined in Table 3}, \\ \frac{1}{3}\mathcal{S}^{\frac{3}{2}}, & \text{if } \frac{10}{3} \leq q < 6. \end{cases}$$

In the case of $b > 0$, there is also a need to establish a similar inequality. However, at present, only one result is available for the range $\frac{14}{3} \leq q < 6$, while the cases of $2 < q < \frac{10}{3}$ and $\frac{10}{3} \leq q < \frac{14}{3}$ remain unresolved due to the strong competitive effect of the term $\|\nabla u\|_2^4$ in Φ, more precisely,

$$M(c) < \begin{cases} ?, & \text{if } 2 < q < \frac{10}{3}, \\ ?, & \text{if } \frac{10}{3} \leq q < \frac{14}{3}, \\ \Theta^* := \frac{ab\mathcal{S}^3}{4} + \frac{b^3\mathcal{S}^6}{24} + \frac{(b^2\mathcal{S}^4+4a\mathcal{S})^{\frac{3}{2}}}{24}, & \text{if } \frac{14}{3} \leq q < 6. \end{cases} \tag{1.9}$$

Hence, the crucial outstanding matter is how to ascertain the compactness threshold for the problem when $2 < q < \frac{10}{3}$ and $\frac{10}{3} \leq q < \frac{14}{3}$, and subsequently develop the appropriate energy estimates to mitigate the unavoidable competitive impact of the term $\|\nabla u\|_2^4$ in the functional Φ.

(iii) Even when the aforementioned difficulties can be addressed, establishing the compactness of (PS) sequences becomes more complicated compared to the case when $b = 0$. This is primarily due to the presence of the term $\|\nabla u\|_2^4$ in Φ, which implies that the weak convergence $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ does not guarantee the convergence

$$\|\nabla u_n\|_2^2 \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla\varphi \mathrm{d}x \to \|\nabla u\|_2^2 \int_{\mathbb{R}^3} \nabla u \cdot \nabla\varphi \mathrm{d}x \text{ for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3).$$

Consequently, when $b > 0$, it becomes even more intricate to rule out the possibility of vanishing and dichotomy for (PS) sequences, preventing its strong convergence in $H^1(\mathbb{R}^3)$.

1.2 Highlights of the paper and main results

Motivated by the aforementioned work, this paper aims to thoroughly investigate the existence and multiplicity of normalized solutions for (1.1), covering the complete range of subcritical perturbations within the interval $2 < q < 6$. In the study of (1.1), we classify the power q into three intervals: $2 < q < \frac{10}{3}$, $\frac{10}{3} \leq q < \frac{14}{3}$, and $\frac{14}{3} \leq q < 6$, taking into account the combined effect of Δu and $(\|\nabla u\|_2^2)\Delta u$. Notably, we use new analytical techniques and ideas to overcome the aforementioned challenges and address two open problems, denoted as (Q1) and (Q2) in Table 3, while also filling the research gap for the interval $\frac{10}{3} \leq q < \frac{14}{3}$. Specifically, for $\mu > 0$ and under suitable conditions on the mass c, we establish the following results:

(i) When $2 < q < \frac{10}{3}$, Φ exhibits a geometry of local minima on $\mathcal{S}_c$, suggesting the existence of an additional mountain pass geometry originating from the local minimizer.

(ii) When $\frac{10}{3} \leq q < \frac{14}{3}$ and $\frac{14}{3} \leq q < 6$, Φ possesses a mountain pass geometry on $\mathcal{S}_c$.

Table 4 Geometry of local minima

q	c $(\mu > 0)$	Type of solutions	Energy level
$2 < q < \frac{10}{3}$	$c \in (0, c_0)$	One local minimizer	$= \inf_{\mathcal{S}_c \cap A_{s_0}} \Phi < 0$
	$c \in (0, \min\{c_0, c_1\})$	A second solution of mountain pass type	$< \inf_{\mathcal{S}_c \cap A_{s_0}} \Psi + \Theta^*$

Table 5 Mountain pass geometry

q	c $(\mu > 0)$	Type of solutions	Energy level
$\frac{10}{3} < q < \frac{14}{3}$	$c \in (0, \min\{c_2, c_3\})$	A mountain pass type solution	$< \Theta^*$, defined by (1.9)
$\frac{14}{3} \le q < 6$	$c > 0$	A ground state solution	

Based on these observations, our research is divided into two parts, which are summarized in Tables 4 and 5. Additionally, we establish the non-existence result for $\mu \le 0$ and $2 < q < 6$.

Here the number Θ^* is defined by (1.9), the numbers s_0, c_0, c_1, c_2 and c_3 are defined by:

$$s_0 := \left[\frac{(10 - 3q)a\mathcal{S}^3}{6 - q} \right]^{\frac{1}{2}}, \tag{1.10}$$

$$c_0 := \left[\frac{4aq}{3(6 - q)\mu\mathcal{C}_q^q} \left(\frac{4q}{3(10 - 3q)\mu\mathcal{C}_q^q \mathcal{S}^3} \right)^{\frac{3q-10}{3(6-q)}} \right]^{\frac{3}{2}}, \tag{1.11}$$

$$c_1 := \left\{ \left[\frac{4q}{\mu(6 - q)\mathcal{C}_q^q} \right] \left[\left(\frac{a}{3} + \frac{b^2\mathcal{S}^3}{6} + \frac{b\mathcal{S}\sqrt{b^2\mathcal{S}^4 + 4(a + bs_0)\mathcal{S}}}{12} \right) s_0^{(10-3q)/4} + \frac{b}{12} s_0^{(14-3q)/4} \right] \right\}^{\frac{4}{6-q}}, \tag{1.12}$$

$$c_2 := \left(\frac{5a}{3\mu\mathcal{C}_{10/3}^{10/3}} \right)^{\frac{3}{2}} \tag{1.13}$$

and

$$c_3 := \left[\frac{4q}{\mu(6 - q)\mathcal{C}_q^q} \right]^{\frac{4}{6-q}} \left[\frac{b}{3(3q - 10)} \right]^{\frac{3q-10}{6-q}} \times \left[\frac{4}{14 - 3q} \left(\frac{a}{3} + \frac{b^2\mathcal{S}^3}{6} + \frac{b\mathcal{S}}{12} \sqrt{b^2 S^4 + 4a\mathcal{S}} \right) \right]^{\frac{14-3q}{6-q}}, \tag{1.14}$$

the functional $\Psi : H^1(\mathbb{R}^3) \to \mathbb{R}$ is defined by

$$\Psi(u) := \frac{(b^2\mathcal{S}^4 + 4a\mathcal{S})^{\frac{3}{2}}}{24}\left[\left(1 + \frac{4b}{b^2\mathcal{S}^3 + 4a}\|\nabla u\|_2^2\right)^{\frac{3}{2}} - 1\right] + \left(\frac{a}{2} + \frac{b^2\mathcal{S}^3}{4}\right)\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{6}\|u\|_6^6 - \frac{\mu}{q}\|u\|_q^q, \tag{1.15}$$

and the set A_ρ is defined by

$$A_\rho := \left\{u \in H^1(\mathbb{R}^3) : \|\nabla u\|_2^2 < \rho\right\},$$

where, and in the rest of the paper, $\mathcal{C}_s$, determined by s, denotes the best constant for the Gagliardo–Nirenberg inequality in $\mathbb{R}^3$ (see [1]),

$$\|u\|_s \le \mathcal{C}_s\|u\|_2^{(6-s)/2s}\|\nabla u\|_2^{3(s-2)/2s} \text{ for } 2 < s < 6. \text{ (Gagliardo-Nirenberg inequality)} \tag{1.16}$$

To state our main results, we define the L^2-Pohozaev functional

$$\mathcal{P}(u) = a\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x + b\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^2 - \int_{\mathbb{R}^3}u^6\mathrm{d}x - \frac{3\mu(q-2)}{2q}\int_{\mathbb{R}^3}|u|^q\mathrm{d}x. \tag{1.17}$$

It is well known that any solution to (1.1) belongs to the L^2-Pohozaev manifold defined by

$$\mathcal{M}(c) := \{u \in \mathcal{S}_c : \mathcal{P}(u) = 0\}. \tag{1.18}$$

We recall a solution u to be a ground state solution on $\mathcal{S}_c$ if u minimizes the functional Φ among all the solutions to (1.1), i.e.,

$$\Phi|'_{\mathcal{S}_c}(u) = 0 \text{ and } \Phi(u) = \inf\left\{\Phi(u) : \|u\|_2^2 = c, \ \Phi|'_{\mathcal{S}_c}(u) = 0\right\}.$$

Our results are as follows.

Theorem 1.1 *Let* $2 < q < \frac{10}{3}$, $\mu > 0$ *and* $c \in (0, c_0)$. *Then* (1.1) *has a couple solution* $(\tilde{u}_c, \tilde{\lambda}_c) \in (\mathcal{S}_c \cap H^1(\mathbb{R}^3)) \times (0, +\infty)$ *such that*

$$\tilde{u}_c \in \mathcal{S}_c \cap A_{s_0}, \quad \tilde{u}_c > 0, \quad \Phi(\tilde{u}_c) = m(c) := \inf_{\mathcal{S}_c \cap A_{s_0}} \Phi < 0. \tag{1.19}$$

Theorem 1.2 *Let* $2 < q < \frac{10}{3}$, $\mu > 0$ *and* $c \in (0, \min\{c_0, c_1\})$. *Then* (1.1) *has a second couple solution* $(\bar{u}_c, \bar{\lambda}_c) \in (\mathcal{S}_c \cap H^1_{rad}(\mathbb{R}^3)) \times (0, +\infty)$ *such that*

$$0 < \Phi(\bar{u}_c) < \inf_{\mathcal{S}_c \cap A_{s_0}} \Psi + \Theta^*. \tag{1.20}$$

Theorem 1.3 *Let* $\frac{10}{3} \leq q < \frac{14}{3}$, $\mu > 0$ *and* $c \in (0, \min\{c_2, c_3\})$. *Then* (1.1) *has a couple solution* $(\bar{u}_c, \lambda_c) \in (\mathcal{S}_c \cap H^1_{rad}(\mathbb{R}^3)) \times (0, +\infty)$ *such that*

$$0 < \Phi(\bar{u}_c) < \Theta^*. \tag{1.21}$$

Theorem 1.4 *Let* $\frac{14}{3} \leq q < 6$, $\mu > 0$ *and* $c \in (0, +\infty)$. *Then* (1.1) *has a couple solution* $(\bar{u}_c, \lambda_c) \in H^1(\mathbb{R}^3) \times (0, +\infty)$ *such that*

$$\Phi(\bar{u}_c) = \inf_{\mathcal{M}(c)} \Phi. \tag{1.22}$$

Theorem 1.5 *Let* $2 < q < 6$, $\mu \leq 0$ *and* $c \in (0, +\infty)$. *Then* (1.1) *has no solutions in* $H^1(\mathbb{R}^3) \times (0, +\infty)$.

Remark 1.6 Our research can be considered as a counterpart of the Brezis-Nirenberg problem in the context of normalized solutions to Kirchhoff equations, and appears to be a significant contribution in this regard. This is particularly noteworthy because our study covers the entire interval of $2 < q < 6$ with subcritical lower exponents. To be more specific, Theorems 1.2 and 1.3 address the open problems (Q1) and (Q2) mentioned in Table 2, respectively, while filling the research gap in the interval $\frac{10}{3} \leq q < \frac{14}{3}$. The statements highlighted in red in Tables 4 and 5 further illustrate this point. Additionally, Theorem 1.5 establishes the first result of nonexistence for (1.1) when $\mu < 0$.

Remark 1.7 (i) Our approach to constructing (PS) sequences in the proofs of Theorems 1.1–1.4 is fundamentally different from the work of [11]. It is based on several critical point theorems on manifolds that we have recently developed in [5] for the study of (1.7). Our method offers several advantages over Ghoussoub's minimax approach introduced in [7], as it is technically simpler and does not rely on the decomposition of Pohozaev manifolds. Consequently, it is applicable to a wider range of nonlinear terms.

(ii) From Theorem 1.2, one might wonder why it is necessary to introduce a new functional Ψ defined by (1.15). In fact, it plays a crucial role in proving the compactness of (PS) sequences. By using new analytical techniques and refined energy estimates, we establish rigorous inequalities concerning the energy levels, which are given as follows:

$$M(c) < \begin{cases} \inf_{\mathcal{S}_c \cap A_{s_0}} \Psi + \Theta^*, & \text{if } 2 < q < \frac{10}{3}, \\ \Theta^*, & \text{if } \frac{10}{3} \leq q < \frac{14}{3}, \\ \Theta^*, & \text{if } \frac{14}{3} \leq q < 6, \end{cases} \tag{1.23}$$

complementing the corresponding result from previous studies, namely the inequality (1.9). The right-hand side of the inequalities represents the compactness threshold of the problem, below which the (PS) condition holds. The derivation of these inequalities is one of the noteworthy highlights of this paper. The argument in the case where $2 < q < \frac{10}{3}$ is the most delicate, making it the key and pivotal element in proving Theorem 1.2.

Let us now highlight the key difficulties encountered and outline our research strategy for proving Theorem 1.2, which we believe is the most inspiring part of this paper.

Motivated by the results on (1.7) in Table 3, it is natural to expect that (1.1) has a second solution of the mountain pass type when $2 < q < \frac{10}{3}$. However, achieving this result poses the greatest challenge, as mentioned in Remark 1.7-(ii). Drawing upon our experience studying (1.7) in [5], we conjecture that the value $m(c) + \Theta^*$ may serve as a potential candidate for the compactness threshold in the case $2 < q < \frac{10}{3}$, where $m(c)$ is given by (1.19). Following our ideas in [5], in order to establish the strict inequality $M(c) < m(c) + \Theta^*$, we consider a superposition of the minimizer of $m(c)$ and the Aubin-Talenti bubbles associated with the Sobolev inequality, while ensuring that the resulting function remains constrained to $\mathcal{S}_c$ through appropriate technical modifications. The interplay between these components is expected to lead to a decrease in the corresponding energy value, ultimately yielding $M(c) < m(c) + \Theta^*$. Unfortunately, unlike in the study of (1.7), the additional term $\|\nabla u\|_2^4$ in Φ causes the energy value to exceed the anticipated compactness threshold. Specifically, considering $\Phi(u) := \phi(u) + \frac{b}{4}\|\nabla u\|_2^4$, we can observe from (1.24) that controlling the mountain pass level from above using $m(c) + \Theta^*$ is not feasible due to the presence of undesirable cross-term interferences:

$$
\begin{aligned}
\Phi(u_1 + u_2) = \phi(u_1 + u_2) + \frac{b}{4}\left(\|\nabla u_1\|_2^4 + \|\nabla u_2\|_2^4\right) \\
+b\left(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2 + \int_{\mathbb{R}^3} \nabla u_1 \cdot \nabla u_2 \mathrm{d}x\right) \int_{\mathbb{R}^3} \nabla u_1 \cdot \nabla u_2 \mathrm{d}x \\
+\frac{b}{2}\|\nabla u_1\|_2^2 \|\nabla u_2\|_2^2, \quad \forall\, u_1, u_2 \in H^1(\mathbb{R}^3). \qquad (1.24)
\end{aligned}
$$

This observation indicates that the aforementioned conjecture does not hold, necessitating the implementation of new ideas to address this problem. Precisely, instead of starting from the local minimizer of $m(c)$, we introduce the auxiliary functional Ψ and search for a local minimizer of Ψ as the first step, as follows:

Step 1: Prove the existence of $\hat{u}_c \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ such that $\Psi(\hat{u}_c) = \hat{m}(c) := \inf_{\mathcal{S}_c \cap A_{s_0}} \Psi$.

Step 2: Using the function $\hat{u}_c$ obtained in Step 1 as the starting point, construct a path set of the mountain pass type:

$$
\Gamma_c = \left\{\gamma \in \mathcal{C}([0,1], \mathcal{S}_c) : \gamma(0) = \hat{u}_c, \Phi(\gamma(1)) < 2m(c)\right\},
$$

and prove that for $c \in (0, c_0)$, there exists $\kappa > 0$ such that

$$M(c) = \inf_{\gamma \in \Gamma_c} \max_{t \in [0,1]} \Phi(\gamma(t)) \geq \kappa > \sup_{\gamma \in \Gamma_c} \max \{\Phi(\gamma(0)), \Phi(\gamma(1))\}.$$

Remarkably, the combination of this inequality and the next step will allow us to obtain a good (PS) sequence $\{u_n\} \subset \mathcal{S}_c$ such that

$$\Phi(u_n) \to M(c) \in (0, \hat{m}(c) + \Theta^*), \quad \Phi|'_{\mathcal{S}_c}(u_n) \to 0 \text{ and } \mathcal{P}(u_n) \to 0. \quad (1.25)$$

Step 3: For each $n \in \mathbb{N}$ and $t > 0$, construct a family of new sequences of testing functions restricted on $\mathcal{S}_c$:

$$W_{n,t}(x) := \sqrt{\tau}[\hat{u}_c(\tau x) + tU_n(\tau x)]$$

with $\tau = \tau_{n,t} := \|\hat{u}_c + tU_n\|_2/\sqrt{c}$ and $U_n(x) := \Theta_n(|x|)$ and

$$\Theta_n(r) = \sqrt[4]{3} \begin{cases} \sqrt{\frac{n}{1+n^2r^2}}, & 0 \leq r < 1; \\ \sqrt{\frac{n}{1+n^2}}(2-r), & 1 \leq r < 2; \\ 0, & r \geq 2, \end{cases}$$

and prove that

$$\Phi(W_{n,t}) < \Theta^* + \Psi(\hat{u}_c) - O\left(\frac{1}{\sqrt{n}}\right), \quad \forall\, t > 0.$$

This novel inequality allows us to find large two numbers $\bar{n} \in \mathbb{N}$ and $\hat{t} > 0$ such that

$$W_{\bar{n},0} = \hat{u}_c \text{ and } \Phi(W_{\bar{n},\hat{t}}) < 2m(c).$$

In this way, we find a suitable path $\gamma_{\bar{n}}(t) := W_{\bar{n},t\hat{t}}$ such that $\gamma_{\bar{n}} \in \Gamma_c$, and thus $M(c) \leq \max_{t\in[0,1]} \Phi(\gamma_{\bar{n}}(t)) < \hat{m}(c) + \Theta^*$, see Lemmas 3.11 and 3.12 for more details.

Step 4: Prove the compactness of the (PS) sequence $\{u_n\}$ obtained in (1.25). The boundedness of $\{u_n\}$ can be deduced from the additional property $\mathcal{P}(u_n) \to 0$. By contradiction and using the strict inequality $M(c) < \hat{m}(c) + \Theta^* < \Theta^*$, we establish two key elements: (i) excluding the possibility of vanishing, which implies the existence of $\bar{u} \in H^1_{\text{rad}}(\mathbb{R}^3)$ with $0 < \|\bar{u}\|_2^2 \leq c$ such that $u_n \rightharpoonup \bar{u}$ in $H^1_{\text{rad}}(\mathbb{R}^3)$; (ii) showing $\|\nabla(u_n - \bar{u})\|_2^2 \to 0$, which is necessary to verify that $\bar{u} \in \mathcal{S}_c$ is a second solution of (1.1). The proof of the former is not difficult since, if $\bar{u} = 0$, a standard argument yields $M(c) + o(1) = \Phi(u_n) \geq \Theta^*$, contradicting the strict inequality. The essential difficulty lies in deducing $\|\nabla(u_n - \bar{u})\|_2^2 \to 0$. To derive a contradiction with $M(c) < \hat{m}(c) + \Theta^*$, we need to establish the relationship between $\Phi(\bar{u})$, $\Psi(\bar{u})$, and Θ^* based on the definition of $\hat{m}(c)$. To accomplish this, we employ fresh analytical techniques by distinguishing two cases: $\|\nabla \bar{u}\|_2^2 < s_0$ and $\|\nabla \bar{u}\|_2^2 \geq s_0$. This process also sheds light on why the value $\hat{m}(c) + \Theta^*$ appears as the compactness threshold of the problem.

The paper is organized as follows. Section 2 is devoted to some preliminary results, which will be used in the rest of paper. In Sect. 3, we study the case $2 < q < \frac{10}{3}$, and give the proofs of Theorems 1.1 and 1.2. In Sect. 4, we study the case $\frac{10}{3} \leq q < \frac{14}{3}$, and finish the proof of Theorem 1.3. In Sect. 4, we study the case $\frac{14}{3} \leq q < 6$, and finish the proof of Theorem 1.4, moreover, Theorem 1.5 is proved in this section.

Throughout the paper, we make use of the following notations:

- $H^1_{\rm rad}(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) \mid u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^3\}$;
- $L^s(\mathbb{R}^3)(1 \leq s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s \mathrm{d}x\right)^{1/s}$;
- For any $u \in H^1(\mathbb{R}^3)$ and $t > 0$, we set $u_t(x) := u(tx)$;
- For any $x \in \mathbb{R}^3$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$ and $B_r = B_r(0)$;
- $C_1, C_2, \ldots$ denote positive constants possibly different in different places.

2 Preliminary results

Let H be a real Hilbert space whose norm and scalar product will be denoted respectively by $\|\cdot\|_H$ and $(\cdot,\cdot)_H$. Let E be a real Banach space with norm $\|\cdot\|_E$. We assume throughout this section that

$$E \hookrightarrow H \hookrightarrow E^* \tag{2.1}$$

with continuous injections, where E^* is the dual space of E. Thus H is identified with its dual space. We will always assume in the sequel that E and H are infinite dimensional spaces. We consider the manifold

$$M := \{u \in E : \|u\|_H = 1\}. \tag{2.2}$$

M is the trace of the unit sphere of H in E and is, in general, unbounded. Throughout the paper, M will be endowed with the topology inherited from E. Moreover M is a submanifold of E of codimension 1 and its tangent space at a given point $u \in M$ can be considered as a closed subspace of E of codimension 1, namely

$$T_u M := \{v \in E : (u, v)_H = 0\}. \tag{2.3}$$

We consider a functional $\varphi : E \to \mathbb{R}$ which is of class $\mathcal{C}^1$ on E. We denote by $\varphi|_M$ the trace of φ on M. Then $\varphi|_M$ is a $\mathcal{C}^1$ functional on M, and for any $u \in M$,

$$\langle \varphi|_M'(u), v\rangle = \langle \varphi'(u), v\rangle, \quad \forall\, v \in T_u M. \tag{2.4}$$

In the sequel, for any $u \in M$, we define the norm $\left\|\varphi|'_M(u)\right\|$ by

$$\left\|\varphi|'_M(u)\right\| = \sup_{v \in T_u M, \|v\|_E = 1} |\langle \varphi'(u), v \rangle|. \tag{2.5}$$

Let $E \times \mathbb{R}$ be equipped with the scalar product

$$((u,\tau),(v,\sigma))_{E\times\mathbb{R}} := (u,v)_H + \tau\sigma, \quad \forall\, (u,\tau),(v,\sigma) \in E \times \mathbb{R},$$

and corresponding norm

$$\|(u,\tau)\|_{E\times\mathbb{R}} := \sqrt{\|u\|_H^2 + \tau^2}, \quad \forall\, (u,\tau) \in E \times \mathbb{R}.$$

Next, we consider a functional $\tilde{\varphi} : E \times \mathbb{R} \to \mathbb{R}$ which is of class $\mathcal{C}^1$ on $E \times \mathbb{R}$. We denote by $\tilde{\varphi}|_{M\times\mathbb{R}}$ the trace of $\tilde{\varphi}$ on $M \times \mathbb{R}$. Then $\tilde{\varphi}|_{M\times\mathbb{R}}$ is a $\mathcal{C}^1$ functional on $M \times \mathbb{R}$, and for any $(u,\tau) \in M \times \mathbb{R}$,

$$\langle \tilde{\varphi}|'_{M\times\mathbb{R}}(u,\tau),(v,\sigma)\rangle := \langle \tilde{\varphi}'(u,\tau),(v,\sigma)\rangle, \quad \forall\, (v,\sigma) \in \tilde{T}_{(u,\tau)}(M\times\mathbb{R}), \tag{2.6}$$

where

$$\tilde{T}_{(u,\tau)}(M\times\mathbb{R}) := \{(v,\sigma) \in E \times \mathbb{R} : (u,v)_H = 0\}. \tag{2.7}$$

In the sequel, for any $(u,\tau) \in M \times \mathbb{R}$, we define the norm $\left\|\tilde{\varphi}|'_{M\times\mathbb{R}}(u,\tau)\right\|$ by

$$\left\|\tilde{\varphi}|'_{M\times\mathbb{R}}(u,\tau)\right\| = \sup_{(v,\sigma)\in\tilde{T}_{(u,\tau)}(M\times\mathbb{R}), \|(v,\sigma)\|_{E\times\mathbb{R}}=1} |\langle \tilde{\varphi}'(u,\tau),(v,\sigma)\rangle|. \tag{2.8}$$

Lemma 2.1 [5] *Let* $\varphi \in \mathcal{C}^1(E,\mathbb{R})$, $S \subset M$, $\tilde{a} \in \mathbb{R}$, $\varepsilon, \delta > 0$ *such that*

$$u \in M \cap \varphi^{-1}([\tilde{a}-2\varepsilon, \tilde{a}+2\varepsilon]) \cap S_{2\delta} \Rightarrow \left\|\varphi|'_M(u)\right\| \geq \frac{8\varepsilon}{\delta}. \tag{2.9}$$

Then, there exists $\eta \in \mathcal{C}([0,1]\times M, M)$ *such that*

(i) $\eta(t,u) = u$, *if* $t = 0$, *or if* $u \notin M \cap \varphi^{-1}([\tilde{a}-2\varepsilon, \tilde{a}+2\varepsilon]) \cap S_{2\delta}$;
(ii) $\eta\left(1, \varphi^{\tilde{a}+\varepsilon} \cap S\right) \subset \varphi^{\tilde{a}-\varepsilon}$;
(iii) *for every* $t \in [0,1]$, $\eta(t,\cdot) : M \to M$ *is a homeomorphism;*
(iv) $\|\eta(t,u) - u\| \leq \delta, \ \forall\, u \in M, \ t \in [0,1]$;
(v) *for every* $u \in M$, $\varphi(\eta(t,u))$ *is non-increasing on* $t \in [0,1]$;
(vi) $\varphi(\eta(t,u)) < \tilde{a}, \ \forall\, u \in M \cap \varphi^{\tilde{a}} \cap S_\delta, \ t \in [0,1]$.

Lemma 2.2 [3] *Let* $\{u_n\} \subset M$ *be a bounded sequence in* E. *Then the following are equivalent:*

(i) $\|\varphi|'_M(u_n)\| \to 0$ *as* $n \to \infty$;
(ii) $\varphi'(u_n) - \langle \varphi'(u_n), u_n\rangle u_n \to 0$ *in* E' *as* $n \to \infty$.

Lemma 2.3 [5] *Let $\varphi \in \mathcal{C}^1(E, \mathbb{R})$ and $K \subset E$. If there exists $\rho > 0$ such that*

$$\tilde{a} := \inf_{v \in M \cap K} \varphi(v) < \tilde{b} := \inf_{v \in M \cap (K_\rho \setminus K)} \varphi(v), \tag{2.10}$$

where $K_\rho := \{v \in E : \|v - u\|_E < \rho,\ u \in K\}$, then, for every $\varepsilon \in (0, (\tilde{b} - \tilde{a})/2)$, $\delta \in (0, \rho/2)$ and $w \in M \cap K$ such that

$$\varphi(w) \leq \tilde{a} + \varepsilon, \tag{2.11}$$

there exists $u \in M$ such that

(i) $\tilde{a} - 2\varepsilon \leq \varphi(u) \leq \tilde{a} + 2\varepsilon$;
(ii) $\|u - w\|_E \leq 2\delta$;
(iii) $\left\|\varphi|'_M(u)\right\| \leq 8\varepsilon/\delta$.

Corollary 2.4 [4] *Let $\varphi \in \mathcal{C}^1(E, \mathbb{R})$ and $K \subset E$. If there exist $\rho > 0$ and $\bar{u} \in M \cap K$ such that*

$$\varphi(\bar{u}) = \inf_{v \in M \cap K} \varphi(v) < \inf_{v \in M \cap (K_\rho \setminus K)} \varphi(v), \tag{2.12}$$

then $\varphi|'_M(\bar{u}) = 0$.

Lemma 2.5 [5] *Assume that $\theta_1, \theta_2 \in \mathbb{R}$ and $\tilde{\varphi} \in \mathcal{C}^1(E \times \mathbb{R}, \mathbb{R})$ satisfies*

$$\tilde{a} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{\varphi}(\tilde{\gamma}(t)) > \tilde{b} := \sup_{\tilde{\gamma} \in \tilde{\Gamma}} \max \left\{\tilde{\varphi}(\tilde{\gamma}(0)), \tilde{\varphi}(\tilde{\gamma}(1))\right\}, \tag{2.13}$$

where

$$\tilde{\Gamma} := \left\{\tilde{\gamma} \in \mathcal{C}([0,1], M \times \mathbb{R}) : \tilde{\varphi}(\tilde{\gamma}(0)) \leq \theta_1, \tilde{\varphi}(\tilde{\gamma}(1)) < \theta_2\right\}.$$

Let $\{\tilde{\gamma}_n\} \subset \tilde{\Gamma}$ be such that

$$\sup_{t \in [0,1]} \tilde{\varphi}(\tilde{\gamma}_n(t)) \leq \tilde{a} + \frac{1}{n}, \quad \forall\, n \in \mathbb{N}. \tag{2.14}$$

Then there exists a sequence $\{(v_n, \tau_n)\} \subset M \times \mathbb{R}$ satisfying

(i) $\tilde{a} - \frac{2}{n} \leq \tilde{\varphi}(v_n, \tau_n) \leq \tilde{a} + \frac{2}{n}$;
(ii) $\min_{t \in [0,1]} \|(v_n, \tau_n) - \tilde{\gamma}_n(t)\|_{E \times \mathbb{R}} \leq \frac{2}{\sqrt{n}}$;
(iii) $\left\|\tilde{\varphi}|'_{M \times \mathbb{R}}(v_n, \tau_n)\right\| \leq \frac{8}{\sqrt{n}}$.

3 The Case when $2 < q < \frac{10}{3}$

In this section, we study the case $2 < q < \frac{10}{3}$, and give the proofs of Theorems 1.1 and 1.2.

For any $c > 0$, we consider the function $g_c(s)$ defined on $s \in (0, +\infty)$ by

$$g_c(s) := \frac{a}{2} - \frac{\mu \mathcal{C}_q^q}{q} c^{(6-q)/4} s^{(3q-10)/4} - \frac{s^2}{6\mathcal{S}^3}. \tag{3.1}$$

By some simple calculations, we easily verify the following lemma.

Lemma 3.1 *There hold*

(i) $(1+st)^{\frac{3}{2}} - 1 \le t^{\frac{3}{2}} \left[(1+s)^{\frac{3}{2}} - 1\right], \quad \forall\, s \ge 0,\ t \ge 1;$

(ii) $(1+s+t)^{\frac{3}{2}} - 1 \ge \left[(1+s)^{\frac{3}{2}} - 1\right] + \left[(1+t)^{\frac{3}{2}} - 1\right], \quad \forall\, s, t \ge 0.$

Similar to [8, Lemma 2.1], we can prove the following lemma.

Lemma 3.2 *Let $2 < q < \frac{10}{3}$ and $\mu > 0$. Then for each $c > 0$, the function $g_c(s)$ has a unique global maximum and the maximum value satisfies*

$$\max_{0<s<+\infty} g_c(s) = g_c(s_c) \begin{cases} > 0, & \text{if } c < c_0, \\ = 0, & \text{if } c = c_0, \\ < 0, & \text{if } c > c_0, \end{cases} \tag{3.2}$$

where c_0 is defined by (1.11), and

$$s_c := \left[\frac{3(10-3q)\mu \mathcal{C}_q^q \mathcal{S}^3}{4q}\right]^{\frac{4}{3(6-q)}} c^{\frac{1}{3}}. \tag{3.3}$$

In particular, we have $s_{c_0} = s_0$.

Lemma 3.3 *Let $2 < q < \frac{10}{3}$ and $\mu > 0$. Then for each $c > 0$, we have that*

$$\Psi(u) \ge \Phi(u) \ge \|\nabla u\|_2^2\, g_c(\|\nabla u\|_2^2), \quad \forall\, u \in \mathcal{S}_c. \tag{3.4}$$

Proof From (1.2), (1.8), (1.15), (1.16) and (3.1), one has

$$\begin{aligned} \Psi(u) &\ge \Phi(u) \\ &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{6}\|u\|_6^6 - \frac{\mu}{q}\|u\|_q^q \\ &\ge \frac{a}{2}\|\nabla u\|_2^2 - \frac{1}{6\mathcal{S}^3}\|\nabla u\|_2^6 - \frac{\mu \mathcal{C}_q^q}{q} c^{(6-q)/4} \|\nabla u\|_2^{3(q-2)/2} \\ &= \|\nabla u\|_2^2\, g_c(\|\nabla u\|_2^2), \quad \forall\, u \in \mathcal{S}_c. \end{aligned}$$

□

Set

$$A_\rho := \left\{u \in H^1(\mathbb{R}^3) : \|\nabla u\|_2^2 < \rho\right\},$$
$$m(c) := \inf_{u \in \mathcal{S}_c \cap A_{s_0}} \Phi(u), \quad \hat{m}(c) := \inf_{u \in \mathcal{S}_c \cap A_{s_0}} \Psi(u).$$

Lemma 3.4 *Let $2 < q < \frac{10}{3}$ and $\mu > 0$. Then for any $c \in (0, c_0)$, the following properties hold,*

$$m(c) = \inf_{u \in \mathcal{S}_c \cap A_{s_0}} \Phi(u) < 0 < \inf_{u \in \partial(\mathcal{S}_c \cap A_{s_0})} \Phi(u) \tag{3.5}$$

and

$$\hat{m}(c) = \inf_{u \in \mathcal{S}_c \cap A_{s_0}} \Psi(u) < 0 < \inf_{u \in \partial(\mathcal{S}_c \cap A_{s_0})} \Psi(u). \tag{3.6}$$

Proof For any $u \in \mathcal{S}_c$, since $t^{3/2}u_t \in \mathcal{S}_c$ and $\|\nabla(t^{3/2}u_t)\|_2^2 = t^2\|\nabla u\|_2^2 < s_0$ for small $t > 0$, it follows that $t^{3/2}u_t \in \mathcal{S}_c \cap A_{s_0}$ for small $t > 0$. Furthermore, we have

$$\begin{aligned}
\Phi(t^{3/2}u_t) &\le \Psi(t^{3/2}u_t) \\
&= \frac{(b^2\mathcal{S}^4 + 4a\mathcal{S})^{\frac{3}{2}}}{24}\left[\left(1 + \frac{4bt^2}{b^2\mathcal{S}^3 + 4a}\|\nabla u\|_2^2\right)^{\frac{3}{2}} - 1\right] \\
&\quad + \left(\frac{a}{2} + \frac{b^2\mathcal{S}^3}{4}\right)t^2\|\nabla u\|_2^2 \\
&\quad + \frac{bt^4}{4}\|\nabla u\|_2^4 - \frac{t^6}{6}\|u\|_6^6 - \frac{\mu t^{3(q-2)/2}}{q}\|u\|_q^q \\
&\le \left(\frac{a}{2} + \frac{b^2\mathcal{S}^3}{4} + \frac{b\mathcal{S}\sqrt{b^2\mathcal{S}^4 + 4a\mathcal{S}}}{3}\right)t^2\|\nabla u\|_2^2 + \frac{bt^4}{4}\|\nabla u\|_2^4 \\
&\quad - \frac{t^6}{6}\|u\|_6^6 - \frac{\mu t^{3(q-2)/2}}{q}\|u\|_q^q < 0, \quad \text{for small} \quad t > 0,
\end{aligned} \tag{3.7}$$

due to $2 < q < \frac{10}{3}$. In the above second inequality, we have used the following fact:

$$(1 + s)^{\frac{3}{2}} \le 1 + 2s, \quad \text{for small} \quad s > 0.$$

(3.7) shows that $\inf_{u \in \mathcal{S}_c \cap A_{s_0}} \Phi(u) \le \inf_{u \in \mathcal{S}_c \cap A_{s_0}} \Psi(u) < 0$. Therefore, (3.5) and (3.6) follow from Lemmas 3.2 and 3.3. □

Lemma 3.5 *Let $2 < q < \frac{10}{3}$ and $\mu > 0$. Then it holds that*

(i) *Let $c \in (0, c_0)$. Then for all $\alpha \in (0, c)$, we have $m(c) \le m(\alpha) + m(c - \alpha)$, and if $m(\alpha)$ or $m(c - \alpha)$ is reached then the inequality is strict.*

(ii) *The function* $c \mapsto m(c)$ *is continuous on* $(0, c_0)$.

Lemma 3.5 can be proved by the similar arguments as the following lemma, so we omit it.

Lemma 3.6 *Let* $2 < q < \frac{10}{3}$ *and* $\mu > 0$. *Then it holds that*

(i) *Let* $c \in (0, c_0)$. *Then for all* $\alpha \in (0, c)$, *we have* $\hat{m}(c) \le \hat{m}(\alpha) + \hat{m}(c - \alpha)$, *and if* $\hat{m}(\alpha)$ *or* $\hat{m}(c - \alpha)$ *is reached then the inequality is strict.*
(ii) *The function* $c \mapsto \hat{m}(c)$ *is continuous on* $(0, c_0)$.

Proof (i) Fix $\alpha \in (0, c)$. By (3.1) and (3.2), we have

$$\begin{aligned} g_\alpha\left(\frac{\theta\alpha}{c}s_0\right) &= \frac{a}{2} - \frac{\mu C_q^q}{q}\alpha^{(6-q)/4}\left(\frac{\theta\alpha}{c}s_0\right)^{(3q-10)/4} - \frac{(\theta\alpha)^2 s_0^2}{6c^2\mathcal{S}^3} \\ &\ge \frac{a}{2} - \frac{\mu C_q^q}{q}\left(\frac{\alpha}{c}\right)^{(q-2)/2} c^{(6-q)/4} s_0^{(3q-10)/4} - \frac{(\theta\alpha)^2 s_0^2}{6c^2\mathcal{S}^3} \\ &\ge g_c(s_0) = g_c(s_{c_0}) > g_{c_0}(s_{c_0}) = 0, \quad \forall\, \theta \in [1, c/\alpha]. \end{aligned} \tag{3.8}$$

Let $\{u_n\} \subset \mathcal{S}_\alpha \cap A_{s_0}$ be such that $\lim_{n\to\infty} \Psi(u_n) = \hat{m}(\alpha)$. Since $\hat{m}(\alpha) < 0$, it follows from (3.4) that for large $n \in \mathbb{N}$,

$$0 > \Psi(u_n) \ge \|\nabla u_n\|_2^2 g_\alpha(\|\nabla u_n\|_2^2),$$

which, together with (3.8), implies that for large $n \in \mathbb{N}$,

$$\|\nabla u_n\|_2^2 < \frac{\alpha}{c}s_0. \tag{3.9}$$

For any $\theta \in (1, c/\alpha]$. Set $v_n(x) := u_n(\theta^{-1/3}x)$. Then $\|v_n\|_2^2 = \theta\|u_n\|_2^2 = \theta\alpha$, $\|v_n\|_p^p = \theta\|u_n\|_p^p$ for $2 \le p \le 6$, and

$$\|\nabla v_n\|_2^2 = \theta^{\frac{1}{3}}\|\nabla u_n\|_2^2 \le \left(\frac{c}{\alpha}\right)^{\frac{1}{3}}\frac{\alpha}{c}s_0 < s_0. \tag{3.10}$$

Hence, it follows from (1.15), (3.6), (3.10) and Lemma 3.1 (i) that

$$\begin{aligned} \hat{m}(\theta\alpha) &\le \Psi(v_n) \\ &= \frac{(b^2\mathcal{S}^4 + 4a\mathcal{S})^{\frac{3}{2}}}{24}\left[\left(1 + \frac{4b\theta^{\frac{1}{3}}}{b^2\mathcal{S}^3 + 4a}\|\nabla u_n\|_2^2\right)^{\frac{3}{2}} - 1\right] \\ &\quad + \left(\frac{a}{2} + \frac{b^2\mathcal{S}^3}{4}\right)\theta^{\frac{1}{3}}\|\nabla u_n\|_2^2 \\ &\quad + \frac{b\theta^{\frac{2}{3}}}{4}\|\nabla u_n\|_2^4 - \frac{\theta}{6}\|u_n\|_6^6 - \frac{\mu\theta}{q}\|u_n\|_q^q \end{aligned}$$

$$
\begin{aligned}
&< \frac{\theta(b^2\mathcal{S}^4+4a\mathcal{S})^{\frac{3}{2}}}{24}\left[\left(1+\frac{4b}{b^2\mathcal{S}^3+4a}\|\nabla u_n\|_2^2\right)^{\frac{3}{2}}-1\right] \\
&\quad +\theta\left(\frac{a}{2}+\frac{b^2\mathcal{S}^3}{4}\right)\|\nabla u_n\|_2^2 \\
&\quad +\frac{b\theta}{4}\|\nabla u_n\|_2^4-\frac{\theta}{6}\|u_n\|_6^6-\frac{\mu\theta}{q}\|u_n\|_q^q \\
&= \theta\Psi(u_n)=\theta\hat{m}(\alpha)+o(1),
\end{aligned}
\tag{3.11}
$$

which implies that

$$
\theta\in\left(1,\frac{c}{\alpha}\right]\Rightarrow\hat{m}(\theta\alpha)\le\theta\hat{m}(\alpha). \tag{3.12}
$$

If $\hat{m}(\alpha)$ is reached by $u\in\mathcal{S}_\alpha\cap A_{s_0}$, then we choose $u_n\equiv u$ in (3.11), and thus the strict inequality follows. Hence, it follows from (3.12) that

$$
\hat{m}(c)=\frac{c-\alpha}{c}\hat{m}(c)+\frac{\alpha}{c}\hat{m}(c)\le\hat{m}(c-\alpha)+\hat{m}(\alpha),
$$

with a strict inequality if $\hat{m}(\alpha)$ or $\hat{m}(c-\alpha)$ is reached.

(ii) Let $c\in(0,c_0)$ be arbitrary and $\{\tilde{c}_n\}\subset(0,c_0)$ be such that $\tilde{c}_n\to c$. For any $\alpha\in(0,c_0)$, by the definition of $\hat{m}(\alpha)$ and Lemma 3.4, one has $\hat{m}(\alpha)<0$. If $\tilde{c}_n<c$, then it follows from (i) that

$$
\hat{m}(c)\le\hat{m}(\tilde{c}_n)+\hat{m}(c-\tilde{c}_n)<\hat{m}(\tilde{c}_n). \tag{3.13}
$$

If $\tilde{c}_n\ge c$, we let $u_n\in\mathcal{S}_{\tilde{c}_n}\cap A_{s_0}$ be such that $\Psi(u_n)\le\hat{m}(\tilde{c}_n)+\frac{1}{n}$. Set $v_n=\sqrt{\frac{c}{\tilde{c}_n}}u_n$. Then $v_n\in\mathcal{S}_c\cap A_{s_0}$. Furthermore, we have

$$
\begin{aligned}
\hat{m}(c)&\le\Psi(v_n)=\Psi(u_n)+[\Psi(v_n)-\Psi(u_n)] \\
&=\Psi(u_n) \\
&\quad +\frac{(b^2\mathcal{S}^4+4a\mathcal{S})^{\frac{3}{2}}}{24}\left[\left(1+\frac{4bc\|\nabla u_n\|_2^2}{\tilde{c}_n\left(b^2\mathcal{S}^3+4a\right)}\right)^{\frac{3}{2}}-\left(1+\frac{4b\|\nabla u_n\|_2^2}{b^2\mathcal{S}^3+4a}\right)^{\frac{3}{2}}\right] \\
&\quad +\left(\frac{a}{2}+\frac{b^2\mathcal{S}^3}{4}\right)\frac{c-\tilde{c}_n}{\tilde{c}_n}\|\nabla u_n\|_2^2+\frac{b\left(c^2-\tilde{c}_n^2\right)}{4\tilde{c}_n^2}\|\nabla u_n\|_2^4 \\
&\quad -\frac{c^3-\tilde{c}_n^3}{6\tilde{c}_n^3}\|u_n\|_6^6-\frac{\mu\left(c^{q/2}-\tilde{c}_n^{q/2}\right)}{q\tilde{c}_n^{q/2}}\|u_n\|_q^q \\
&=\Psi(u_n)+o(1)\le\hat{m}(\tilde{c}_n)+o(1).
\end{aligned}
\tag{3.14}
$$

Combining (3.13) with (3.14), we have

$$\hat{m}(c) \leq \hat{m}(\tilde{c}_n) + o(1). \tag{3.15}$$

Now, for any $\varepsilon > 0$ sufficiently small, there exists $u \in \mathcal{S}_c \cap A_{s_0}$ such that

$$\Psi(u) < \hat{m}(c) + \varepsilon. \tag{3.16}$$

Set $w_n = \sqrt{\frac{\tilde{c}_n}{c}}u$. Then $w_n \in \mathcal{S}_{\tilde{c}_n} \cap A_{s_0}$ for n large enough. Since $\Psi(w_n) \to \Psi(u)$, then

$$\hat{m}(\tilde{c}_n) \leq \Psi(w_n) = \Psi(u) + [\Psi(w_n) - \Psi(u)] = \Psi(u) + o(1) < \hat{m}(c) + \varepsilon + o(1).$$

Therefore, since $\varepsilon > 0$ is arbitrary, we deduce that $\hat{m}(\tilde{c}_n) \to \hat{m}(c)$ from the above inequality and (3.15). □

Proof of Theorem 1.1 Let $\{u_n\} \subset \mathcal{S}_c \cap A_{s_0}$ be a minimizing sequence for $m(c)$. Since $\{|u_n|\} \subset \mathcal{S}_c \cap A_{s_0}$ is also a minimizing sequence for $m(c)$, so we can assume that $u_n \geq 0$. Then by Lemma 3.4, we have

$$\|u_n\|_2^2 = c, \quad \|\nabla u_n\|_2^2 < s_0 < +\infty, \quad \Phi(u_n) = m(c) + o(1) < 0. \tag{3.17}$$

To obtain the existence of solutions for (1.1), we split the proof into several steps.

Step 1. Set $\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 \mathrm{d}x$. If $\delta = 0$, then by Lions' concentration compactness principle [17, Lemma 1.21], we have $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$. It follows that

$$\int_{\mathbb{R}^3} |u_n|^q \mathrm{d}x = o(1). \tag{3.18}$$

From (1.2), (1.8), (3.1), (3.2), (3.17) and (3.18), one has

$$\begin{aligned} m(c) + o(1) &= \frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{\mu}{q}\|u_n\|_q^q - \frac{1}{6}\|u_n\|_6^6 \\ &\geq \frac{a}{2}\|\nabla u_n\|_2^2 - \frac{1}{6\mathcal{S}^3}\|\nabla u_n\|_2^6 + o(1) \\ &\geq \|\nabla u_n\|_2^2 \left(\frac{a}{2} - \frac{s_0^2}{6\mathcal{S}^3}\right) + o(1) \\ &= \|\nabla u_n\|_2^2 \left[g_c(s_0) + \frac{\mu C_q^q}{q} c^{(6-q)/4} s_0^{(3q-10)/2}\right] + o(1) \\ &\geq o(1). \end{aligned} \tag{3.19}$$

This contradiction shows that $\delta > 0$ due to $m(c) < 0$.

Going if necessary to a subsequence, we may assume the existence of $y_n \in \mathbb{R}^3$ such that

$$\int_{B_1(y_n)} |u_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{3.20}$$

Let $\tilde{u}_n(x) = u_n(x + y_n)$. Then

$$\int_{B_1(0)} |\tilde{u}_n|^2 \mathrm{d}x > \frac{\delta}{2}, \tag{3.21}$$

and so there exists $\tilde{u} \in H^1(\mathbb{R}^3)\backslash\{0\}$ with $\tilde{u} \geq 0$ such that, passing to a subsequence,

$$\tilde{u}_n \rightharpoonup \tilde{u} \ \text{ in } H^1(\mathbb{R}^3), \quad \tilde{u}_n \to \tilde{u} \ \text{ in } L^s_{\mathrm{loc}}(\mathbb{R}^3) \text{ for } s \in (1, 6), \quad \tilde{u}_n \to \tilde{u} \text{ a.e. on } \mathbb{R}^3. \tag{3.22}$$

Moreover, (3.17) gives

$$0 < \|\tilde{u}\|_2^2 \leq \|\tilde{u}_n\|_2^2 = c, \quad \|\nabla \tilde{u}_n\|_2^2 < s_0, \quad \Phi(\tilde{u}_n) = m(c) + o(1). \tag{3.23}$$

Step 2. Set $v_n := \tilde{u}_n - \tilde{u}$. By (3.22), we have

$$\|\nabla \tilde{u}_n\|_2^2 = \|\nabla \tilde{u}\|_2^2 + \|\nabla v_n\|_2^2 + o(1) \tag{3.24}$$

and

$$\|\nabla \tilde{u}_n\|_2^4 = \|\nabla \tilde{u}\|_2^4 + \|\nabla v_n\|_2^4 + 2\|\nabla \tilde{u}\|_2^2 \|\nabla v_n\|_2^2 + o(1). \tag{3.25}$$

Hence, by (1.2), (3.24), (3.25) and the Brezis–Lieb lemma, we have

$$\Phi(\tilde{u}_n) = \Phi(\tilde{u}) + \Phi(v_n) + \frac{b}{2}\|\nabla \tilde{u}\|_2^2 \|\nabla v_n\|_2^2 + o(1). \tag{3.26}$$

Step 3. By (3.22) and (3.23), we have

$$\|v_n\|_2^2 = \|\tilde{u}_n\|_2^2 - \|\tilde{u}\|_2^2 + o(1) = c - \|\tilde{u}\|_2^2 + o(1). \tag{3.27}$$

Now, we claim that $\|v_n\|_2^2 \to 0$. In order to prove this, let us denote $\tilde{c} := \|\tilde{u}\|_2^2 > 0$. By (3.27), if we show that $\tilde{c} = c$ then the claim follows. We assume by contradiction that $\tilde{c} < c$. In view of (3.24) and (3.27), for $n \in \mathbb{N}$ large enough, we have

$$\alpha_n := \|v_n\|_2^2 \leq c, \quad \|\nabla v_n\|_2^2 \leq \|\nabla \tilde{u}_n\|_2^2 < s_0. \tag{3.28}$$

Hence, we obtain that

$$v_n \in \mathcal{S}_{\alpha_n} \cap A_{s_0}, \quad \Phi(v_n) \geq m(\alpha_n) := \inf_{u \in \mathcal{S}_{\alpha_n} \cap A_{s_0}} \Phi(u). \tag{3.29}$$

From (3.23), (3.26) and (3.29), we have

$$
\begin{aligned}
m(c)+o(1)=\Phi(\tilde{u}_n) &= \Phi(\tilde{u})+\Phi(v_n)+\frac{b}{2}\|\nabla\tilde{u}\|_2^2\|\nabla v_n\|_2^2+o(1) \\
&\geq \Phi(\tilde{u})+m(\alpha_n)+\frac{b}{2}\|\nabla\tilde{u}\|_2^2\|\nabla v_n\|_2^2+o(1). \qquad (3.30)
\end{aligned}
$$

Since the map $c\mapsto m(c)$ is continuous (see Lemma 3.5 (ii)) and in view of (3.27), we deduce

$$
m(c)\geq \Phi(\tilde{u})+m(c-\tilde{c}). \qquad (3.31)
$$

We also have that $\tilde{u}\in\mathcal{S}_{\tilde{c}}\cap\overline{A_{s_0}}$ by the weak limit. This implies that $\Phi(\tilde{u})\geq m(\tilde{c})$. If $\Phi(\tilde{u})>m(\tilde{c})$, then it follows from (3.31) and Lemma 3.5 (i) that

$$
m(c)>m(\tilde{c})+m(c-\tilde{c})\geq m(c),
$$

which is impossible. Hence, we have $\Phi(\tilde{u})=m(\tilde{c})$. So, using Lemma 3.5 (i) with the strict inequality, we deduce from (3.31) that

$$
m(c)\geq m(\tilde{c})+m(c-\tilde{c})>m(c),
$$

which is impossible. Thus, the claim follows and from (3.27) we deduce that $\|\tilde{u}\|_2^2=c$ and so $\tilde{u}\in\mathcal{S}_c\cap\overline{A_{s_0}}$ by the weak limit. It follows from (1.8), (3.1), (3.2), (3.23), (3.26) and $\Phi(\tilde{u})\geq m(c)$ that

$$
\begin{aligned}
o(1) &\geq \frac{a}{2}\|\nabla v_n\|_2^2+\frac{b}{4}\|\nabla v_n\|_2^4-\frac{\mu}{q}\|v_n\|_q^q-\frac{1}{6}\|v_n\|_6^6+\frac{b}{2}\|\nabla\tilde{u}\|_2^2\|\nabla v_n\|_2^2 \\
&\geq \frac{a}{2}\|\nabla v_n\|_2^2-\frac{1}{6\mathcal{S}^3}\|\nabla v_n\|_2^6+o(1) \\
&\geq \|\nabla v_n\|_2^2\left(\frac{a}{2}-\frac{s_0^2}{6\mathcal{S}^3}\right)+o(1) \\
&= \|\nabla v_n\|_2^2\left[g_c(s_0)+\frac{\mu C_q^q}{q}c^{(6-q)/4}s_0^{(3q-10)/2}\right]+o(1). \qquad (3.32)
\end{aligned}
$$

It follows from that $\|\nabla v_n\|_2^2=o(1)$. Since $\|v_n\|_2^2=o(1)$, we have $\tilde{u}_n\to\tilde{u}$ in $H^1(\mathbb{R}^3)$. Hence,

$$
\|\tilde{u}\|_2^2=c,\quad \|\nabla\tilde{u}\|_2^2\leq s_0,\quad \Phi(\tilde{u})=m(c),
$$

which, together with Lemma 3.4, implies $\|\nabla\tilde{u}\|_2^2<s_0$. Hence, Corollary 2.4 implies that $\Phi|'_{\mathcal{S}_c}(\tilde{u})=0$, and so there exists a Lagrange multiplier $\tilde{\lambda}_c\in\mathbb{R}$ such that

$$
-\left(a+b\|\nabla\tilde{u}\|_2^2\right)\Delta\tilde{u}+\tilde{\lambda}_c\tilde{u}=\tilde{u}^5+\mu|\tilde{u}|^{q-2}\tilde{u},\ \ x\in\mathbb{R}^3.
$$

It is easy to verify that $\tilde{\lambda}_c > 0$. Since $\tilde{u} \geq 0$ and $\tilde{u} \neq 0$, the strong maximum principle implies that $\tilde{u} > 0$. □

Lemma 3.7 *Let $2 < q < \frac{10}{3}$, $\mu > 0$ and $c \in (0, c_0)$. Then $\hat{m}(c)$ is reached by a positive, radially symmetric function, denoted $\hat{u}_c \in \mathcal{S}_c \cap A_{s_0}$ that satisfies, for a $\lambda_c \in \mathbb{R}$,*

$$
\begin{aligned}
&-\left[a + \frac{b^2\mathcal{S}^3}{2} + b\|\nabla \hat{u}_c\|_2^2 + \frac{b\mathcal{S}}{2}\sqrt{b^2\mathcal{S}^4 + 4\left(a + b\|\nabla \hat{u}_c\|_2^2\right)\mathcal{S}}\right]\Delta \hat{u}_c + \lambda_c \hat{u}_c \\
&\quad = \hat{u}_c^5 + \mu|\hat{u}_c|^{q-2}\hat{u}_c. \qquad (3.33)
\end{aligned}
$$

Proof Let $\{u_n\} \subset \mathcal{S}_c \cap A_{s_0}$ be a minimizing sequence for $\hat{m}(c)$. It is not restrictive to assume that $\{u_n\}$ is radially decreasing for every n (if this is not the case, we can replace u_n with $|u_n|^*$, the Schwarz rearrangement of $|u_n|$). Then by Lemma 3.4, we have

$$
\|u_n\|_2^2 = c, \quad \|\nabla u_n\|_2^2 < s_0, \quad \Psi(u_n) = \hat{m}(c) + o(1) < 0. \qquad (3.34)
$$

Since $\{u_n\} \subset H^1_{\mathrm{rad}}(\mathbb{R}^3)$ is bounded, we may thus assume, passing to a subsequence if necessary, that

$$
\begin{cases}
u_n \rightharpoonup \hat{u}, & \text{in } H^1_{\mathrm{rad}}(\mathbb{R}^3); \\
u_n \to \hat{u}, & \text{in } L^s(\mathbb{R}^3), \ \forall\, s \in (2, 6); \\
u_n \to \hat{u}, & \text{a.e. on } \mathbb{R}^3.
\end{cases} \qquad (3.35)
$$

To prove the lemma, we split the proof into several steps.

Step 1. $\hat{u} \neq 0$. Otherwise, we have $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $s \in (2, 6)$. It follows that

$$
\int_{\mathbb{R}^3} |u_n|^q \mathrm{d}x = o(1). \qquad (3.36)
$$

From (1.8), (1.15), (3.1), (3.2), (3.34), (3.35) and (3.36), one has

$$
\begin{aligned}
\hat{m}(c) + o(1) &= \Psi(u_n) \\
&\geq \frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{1}{6}\|u_n\|_6^6 - \frac{\mu}{q}\|u_n\|_q^q \\
&\geq \frac{a}{2}\|\nabla u_n\|_2^2 - \frac{1}{6\mathcal{S}^3}\|\nabla u_n\|_2^6 + o(1) \\
&\geq \|\nabla u_n\|_2^2\left(\frac{a}{2} - \frac{s_0^2}{6\mathcal{S}^3}\right) + o(1) \\
&= \|\nabla u_n\|_2^2\left[g_c(s_0) + \frac{\mu\mathcal{C}_q^q}{q}c^{(6-q)/4}s_0^{(3q-10)/2}\right] + o(1) \\
&\geq o(1). \qquad (3.37)
\end{aligned}
$$

This contradiction shows that $\hat{u} \neq 0$ due to $\hat{m}(c) < 0$.

Step 2. Set $v_n := u_n - \hat{u}$. By (3.35), we have

$$\|\nabla u_n\|_2^2 = \|\nabla \hat{u}\|_2^2 + \|\nabla v_n\|_2^2 + o(1) \tag{3.38}$$

and

$$\|\nabla u_n\|_2^4 = \|\nabla \hat{u}\|_2^4 + \|\nabla v_n\|_2^4 + 2\|\nabla \hat{u}\|_2^2\|\nabla v_n\|_2^2 + o(1). \tag{3.39}$$

Hence, by (1.15), (3.38), (3.39), Lemma 3.1 (ii) and the Brezis-Lieb lemma, we have

$$\Psi(u_n) \geq \Psi(\hat{u}) + \Psi(v_n) + \frac{b}{2}\|\nabla \hat{u}\|_2^2\|\nabla v_n\|_2^2 + o(1). \tag{3.40}$$

Step 3. By (3.34) and (3.35), we have

$$\|v_n\|_2^2 = \|u_n\|_2^2 - \|\hat{u}\|_2^2 + o(1) = c - \|\hat{u}\|_2^2 + o(1). \tag{3.41}$$

Now, we claim that $\|v_n\|_2^2 \to 0$. In order to prove this, let us denote $\tilde{c} := \|\hat{u}\|_2^2 > 0$. By (3.41), if we show that $\tilde{c} = c$ then the claim follows. We assume by contradiction that $\tilde{c} < c$. In view of (3.38) and (3.41), for $n \in \mathbb{N}$ large enough, we have

$$\alpha_n := \|v_n\|_2^2 \leq c, \quad \|\nabla v_n\|_2^2 \leq \|\nabla u_n\|_2^2 < s_0. \tag{3.42}$$

Hence, we obtain that

$$v_n \in \mathcal{S}_{\alpha_n} \cap A_{s_0}, \quad \Psi(v_n) \geq \hat{m}(\alpha_n) := \inf_{u \in \mathcal{S}_{\alpha_n} \cap A_{s_0}} \Psi(u). \tag{3.43}$$

From (3.34), (3.40) and (3.43), we have

$$\begin{aligned} \hat{m}(c) + o(1) &= \Psi(u_n) \\ &\geq \Psi(\hat{u}) + \Psi(v_n) + \frac{b}{2}\|\nabla \hat{u}\|_2^2\|\nabla v_n\|_2^2 + o(1) \\ &\geq \Psi(\hat{u}) + \hat{m}(\alpha_n) + \frac{b}{2}\|\nabla \hat{u}\|_2^2\|\nabla v_n\|_2^2 + o(1). \end{aligned} \tag{3.44}$$

Since the map $c \mapsto \hat{m}(c)$ is continuous (see Lemma 3.6 (ii)) and (3.41), we deduce

$$\hat{m}(c) \geq \Psi(\hat{u}) + \hat{m}(c - \tilde{c}). \tag{3.45}$$

We also have that $\hat{u} \in \mathcal{S}_{\tilde{c}} \cap \overline{A_{s_0}}$ by the weak limit. This implies that $\Psi(\hat{u}) \geq \hat{m}(\tilde{c})$. If $\Psi(\hat{u}) > \hat{m}(\tilde{c})$, then

$$\hat{m}(c) > \hat{m}(\tilde{c}) + \hat{m}(c - \tilde{c}) \geq \hat{m}(c),$$

which is impossible. Hence, we have $\Psi(\hat{u}) = \hat{m}(\tilde{c})$. So, using Lemma 3.6 (i) with the strict inequality, we deduce from (3.45) that

$$\hat{m}(c) \geq \hat{m}(\tilde{c}) + \hat{m}(c - \tilde{c}) > \hat{m}(c),$$

which is impossible. Thus, the claim follows and from (3.41) we deduce that $\|\hat{u}\|_2^2 = c$ and so $\hat{u} \in \mathcal{S}_c \cap \overline{A_{s_0}}$ by the weak limit. It follows from (1.8), (3.1), (3.2), (3.34), (3.40), (3.41) and $\Psi(\hat{u}) \geq \hat{m}(c)$ that

$$\begin{aligned} o(1) &\geq \Psi(v_n) + \frac{b}{2}\|\nabla\hat{u}\|_2^2\|\nabla v_n\|_2^2 + o(1) \\ &\geq \frac{a}{2}\|\nabla v_n\|_2^2 + \frac{b}{4}\|\nabla v_n\|_2^4 - \frac{\mu}{q}\|v_n\|_q^q - \frac{1}{6}\|v_n\|_6^6 \\ &\geq \frac{a}{2}\|\nabla v_n\|_2^2 - \frac{1}{6\mathcal{S}^3}\|\nabla v_n\|_2^6 + o(1) \\ &\geq \|\nabla v_n\|_2^2\left(\frac{a}{2} - \frac{s_0^2}{6\mathcal{S}^3}\right) + o(1) \\ &= \|\nabla v_n\|_2^2\left[g_c(s_0) + \frac{\mu \mathcal{C}_q^q}{q}c^{(6-q)/4}s_0^{(3q-10)/2}\right] + o(1). \end{aligned} \tag{3.46}$$

It follows from that $\|\nabla v_n\|_2^2 = o(1)$. Since $\|v_n\|_2^2 = o(1)$, we have $u_n \to \hat{u}$ in $H^1_{\text{rad}}(\mathbb{R}^3)$. Hence,

$$\|\hat{u}\|_2^2 = c, \quad \|\nabla\hat{u}\|_2^2 \leq s_0, \quad \Psi(\hat{u}) = \hat{m}(c),$$

which, together with Lemma 3.4, implies $\|\nabla\hat{u}\|_2^2 < s_0$. Hence, Corollary 2.4 implies that $\Psi|'_{\mathcal{S}_c}(\hat{u}) = 0$, and so there exists a Lagrange multiplier $\lambda_c \in \mathbb{R}$ such that $\Psi'(\hat{u}) + \lambda_c\hat{u} = 0$, which implies (3.33) holds with $\hat{u}_c = \hat{u}$. Since $\hat{u}_c \geq 0$ and $\hat{u}_c \neq 0$, the strong maximum principle implies that $\hat{u}_c > 0$. □

Since $\Psi'(\hat{u}_c) + \lambda_c\hat{u}_c = 0$, by a standard argument, we have the following lemma immediately.

Lemma 3.8 *Let* $2 < q < \frac{10}{3}$, $\mu > 0$ *and* $c \in (0, c_0)$. *Then there holds*

$$\begin{aligned} &\left[a + \frac{b^2\mathcal{S}^3}{2} + \frac{b\mathcal{S}}{2}\sqrt{b^2\mathcal{S}^4 + 4\left(a + b\|\nabla\hat{u}_c\|_2^2\right)\mathcal{S}}\right]\|\nabla\hat{u}_c\|_2^2 \\ &\quad + b\|\nabla\hat{u}_c\|_2^4 - \|\hat{u}_c\|_6^6 - \frac{3\mu(q-2)}{2q}\|\hat{u}_c\|_q^q = 0. \end{aligned} \tag{3.47}$$

To apply Lemma 2.5, we let $E = H^1_{\text{rad}}(\mathbb{R}^3)$ and $H = L^2(\mathbb{R}^3)$. Define the norms of E and H by

$$\|u\|_E := \left[\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x\right]^{1/2}, \quad \|u\|_H := \frac{1}{\sqrt{c}} \left(\int_{\mathbb{R}^3} u^2 \mathrm{d}x\right)^{1/2}, \quad \forall\, u \in E. \tag{3.48}$$

After identifying H with its dual, we have $E \hookrightarrow H \hookrightarrow E^*$ with continuous injections. Set

$$M := \left\{u \in E : \|u\|_2^2 = \int_{\mathbb{R}^3} u^2 \mathrm{d}x = c\right\}. \tag{3.49}$$

Let us define a continuous map $\beta : H^1_{\mathrm{rad}}(\mathbb{R}^3) \times \mathbb{R} \to H^1(\mathbb{R}^3)$ by

$$\beta(v,t)(x) := e^{3t/2} v(e^t x) \;\text{ for } v \in H^1_{\mathrm{rad}}(\mathbb{R}^3), \quad \forall\, t \in \mathbb{R},\; x \in \mathbb{R}^3, \tag{3.50}$$

and consider the following auxiliary functional:

$$\begin{aligned}\tilde{\Phi}(v,t) &:= \Phi(\beta(v,t)) \\ &= \frac{ae^{2t}}{2}\|\nabla v\|_2^2 + \frac{be^{4t}}{4}\|\nabla v\|_2^4 - \frac{e^{6t}}{6}\|v\|_6^6 - \frac{\mu e^{3(q-2)t/2}}{q}\|v\|_q^q.\end{aligned} \tag{3.51}$$

We see that $\tilde{\Phi}$ is of class $\mathcal{C}^1$, and for any $(w,s) \in H^1_{\mathrm{rad}}(\mathbb{R}^3) \times \mathbb{R}$,

$$\begin{aligned}\left\langle \tilde{\Phi}'(v,t),(w,s)\right\rangle &= \left\langle \tilde{\Phi}'(v,t),(w,0)\right\rangle + \left\langle \tilde{\Phi}'(v,t),(0,s)\right\rangle \\ &= e^{2t}\left(a + e^{2t} b\|\nabla v\|_2^2\right)\int_{\mathbb{R}^3} \nabla v \cdot \nabla w \mathrm{d}x \\ &\quad + e^{2t} s\left(a + e^{2t} b\|\nabla v\|_2^2\right)\|\nabla v\|_2^2 \\ &\quad - \int_{\mathbb{R}^3}\left[e^{6t} v^5 w + \mu e^{3(q-2)t/2}|v|^{q-2} v w\right]\mathrm{d}x \\ &\quad - s\int_{\mathbb{R}^3}\left[e^{6t} v^6 + \frac{3\mu(q-2)}{2q} e^{3(q-2)t/2}|v|^q\right]\mathrm{d}x \\ &= \left\langle \Phi'(\beta(v,t)),\beta(w,t)\right\rangle + s\mathcal{P}(\beta(v,t)).\end{aligned} \tag{3.52}$$

Let

$$u(x) := \beta(v,t)(x) = e^{3t/2} v(e^t x), \;\; \phi(x) := \beta(w,t)(x) = e^{3t/2} w(e^t x). \tag{3.53}$$

Then

$$(u,\phi)_H = \frac{1}{c}\int_{\mathbb{R}^3} u(x)\phi(x)\mathrm{d}x = \frac{1}{c}\int_{\mathbb{R}^3} v(x) w(x)\mathrm{d}x = (v,w)_H. \tag{3.54}$$

This shows that

$$\phi \in T_u(\mathcal{S}_c) \ \Leftrightarrow \ (w, s) \in \tilde{T}_{(v,t)}(\mathcal{S}_c \times \mathbb{R}), \quad \forall\, t, s \in \mathbb{R}. \tag{3.55}$$

It follows from (3.52), (3.53) and (3.55) that

$$|\mathcal{P}(u)| = \left|\left\langle \tilde{\Phi}'(v,t), (0,1)\right\rangle\right| \le \left\| \tilde{\Phi}|'_{\mathcal{S}_c\times\mathbb{R}}(v,t)\right\| \tag{3.56}$$

and

$$\begin{aligned}
\left\|\Phi|'_{\mathcal{S}_c}(u)\right\| &= \sup_{\phi\in T_u(\mathcal{S}_c)} \frac{1}{\|\phi\|_E} \left|\left\langle \Phi'(u), \phi\right\rangle\right| \\
&= \sup_{\phi\in T_u(\mathcal{S}_c)} \frac{1}{\sqrt{\|\nabla\phi\|_2^2 + \|\phi\|_2^2}} \left|\left\langle \Phi'(\beta(v,t)), \beta(w,t)\right\rangle\right| \\
&= \sup_{\phi\in T_u(\mathcal{S}_c)} \frac{1}{\sqrt{\|\nabla\phi\|_2^2 + \|\phi\|_2^2}} \left|\left\langle \tilde{\Phi}'(v,t), (w,0)\right\rangle\right| \\
&\le \sup_{(w,0)\in\tilde{T}_{(v,t)}(\mathcal{S}_c\times\mathbb{R})} \frac{e^{|t|}}{\|(w,0)\|_{E\times\mathbb{R}}} \left|\left\langle \tilde{\Phi}'(v,t), (w,0)\right\rangle\right| \\
&\le e^{|t|} \left\|\tilde{\Phi}|'_{\mathcal{S}_c\times\mathbb{R}}(v,t)\right\|.
\end{aligned} \tag{3.57}$$

Lemma 3.9 *Let* $2 < q < \frac{10}{3}$, $\mu > 0$ *and* $c \in (0, c_0)$. *Then there exists* $\kappa > 0$ *such that*

$$M(c) := \inf_{\gamma\in\Gamma_c} \max_{t\in[0,1]} \Phi(\gamma(t)) \ge \kappa > \sup_{\gamma\in\Gamma_c} \max\{\Phi(\gamma(0)), \Phi(\gamma(1))\}, \tag{3.58}$$

where

$$\Gamma_c = \left\{\gamma \in \mathcal{C}([0,1], \mathcal{S}_c \cap H^1_{\mathrm{rad}}(\mathbb{R}^3)) : \gamma(0) = \hat{u}_c, \Phi(\gamma(1)) < 2m(c)\right\}. \tag{3.59}$$

Proof Set $\kappa := \inf_{u\in\partial(\mathcal{S}_c\cap A_{s_0})} \Phi(u)$. By (3.5), $\kappa > 0$. Let $\gamma \in \Gamma_c$ be arbitrary. By Lemma 3.7, $\gamma(0) = \hat{u}_c \in (\mathcal{S}_c\cap A_{s_0})\backslash(\partial(\mathcal{S}_c\cap A_{s_0}))$, and $\Phi(\gamma(1)) < 2m(c) < m(c) < 0$, necessarily in view of (3.5), $\gamma(1) \notin \mathcal{S}_c \cap \overline{A_{s_0}}$. By continuity of $\gamma(t)$ on $[0, 1]$, there exists a $t_0 \in (0, 1)$ such that $\gamma(t_0) \in \partial(\mathcal{S}_c \cap A_{s_0})$, and so $\max_{t\in[0,1]} \Phi(\gamma(t)) \ge \kappa$. Since $\Phi(\gamma(0)) = \Phi(\hat{u}_c) < \Psi(\hat{u}_c) = \hat{m}(c) < 0$. Thus, (3.58) holds. □

Remark 3.10 In Lemma 3.9, one may wonder why the starting point of the path set Γ_c, defined by (3.59), is chosen as $\hat{u}_c$ (the solution of the auxiliary problem (3.33)), rather than the solution of the original constraint problem (1.1) as we did previously in the case of $b = 0$ ([5, Lemma 4.2]). It is worth noting that when $2 < q < \frac{10}{3}$, the new compactness threshold for the constraint problem (1.1) is $\hat{m}(c) + \Theta^*$, not $m(c) + \Theta^*$ as in the case of $b = 0$, as we mentioned in Remark 1.7 and subsequent remarks after it. Importantly, $\hat{u}_c$ is precisely the minimizer of $\hat{m}(c)$, which will be crucial in our subsequent proof of Lemma 3.12 that the mountain pass level is below the compactness threshold. Therefore, the solution of the original constraint problem (1.1) is not suitable as the starting point of the path set. This reveals an essential difference between the constraint problem (1.1) in the case of $b = 0$ and $b > 0$, and also explains why the methods developed for the study of the case $b = 0$ cannot be directly applied to the case $b > 0$.

Lemma 3.11 *Let* $2 < q < \frac{10}{3}$, $\mu > 0$ *and* $c \in (0, c_0)$. *Then there exists a sequence* $\{u_n\} \subset \mathcal{S}_c \cap H^1_{\text{rad}}(\mathbb{R}^3)$ *such that*

$$\Phi(u_n) \to M(c) > 0, \quad \Phi|'_{\mathcal{S}_c}(u_n) \to 0 \ \text{ and } \ \mathcal{P}(u_n) \to 0. \tag{3.60}$$

Proof By Lemma 3.7, $\hat{u}_c \in \mathcal{S}_c \cap H^1_{\text{rad}}(\mathbb{R}^3)$. Let $\tilde{\Phi}$ be defined by (3.51),

$$\tilde{\Gamma}_c := \left\{\tilde{\gamma} \in \mathcal{C}([0,1], (\mathcal{S}_c \cap H^1_{\text{rad}}(\mathbb{R}^3)) \times \mathbb{R}) : \tilde{\gamma}(0) = (\hat{u}_c, 0), \tilde{\Phi}(\tilde{\gamma}(1)) < 2m(c)\right\} \tag{3.61}$$

and

$$\tilde{M}(c) := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max_{t \in [0,1]} \tilde{\Phi}(\tilde{\gamma}(t)). \tag{3.62}$$

For any $\tilde{\gamma} \in \tilde{\Gamma}_c$, it is easy to see that $\gamma = \beta \circ \tilde{\gamma} \in \Gamma_c$ defined by (3.59). By (3.58), there exists $\kappa'_c > 0$ such that

$$\begin{aligned}\max_{t \in [0,1]} \tilde{\Phi}(\tilde{\gamma}(t)) &= \max_{t \in [0,1]} \Phi(\gamma(t)) \geq \kappa_c > \kappa'_c > \max\{\Phi(\gamma(0)), \Phi(\gamma(1))\} \\ &= \max\left\{\tilde{\Phi}(\tilde{\gamma}(0)), \tilde{\Phi}(\tilde{\gamma}(1))\right\}.\end{aligned}$$

It follows that $\tilde{M}(c) \geq M(c)$, and

$$\inf_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max_{t \in [0,1]} \tilde{\Phi}(\tilde{\gamma}(t)) \geq \kappa_c > \kappa'_c \geq \sup_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max\left\{\tilde{\Phi}(\tilde{\gamma}(0)), \Phi(\tilde{\gamma}(1))\right\}. \tag{3.63}$$

This shows that (2.13) holds with $\tilde{\varphi} = \tilde{\Phi}$.

On the other hand, for any $\gamma \in \Gamma_c$, let $\tilde{\gamma}(t) := (\gamma(t), 0)$. It is easy to verify that $\tilde{\gamma} \in \tilde{\Gamma}_c$ and $\Phi(\gamma(t)) = \tilde{\Phi}(\tilde{\gamma}(t))$, and so, we trivially have $\tilde{M}(c) \leq M(c)$. Thus $\tilde{M}(c) = M(c)$.

For any $n \in \mathbb{N}$, (3.59) implies that there exists $\gamma_n \in \Gamma_c$ such that

$$\max_{t \in [0,1]} \Phi(\gamma_n(t)) \leq M(c) + \frac{1}{n}. \tag{3.64}$$

Set $\tilde{\gamma}_n(t) := (\gamma_n(t), 0)$. Then applying Lemma 2.5 to $\tilde{\Phi}$, there exists a sequence $\{(v_n, t_n)\} \subset (\mathcal{S}_c \cap H^1_{\rm rad}(\mathbb{R}^3)) \times \mathbb{R}$ satisfying

(i) $M(c) - \frac{2}{n} \le \tilde{\Phi}(v_n, t_n) \le M(c) + \frac{2}{n}$;
(ii) $\min_{t\in[0,1]} \|(v_n, t_n) - (\gamma_n(t), 0)\|_{E\times\mathbb{R}} \le \frac{2}{\sqrt{n}}$;
(iii) $\left\| \tilde{\Phi}|'_{\mathcal{S}_c\times\mathbb{R}}(v_n, t_n) \right\| \le \frac{8}{\sqrt{n}}$.

Let $u_n = \beta(v_n, t_n)$. It follows from (3.56), (3.57) and (i)–(iii) that (3.60) holds. □

Now we define functions $U_n(x) := \Theta_n(|x|)$, where

$$\Theta_n(r) = \sqrt[4]{3} \begin{cases} \sqrt{\frac{n}{1+n^2r^2}}, & 0 \le r < 1; \\ \sqrt{\frac{n}{1+n^2}}(2-r), & 1 \le r < 2; \\ 0, & r \ge 2. \end{cases} \tag{3.65}$$

Computing directly, we have

$$\begin{aligned} \|U_n\|_2^2 &= \int_{\mathbb{R}^3} |U_n|^2 \mathrm{d}x = 4\pi \int_0^{+\infty} r^2 |\Theta_n(r)|^2 \mathrm{d}r \\ &= 4\sqrt{3}\pi \left[\int_0^1 \frac{nr^2}{(1+n^2r^2)} \mathrm{d}r + \left(\frac{n}{1+n^2}\right) \int_1^2 r^2(2-r)^2 \mathrm{d}r \right] \\ &= 4\sqrt{3}\pi \left[\frac{n - \arctan n}{n^2} + \frac{8}{15}\left(\frac{n}{1+n^2}\right)\right] = O\left(\frac{1}{n}\right), \quad n\to\infty, \end{aligned} \tag{3.66}$$

$$\begin{aligned} \|\nabla U_n\|_2^2 &= \int_{\mathbb{R}^3} |\nabla U_n|^2 \mathrm{d}x = 4\pi \int_0^{+\infty} r^2 |\Theta_n'(r)|^2 \mathrm{d}r \\ &= 4\sqrt{3}\pi \left[\int_0^1 \frac{n^5 r^4}{(1+n^2r^2)^3} \mathrm{d}r + \frac{n}{1+n^2} \int_1^2 r^2 \mathrm{d}r \right] \\ &= \mathcal{S}^{3/2} + 4\sqrt{3}\pi \left[-\int_n^{+\infty} \frac{s^4}{(1+s^2)^3} \mathrm{d}s + \frac{7n}{3(1+n^2)} \right] \\ &= \mathcal{S}^{3/2} + O\left(\frac{1}{n}\right), \quad n\to\infty \end{aligned} \tag{3.67}$$

and

$$\begin{aligned} \|U_n\|_6^6 &= \int_{\mathbb{R}^3} |U_n|^6 \mathrm{d}x = 4\pi \int_0^{+\infty} r^2 |\Theta_n(r)|^6 \mathrm{d}r \\ &= 12\sqrt{3}\pi \left[\int_0^1 \frac{n^3 r^2}{(1+n^2r^2)^3} \mathrm{d}r + \left(\frac{n}{1+n^2}\right)^3 \int_1^2 r^2(2-r)^6 \mathrm{d}r \right] \end{aligned}$$

$$= 12\sqrt{3}\pi \left[\int_0^n \frac{s^2}{(1+s^2)^3} \mathrm{d}s + \left(\frac{n}{1+n^2} \right)^3 \int_0^1 s^6 (2-s)^2 \mathrm{d}s \right]$$
$$= \mathcal{S}^{3/2} + O\left(\frac{1}{n^3}\right), \quad n \to \infty. \tag{3.68}$$

Both (3.66) and (3.67) imply that $U_n \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$.

Lemma 3.12 *Let* $2 < q < \frac{10}{3}$, $\mu > 0$ *and* $c \in (0, c_0)$. *Then there holds:*

$$M(c) < \hat{m}(c) + \Theta^*. \tag{3.69}$$

Proof Let $\hat{u}_c \in \mathcal{S}_c \cap H^1_{\mathrm{rad}}(\mathbb{R}^3)$ be given in Lemma 3.7. Then by Lemmas 3.7 and 3.8, we have

$$\|\hat{u}_c\|_2^2 = c, \quad \Psi(\hat{u}_c) = \hat{m}(c), \quad \lambda_c \|\hat{u}_c\|_2^2 = \frac{\mu(6-q)}{2q} \|\hat{u}_c\|_q^q, \quad \hat{u}_c(x) > 0, \quad \forall\, x \in \mathbb{R}^3 \tag{3.70}$$

and

$$\left[a + \frac{b^2\mathcal{S}^3}{2} + b\|\nabla \hat{u}_c\|_2^2 + \frac{b\mathcal{S}}{2}\sqrt{b^2\mathcal{S}^4 + 4\left(a + b\|\nabla \hat{u}_c\|_2^2\right)\mathcal{S}} \right] \int_{\mathbb{R}^3} \nabla \hat{u}_c \cdot \nabla U_n \mathrm{d}x$$
$$= \int_{\mathbb{R}^3} \left(\hat{u}_c^5 + \mu \hat{u}_c^{q-1} - \lambda_c \hat{u}_c \right) U_n \mathrm{d}x. \tag{3.71}$$

Set $B := \inf_{|x| \le 1} \hat{u}_c(x)$. Then $B > 0$. Hence, it follows from (3.65), (3.66) and (3.71) that

$$\int_{\mathbb{R}^3} \hat{u}_c U_n \mathrm{d}x = O\left(\frac{1}{\sqrt{n}}\right), \quad n \to \infty, \tag{3.72}$$

$$\left| \int_{\mathbb{R}^3} \nabla \hat{u}_c \cdot \nabla U_n \mathrm{d}x \right| = \left| \int_{\mathbb{R}^3} U_n \Delta \hat{u}_c \mathrm{d}x \right| = O\left(\frac{1}{\sqrt{n}}\right), \quad n \to \infty, \tag{3.73}$$

$$\int_{\mathbb{R}^3} \hat{u}_c^{q-1} U_n \mathrm{d}x \le \left[\int_{\mathbb{R}^3} \hat{u}_c^{2(q-1)} \mathrm{d}x \int_{|x|\le 2} U_n^2 \mathrm{d}x \right]^{\frac{1}{2}} = O\left(\frac{1}{\sqrt{n}}\right), \quad n \to \infty \tag{3.74}$$

and

$$\int_{\mathbb{R}^3} \hat{u}_c U_n^5 \mathrm{d}x \ge 4\pi B \int_0^1 r^2 |\Theta_n(r)|^5 \mathrm{d}r$$

$$
\begin{aligned}
&= 12\pi\sqrt[4]{3}B\int_0^1 \frac{n^{5/2}r^2}{\left(1+n^2r^2\right)^{5/2}}\mathrm{d}r \\
&\geq \frac{12\pi\sqrt[4]{3}B}{\sqrt{n}}\int_0^1 \frac{s^2}{\left(1+s^2\right)^{5/2}}\mathrm{d}s := \frac{B_0}{\sqrt{n}}.
\end{aligned} \tag{3.75}
$$

By (3.66) and (3.70), one has

$$
\begin{aligned}
\|\hat{u}_c + tU_n\|_2^2 &= c + t^2\|U_n\|_2^2 + 2t\int_{\mathbb{R}^3}\hat{u}_c U_n \mathrm{d}x \\
&= c + 2t\int_{\mathbb{R}^3}\hat{u}_c U_n \mathrm{d}x + t^2\left[O\left(\frac{1}{n}\right)\right], \quad n\to\infty.
\end{aligned} \tag{3.76}
$$

Let $\tau = \tau_{n,t} := \|\hat{u}_c + tU_n\|_2/\sqrt{c}$. Then

$$
\tau^2 = 1 + \frac{2t}{c}\int_{\mathbb{R}^3}\hat{u}_c U_n \mathrm{d}x + t^2\left[O\left(\frac{1}{n}\right)\right], \quad n\to\infty. \tag{3.77}
$$

Now, we define

$$
W_{n,t}(x) := \sqrt{\tau}[\hat{u}_c(\tau x) + tU_n(\tau x)]. \tag{3.78}
$$

Then one has

$$
\|\nabla W_{n,t}\|_2^2 = \|\nabla(\hat{u}_c + tU_n)\|_2^2, \quad \|W_{n,t}\|_6^6 = \|\hat{u}_c + tU_n\|_6^6 \tag{3.79}
$$

and

$$
\|W_{n,t}\|_2^2 = \tau^{-2}\|\hat{u}_c + tU_n\|_2^2 = c, \quad \|W_{n,t}\|_q^q = \tau^{(q-6)/2}\|\hat{u}_c + tU_n\|_q^q. \tag{3.80}
$$

Set

$$
t_*^2 = \frac{1}{2}\left[b\mathcal{S}^{\frac{3}{2}} + \sqrt{b^2S^3 + 4\left(a + b\|\nabla\hat{u}_c\|_2^2\right)}\right]. \tag{3.81}
$$

Then (3.71) can be rewritten as

$$
\left(a + b\|\nabla\hat{u}_c\|_2^2 + b\mathcal{S}^{\frac{3}{2}}t_*^2\right)\int_{\mathbb{R}^3}\nabla\hat{u}_c\cdot\nabla U_n \mathrm{d}x = \int_{\mathbb{R}^3}\left(\hat{u}_c^5 + \mu|\hat{u}_c|^{q-2}\hat{u}_c - \lambda_c\hat{u}_c\right)U_n \mathrm{d}x. \tag{3.82}
$$

By (1.9) and (3.81), we can deduce

$$
\begin{aligned}
&\mathcal{S}^{\frac{3}{2}}\left[\frac{\left(a+b\|\nabla \hat{u}_c\|_2^2\right)}{2}t^2+\frac{b\mathcal{S}^{\frac{3}{2}}}{4}t^4-\frac{1}{6}t^6\right]\\
&\quad<\mathcal{S}^{\frac{3}{2}}\left[\frac{\left(a+b\|\nabla \hat{u}_c\|_2^2\right)}{2}t_*^2+\frac{b\mathcal{S}^{\frac{3}{2}}}{4}t_*^4-\frac{1}{6}t_*^6\right]\\
&\quad=\mathcal{S}^{\frac{3}{2}}\left[\frac{\left(a+b\|\nabla \hat{u}_c\|_2^2\right)}{3}t_*^2+\frac{b\mathcal{S}^{\frac{3}{2}}}{12}t_*^4\right]\\
&\quad=\frac{\left(a+b\|\nabla \hat{u}_c\|_2^2\right)}{6}\mathcal{S}^{\frac{3}{2}}\left[b\mathcal{S}^{\frac{3}{2}}+\sqrt{b^2S^3+4\left(a+b\|\nabla \hat{u}_c\|_2^2\right)}\right]\\
&\qquad+\frac{b\mathcal{S}^3}{48}\left[b\mathcal{S}^{\frac{3}{2}}+\sqrt{b^2S^3+4\left(a+b\|\nabla \hat{u}_c\|_2^2\right)}\right]^2\\
&\quad=\frac{b\mathcal{S}^3\left(a+b\|\nabla \hat{u}_c\|_2^2\right)}{4}+\frac{b^3\mathcal{S}^6}{24}+\frac{\left[b^2\mathcal{S}^4+4\left(a+b\|\nabla \hat{u}_c\|_2^2\right)\mathcal{S}\right]^{\frac{3}{2}}}{24}\\
&\quad=\Theta^*+\frac{(b^2\mathcal{S}^4+4a\mathcal{S})^{\frac{3}{2}}}{24}\left[\left(1+\frac{4b}{b^2\mathcal{S}^3+4a}\|\nabla \hat{u}_c\|_2^2\right)^{\frac{3}{2}}-1\right]\\
&\qquad+\frac{b^2\mathcal{S}^3}{4}\|\nabla \hat{u}_c\|_2^2,\ \forall\, t\in(0,t_*)\cup(t_*,+\infty). \qquad (3.83)
\end{aligned}
$$

It is easy to verify that

$$
(1+t)^p\geq 1+pt+pt^{p-1}+t^p,\quad \forall p\geq 3,\quad t\geq 0 \qquad (3.84)
$$

and

$$
(1+t)^p\geq 1+pt^{p-1}+t^p,\ \ \forall p\geq 2,\ \ t\geq 0. \qquad (3.85)
$$

From (1.2), (1.15), (3.66)–(3.68), (3.70) and (3.72)–(3.85), we have

$$
\begin{aligned}
&\Phi(W_{n,t})\\
&\quad=\frac{a}{2}\|\nabla W_{n,t}\|_2^2+\frac{b}{4}\|\nabla W_{n,t}\|_2^4-\frac{1}{6}\|W_{n,t}\|_6^6-\frac{\mu}{q}\|W_{n,t}\|_q^q\\
&\quad=\frac{a}{2}\|\nabla(\hat{u}_c+tU_n)\|_2^2+\frac{b}{4}\|\nabla(\hat{u}_c+tU_n)\|_2^4-\frac{1}{6}\|\hat{u}_c+tU_n\|_6^6-\frac{\mu\tau^{(q-6)/2}}{q}\|\hat{u}_c+tU_n\|_q^q\\
&\quad\leq\frac{a}{2}\|\nabla\hat{u}_c\|_2^2+\frac{b}{4}\|\nabla\hat{u}_c\|_2^4-\frac{1}{6}\|\hat{u}_c\|_6^6-\frac{\mu\tau^{(q-6)/2}}{q}\|\hat{u}_c\|_q^q+\frac{at^2}{2}\|\nabla U_n\|_2^2\\
&\qquad+\frac{bt^4}{4}\|\nabla U_n\|_2^4-\frac{t^6}{6}\|U_n\|_6^6-t\int_{\mathbb{R}^3}\hat{u}_c^5U_n\mathrm{d}x-t^5\int_{\mathbb{R}^3}\hat{u}_cU_n^5\mathrm{d}x\\
&\qquad-\mu\tau^{(q-6)/2}t\int_{\mathbb{R}^3}\hat{u}_c^{q-1}U_n\mathrm{d}x+\frac{bt^2}{2}\|\nabla\hat{u}_c\|_2^2\|\nabla U_n\|_2^2\\
&\qquad+\left(a+b\|\nabla\hat{u}_c\|_2^2+bt^2\|\nabla U_n\|_2^2\right)t\int_{\mathbb{R}^3}\nabla\hat{u}_c\cdot\nabla U_n\mathrm{d}x+bt^2\left(\int_{\mathbb{R}^3}\nabla\hat{u}_c\cdot\nabla U_n\mathrm{d}x\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a}{2}\|\nabla \hat{u}_c\|_2^2 + \frac{b}{4}\|\nabla \hat{u}_c\|_2^4 - \frac{1}{6}\|\hat{u}_c\|_6^6 - \frac{\mu}{q}\|\hat{u}_c\|_q^q + \frac{at^2}{2}\|\nabla U_n\|_2^2 + \frac{bt^4}{4}\|\nabla U_n\|_2^4 - \frac{t^6}{6}\|U_n\|_6^6 \\
&\quad + \frac{\mu\left(1-\tau^{(q-6)/2}\right)}{q}\|\hat{u}_c\|_q^q + \mu\left(1-\tau^{(q-6)/2}\right) t\int_{\mathbb{R}^3} \hat{u}_c^{q-1} U_n \mathrm{d}x - \lambda_c t \int_{\mathbb{R}^3} \hat{u}_c U_n \mathrm{d}x \\
&\quad + \frac{bt^2}{2}\|\nabla \hat{u}_c\|_2^2 \|\nabla U_n\|_2^2 + b\left(t^2\|\nabla U_n\|_2^2 - t_*^2 \mathcal{S}^{\frac{3}{2}}\right) t \int_{\mathbb{R}^3} \nabla \hat{u}_c \cdot \nabla U_n \mathrm{d}x - t^5 \int_{\mathbb{R}^3} \hat{u}_c U_n^5 \mathrm{d}x \\
&\quad + t^2\left[O\left(\frac{1}{n}\right)\right] \\
&\le \frac{a}{2}\|\nabla \hat{u}_c\|_2^2 + \frac{b}{4}\|\nabla \hat{u}_c\|_2^4 - \frac{1}{6}\|\hat{u}_c\|_6^6 - \frac{\mu}{q}\|\hat{u}_c\|_q^q + \mathcal{S}^{\frac{3}{2}}\left[\frac{\left(a+b\|\nabla \hat{u}_c\|_2^2\right)}{2}t^2 + \frac{b\mathcal{S}^{\frac{3}{2}}}{4}t^4 - \frac{1}{6}t^6\right] \\
&\quad + \frac{\mu\|\hat{u}_c\|_q^q}{q}\left\{1 - \left[1 + \frac{2t}{c}\int_{\mathbb{R}^3} \hat{u}_c U_n \mathrm{d}x + t^2\left(O\left(\frac{1}{n}\right)\right)\right]^{(q-6)/2}\right\} - \lambda_c t \int_{\mathbb{R}^3} \hat{u}_c U_n \mathrm{d}x \\
&\quad + \mu\left\{1 - \left[1 + \frac{2t}{c}\int_{\mathbb{R}^3} \hat{u}_c U_n \mathrm{d}x + t^2\left(O\left(\frac{1}{n}\right)\right)\right]^{(q-6)/2}\right\} t \int_{\mathbb{R}^3} \hat{u}_c^{q-1} U_n \mathrm{d}x \\
&\quad + b\mathcal{S}^{\frac{3}{2}}\left(t^2 - t_*^2\right) t \int_{\mathbb{R}^3} \nabla \hat{u}_c \cdot \nabla U_n \mathrm{d}x - t^5 \int_{\mathbb{R}^3} \hat{u}_c U_n^5 \mathrm{d}x + \left(t^2 + t^6\right)\left[O\left(\frac{1}{n}\right)\right] \\
&\le \frac{a}{2}\|\nabla \hat{u}_c\|_2^2 + \frac{b}{4}\|\nabla \hat{u}_c\|_2^4 - \frac{1}{6}\|\hat{u}_c\|_6^6 - \frac{\mu}{q}\|\hat{u}_c\|_q^q + \mathcal{S}^{\frac{3}{2}}\left[\frac{\left(a+b\|\nabla \hat{u}_c\|_2^2\right)}{2}t^2 + \frac{b\mathcal{S}^{\frac{3}{2}}}{4}t^4 - \frac{1}{6}t^6\right] \\
&\quad - \frac{B_0 t^5}{\sqrt{n}} + b\mathcal{S}^{\frac{3}{2}}\left(t^2 - t_*^2\right) t\left[O\left(\frac{1}{\sqrt{n}}\right)\right] + \left(t^2 + t^6\right)\left[O\left(\frac{1}{n}\right)\right] \qquad (3.86) \\
&\le \Theta^* + \frac{(b^2\mathcal{S}^4 + 4a\mathcal{S})^{\frac{3}{2}}}{24}\left[\left(1 + \frac{4b}{b^2\mathcal{S}^3 + 4a}\|\nabla \hat{u}_c\|_2^2\right)^{\frac{3}{2}} - 1\right] + \left(\frac{a}{2} + \frac{b^2\mathcal{S}^3}{4}\right)\|\nabla \hat{u}_c\|_2^2 \\
&\quad + \frac{b}{4}\|\nabla \hat{u}_c\|_2^4 - \frac{1}{6}\|\hat{u}_c\|_6^6 - \frac{\mu}{q}\|\hat{u}_c\|_q^q - O\left(\frac{1}{\sqrt{n}}\right) \\
&= \Theta^* + \Psi(\hat{u}_c) - O\left(\frac{1}{\sqrt{n}}\right) \\
&= \hat{m}(c) + \Theta^* - O\left(\frac{1}{\sqrt{n}}\right), \quad \forall\, t > 0. \qquad (3.87)
\end{aligned}
$$

Hence, it follows from (3.87) that there exists $\bar{n} \in \mathbb{N}$ such that

$$\sup_{t>0} \Phi(W_{\bar{n},t}) < \hat{m}(c) + \Theta^*. \tag{3.88}$$

Next, we prove that (3.69) holds. Let $\bar{n} \in \mathbb{N}$ be given in (3.88). By (3.76), (3.78), (3.79) and (3.80), we have

$$W_{\bar{n},t}(x) := \bar{\tau}^{1/2}[\hat{u}_c(\bar{\tau}x) + tU_{\bar{n}}(\tau x)], \quad \|W_{\bar{n},t}\|_2^2 = c \tag{3.89}$$

and

$$\begin{aligned}\|\nabla W_{\bar{n},t}\|_2^2 &= \|\nabla(\hat{u}_c + tU_{\bar{n}})\|_2^2 \\ &= \|\nabla\hat{u}_c\|_2^2 + t^2\|\nabla U_{\bar{n}}\|_2^2 + 2t\int_{\mathbb{R}^3}\nabla\hat{u}_c\cdot\nabla U_{\bar{n}}\mathrm{d}x,\end{aligned} \tag{3.90}$$

where

$$\bar{\tau}^2 = \|\hat{u}_c + tU_{\bar{n}}\|_2^2/c = 1 + \frac{2t}{c}\int_{\mathbb{R}^3}\hat{u}_c U_{\bar{n}}\mathrm{d}x + t^2\|U_{\bar{n}}\|_2^2. \tag{3.91}$$

It follows from (3.86), (3.89) and (3.90) that $W_{\bar{n},t} \in \mathcal{S}_c$ for all $t > 0$, $W_{\bar{n},0} = \hat{u}_c$ and $\Phi(W_{\bar{n},t}) < 2m(c)$ for large $t > 0$. Thus, there exists $\hat{t} > 0$ such that

$$\Phi(W_{\bar{n},\hat{t}}) < 2m(c). \tag{3.92}$$

Let $\gamma_{\bar{n}}(t) := W_{\bar{n},t\hat{t}}$. Then $\gamma_{\bar{n}} \in \Gamma_c$ defined by (3.59). Hence, it follows from (3.58) and (3.88) that (3.69) holds. □

Proof of Theorems 1.2 In view of Lemmas 3.11 and 3.12, there exists $\{u_n\} \subset \mathcal{S}_c \cap H^1_{\mathrm{rad}}(\mathbb{R}^3)$ such that

$$\|u_n\|_2^2 = c, \quad \Phi(u_n) \to M(c) \in (0, \hat{m}(c) + \Theta^*), \quad \Phi|'_{\mathcal{S}_c}(u_n) \to 0, \quad \mathcal{P}(u_n) \to 0. \tag{3.93}$$

It follows from (1.2), (1.17) and (3.93) that

$$M(c) + o(1) = \frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{1}{6}\|u_n\|_6^6 - \frac{\mu}{q}\|u_n\|_q^q \tag{3.94}$$

and

$$o(1) = a\|\nabla u_n\|_2^2 + b\|\nabla u_n\|_2^4 - \|u_n\|_6^6 - \frac{3\mu(q-2)}{2q}\|u_n\|_q^q. \tag{3.95}$$

Both (3.94) and (3.95), together with (1.16), show that

$$\begin{aligned}M(c) + o(1) &= \frac{a}{3}\|\nabla u_n\|_2^2 + \frac{b}{12}\|\nabla u_n\|_2^4 - \frac{\mu(6-q)}{4q}\|u_n\|_q^q \\ &\ge \frac{a}{3}\|\nabla u_n\|_2^2 + \frac{b}{12}\|\nabla u_n\|_2^4 - \frac{\mu(6-q)}{4q}\mathcal{C}_q^q c^{(6-q)/4}\|\nabla u_n\|_2^{3(q-2)/2}.\end{aligned} \tag{3.96}$$

Since $2 < q < \frac{10}{3}$, it follows that $\{\|u_n\|\}$ is bounded. By Lemma 2.2, one has

$$\Phi'(u_n) + \lambda_n u_n \to 0, \tag{3.97}$$

where

$$-\lambda_n = \frac{1}{\|u_n\|_2^2}\langle \Phi'(u_n), u_n\rangle = \frac{1}{c}\left[\left(a + b\|\nabla u_n\|_2^2\right)\|\nabla u_n\|_2^2 - \mu\|u_n\|_q^q - \|u_n\|_6^6\right]. \tag{3.98}$$

Since $\{\|u_n\|\}$ is bounded, it follows from (3.98) that $\{|\lambda_n|\}$ is also bounded. Thus, we may thus assume, passing to a subsequence if necessary, that

$$\begin{cases} \lambda_n \to \lambda_c, & \|\nabla u_n\|_2^2 \to A^2; \\ u_n \rightharpoonup \bar{u}, & \text{in } H^1_{\text{rad}}(\mathbb{R}^3); \\ u_n \to \bar{u}, & \text{in } L^s(\mathbb{R}^3), \ \forall\, s \in (2, 6); \\ u_n \to \bar{u}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{3.99}$$

First, we prove that $\bar{u} \neq 0$. Otherwise, we assume that $\bar{u} = 0$. Then $\|u_n\|_q^q \to 0$. It follows from (3.95) that

$$o(1) = a\|\nabla u_n\|_2^2 + b\|\nabla u_n\|_2^4 - \|u_n\|_6^6. \tag{3.100}$$

Up to a subsequence, we assume that

$$\|\nabla u_n\|_2^2 \to \hat{l}_1 \geq 0, \quad \|u_n\|_6^6 \to \hat{l}_2 \geq 0. \tag{3.101}$$

Then it follows from (1.8), (3.100) and (3.101) that $a\hat{l}_1 + b\hat{l}_1^2 = \hat{l}_2 \leq \mathcal{S}^{-3}\hat{l}_1^3$. If $\hat{l}_1 > 0$, an elementary calculation yields that

$$\hat{l}_1 \geq \frac{\mathcal{S}}{2}\left[b\mathcal{S}^2 + \sqrt{b^2\mathcal{S}^4 + 4a\mathcal{S}}\right]. \tag{3.102}$$

From (3.94), (3.100), (3.101) and (3.102), we obtain

$$\begin{aligned} M(c) + o(1) &= \frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{1}{6}\|u_n\|_6^6 \\ &= \frac{a}{3}\|\nabla u_n\|_2^2 + \frac{b}{12}\|\nabla u_n\|_2^4 \\ &\geq \frac{ab\mathcal{S}^3}{4} + \frac{b^3\mathcal{S}^6}{24} + \frac{\left(b^2\mathcal{S}^4 + 4a\mathcal{S}\right)^{3/2}}{24} + o(1) = \Theta^* + o(1), \end{aligned}$$

which contradicts with (3.93). Thus, $\|\nabla u_n\|_2^2 \to 0$, and so it follows from (3.94) that $M(c) = 0$, which contradicts with (3.93) also. Therefore, $\bar{u} \neq 0$.

Define $I(u)$ as follows:

$$I(u) := \frac{a + bA^2}{2}\|\nabla u\|_2^2 - \frac{1}{6}\|u\|_6^6 - \frac{\mu}{q}\|u\|_q^q. \tag{3.103}$$

By (3.97), (3.98), (3.99) and (3.103) and a standard argument, we can deduce

$$I'(\bar{u}) + \lambda_c \bar{u} = 0. \tag{3.104}$$

It follows that

$$\left(a + bA^2\right) \|\nabla \bar{u}\|_2^2 + \lambda_c \|\bar{u}\|_2^2 - \mu \|\bar{u}\|_q^q - \|\bar{u}\|_6^6 = 0. \tag{3.105}$$

By the Pohozaev type identity for the functional (3.103), one has

$$\left(a + bA^2\right) \|\nabla \bar{u}\|_2^2 + 3\lambda_c \|\bar{u}\|_2^2 - \frac{6\mu}{q} \|\bar{u}\|_q^q - \|\bar{u}\|_6^6 = 0. \tag{3.106}$$

Combining (3.105) with (3.106), one has

$$\mathcal{P}_I(\bar{u}) := (a + bA^2)\|\nabla \bar{u}\|_2^2 - \|\bar{u}\|_6^6 - \frac{3\mu(q-2)}{2q} \|\bar{u}\|_q^q = 0 \tag{3.107}$$

and

$$\lambda_c \|\bar{u}\|_2^2 = \frac{\mu(6-q)}{2q} \|\bar{u}\|_q^q. \tag{3.108}$$

Let $v_n := u_n - \bar{u}$. Then $v_n \rightharpoonup 0$ in $H^1_{\text{rad}}(\mathbb{R}^3)$ and $v_n \to 0$ in $L^s(\mathbb{R}^3)$ for all $s \in (2, 6)$. Using Brezis–Lieb lemma, one has

$$\begin{cases} \|v_n\|_2^2 = \|u_n\|_2^2 - \|\bar{u}\|_2^2 + o(1); \\ \|v_n\|_6^6 = \|u_n\|_6^6 - \|\bar{u}\|_6^6 + o(1); \\ A^2 = \|\nabla u_n\|_2^2 + o(1) = \|\nabla \bar{u}\|_2^2 + \|\nabla v_n\|_2^2 + o(1). \end{cases} \tag{3.109}$$

From (3.95), (3.107), and (3.109), we deduce

$$\begin{aligned} o(1) &= \left(a + b\|\nabla u_n\|_2^2\right) \|\nabla u_n\|_2^2 - \|u_n\|_6^6 - \frac{3\mu(q-2)}{2q} \|u_n\|_q^q \\ &= \left(a + bA^2\right) \|\nabla \bar{u}\|_2^2 - \|\bar{u}\|_6^6 - \frac{3\mu(q-2)}{2q} \|\bar{u}\|_q^q \\ &\quad + \left(a + bA^2\right) \|\nabla v_n\|_2^2 - \|v_n\|_6^6 + o(1) \\ &= \left(a + bA^2\right) \|\nabla v_n\|_2^2 - \|v_n\|_6^6 + o(1) \\ &= \left(a + b\|\nabla \bar{u}\|_2^2\right) \|\nabla v_n\|_2^2 + b\|\nabla v_n\|_2^4 - \|v_n\|_6^6 + o(1). \end{aligned} \tag{3.110}$$

Up to a subsequence, we assume that

$$\|\nabla v_n\|_2^2 \to l_1 \geq 0, \quad \|v_n\|_6^6 \to l_2 \geq 0. \tag{3.111}$$

Then it follows from (3.110) and (3.111) that

$$\left(a+b\|\nabla\bar{u}\|_2^2\right)l_1+bl_1^2=l_2. \tag{3.112}$$

If $l_1>0$, by (1.8), (3.111) and (3.112), we have

$$l_1\geq\mathcal{S}\left[\left(a+b\|\nabla\bar{u}\|_2^2\right)l_1+bl_1^2\right]^{1/3},$$

which implies

$$l_1\geq\frac{\mathcal{S}}{2}\left[b\mathcal{S}^2+\sqrt{b^2\mathcal{S}^4+4\left(a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}}\right]. \tag{3.113}$$

From (1.2), (3.94), (3.109) and (3.110), we obtain

$$\begin{aligned}M(c)+o(1)&=\frac{a}{2}\|\nabla u_n\|_2^2+\frac{b}{4}\|\nabla u_n\|_2^4-\frac{1}{6}\|u_n\|_6^6-\frac{\mu}{q}\|u_n\|_q^q\\&=\frac{a}{2}\|\nabla v_n\|_2^2+\frac{b}{4}\|\nabla v_n\|_2^4-\frac{1}{6}\|v_n\|_6^6+\frac{b}{2}\|\nabla\bar{u}\|_2^2\|\nabla v_n\|_2^2+\Phi(\bar{u})+o(1)\\&=\frac{a}{3}\|\nabla v_n\|_2^2+\frac{b}{12}\|\nabla v_n\|_2^4+\frac{b}{3}\|\nabla\bar{u}\|_2^2\|\nabla v_n\|_2^2+\Phi(\bar{u})+o(1).\end{aligned} \tag{3.114}$$

There are two cases to distinguish.

Case 1). $\|\nabla\bar{u}\|_2^2<s_0$. Then it follows from Lemmas 3.4 and 3.6 that

$$\Psi(\bar{u})\geq\hat{m}(\|\bar{u}\|_2^2)\geq\hat{m}(c). \tag{3.115}$$

From (1.15), (3.111), (3.113), (3.114) and (3.115), we obtain

$$\begin{aligned}&M(c)+o(1)\\&\quad=\frac{a}{3}\|\nabla v_n\|_2^2+\frac{b}{12}\|\nabla v_n\|_2^4+\frac{b}{3}\|\nabla\bar{u}\|_2^2\|\nabla v_n\|_2^2+\Phi(\bar{u})+o(1)\\&\quad=\frac{a+b\|\nabla\bar{u}\|_2^2}{3}l_1+\frac{b}{12}l_1^2+\Phi(\bar{u})+o(1)\\&\quad\geq\frac{(a+b\|\nabla\bar{u}\|_2^2)\mathcal{S}}{6}\left[b\mathcal{S}^2+\sqrt{b^2\mathcal{S}^4+4\left(a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}}\right]\\&\qquad+\frac{b\mathcal{S}^2}{48}\left[b\mathcal{S}^2+\sqrt{b^2\mathcal{S}^4+4\left(a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}}\right]^2+\Phi(\bar{u})+o(1)\\&\quad=\frac{ab\mathcal{S}^3}{4}+\frac{b^3\mathcal{S}^6}{24}+\frac{\left[b^2\mathcal{S}^4+4\left(a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}\right]^{3/2}}{24}+\frac{b^2\mathcal{S}^3}{4}\|\nabla\bar{u}\|_2^2+\Phi(\bar{u})+o(1)\\&\quad=\frac{ab\mathcal{S}^3}{4}+\frac{b^3\mathcal{S}^6}{24}+\frac{\left(b^2\mathcal{S}^4+4a\mathcal{S}\right)^{3/2}}{24}\end{aligned}$$

$$
\begin{aligned}
&+\frac{(b^2\mathcal{S}^4+4a\mathcal{S})^{\frac{3}{2}}}{24}\left[\left(1+\frac{4b}{b^2\mathcal{S}^3+4a}\|\nabla\bar{u}\|_2^2\right)^{\frac{3}{2}}-1\right]\\
&+\left(\frac{a}{2}+\frac{b^2\mathcal{S}^3}{4}\right)\|\nabla\bar{u}\|_2^2+\frac{b}{4}\|\nabla\bar{u}\|_2^4-\frac{1}{6}\|\bar{u}\|_6^6-\frac{\mu}{q}\|\bar{u}\|_q^q+o(1)\\
&=\Theta^*+\Psi(\bar{u})+o(1)\geq\Theta^*+\hat{m}(c)+o(1),
\end{aligned}
$$

which contradicts with (3.93).

Case 2). $\|\nabla\bar{u}\|_2^2\geq s_0$. Then it follows from (1.2), (1.12), (1.16), (3.107), (3.109), (3.111) and (3.114) that

$$
\begin{aligned}
&M(c)+o(1)\\
&=\frac{a}{3}\|\nabla v_n\|_2^2+\frac{b}{12}\|\nabla v_n\|_2^4+\frac{b}{3}\|\nabla\bar{u}\|_2^2\|\nabla v_n\|_2^2+\Phi(\bar{u})+o(1)\\
&=\frac{a}{3}\|\nabla v_n\|_2^2+\frac{b}{12}\|\nabla v_n\|_2^4+\frac{b}{6}\|\nabla\bar{u}\|_2^2\|\nabla v_n\|_2^2+\frac{a}{3}\|\nabla\bar{u}\|_2^2+\frac{b}{12}\|\nabla\bar{u}\|_2^4\\
&\quad-\frac{\mu(6-q)}{4q}\|\bar{u}\|_q^q+o(1)\\
&=\frac{2a+b\|\nabla\bar{u}\|_2^2}{6}l_1+\frac{b}{12}l_1^2+\frac{a}{3}\|\nabla\bar{u}\|_2^2+\frac{b}{12}\|\nabla\bar{u}\|_2^4-\frac{\mu(6-q)}{4q}\|\bar{u}\|_q^q+o(1)\\
&\geq\frac{(2a+b\|\nabla\bar{u}\|_2^2)\mathcal{S}}{12}\left[b\mathcal{S}^2+\sqrt{b^2\mathcal{S}^4+4\left(a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}}\right]\\
&\quad+\frac{b\mathcal{S}^2}{48}\left[b\mathcal{S}^2+\sqrt{b^2\mathcal{S}^4+4\left(a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}}\right]^2+\frac{a}{3}\|\nabla\bar{u}\|_2^2+\frac{b}{12}\|\nabla\bar{u}\|_2^4\\
&\quad-\frac{\mu(6-q)}{4q}\|\bar{u}\|_q^q+o(1)\\
&=\frac{ab\mathcal{S}^3}{4}+\frac{b^3\mathcal{S}^6}{24}+\frac{b^2S^4+2\left(2a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}}{24}\sqrt{b^2S^4+4\left(a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}}\\
&\quad+\left(\frac{a}{3}+\frac{b^2\mathcal{S}^3}{6}\right)\|\nabla\bar{u}\|_2^2+\frac{b}{12}\|\nabla\bar{u}\|_2^4-\frac{\mu(6-q)}{4q}\|\bar{u}\|_q^q+o(1)\\
&\geq\frac{ab\mathcal{S}^3}{4}+\frac{b^3\mathcal{S}^6}{24}+\frac{\left(b^2S^4+4a\mathcal{S}\right)^{3/2}}{24}\\
&\quad+\left(\frac{a}{3}+\frac{b^2\mathcal{S}^3}{6}+\frac{b\mathcal{S}\sqrt{b^2S^4+4(a+bs_0)\mathcal{S}}}{12}\right)\|\nabla\bar{u}\|_2^2\\
&\quad+\frac{b}{12}\|\nabla\bar{u}\|_2^4-\frac{\mu(6-q)}{4q}\|\bar{u}\|_q^q+o(1)\\
&\geq\Theta^*+\left(\frac{a}{3}+\frac{b^2\mathcal{S}^3}{6}+\frac{b\mathcal{S}\sqrt{b^2S^4+4(a+bs_0)\mathcal{S}}}{12}\right)\|\nabla\bar{u}\|_2^2+\frac{b}{12}\|\nabla\bar{u}\|_2^4\\
&\quad-\frac{\mu(6-q)}{4q}\mathcal{C}_q^q c^{(6-q)/4}\|\nabla\bar{u}\|_2^{3(q-2)/2}+o(1)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \Theta^* + \left[\left(\frac{a}{3} + \frac{b^2\mathcal{S}^3}{6} + \frac{b\mathcal{S}\sqrt{b^2 S^4 + 4(a+bs_0)\mathcal{S}}}{12}\right) s_0^{(10-3q)/4} + \frac{b}{12} s_0^{(14-3q)/4}\right.\\
&\quad \left. - \frac{\mu(6-q)}{4q}\mathcal{C}_q^q c^{(6-q)/4}\right] \|\nabla \bar{u}\|_2^{3(q-2)/2} + o(1)\\
&\geq \Theta^* + o(1),
\end{aligned}
$$

which contradicts with (3.93). Both Cases 1) and 2) show that $l_1 = 0$, i.e. $\|\nabla v_n\| \to 0$, and so

$$
\|\nabla u_n\|_2^2 \to \|\nabla \bar{u}\|_2^2, \quad \|u_n\|_6^6 \to \|\bar{u}\|_6^6. \tag{3.116}
$$

Now from (1.2), (3.93), (3.94), (3.97), (3.98), (3.99), (3.105), (3.108) and (3.116), it is easy to deduce that

$$
\lambda_c > 0, \quad \|\bar{u}\|_2^2 = c, \quad \Phi'(\bar{u}) + \lambda_c \bar{u} = 0, \quad \Phi(\bar{u}) = M(c).
$$

□

4 The case when $\frac{10}{3} \leq q < \frac{14}{3}$

In this section, we study the case $\frac{10}{3} \leq q < \frac{14}{3}$, and finish the proof of Theorem 1.3.

Lemma 4.1 *Let* $\frac{10}{3} \leq q < \frac{14}{3}$, $\mu > 0$ *and* $c \in (0, c_2]$. *Then*

(i) *there exist* $\vartheta_c' > \vartheta_c > 0$ *such that* $\Phi(u) > 0$ *if* $u \in A_{\vartheta_c'}$, *and*

$$
0 < \sup_{u \in A_{\vartheta_c}} \Phi(u) < \inf\left\{\Phi(u) : u \in \mathcal{S}_c,\ \|\nabla u\|_2^2 = \vartheta_c'\right\}, \tag{4.1}
$$

where

$$
A_{\vartheta_c} = \left\{u \in \mathcal{S}_c : \|\nabla u\|_2^2 < \vartheta_c\right\} \ \ \textit{and} \ \ A_{\vartheta_c'} = \left\{u \in \mathcal{S}_c : \|\nabla u\|_2^2 < \vartheta_c'\right\}; \tag{4.2}
$$

(ii) $\hat{\Gamma}_c = \{\gamma \in \mathcal{C}([0,1], \mathcal{S}_c \cap H^1_{\mathrm{rad}}(\mathbb{R}^3)) : \|\nabla \gamma(0)\|_2^2 < \vartheta_c, \Phi(\gamma(1)) < 0\} \neq \emptyset$ *and*

$$
\begin{aligned}
\hat{M}(c) := \inf_{\gamma \in \hat{\Gamma}_c} \max_{t \in [0,1]} \Phi(\gamma(t)) \geq \hat{\kappa}_c := \inf\left\{\Phi(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = \vartheta_c'\right\}\\
> \max_{\gamma \in \hat{\Gamma}_c} \max\{\Phi(\gamma(0)), \Phi(\gamma(1))\}.
\end{aligned} \tag{4.3}
$$

Proof (i) We distinguish two cases.

Case 1). $\frac{10}{3} < q < \frac{14}{3}$. In this case, one has $0 < \frac{3q-10}{2} < 2$. By (1.2), (1.8) and (1.16), one has

$$\begin{aligned}\Phi(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{6}\|u\|_6^6 \\ &\leq \|\nabla u\|_2^2\left(\frac{a}{2} + \frac{b}{4}\|\nabla u\|_2^2\right), \quad \forall\, u \in \mathcal{S}_c\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\Phi(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{6}\|u\|_6^6 \\ &\geq \|\nabla u\|_2^2\left[\frac{a}{2} - \frac{\mu c^{(6-q)/4}\mathcal{C}_q^q}{q}\|\nabla u\|_2^{(3q-10)/2} - \frac{1}{6\mathcal{S}^3}\|\nabla u\|_2^4\right], \quad \forall\, u \in \mathcal{S}_c.\end{aligned}$$

Since $0 < \frac{3q-10}{2} < 2$, the above inequalities show that there exist $\vartheta_c' > \vartheta_c > 0$ such that (i) holds.

Case 2). $q = \frac{10}{3}$. By (1.2), (1.8) and (1.16), one has

$$\begin{aligned}\Phi(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{3\mu}{10}\|u\|_{10/3}^{10/3} - \frac{1}{6}\|u\|_6^6 \\ &\geq \|\nabla u\|_2^2\left(\frac{a}{2} + \frac{b}{4}\|\nabla u\|_2^2 - \frac{3\mu}{10}\mathcal{C}_{10/3}^{10/3}c^{2/3} - \frac{1}{6\mathcal{S}^3}\|\nabla u\|_2^4\right), \quad \forall\, u \in \mathcal{S}_c.\end{aligned}$$

Since $c \leq c_2$, the above inequality and (4.4) show that there exist $\vartheta_c' > \vartheta_c > 0$ such that (i) holds also.

(ii) For any given $w \in \mathcal{S}_c \cap H^1_{\text{rad}}(\mathbb{R}^3)$, we have $\|t^{3/2}w_t\|_2 = \|w\|_2$, and so $t^{3/2}w_t \in \mathcal{S}_c \cap H^1_{\text{rad}}(\mathbb{R}^3)$ for every $t > 0$. Then (1.2) yields

$$\begin{aligned}\Phi\left(t^{3/2}w_t\right) &= \frac{at^2}{2}\|\nabla w\|_2^2 + \frac{bt^4}{4}\|\nabla w\|_2^4 - \frac{\mu t^{3(q-2)/2}}{q}\|w\|_q^q \\ &\quad - \frac{t^6}{6}\|w\|_6^6 \to -\infty \text{ as } t \to +\infty.\end{aligned} \tag{4.5}$$

Thus we can deduce that there exist $t_1 > 0$ small enough and $t_2 > 0$ large enough such that

$$\left\|\nabla\left(t_1^{3/2}w_{t_1}\right)\right\|_2^2 = t_1^2\|\nabla w\|_2^2 < \vartheta_c, \text{ and } \Phi\left(t_2^{3/2}w_{t_2}\right) < 0. \tag{4.6}$$

Let $\gamma_0(t) := [t_1 + (t_2 - t_1)t]^{3/2}w_{t_1+(t_2-t_1)t}$. Then $\gamma_0 \in \hat{\Gamma}_c$, and so $\hat{\Gamma}_c \neq \emptyset$. Now using the intermediate value theorem, for any $\gamma \in \hat{\Gamma}_c$, there exists $t_0 \in (0, 1)$, depending on γ, such that $\|\nabla\gamma(t_0)\|_2^2 = \vartheta_c'$ and

$$\max_{t\in[0,1]} \Phi(\gamma(t)) \geq \Phi(\gamma(t_0)) \geq \inf\left\{\Phi(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = \vartheta_c'\right\},$$

which, together with the arbitrariness of $\gamma \in \hat{\Gamma}_c$, implies

$$\hat{M}(c) = \inf_{\gamma \in \hat{\Gamma}} \max_{t\in[0,1]} \Phi(\gamma(t)) \geq \inf \left\{ \Phi(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = \vartheta_c' \right\}. \tag{4.7}$$

Hence, (4.3) follows directly from (4.1) and (4.7), and the proof is completed. □

Lemma 4.2 *Let* $\frac{10}{3} \leq q < \frac{14}{3}$, $\mu > 0$ *and* $c \in (0, c_2]$. *Then there exists a sequence* $\{u_n\} \subset \mathcal{S}_c \cap H^1_{\text{rad}}(\mathbb{R}^3)$ *such that*

$$\Phi(u_n) \to \hat{M}(c) > 0, \quad \Phi|'_{\mathcal{S}_c}(u_n) \to 0 \ \text{ and } \ \mathcal{P}(u_n) \to 0. \tag{4.8}$$

Proof Let $\tilde{\Phi}$ be defined by (3.51),

$$\tilde{\Gamma}_c := \Big\{ \tilde{\gamma} \in \mathcal{C}([0,1], (\mathcal{S}_c \cap H^1_{\text{rad}}(\mathbb{R}^3)) \times \mathbb{R}) : \tilde{\gamma}(0) = (\tilde{\gamma}_1(0), 0), \|\nabla \tilde{\gamma}_1(0)\|_2^2 < \vartheta_c, \\ \tilde{\Phi}(\tilde{\gamma}(1)) < 0 \Big\} \tag{4.9}$$

and

$$\tilde{M}(c) := \inf_{\tilde{\gamma}\in\tilde{\Gamma}_c} \max_{t\in[0,1]} \tilde{\Phi}(\tilde{\gamma}(t)). \tag{4.10}$$

For any $\tilde{\gamma} \in \tilde{\Gamma}_c$, it is easy to see that $\gamma = \beta \circ \tilde{\gamma} \in \hat{\Gamma}_c$. By (4.3), there exists $\hat{\kappa}_c' \in (0, \hat{\kappa}_c)$ such that

$$\begin{aligned} \max_{t\in[0,1]} \tilde{\Phi}(\tilde{\gamma}(t)) &= \max_{t\in[0,1]} \Phi(\gamma(t)) \geq \hat{\kappa}_c > \hat{\kappa}_c' > \max\{\Phi(\gamma(0)), \Phi(\gamma(1))\} \\ &= \max\left\{ \tilde{\Phi}(\tilde{\gamma}(0)), \tilde{\Phi}(\tilde{\gamma}(1)) \right\}. \end{aligned}$$

It follows that $\tilde{M}(c) \geq \hat{M}(c)$, and

$$\inf_{\tilde{\gamma}\in\tilde{\Gamma}_c} \max_{t\in[0,1]} \tilde{\Phi}(\tilde{\gamma}(t)) \geq \kappa_c > \kappa_c' \geq \sup_{\tilde{\gamma}\in\tilde{\Gamma}_c} \max\left\{ \tilde{\Phi}(\tilde{\gamma}(0)), \tilde{\Phi}(\tilde{\gamma}(1)) \right\}. \tag{4.11}$$

This shows that (2.13) holds with $\tilde{\varphi} = \tilde{\Phi}$.

On the other hand, for any $\gamma \in \hat{\Gamma}_c$, let $\tilde{\gamma}(t) := (\gamma(t), 0)$. It is easy to verify that $\tilde{\gamma} \in \tilde{\Gamma}_c$ and $\Phi(\gamma(t)) = \tilde{\Phi}(\tilde{\gamma}(t))$, and so, we trivially have $\tilde{M}(c) \leq \hat{M}(c)$. Thus $\tilde{M}(c) = \hat{M}(c)$.

For any $n \in \mathbb{N}$, (4.3) implies that there exists $\gamma_n \in \hat{\Gamma}_c$ such that

$$\max_{t\in[0,1]} \Phi(\gamma_n(t)) \leq \hat{M}(c) + \frac{1}{n}. \tag{4.12}$$

Set $\tilde{\gamma}_n(t) := (\gamma_n(t), 0)$. Then applying Lemma 2.5 to $\tilde{\Phi}$, there exists a sequence $\{(v_n, t_n)\} \subset (\mathcal{S}_c \cap H^1_{\text{rad}}(\mathbb{R}^3)) \times \mathbb{R}$ satisfying

(i) $\hat{M}(c) - \frac{2}{n} \leq \tilde{\Phi}(v_n, t_n) \leq \hat{M}(c) + \frac{2}{n}$;
(ii) $\min_{t\in[0,1]} \|(v_n, t_n) - (\gamma_n(t), 0)\|_{E\times\mathbb{R}} \leq \frac{2}{\sqrt{n}}$;
(iii) $\left\| \tilde{\Phi}|'_{\mathcal{S}_c\times\mathbb{R}}(v_n, t_n) \right\| \leq \frac{8}{\sqrt{n}}$.

Let $u_n = \beta(v_n, t_n)$. It follows from (3.56), (3.57) and (i)–(iii) that (4.8) holds. □

Next, we give a precise estimation for the energy level $\hat{M}(c)$ given by (4.3) when $\frac{10}{3} \leq q < \frac{14}{3}$. To this end, for any fixed $c > 0$, we choose $\max\{(14-3q)/8, 0\} < \alpha < 1$ and $R_n > n^\alpha$ to be such that

$$c = 4\sqrt{3}\pi \left\{ \frac{n^{1+\alpha} - \arctan\left(n^{1+\alpha}\right)}{n^2} + \frac{R_n^5 n - [10R_n^2 - 15R_n n^\alpha + 6n^{2\alpha}]n^{1+3\alpha}}{30(R_n - n^\alpha)^2(1 + n^{2(1+\alpha)})} \right\}. \tag{4.13}$$

From (4.13), one can deduce that

$$\lim_{n\to\infty} \frac{R_n}{n^{(1+2\alpha)/3}} = \sqrt[3]{\frac{15c}{2\sqrt{3}\pi}}. \tag{4.14}$$

Now, we define function $\tilde{U}_n(x) := \tilde{\Theta}_n(|x|)$, where

$$\tilde{\Theta}_n(r) = \sqrt[4]{3} \begin{cases} \sqrt{\frac{n}{1+n^2r^2}}, & 0 \leq r < n^\alpha; \\ \sqrt{\frac{n}{1+n^{2(1+\alpha)}}} \frac{R_n - r}{R_n - n^\alpha}, & n^\alpha \leq r < R_n; \\ 0, & r \geq R_n. \end{cases} \tag{4.15}$$

Computing directly, we have

$$\begin{aligned} \|\tilde{U}_n\|_2^2 &= \int_{\mathbb{R}^3} |\tilde{U}_n|^2 \mathrm{d}x = 4\pi \int_0^{+\infty} r^2 |\tilde{\Theta}_n(r)|^2 \mathrm{d}r \\ &= 4\sqrt{3}\pi \left[\int_0^{n^\alpha} \frac{nr^2}{1+n^2r^2} \mathrm{d}r + \frac{n}{1+n^{2(1+\alpha)}} \int_{n^\alpha}^{R_n} \frac{r^2(R_n - r)^2}{(R_n - n^\alpha)^2} \mathrm{d}r \right] \\ &= 4\sqrt{3}\pi \left\{ \frac{1}{n^2} \int_0^{n^{1+\alpha}} \frac{s^2}{1+s^2} \mathrm{d}s + \frac{n}{1+n^{2(1+\alpha)}} \frac{R_n^5 - [10R_n^2 - 15R_n n^\alpha + 6n^{2\alpha}]n^{3\alpha}}{30(R_n - n^\alpha)^2} \right\} \\ &= 4\sqrt{3}\pi \left\{ \frac{n^{1+\alpha} - \arctan\left(n^{1+\alpha}\right)}{n^2} + \frac{R_n^5 n - [10R_n^2 - 15R_n n^\alpha + 6n^{2\alpha}]n^{1+3\alpha}}{30(R_n - n^\alpha)^2(1+n^{2(1+\alpha)})} \right\} \\ &= c, \end{aligned} \tag{4.16}$$

$$\begin{aligned} \|\nabla\tilde{U}_n\|_2^2 &= \int_{\mathbb{R}^3} |\nabla\tilde{U}_n|^2 \mathrm{d}x = 4\pi \int_0^{+\infty} r^2 |\tilde{\Theta}'_n(r)|^2 \mathrm{d}r \\ &= 4\sqrt{3}\pi \left[\int_0^{n^\alpha} \frac{n^5 r^4}{(1+n^2r^2)^3} \mathrm{d}r + \frac{n}{1+n^{2(1+\alpha)}} \int_{n^\alpha}^{R_n} \frac{r^2}{(R_n - n^\alpha)^2} \mathrm{d}r \right] \\ &= 4\sqrt{3}\pi \left[\int_0^{n^{1+\alpha}} \frac{s^4}{(1+s^2)^3} \mathrm{d}s + \frac{R_n^3 - n^{3\alpha}}{3(R_n - n^\alpha)^2} \frac{n}{1+n^{2(1+\alpha)}} \right] \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{S}^{\frac{3}{2}} + 4\sqrt{3}\pi\left[-\int_{n^{1+\alpha}}^{+\infty}\frac{s^4}{\left(1+s^2\right)^3}\mathrm{d}s + \frac{n(R_n^3-n^{3\alpha})}{3(R_n-n^\alpha)^2(1+n^{2(1+\alpha)})}\right]\\
&= \mathcal{S}^{\frac{3}{2}} + O\left(\frac{1}{n^{2(1+2\alpha)/3}}\right), \quad n\to\infty, \qquad (4.17)
\end{aligned}
$$

$$
\begin{aligned}
\|\tilde{U}_n\|_6^6 &= \int_{\mathbb{R}^3}|\tilde{U}_n|^6\mathrm{d}x = 4\pi\int_0^{+\infty}r^2|\tilde{\Theta}_n(r)|^6\mathrm{d}r\\
&= 12\sqrt{3}\pi\left[\int_0^{n^\alpha}\frac{n^3r^2}{\left(1+n^2r^2\right)^3}\mathrm{d}r + \left(\frac{n}{1+n^{2(1+\alpha)}}\right)^3\int_{n^\alpha}^{R_n}\frac{r^2(R_n-r)^6}{(R_n-n^\alpha)^6}\mathrm{d}r\right]\\
&= 12\sqrt{3}\pi\left[\int_0^{n^{1+\alpha}}\frac{s^2}{\left(1+s^2\right)^3}\mathrm{d}s + \left(\frac{n}{1+n^{2(1+\alpha)}}\right)^3\int_0^{R_n-n^\alpha}\frac{s^6(R_n-s)^2}{(R_n-n^\alpha)^6}\mathrm{d}s\right]\\
&= \mathcal{S}^{\frac{3}{2}} + 12\sqrt{3}\pi\left[-\int_{n^{1+\alpha}}^{+\infty}\frac{s^2}{\left(1+s^2\right)^3}\mathrm{d}s\right.\\
&\qquad\left. + R_n^3\left(\frac{n}{1+n^{2(1+\alpha)}}\right)^3\int_0^{1-n^\alpha/R_n}\frac{s^6(1-s)^2}{(1-n^\alpha/R_n)^6}\mathrm{d}s\right]\\
&= \mathcal{S}^{\frac{3}{2}} + O\left(\frac{1}{n^{2(1+2\alpha)}}\right), \quad n\to\infty \qquad (4.18)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\tilde{U}_n\|_q^q &= \int_{\mathbb{R}^3}|\tilde{U}_n|^q\mathrm{d}x = 4\pi\int_0^{+\infty}r^2|\tilde{\Theta}_n(r)|^q\mathrm{d}r\\
&\ge 4\cdot 3^{q/4}\pi\int_0^{n^\alpha}\frac{n^{q/2}r^2}{\left(1+n^2r^2\right)^{q/2}}\mathrm{d}r\\
&\ge \frac{4\cdot 3^{q/4}\pi}{n^{(6-q)/2}}\int_0^1\frac{s^2}{\left(1+s^2\right)^{q/2}}\mathrm{d}s := \frac{K_0}{n^{3-q/2}}. \qquad (4.19)
\end{aligned}
$$

Both (4.16) and (4.18) imply that $\tilde{U}_n\in\mathcal{S}_c$.

Lemma 4.3 *Let $\frac{10}{3}\le q<6$, $\mu>0$ and $c>0$. Then there exists $\bar{n}\in\mathbb{N}$ such that*

$$
\sup_{t>0}\Phi\left(t^{3/2}(\tilde{U}_{\bar{n}})_t\right) < \Theta^*. \qquad (4.20)
$$

Proof Set

$$
t_{**}^2 = \frac{1}{2}\left[b\mathcal{S}^{\frac{3}{2}} + \sqrt{b^2S^3+4a}\right]. \qquad (4.21)
$$

By (1.9) and (4.21), we can deduce

$$
\begin{aligned}
\frac{\mathcal{S}^{\frac{3}{2}}}{2}\left(at^2+\frac{b\mathcal{S}^{\frac{3}{2}}}{2}t^4-\frac{1}{3}t^6\right) &< \frac{\mathcal{S}^{\frac{3}{2}}}{2}\left(at_{**}^2+\frac{b\mathcal{S}^{\frac{3}{2}}}{2}t_{**}^4-\frac{1}{3}t_{**}^6\right)\\
&= \Theta^*, \quad \forall\, t\in(0,t_{**})\cup(t_{**},+\infty). \qquad (4.22)
\end{aligned}
$$

From (1.2), (4.17), (4.18) and (4.19), we have

$$\begin{aligned}
\Phi\left(t^{3/2}(\tilde{U}_n)_t\right) &= \frac{at^2}{2}\|\nabla\tilde{U}_n\|_2^2 + \frac{bt^4}{4}\|\nabla\tilde{U}_n\|_2^4 - \frac{\mu t^{3(q-2)/2}}{q}\|\tilde{U}_n\|_q^q - \frac{t^6}{6}\|\tilde{U}_n\|_6^6 \\
&\le \frac{at^2}{2}\left[\mathcal{S}^{\frac{3}{2}} + O\left(\frac{1}{n^{2(1+2\alpha)/3}}\right)\right] + \frac{bt^4}{4}\left[\mathcal{S}^{\frac{3}{2}} + O\left(\frac{1}{n^{2(1+2\alpha)/3}}\right)\right]^2 \\
&\quad - \frac{t^6}{6}\left[\mathcal{S}^{\frac{3}{2}} + O\left(\frac{1}{n^{2(1+2\alpha)}}\right)\right] - \frac{K_0\mu t^{3(q-2)/2}}{qn^{3-q/2}} \\
&= \frac{\mathcal{S}^{\frac{3}{2}}}{2}\left(at^2 + \frac{b\mathcal{S}^{\frac{3}{2}}}{2}t^4 - \frac{1}{3}t^6\right) + \frac{2at^2+bt^4}{4}\left[O\left(\frac{1}{n^{2(1+2\alpha)/3}}\right)\right] \\
&\quad - \frac{t^6}{6}\left[O\left(\frac{1}{n^{2(1+2\alpha)}}\right)\right] - \frac{K_0\mu t^{3(q-2)/2}}{qn^{3-q/2}}, \quad \forall\, t > 0. \qquad (4.23)
\end{aligned}$$

Hence, it follows from (4.22), (4.23) and the fact $\max\{(14-3q)/8, 0\} < \alpha < 1$ that there exists $\bar{n} \in \mathbb{N}$ such that (4.20) holds. □

Lemma 4.4 *Let* $\frac{10}{3} \le q < \frac{14}{3}$, $\mu > 0$ *and* $c \in (0, c_2]$. *Then there holds*

$$\hat{M}(c) < \Theta^*. \qquad (4.24)$$

Proof Let $\bar{n} \in \mathbb{N}$ be given by (4.20). Then it follows from (1.2) that

$$\begin{aligned}
\Phi\left(t^{3/2}(\tilde{U}_{\bar{n}})_t\right) &= \frac{at^2}{2}\|\nabla\tilde{U}_{\bar{n}}\|_2^2 + \frac{bt^4}{4}\|\nabla\tilde{U}_{\bar{n}}\|_2^4 \\
&\quad - \frac{\mu t^{3(q-2)/2}}{q}\|\tilde{U}_{\bar{n}}\|_q^q - \frac{t^6}{6}\|\tilde{U}_{\bar{n}}\|_6^6, \quad \forall\, t > 0. \qquad (4.25)
\end{aligned}$$

By (4.25), we can deduce that there exist $t_1 > 0$ small enough and $t_2 > 0$ large enough such that

$$\left\|\nabla\left(t_1^{3/2}(\tilde{U}_{\bar{n}})_{t_1}\right)\right\|_2^2 = t_1^2\|\nabla\tilde{U}_{\bar{n}}\|_2^2 < \vartheta_c, \quad \text{and} \quad \Phi\left(t_2^{3/2}(\tilde{U}_{\bar{n}})_{t_2}\right) < 0. \qquad (4.26)$$

Let $\gamma_0(t) := [t_1 + (t_2 - t_1)t]^{3/2}(\tilde{U}_{\bar{n}})_{t_1+(t_2-t_1)t}$. Then $\gamma_0 \in \hat{\Gamma}_c$ which is defined by Lemma 4.1. Therefore, we have by Lemma 4.3

$$\hat{M}(c) \le \sup_{t>0} \Phi\left(t^{3/2}(\tilde{U}_{\bar{n}})_t\right) < \Theta^*.$$

This shows (4.24) holds. □

Proof of Theorems 1.3 In view of Lemmas 4.2 and 4.4, there exists $\{u_n\} \subset \mathcal{S}_c \cap H^1_{\text{rad}}(\mathbb{R}^3)$ such that

$$\|u_n\|_2^2 = c, \quad \Phi(u_n) \to \hat{M}(c) \in (0, \Theta^*), \quad \Phi(u_n)|'_{\mathcal{S}_c} \to 0, \quad \mathcal{P}(u_n) \to 0. \qquad (4.27)$$

It follows from (1.2), (1.16), (1.17) and (4.27) that

$$\begin{aligned}\hat{M}(c)+o(1) &= \Phi(u_n)-\frac{1}{6}\mathcal{P}(u_n)\\ &= \frac{a}{3}\|\nabla u_n\|_2^2+\frac{b}{12}\|\nabla u_n\|_2^4-\frac{\mu(6-q)}{4q}\|u_n\|_q^q\\ &\geq \frac{a}{3}\|\nabla u_n\|_2^2+\frac{b}{12}\|\nabla u_n\|_2^4-\frac{\mu(6-q)}{4q}\mathcal{C}_q^q c^{(6-q)/4}\|\nabla u_n\|_2^{3(q-2)/2}.\end{aligned} \tag{4.28}$$

Since $\frac{10}{3}\leq q<\frac{14}{3}$, it follows that $\{\|u_n\|\}$ is bounded. Similar to the proof of Theorem 1.2, one has (3.96)–(3.114) instead of $M(c)$ by $\hat{M}(c)$. From (1.14), (3.107), (3.110), (3.113) and (3.114), we have

$$\begin{aligned}&\hat{M}(c)+o(1)\\ &= \frac{a}{3}\|\nabla v_n\|_2^2+\frac{b}{12}\|\nabla v_n\|_2^4+\frac{b}{3}\|\nabla\bar{u}\|_2^2\|\nabla v_n\|_2^2+\Phi(\bar{u})+o(1)\\ &= \frac{a}{3}\|\nabla v_n\|_2^2+\frac{b}{12}\|\nabla v_n\|_2^4+\frac{b}{6}\|\nabla\bar{u}\|_2^2\|\nabla v_n\|_2^2\\ &\quad+\frac{a}{3}\|\nabla\bar{u}\|_2^2+\frac{b}{12}\|\nabla\bar{u}\|_2^4-\frac{\mu}{4q}(6-q)\|\bar{u}\|_q^q+o(1)\\ &= \frac{2a+b\|\nabla\bar{u}\|_2^2}{6}l_1+\frac{b}{12}l_2^2+\frac{a}{3}\|\nabla\bar{u}\|_2^2+\frac{b}{12}\|\nabla\bar{u}\|_2^4-\frac{\mu(6-q)}{4q}\|\bar{u}\|_q^q+o(1)\\ &\geq \frac{(2a+b\|\nabla\bar{u}\|_2^2)\mathcal{S}}{12}\left[b\mathcal{S}^2+\sqrt{b^2\mathcal{S}^4+4\left(a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}}\right]\\ &\quad+\frac{b\mathcal{S}^2}{48}\left[b\mathcal{S}^2+\sqrt{b^2\mathcal{S}^4+4\left(a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}}\right]^2+\frac{a}{3}\|\nabla\bar{u}\|_2^2+\frac{b}{12}\|\nabla\bar{u}\|_2^4\\ &\quad-\frac{\mu(6-q)}{4q}\|\bar{u}\|_q^q+o(1)\\ &=\frac{ab\mathcal{S}^3}{4}+\frac{b^3\mathcal{S}^6}{24}+\frac{b^2S^4+4a\mathcal{S}}{24}\sqrt{b^2S^4+4\left(a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}}+\frac{b^2\mathcal{S}^3}{6}\|\nabla\bar{u}\|_2^2\\ &\quad+\frac{b\mathcal{S}}{12}\sqrt{b^2S^4+4\left(a+b\|\nabla\bar{u}\|_2^2\right)\mathcal{S}}\,\|\nabla\bar{u}\|_2^2+\frac{a}{3}\|\nabla\bar{u}\|_2^2+\frac{b}{12}\|\nabla\bar{u}\|_2^4\\ &\quad-\frac{\mu(6-q)}{4q}\mathcal{C}_q^q c^{(6-q)/4}\|\nabla\bar{u}\|_2^{3(q-2)/2}+o(1)\\ &\geq\frac{ab\mathcal{S}^3}{4}+\frac{b^3\mathcal{S}^6}{24}+\frac{\left(b^2S^4+4a\mathcal{S}\right)^{3/2}}{24}+\left(\frac{a}{3}+\frac{b^2\mathcal{S}^3}{6}+\frac{b\mathcal{S}}{12}\sqrt{b^2S^4+4a\mathcal{S}}\right)\|\nabla\bar{u}\|_2^2\\ &\quad+\frac{b}{12}\|\nabla\bar{u}\|_2^4-\frac{\mu(6-q)}{4q}\mathcal{C}_q^q c^{(6-q)/4}\|\nabla\bar{u}\|_2^{3(q-2)/2}+o(1)\end{aligned}$$

$$\geq \Theta^* + \left\{ \left[\frac{4}{14-3q} \left(\frac{a}{3} + \frac{b^2 \mathcal{S}^3}{6} + \frac{b\mathcal{S}}{12}\sqrt{b^2 S^4 + 4a\mathcal{S}} \right) \right]^{\frac{14-3q}{4}} \left[\frac{b}{3(3q-10)} \right]^{\frac{3q-10}{4}} \right.$$
$$\left. - \frac{\mu(6-q)}{4q} C_q^q c^{(6-q)/4} \right\} \|\nabla \bar{u}\|_2^{3(q-2)/2} + o(1)$$
$$\geq \Theta^* + o(1), \tag{4.29}$$

which contradicts with (4.27). This shows that $l_1 = 0$, i.e. $\|\nabla v_n\| \to 0$, and so

$$\|\nabla u_n\|_2^2 \to \|\nabla \bar{u}\|_2^2, \quad \|u_n\|_6^6 \to \|\bar{u}\|_6^6. \tag{4.30}$$

Now from (1.2), (3.97), (3.98), (3.99), (3.105), (3.108), (4.27) and (4.30), it is easy to deduce that

$$\lambda_c > 0, \quad \|\bar{u}\|_2^2 = c, \quad \Phi'(\bar{u}) + \lambda_c \bar{u} = 0, \quad \Phi(\bar{u}) = \hat{M}(c).$$

□

5 The case when $\frac{14}{3} \leq q < 6$

In this section, we study the case $\frac{14}{3} \leq q < 6$, and finish the proofs of Theorems 1.4 and 1.5.

Let us define the following function

$$h(t) := \frac{1-t^4}{4} - \frac{1-t^6}{6}, \quad \forall\, t \geq 0. \tag{5.1}$$

It is easy to see that $h(t) > h(1) = 0$ for all $t \in [0, 1) \cup (1, +\infty)$. With it, we establish the following crucial inequality,

Lemma 5.1 *Let $\frac{14}{3} \leq q < 6$, $\mu > 0$ and $c > 0$. Then there holds*

$$\Phi(u) \geq \Phi\left(t^{3/2} u_t\right) + \frac{1-t^4}{4}\mathcal{P}(u) + \frac{a(1-t^2)^2}{4}\|\nabla u\|_2^2$$
$$+ h(t)\|u\|_6^6, \quad \forall\, u \in \mathcal{S}_c, \ t > 0. \tag{5.2}$$

Proof Since $\frac{14}{3} \leq q < 6$, it is easy to see that

$$\frac{3(q-2)(1-t^4)}{8q} - \frac{1-t^{3(q-2)/2}}{q} \geq 0, \quad \forall\, t \geq 0. \tag{5.3}$$

From (1.2), (1.17), (5.1) and (5.3), one has

$$
\begin{aligned}
\Phi(u) - \Phi\left(t^{3/2}u_t\right) &= \frac{a(1-t^2)}{2}\|\nabla u\|_2^2 + \frac{b(1-t^4)}{4}\|\nabla u\|_2^4 \\
&\quad - \frac{\mu(1-t^{3(q-2)/2})}{q}\|u\|_q^q - \frac{1-t^6}{6}\|u\|_6^6 \\
&= \frac{1-t^4}{4}\left[a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \frac{3\mu(q-2)}{2q}\|u\|_q^q - \|u\|_6^6\right] \\
&\quad + \frac{a(1-t^2)^2}{4}\|\nabla u\|_2^2 + \mu\left(\frac{3(q-2)(1-t^4)}{8q} - \frac{1-t^{3(q-2)/2}}{q}\right)\|u\|_q^q \\
&\quad + \left(\frac{1-t^4}{4} - \frac{1-t^6}{6}\right)\|u\|_6^6 \\
&\geq \frac{1-t^4}{4}\mathcal{P}(u) + \frac{a(1-t^2)^2}{4}\|\nabla u\|_2^2 + h(t)\|u\|_6^6.
\end{aligned}
$$

□

From Lemma 5.1, we have the following corollary.

Corollary 5.2 *Let* $\frac{14}{3} \leq q < 6$*,* $\mu > 0$ *and* $c > 0$*. Then for* $u \in \mathcal{M}(c)$*, there holds*

$$
\Phi(u) = \max_{t>0} \Phi\left(t^{3/2}u_t\right). \tag{5.4}
$$

Lemma 5.3 *Let* $\frac{14}{3} \leq q < 6$*,* $\mu > 0$ *and* $c > 0$*. Then for any* $u \in \mathcal{S}_c$*, there exists a unique* $t_u > 0$ *such that* $t_u^{3/2}u_{t_u} \in \mathcal{M}(c)$*.*

The proof of Lemma 5.3 is standard, so we omit it.

From Corollary 5.2 and Lemma 5.3, we have the following lemma.

Lemma 5.4 *Let* $\frac{14}{3} \leq q < 6$*,* $\mu > 0$ *and* $c > 0$*. Then*

$$
\inf_{u\in\mathcal{M}(c)} \Phi(u) := \tilde{m}(c) = \inf_{u\in\mathcal{S}_c} \max_{t>0} \Phi\left(t^{3/2}u_t\right). \tag{5.5}
$$

By the Brezis–Lieb lemma, we have the following lemma.

Lemma 5.5 *Let* $\frac{14}{3} \leq q < 6$*,* $\mu > 0$ *and* $c > 0$*. If* $u_n \rightharpoonup \bar{u}$ *in* $H^1(\mathbb{R}^3)$*, then*

$$
\Phi(u_n) = \Phi(\bar{u}) + \Phi(u_n - \bar{u}) + \frac{b}{2}\|\nabla\bar{u}\|_2^2\|\nabla(u_n - \bar{u})\|_2^2 + o(1) \tag{5.6}
$$

and

$$
\mathcal{P}(u_n) = \mathcal{P}(\bar{u}) + \mathcal{P}(u_n - \bar{u}) + 2b\|\nabla\bar{u}\|_2^2\|\nabla(u_n - \bar{u})\|_2^2 + o(1). \tag{5.7}
$$

Lemma 5.6 *Let* $\frac{14}{3} \leq q < 6$*,* $\mu > 0$ *and* $c > 0$*. Then*

(i) *there exists* $\vartheta_0 > 0$ *such that* $\|\nabla u\|_2 \geq \vartheta_0, \ \forall\, u \in \mathcal{M}(c)$;
(ii) $\tilde{m}(c) > 0$.

Proof (i) Since $\mathcal{P}(u) = 0, \ \forall\, u \in \mathcal{M}(c)$, by (1.8), (1.16) and (1.17), one has

$$\begin{aligned} a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 &= \|u\|_6^6 + \frac{3\mu(q-2)}{2q}\|u\|_q^q \\ &\leq \frac{1}{\mathcal{S}^3}\|\nabla u\|_2^6 \\ &\quad + \frac{3\mu(q-2)}{2q}\mathcal{C}_q^q c^{(6-q)/4}\|\nabla u\|_2^{3(q-2)/2}, \quad \forall\, u \in \mathcal{M}(c), \end{aligned} \tag{5.8}$$

which implies

$$a \leq \frac{1}{\mathcal{S}^3}\|\nabla u\|_2^4 + \frac{3\mu(q-2)}{2q}\mathcal{C}_q^q c^{(6-q)/4}\|\nabla u\|_2^{(3q-10)/2}, \quad \forall\, u \in \mathcal{M}(c).$$

Since $\frac{14}{3} \leq q < 6$, then the above inequality shows there exists $\vartheta_0 > 0$ such that

$$\|\nabla u\|_2 \geq \vartheta_0, \quad \forall\, u \in \mathcal{M}(c). \tag{5.9}$$

(ii) From (1.2), (1.17) and (5.9), we have

$$\begin{aligned} \Phi(u) &= \Phi(u) - \frac{2}{3(q-2)}\mathcal{P}(u) \\ &= \frac{a(3q-10)}{6(q-2)}\|\nabla u\|_2^2 + \frac{b(3q-14)}{12(q-2)}\|\nabla u\|_2^4 + \frac{6-q}{6(q-2)}\|u\|_6^6 \\ &\geq \frac{a(3q-10)}{6(q-2)}\vartheta_0^2, \quad \forall\, u \in \mathcal{M}(c). \end{aligned} \tag{5.10}$$

This shows that $\tilde{m}(c) = \inf_{u\in\mathcal{M}(c)} \Phi(u) > 0$. □

Lemma 5.7 *Let* $\frac{14}{3} \leq q < 6$, $\mu > 0$ *and* $c > 0$. *Then the function* $c \mapsto \tilde{m}(c)$ *is nonincreasing on* $(0, +\infty)$. *In particular, if* $\tilde{m}(c_0')$ *is achieved, then* $\tilde{m}(c_0') > \tilde{m}(c_2')$ *for any* $c_2' > c_0'$.

Proof For any $c_2' > c_0' > 0$, it follows from the definition of $\tilde{m}(c_0')$ that there exists $\{u_n\} \subset \mathcal{M}(c_0')$ such that

$$\Phi(u_n) < \tilde{m}(c_0') + \frac{1}{n}, \quad \forall\, n \in \mathbb{N}. \tag{5.11}$$

Let $\theta := \sqrt{c_2'/c_0'}$ and $v_n(x) := \theta^{-1/2}u_n(x/\theta)$. Then $\|\nabla v_n\|_2^2 = \|\nabla u_n\|_2^2$, $\|v_n\|_6^6 = \|u_n\|_6^6$, $\|v_n\|_q^q = \theta^{3-q/2}\|u_n\|_q^q$ and $\|v_n\|_2^2 = c_2'$. By Lemma 5.3, there exists $t_n > 0$

such that $t_n^{3/2}(v_n)_{t_n} \in \mathcal{M}(c_2')$. Then it follows from (1.2), (5.11) and Corollary 5.2 that

$$\begin{aligned}\tilde{m}(c_2') &\leq \Phi\left(t_n^{3/2}(v_n)_{t_n}\right)\\ &= \frac{at_n^2}{2}\|\nabla v_n\|_2^2 + \frac{bt_n^4}{4}\|\nabla v_n\|_2^4 - \frac{t_n^6}{6}\|v_n\|_6^6 - \frac{\mu t_n^{3(q-2)/2}}{q}\|v_n\|_q^q\\ &= \frac{at_n^2}{2}\|\nabla u_n\|_2^2 + \frac{bt_n^4}{4}\|\nabla u_n\|_2^4 - \frac{t_n^6}{6}\|u_n\|_6^6 - \frac{\mu\theta^{3-q/2}t_n^{3(q-2)/2}}{q}\|u_n\|_q^q\\ &< \Phi\left(t_n^{3/2}(u_n)_{t_n}\right) \leq \Phi(u_n) < \tilde{m}(c_0') + \frac{1}{n},\end{aligned} \tag{5.12}$$

which shows that $\tilde{m}(c_2') \leq \tilde{m}(c_0')$ by letting $n \to \infty$.

If $\tilde{m}(c_0')$ is achieved, i.e., there exists $\tilde{u} \in \mathcal{M}(c_0')$ such that $\Phi(\tilde{u}) = \tilde{m}(c_0')$. By the same argument as in (5.12), we can obtain that $\tilde{m}(c_2') < \tilde{m}(c_0')$. □

By Lemma 4.3, we have the following lemma.

Lemma 5.8 *Let* $\frac{14}{3} \leq q < 6$, $\mu > 0$ *and* $c > 0$. *Then there holds*

$$\tilde{m}(c) < \Theta^*. \tag{5.13}$$

Lemma 5.9 *Let* $\frac{14}{3} \leq q < 6$, $\mu > 0$ *and* $c > 0$. *Then* $\tilde{m}(c)$ *is achieved.*

Proof In view of Lemmas 5.3 and 5.6, we have $\mathcal{M}(c) \neq \emptyset$ and $\tilde{m}(c) > 0$. Let $\{u_n\} \subset \mathcal{M}(c)$ be such that $\Phi(u_n) \to \tilde{m}(c)$. It follows from (1.2) and (1.17) that

$$\tilde{m}(c) + o(1) = \frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{1}{6}\|u_n\|_6^6 - \frac{\mu}{q}\|u_n\|_q^q \tag{5.14}$$

and

$$0 = a\|\nabla u_n\|_2^2 + b\|\nabla u_n\|_2^4 - \|u_n\|_6^6 - \frac{3\mu(q-2)}{2q}\|u_n\|_q^q. \tag{5.15}$$

From (5.14) and (5.15), one has

$$\tilde{m}(c) + o(1) \geq \frac{a}{4}\|\nabla u_n\|_2^2. \tag{5.16}$$

This shows that $\{\|\nabla u_n\|_2\}$ is bounded, and so $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

Let $\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 \mathrm{d}x$. We show that $\delta > 0$. Otherwise, in light of Lions' concentration compactness principle [17, Lemma 1.21], $\|u_n\|_q \to 0$. Hence, it follows from (5.15) that

$$a\|\nabla u_n\|_2^2 + b\|\nabla u_n\|_2^4 = \|u_n\|_6^6 + \frac{3\mu(q-2)}{2q}\|u_n\|_q^q = \|u_n\|_6^6 + o(1). \tag{5.17}$$

Up to a subsequence, we assume that

$$\|\nabla u_n\|_2^2 \to l, \quad \|u_n\|_6^6 \to al + bl^2. \tag{5.18}$$

If $l = 0$, then it follows from (5.14) and (5.18) that $\tilde{m}(c) = 0$, a contradiction. If $l > 0$, by Sobolev inequality (1.8) and (5.18), we have

$$l \geq \frac{\mathcal{S}}{2}\left[b\mathcal{S}^2 + \sqrt{b^2 S^4 + 4a\mathcal{S}}\right]. \tag{5.19}$$

Hence, it follows from (5.14), (5.15), (5.18), (5.19), the definition of $\{u_n\}$ and $\|u_n\|_q \to 0$ that

$$\begin{aligned}\tilde{m}(c) + o(1) &= \Phi(u_n) - \frac{1}{6}\mathcal{P}(u_n)\\ &= \frac{a}{3}\|\nabla u_n\|_2^2 + \frac{b}{12}\|\nabla u_n\|_2^4 - \frac{\mu(6-q)}{4q}\|u_n\|_q^q + o(1)\\ &= \frac{a}{3}\|\nabla u_n\|_2^2 + \frac{b}{12}\|\nabla u_n\|_2^4 + o(1)\\ &\geq \Theta^* + o(1),\end{aligned} \tag{5.20}$$

which contradicts (5.13). Thus $\delta > 0$. Without loss of generality, we may assume the existence of $y_n \in \mathbb{R}^3$ such that $\int_{B_1(y_n)} |u_n|^2 \mathrm{d}x > \frac{\delta}{2}$. Let $\hat{u}_n(x) = u_n(x + y_n)$. Then we have

$$\|\hat{u}_n\|_2^2 = c, \quad \mathcal{P}(\hat{u}_n) = 0, \quad \Phi(\hat{u}_n) \to \tilde{m}(c), \quad \int_{B_1(0)} |\hat{u}_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{5.21}$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u}, & \text{in } H^1(\mathbb{R}^3);\\ \hat{u}_n \to \hat{u}, & \text{in } L^s_{\text{loc}}(\mathbb{R}^3), \ \forall\, s \in [1, 6);\\ \hat{u}_n \to \hat{u}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{5.22}$$

Let $w_n = \hat{u}_n - \hat{u}$. Then (5.22) and Lemma 5.5 yield

$$\Phi(\hat{u}_n) = \Phi(\hat{u}) + \Phi(w_n) + \frac{b}{2}\|\nabla \hat{u}\|_2^2\|\nabla w_n\|_2^2 + o(1) \tag{5.23}$$

and

$$\mathcal{P}(\hat{u}_n) = \mathcal{P}(\hat{u}) + \mathcal{P}(w_n) + 2b\|\nabla \hat{u}\|_2^2\|\nabla w_n\|_2^2 + o(1). \tag{5.24}$$

Set

$$Q(u) := \frac{a}{4}\|\nabla u\|_2^2 + \frac{1}{12}\|u\|_6^6 + \frac{\mu(3q-14)}{8q}\|u\|_q^q. \tag{5.25}$$

Then it follows from (1.2), (1.17), (5.21), (5.23), (5.24) and (5.25) that

$$\tilde{m}(c) - Q(\hat{u}) + o(1) = Q(w_n) \tag{5.26}$$

and

$$\mathcal{P}(w_n) = -\mathcal{P}(\hat{u}) - 2b\|\nabla\hat{u}\|_2^2\|\nabla w_n\|_2^2 + o(1). \tag{5.27}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then going to this subsequence, we have

$$\|\hat{u}\|_2^2 = c, \quad \Phi(\hat{u}) = \tilde{m}(c), \quad \mathcal{P}(\hat{u}) = 0, \tag{5.28}$$

which implies the conclusion of Lemma 5.9 holds. Next, we assume that $w_n \neq 0$. By (5.21) and (5.22), one has

$$c = \|\hat{u}_n\|_2^2 = \|\hat{u}\|_2^2 + \|w_n\|_2^2 + o(1). \tag{5.29}$$

This implies that $\|\hat{u}\|_2^2 := \hat{c} \leq c$ and $\|w_n\|_2^2 := \tilde{c}_n \leq c$ for large $n \in \mathbb{N}$. We claim that $\mathcal{P}(\hat{u}) \leq 0$. Otherwise, if $\mathcal{P}(\hat{u}) > 0$, then (5.27) implies $\mathcal{P}(w_n) < 0$ for large n. In view of Lemma 5.3, there exists $t_n > 0$ such that $t_n^{3/2}(w_n)_{t_n} \in \mathcal{M}(\tilde{c}_n)$. From (1.2), (1.17), (5.2), (5.26), (5.27), Lemma 5.1 and Lemma 5.7, we obtain

$$\begin{aligned}\tilde{m}(c) - Q(\hat{u}) + o(1) = Q(w_n) &= \Phi(w_n) - \frac{1}{4}\mathcal{P}(w_n)\\ &\geq \Phi\left(t_n^{3/2}(w_n)_{t_n}\right) - \frac{t_n^4}{4}\mathcal{P}(w_n) \geq \tilde{m}(c),\end{aligned}$$

which implies $\mathcal{P}(\hat{u}) \leq 0$ due to $Q(\hat{u}) > 0$. Since $\hat{u} \neq 0$ and $\mathcal{P}(\hat{u}) \leq 0$, in view of Lemma 5.3, there exists $\hat{t} > 0$ such that $\hat{t}^{3/2}\hat{u}_{\hat{t}} \in \mathcal{M}(\hat{c})$. From (1.2), (1.17), (5.2), (5.26), (5.27), Lemmas 5.1, 5.7, Fatou's lemma and the weak semicontinuity of norm, one has

$$\tilde{m}(c) = \lim_{n\to\infty} Q(\hat{u}_n) \geq Q(\hat{u}) = \Phi(\hat{u}) - \frac{1}{4}\mathcal{P}(\hat{u}) \geq \Phi\left(\hat{t}^{3/2}\hat{u}_{\hat{t}}\right) - \frac{\hat{t}^4}{4}\mathcal{P}(\hat{u}) \geq \tilde{m}(c), \tag{5.30}$$

which implies

$$\|\hat{u}\|_2^2 = \hat{c}, \quad \Phi(\hat{u}) = \tilde{m}(\hat{c}) = \tilde{m}(c), \quad \mathcal{P}(\hat{u}) = 0. \tag{5.31}$$

This shows $\tilde{m}(\hat{c})$ is achieved. In view of Lemma 5.7, $\hat{c} = c$. Thus, (5.28) holds also, i.e. the conclusion of Lemma 5.9 holds. □

Lemma 5.10 *Let* $\frac{14}{3} \leq q < 6$, $\mu > 0$ *and* $c > 0$. *If* $\bar{u} \in \mathcal{M}(c)$ *and* $\Phi(\bar{u}) = \tilde{m}(c)$, *then* $\bar{u}$ *is a critical point of* Φ *on* $\mathcal{S}_c$, *i.e.* $\Phi|'_{\mathcal{S}_c}(\bar{u}) = 0$.

Proof Assume that $\Phi|'_{\mathcal{S}_c}(\bar{u}) \neq 0$. Then there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|u - \bar{u}\| \leq 3\delta \Rightarrow \|\Phi|'_{\mathcal{S}_c}(u)\| \geq \varrho. \tag{5.32}$$

It is easy to see that

$$\begin{aligned}\left\|\nabla\left(t^{\frac{3}{2}}\bar{u}_t\right) - \nabla\bar{u}\right\|_2^2 &= \int_{\mathbb{R}^3}\left|\nabla\left(t^{\frac{3}{2}}\bar{u}_t\right) - \nabla\bar{u}\right|^2 \mathrm{d}x \\ &= (t^2+1)\int_{\mathbb{R}^3}|\nabla\bar{u}|^2\mathrm{d}x - 2t^{\frac{3}{2}}\int_{\mathbb{R}^3}\nabla\bar{u}_t\cdot\nabla\bar{u}\mathrm{d}x \to 0, \quad t\to 1.\end{aligned} \tag{5.33}$$

Thus, there exists $\delta_1 \in (0, 1/4)$ such that

$$|t-1| < \delta_1 \Rightarrow \left\|t^{\frac{3}{2}}\bar{u}_t - \bar{u}\right\| < \delta. \tag{5.34}$$

In view of Lemma 5.1, one has

$$\Phi\left(t^{\frac{3}{2}}\bar{u}_t\right) \leq \Phi(\bar{u}) - h(t)\|\bar{u}\|_6^6 = \tilde{m}(c) - h(t)\|\bar{u}\|_6^6, \quad \forall\, t > 0. \tag{5.35}$$

It follows from (1.17) that there exist $T_1 \in (0, 1/2)$ and $T_2 \in (3/2, +\infty)$ such that

$$\mathcal{P}\left(T_1^{\frac{3}{2}}\bar{u}_{T_1}\right) > 0, \quad \mathcal{P}\left(T_2^{\frac{3}{2}}\bar{u}_{T_2}\right) < 0. \tag{5.36}$$

Let

$$\varepsilon := \min\left\{\frac{1}{4}h(T_1)\|\bar{u}\|_6^6, \frac{1}{4}h(T_2)\|\bar{u}\|_6^6, 1, \frac{\varrho\delta}{8}\right\}, \quad S := \{v \in \mathcal{S}_c : \|v - \bar{u}\| < \delta\}.$$

Then Lemma 2.1 yields a deformation $\eta \in \mathcal{C}([0,1]\times\mathcal{S}_c, \mathcal{S}_c)$ such that

(i) $\eta(1,u) = u$ if $\Phi(u) < \tilde{m}(c) - 2\varepsilon$ or $\Phi(u) > \tilde{m}(c) + 2\varepsilon$;
(ii) $\eta\left(1, \Phi^{\tilde{m}(c)+\varepsilon} \cap S\right) \subset \Phi^{\tilde{m}(c)-\varepsilon}$;
(iii) $\Phi(\eta(1,u)) \leq \Phi(u), \ \forall\, u \in \mathcal{S}_c$;
(iv) $\eta(1,u)$ is a homeomorphism of $\mathcal{S}_c$.

By Corollary 5.2, $\Phi\left(t^{\frac{3}{2}}\bar{u}_t\right) \leq \Phi(\bar{u}) = \tilde{m}(c)$ for $t > 0$, then it follows from (5.34) and ii) that

$$\Phi\left(\eta\left(1, t^{\frac{3}{2}}\bar{u}_t\right)\right) \leq \tilde{m}(c) - \varepsilon, \quad \forall\, t > 0, \ \ |t-1| < \delta_1. \tag{5.37}$$

On the other hand, by iii) and (5.35), one has

$$\Phi\left(\eta\left(1, t^{\frac{3}{2}}\bar{u}_t\right)\right) \leq \Phi\left(t^{\frac{3}{2}}\bar{u}_t\right) \leq \tilde{m}(c) - h(t)\|\bar{u}\|_6^6$$
$$\leq \tilde{m}(c) - \delta_2\|\bar{u}\|_6^6, \quad \forall\, t > 0, \quad |t-1| \geq \delta_1, \tag{5.38}$$

where

$$\delta_2 := \min\{h(1-\delta_1), h(1+\delta_1)\} > 0.$$

Combining (5.37) with (5.38), we have

$$\max_{t\in[T_1,T_2]} \Phi\left(\eta\left(1, t^{\frac{3}{2}}\bar{u}_t\right)\right) < \tilde{m}(c). \tag{5.39}$$

Define $\Psi_0(t) := \mathcal{P}\left(\eta\left(1, t^{\frac{3}{2}}\bar{u}_t\right)\right)$ for $t > 0$. It follows from (5.35) and (i) that $\eta\left(1, t^{\frac{3}{2}}\bar{u}_t\right) = t^{\frac{3}{2}}\bar{u}_t$ for $t = T_1$ and $t = T_2$, which, together with (5.36), implies

$$\Psi_0(T_1) = \mathcal{P}\left(T_1^{\frac{3}{2}}\bar{u}_{T_1}\right) > 0, \quad \Psi_0(T_2) = \mathcal{P}\left(T_2^{\frac{3}{2}}\bar{u}_{T_2}\right) < 0.$$

Since $\Psi_0(t)$ is continuous on $(0, \infty)$, then we have that $\eta\left(1, t^{\frac{3}{2}}\bar{u}_t\right) \cap \mathcal{M}(c) \neq \emptyset$ for some $t_0 \in [T_1, T_2]$, contradicting the definition of $\tilde{m}(c)$. □

Proof of Theorem 1.4 It follows directly combining Lemmas 5.9 and 5.10. □

6 The case when $\mu \leq 0$

In this section, we shall prove Theorem 1.5.

Proof of Theorem 1.5 Assume that $(u, \lambda) \in H^1(\mathbb{R}^3) \times (0, +\infty)$ is a solution of Eq. (1.1). Then it follows from (1.1) and the Pohozaev type identity that

$$\left(a + b\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 + \lambda\|u\|_2^2 - \mu\|u\|_q^q - \|u\|_6^6 = 0 \tag{6.1}$$

and

$$\left(a + b\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 + 3\lambda\|u\|_2^2 - \frac{6\mu}{q}\|u\|_q^q - \|u\|_6^6 = 0. \tag{6.2}$$

Combining (6.1) with (6.2), we have

$$0 < \lambda c = \lambda\|u\|_2^2 = \frac{(6-q)\mu}{2q}\|u\|_q^q \leq 0,$$

which is a contradiction. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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