Mathematische Annalen
https://doi.org/10.1007/s00208-024-02971-0

Mathematische Annalen

Multifractality and intermittency in the limit evolution of polygonal vortex filaments

Valeria Banica[1] · **Daniel Eceizabarrena**[2] · **Andrea R. Nahmod**[2] · **Luis Vega**[3,4]

Received: 4 January 2024 / Revised: 31 July 2024 / Accepted: 8 August 2024

Abstract

With the aim of quantifying turbulent behaviors of vortex filaments, we study the multifractality and intermittency of the family of generalized Riemann's non-differentiable functions

$$R_{x_0}(t) = \sum_{n \neq 0} \frac{e^{2\pi i (n^2 t + n x_0)}}{n^2}, \qquad x_0 \in [0, 1].$$

These functions represent, in a certain limit, the trajectory of regular polygonal vortex filaments that evolve according to the binormal flow. When x_0 is rational, we show that R_{x_0} is multifractal and intermittent by completely determining the spectrum of singularities of R_{x_0} and computing the L^p norms of its Fourier high-pass filters, which are analogues of structure functions. We prove that R_{x_0} has a multifractal behavior also when x_0 is irrational. The proofs rely on a careful design of Diophantine sets that depend on x_0, which we study by using the Duffin–Schaeffer theorem and the Mass Transference Principle.

✉ Daniel Eceizabarrena
deceizabarrena@bcamath.org

Valeria Banica
Valeria.Banica@math.cnrs.fr

Andrea R. Nahmod
nahmod@umass.edu

Luis Vega
lvega@bcamath.org

1 Laboratoire Jacques-Louis Lions (LJLL), Sorbonne Universite, CNRS, Université de Paris, Paris, France

2 Department of Mathematics and Statistics, University of Massachusetts Amherst, Amherst, Massachusetts, USA

3 BCAM-Basque Center for Applied Mathematics, Bilbao, Spain

4 Departamento de Matemáticas, Universidad del País Vasco (UPV/EHU), Leioa, Spain

Published online: 17 September 2024

Mathematics Subject Classification 11J82 · 11J83 · 26A27 · 28A78 · 42A16 · 76F99

1 Introduction

Multifractality and intermittency are among the main properties expected in turbulent flows but, as usual in the theory of turbulence, it is challenging to analyze them rigorously. The motivation of this article is to quantify the multifractal and intermittent behavior of regular polygonal vortex filaments that evolve with the binormal flow. This evolution is represented, in a certain limit, by the function $R_{x_0} : \mathbb{R} \to \mathbb{C}$ defined by

$$R_{x_0}(t) = \sum_{n \neq 0} \frac{e^{2\pi i (n^2 t + n x_0)}}{n^2}, \tag{1}$$

for $x_0 \in [0, 1]$ fixed. This function is one of the possible generalizations of the classic Riemann's non-differentiable function, which is recovered when $x_0 = 0$, and it can also be seen as the solution to a periodic Cauchy problem for the free Schrödinger equation. In this article we study the multifractality and intermittency of R_{x_0}, which until now was unknown for $x_0 \neq 0$:

- When $x_0 \in \mathbb{Q}$, we completely describe the multifractality of R_{x_0} by computing its spectrum of singularities (Theorem 1.1). We also compute the L^p norms of its Fourier high-pass filters to deduce its intermittency exponents (Theorem 1.6) and show that R_{x_0} is intermittent.
- When $x_0 \notin \mathbb{Q}$, we give a result that proves multifractality (Theorem 1.3) and strongly suggests that the spectrum of singularities depends on the irrationality of x_0, and hence that it is different from when $x_0 \in \mathbb{Q}$.

The main novelty in this article is a careful design of Diophantine sets and the use of the Duffin–Schaeffer theorem and the Mass Transference Principle to compute their measure and dimension. When $x_0 \in \mathbb{Q}$, we use the partial Duffin–Schaeffer theorem as proved by Duffin and Schaeffer in [21], while when $x_0 \notin \mathbb{Q}$ we need the full strength of the theorem as proved by Koukoulopoulos and Maynard [37]. We give an overview of these arguments in Sect. 2. Before that, we introduce the concepts of multifractality and intermittency in Sect. 1.1, we discuss the connection of R_{x_0} and vortex filaments in Sect. 1.2 and we state our results in Sects. 1.3 and 1.4.

1.1 Multifractality and intermittency

The concepts of multifractality and intermittency arise in the study of three dimensional turbulence of fluids and waves, both characterized by low regularity and a chaotic behavior. These are caused by an energy cascade by which the energy injected in large scales is transferred to small scales. In this setting, large eddies constantly split in smaller eddies, generating sharp changes in the velocity magnitude. Moreover, this cascade is not expected to be uniform in space, and the rate at which these eddies decrease depends on their location.

Mathematically speaking, an option to measure the irregularity of the velocity v is to compute the local Hölder regularity, that is, the largest $\alpha = \alpha(x)$ such that $|v(x+h) - v(x)| \lesssim |h|^\alpha$ when $|h| \to 0$. The lack of uniformity in space suggests that the Hölder level sets $D_\alpha = \{ x : \alpha(x) = \alpha \}$ should be non-empty, and of different size, for many values of α. In this context, the spectrum of singularities is defined as $d(\alpha) = \dim_{\mathcal{H}} D_\alpha$, where $\dim_{\mathcal{H}}$ is the Hausdorff dimension, and the velocity v is said to be **multifractal** if $d(\alpha)$ takes values in multiple Hölder regularities α.

On the other hand, intermittency is a measure of the likelihood of localized bursts or outlier events. One way to quantify it is by analyzing the structure functions $S_p(h) = \langle |v(x+h) - v(x)|^p \rangle$ of the velocity when the scale h tends to zero. More precisely, defining the flatness as

$$F_4(h) = \frac{S_4(h)}{S_2(h)^2}, \qquad \text{for very small } h, \tag{2}$$

we have **small-scale intermittency**[1] if $\lim_{h\to 0} F_4(h) = +\infty$. Assuming the typical power law

$$S_p(h) \simeq |h|^{\zeta_p}, \tag{3}$$

it is usual to rephrase the definition of intermittency as $\zeta_4 - 2\zeta_2 < 0$ for the intermittency exponent[2] ζ_p. This definition, and in particular (2), is inspired by the probabilistic concept of kurtosis,[3] which quantifies how large the tails of the underlying probability distribution are. A large kurtosis implies fat tails, which suggests that outlier events are more likely than for a normal distribution, agreeing with the widespread idea of non-Gaussianity. More generally, moments $F_p(h) = S_p(h)/S_2(h)^{p/2}$ of order $p \geq 4$ can be used to measure the tails of a probability distribution (see [27, p. 124]) and therefore intermittency, so it is common in recent physics literature to measure ζ_p for different p (see [42] and references therein, also [2] for a numeric intermittent model). The intermittency condition is then rewritten as $\zeta_p - p\zeta_2/2 < 0$, a behavior that corresponds to a sublinear ζ_p.

1.2 R_{x_0} as the trajectory of polygonal vortex filaments

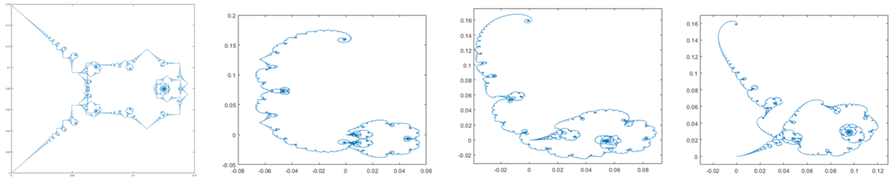
The binormal flow is a model introduced by Da Rios[4] in 1906 [19] as an approximation to the evolution of a vortex filament according to Euler equation and whose validity has been precisely and rigorously described theoretically by Fontelos and Vega in [26] in the setting of the Navier–Stokes equations. This model describes the motion of the filament $\boldsymbol{X} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$, $\boldsymbol{X} = \boldsymbol{X}(x,t)$ by the equation $\boldsymbol{X}_t = \boldsymbol{X}_x \times \boldsymbol{X}_{xx}$. Inspired

[1] Proposed by Frisch [27, p. 122, (8.2)] and Anselmet et al. [1].

[2] In this setting, intermittency is regarded as a nonlinear correction to Kolmogorov's theory (see [12, Section 2.4]) which predicted the exponents ζ_p to be a linear function of p and hence $\zeta_4 - 2\zeta_2 = 0$ and, in general, $\zeta_p - p\zeta_2/2 = 0$.

[3] The fourth standardized moment, sometimes also referred to as *tailedness*.

[4] Explored also by Levi-Civita in [38].



(A) $x_0 = 0$ (B) $x_0 = 1/4$ (C) $x_0 = 1/7$ (D) $x_0 = 1/17$

Fig. 1 Image of ϕ_{x_0}, $t \in [0, 1]$, defined in (5), for some values of x_0

by Jerrard and Smets [33], De la Hoz and Vega [20] observed numerically that if the initial filament $\boldsymbol{X}_M(x, 0)$ is a regular polygon with M corners at the integers $x \in \mathbb{Z}$, then the trajectory of the corners $\boldsymbol{X}_M(0, t)$ is a plane curve which, identifying the plane with $\mathbb{C}$ and when M is large, looks like

$$\phi(t) = \sum_{n \in \mathbb{Z}} \frac{e^{2\pi i n^2 t} - 1}{n^2} = 2\pi i t - \frac{\pi^2}{3} + R_0(t). \tag{4}$$

Moreover, let $\boldsymbol{\chi}_M(x, 0)$ be an infinite polygonal line that loops the polygon of M sides a finite but large number of times and ends in two half-lines, symmetrized at $x = 0$. Banica and Vega rigorously proved in [4] that, under certain hypotheses, its binormal flow evolution $\boldsymbol{\chi}_M(x, t)$ obtained in [3] satisfies

$$\lim_{M \to \infty} M \, \boldsymbol{\chi}_M(x_0, t) = \phi_{x_0}(t) := \sum_{n \in \mathbb{Z}} \frac{e^{2\pi i n^2 t} - 1}{n^2} \, e^{2\pi i n x_0}, \quad \forall x_0 \in [0, 1]. \tag{5}$$

We show in Figs. 1 and 2 the image of ϕ_{x_0} for some values of x_0. Like in (4), noticing that the Fourier series $\sum_{n \neq 0} \frac{e^{2\pi i n x}}{n^2}$ is $2\pi^2 \left(x^2 - x + \frac{1}{6}\right)$, we can write

$$\phi_{x_0}(t) = 2\pi i t - 2\pi^2 \Big(x_0^2 - x_0 + \frac{1}{6}\Big) + R_{x_0}(t),$$

which shows that ϕ_{x_0} and R_{x_0} have the same regularity as functions of t. In other words, R_{x_0} captures the regularity of the limit trajectory of polygonal vortex filaments that evolve with the binormal flow. This connection motivates us to study the multifractality and intermittency of R_{x_0}.

1.3 Definitions and notation

We now rigorously define the concepts discussed above.

1.3.1 Holder regularity

A function $f : \mathbb{R} \to \mathbb{C}$ is α-Hölder at $t \in \mathbb{R}$, which we denote by $f \in C^\alpha(t)$, if there exists a polynomial P_t of degree at most α such that $|f(t+h) - P_t(h)| \leq C|h|^\alpha$

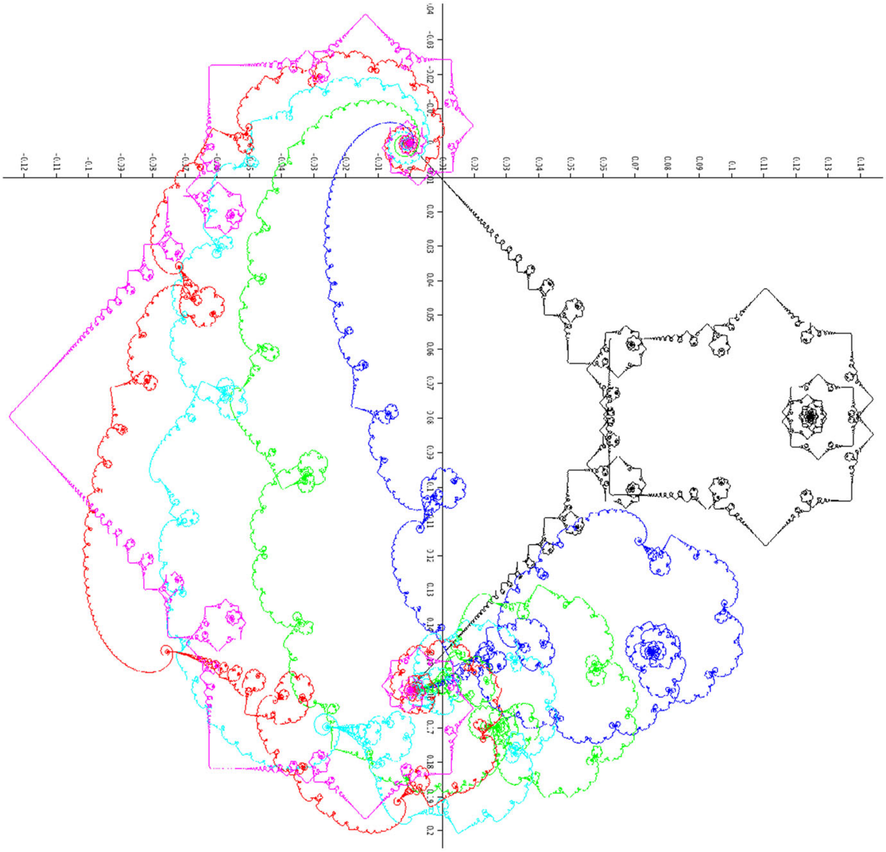


Fig. 2 The images of ϕ_{x_0}, $t \in [0, 1]$, for the values $x_0 = 0, 0.1, 0.2, 0.3, 0.4, 0.5$, from the rightmost to the leftmost

for some constant $C > 0$ and for h small enough. In particular, if $0 < \alpha < 1$, the definition above becomes

$$f \in \mathcal{C}^{\alpha}(t) \quad \Longleftrightarrow \quad |f(t+h) - f(t)| \leq C|h|^{\alpha}, \quad \text{for } h \text{ small enough.}$$

The local Hölder exponent of f at t is $\alpha_f(t) = \sup\{\alpha : f \in \mathcal{C}^{\alpha}(t)\}$. We say f is globally α-Hölder if $f \in \mathcal{C}^{\alpha}(t)$ for all $t \in \mathbb{R}$.

1.3.2 Spectrum of singularities

The spectrum of singularities of f is

$$d_f(\alpha) = \dim_{\mathcal{H}}\{t : \alpha_f(t) = \alpha\},$$

where $\dim_{\mathcal{H}}$ is the Hausdorff dimension,[5] and convene that $d(\alpha) = -\infty$ if $\{t : \alpha_f(t) = \alpha\} = \emptyset$.

1.3.3 Intermittency exponents

As discussed in (3), the exponents ζ_p of the structure functions $S_p(h)$ describe the behavior of the increments of functions in small scales. Here we take the analogous approach of studying the high-frequency behavior of functions. Let $\Phi \in C^\infty(\mathbb{R})$ be a cutoff function such that $\Phi(x) = 0$ in a neighborhood of the origin and $\Phi(x) = 1$ for $|x| \geq 2$. For a periodic function f with Fourier series $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi int}$, define the high-pass filter by

$$P_{\geq N} f(t) = \sum_{n\in\mathbb{Z}} \Phi\Big(\frac{n}{N}\Big)\, a_n\, e^{2\pi int}, \qquad N \in \mathbb{N}.$$

We treat the L^p norms $\|P_{\geq N} f\|_p^p$ as the analytic and Fourier space analogues of the structure functions.[6] Our analogous to the power law (3) is[7]

$$\eta_f(p) = \liminf_{N\to\infty} \frac{\log(\|P_{\geq N} f\|_p^p)}{\log(1/N)}, \tag{6}$$

which means that for any $\epsilon > 0$ we have $\|P_{\geq N} f\|_p^p \leq N^{-\eta_f(p)+\epsilon}$ for $N \gg_\epsilon 1$, and that this is optimal in the sense that there is a subsequence $N_k \to \infty$ such that $\|P_{\geq N_k} f\|_p^p \geq N_k^{-\eta_f(p)-\epsilon}$ for $k \gg_\epsilon 1$. We define the ***p*-flatness** to be

$$F_p(N) = \frac{\|P_{\geq N} f\|_p^p}{\|P_{\geq N} f\|_2^p}, \qquad N \gg 1.$$

The corresponding intermittency exponent[8] is $\eta_f(p) - p\,\eta_f(2)/2$.

1.4 Results

To simplify notation, let us denote $\alpha_{R_{x_0}}(t) = \alpha_{x_0}(t)$, $d_{R_{x_0}}(\alpha) = d_{x_0}(\alpha)$ and $\eta_{R_{x_0}}(p) = \eta_{x_0}(p)$ for our function R_{x_0} defined in (1).

---

5 See [25, Sections 3.1–3.2] for definitions and basic properties of Hausdorff measures and the Hausdorff dimension.

6 We may think of the small scale h to be represented by $1/N$, where N is the frequency parameter.

7 The heuristic exponent ζ_p in (3) and $\eta(p)$ defined in (6) are a priori different. However, the definition of ζ_p can be made rigorous using L^p norms so that it is equal to $\eta(p)$, as shown by Jaffard in [32, Prop. 3.1] The exponent $\eta(p)$ is actually related to the Besov regularity of f. Assuming $\|P_{\geq N} f\|_p \simeq \|P_{\simeq N} f\|_p$ (which is the case for R_{x_0}), where $P_{\simeq N} f$ denotes the band-pass filter defined with the cutoff Φ with the additional assumption of compact support, then $\eta(p) = \sup\{s : f \in B^{s/p}_{p,\infty}\}$, where $f \in B^s_{p,q}$ if and only if $(2^{ks}\|P_{\simeq 2^k} f\|)_k \in \ell^q$.

8 If the liminf in (6) is a limit, then $\|P_{\geq N} f\|_p^p \simeq N^{-\eta_p}$ and hence $F_p(N) \simeq N^{-(\eta_f(p)-p\eta_f(2)/2)}$.

Since Weierstrass [47] announced[9] Riemann's non-differentiable function as the first candidate of a continuous and non-differentiable function in 1872, the regularity of R_0 has been studied by several authors. After Hardy [30] and Gerver [28, 29] proved that it is only almost nowhere differentiable (see also the simplified proof of Smith [45]), Duistermaat [22] launched the study of its Hölder regularity. Jaffard completed the picture in his remarkable work [31, Theorem 1] (see also [11] for a recent alternative proof) by computing

$$\alpha_0(t) = \frac{1}{2} + \frac{1}{2\widetilde{\mu}(t)}, \qquad \text{for } t \notin \mathbb{Q}, \tag{7}$$

where $\widetilde{\mu}(t)$ is the exponent of irrationality of t restricted to denominators $q \not\equiv 2 \pmod 4$.[10] He combined this with an adaptation of the Jarník–Besicovitch theorem to prove

$$d_0(\alpha) = \begin{cases} 4\alpha - 2, & 1/2 \leq \alpha \leq 3/4, \\ 0, & \alpha = 3/2, \\ -\infty, & \text{otherwise.} \end{cases}$$

Our first results concern the spectrum of singularities of R_{x_0} for $x_0 \neq 0$.

Theorem 1.1 *Let $x_0 \in \mathbb{Q}$. Then,*

$$d_{x_0}(\alpha) = \begin{cases} 4\alpha - 2, & 1/2 \leq \alpha \leq 3/4, \\ 0, & \alpha = 3/2, \\ -\infty, & \textit{otherwise.} \end{cases}$$

Remark 1.2 (a) To prove Theorem 1.1, we adapt the classical approach due to Duistermaat [22] and Jaffard [31] by carefully choosing subsets of the irrationals with novel Diophantine restrictions to disprove Hölder regularities. However, the arguments in [31] to compute their Hausdorff dimension do not suffice[11] when $x_0 \neq 0$. We solve this by using the Duffin–Schaeffer theorem and the Mass Transference Principle; see Sect. 2 for the outline of the argument.

(b) Even if $d_{x_0} = d_0$ for all $x_0 \in \mathbb{Q}$, we think that $\alpha_{x_0}(t) \neq \alpha_0(t)$. However, Theorem 1.1 does not require computing $\alpha_{x_0}(t)$ for all $t \in \mathbb{R}$. A full description of the sets $\{ t : \alpha_{x_0}(t) = \alpha \}$ is an interesting and challenging problem because when $x_0 \neq 0$ it is not clear how to characterize the Hölder regularity $\alpha_{x_0}(t)$ in terms of some irrationality exponent like in (7). We do not pursue this problem here, which we leave for a future work.

9 Weierstrass announced $R(t) = \sum_{n=1}^{\infty} \sin(n^2 t)/n^2$; $R_0(t) = \sum_{n\neq 0}^{\infty} e^{2\pi i n^2 t}/n^2$ can be seen as its imaginary part.

10 Precisely, $\widetilde{\mu}(t) = \sup\{\mu > 0 : \left|t - \frac{p}{q}\right| \leq q^{-\mu}$ for infinitely many coprime pairs $(p, q) \in \mathbb{N}^2$ with $q_n \not\equiv 2 \pmod 4\}$.

11 The restriction for denominators in the case $x_0 = 0$ is essentially a parity condition, which is solved in [31] by dividing the set by the factor 2. This does not generalize to the case $x_0 = P/Q$ where the condition for the denominator will be to be a multiple of $4Q$.

Let now $x_0 \notin \mathbb{Q}$. Let p_n/q_n be its approximations by continued fractions, and define the exponents μ_n by $|x_0 - p_n/q_n| = 1/q_n^{\mu_n}$. Define the alternative[12] exponent of irrationality

$$\sigma(x_0) = \limsup_{n\to\infty} \{ \mu_n \ : \ q_n \notin 4\mathbb{N} \}. \tag{8}$$

This exponent always exists and $\sigma(x_0) \geq 2$ (see Proposition 5.2). Our result is the following.

Theorem 1.3 *Let $x_0 \notin \mathbb{Q}$. Let $2 \leq \mu < 2\sigma(x_0)$, with $\sigma(x_0)$ as in (8). Then, for all $\delta > 0$,*

$$\frac{1}{\mu} \leq \dim_{\mathcal{H}} \left\{ t \ : \frac{1}{2} + \frac{1}{4\mu} - \delta \leq \alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu} \right\} \leq \frac{2}{\mu}. \tag{9}$$

Remark 1.4 (a) We show in Fig. 3 a graphic representation of Theorem 1.3.

(b) Theorem 1.3 shows that R_{x_0} is multifractal when $\sigma(x_0) > 2$.

(c) Theorem 1.3 would be strengthened to $1/\mu \leq d_{x_0}(1/2 + 1/2\mu) \leq 2/\mu$ for $\mu < 2\sigma(x_0)$ if we could compute the dimension of some well-identified Diophantine sets, see Remark 5.4. This would give a nontrivial spectrum of singularities in an open interval for all $x_0 \notin \mathbb{Q}$. We leave this for a future work.

(d) The reasons to have an interval $(1/\mu, 2/\mu)$ for the dimension in (9) seem to us deeper in nature. Unlike the upper bound $2/\mu$, which follows from approximating t with rationals p/q with unrestricted $q \in \mathbb{N}$ and with error $q^{-\mu}$ (see the Jarník–Besicovitch theorem 2.2), the lower bound depends on the nature of x_0 which imposes restrictions to q. When $x_0 = P/Q \in \mathbb{Q}$, we require $q \in 4Q\mathbb{N}$, which still results in a set of dimension $2/\mu$. However, when $x_0 \notin \mathbb{Q}$ we require q be restricted to an exponentially growing sequence (given by the denominators of the continued fraction approximations of x_0). This restriction is much stronger and gives a set of t of dimension $1/\mu$. These results follow from the Duffin–Schaeffer theorem and the Mass Transference Principle.

(e) The theorem and its proof (see the heuristic discussion in Sect. 5.2.1) suggest that the spectrum of singularities may be $d_{x_0}(\alpha) = 4\alpha - 2$ in the range $\frac{1}{2} + \frac{1}{4\sigma(x_0)} \leq \alpha \leq \frac{3}{4}$, and possibly something different outside of this range. In particular, we expect the segment of the spectrum in $5/8 \leq \alpha \leq 3/4$ to be present for all x_0.

Remark 1.5 Our results suggest that the trajectories of the binormal flow do not have a generic behavior in terms of regularity. Indeed, if X_n is a sequence of independent and identically distributed complex Gaussian random variables, then the random function

$$S(t) = \sum_{n=1}^{\infty} X_n \frac{e^{2\pi i n^2 t}}{n^2} \tag{10}$$

[12] The usual exponent of irrationality is $\mu(x_0) = \limsup_{n\to\infty} \mu_n$.

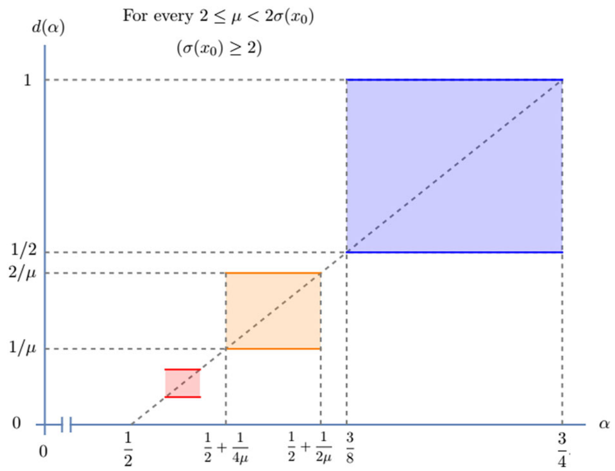


Fig. 3 A graphic representation of Theorem 1.3. We have a continuum of Whitney-type boxes parametrized by μ along the dashed diagonal line $d(\alpha) = 4\alpha - 2$. The graph of $d_{x_0}(\alpha)$ has at least a point in each of those boxes

has[13] almost surely $\alpha_S(t) = 3/4$ for all $t \in \mathbb{R}$ [34]. Hence the generic behavior of (10) is monofractal. In contrast, the fine structure of the linear phase nx_0 of R_{x_0} causes a multifractal behavior.

Regarding intermittency, we compute the L^p norms of the Fourier high-pass filters of R_{x_0} and the intermittency exponents $\eta_{x_0}(p)$ when $x_0 \in \mathbb{Q}$, from which we deduce that R_{x_0} is intermittent.

Theorem 1.6 *Let $x_0 \in \mathbb{Q}$. Let $1 < p < \infty$. Then,*

$$\| P_{\ge N} R_{x_0} \|_p^p \simeq \begin{cases} N^{-\frac{p}{2}-1}, & p > 4, \\ N^{-3} \log N, & p = 4, \\ N^{-3p/4}, & p < 4, \end{cases} \tag{11}$$

and therefore

$$\eta_{x_0}(p) = \lim_{N \to \infty} \frac{\log(\| P_{\ge N} f \|_p^p)}{\log(1/N)} = \begin{cases} p/2 + 1, & p > 4, \\ 3p/4, & p \le 4. \end{cases}$$

Consequently, $\lim_{N\to\infty} F_p(N) = +\infty$ *for* $p \ge 4$*. In particular,* R_{x_0} *is intermittent.*

Remark 1.7 (a) The $p = 4$ intermittency exponent in (11) is $\eta(4) - 2\eta(2) = 0$, but the fact that $\| P_{\ge N} R_{x_0} \|_4^4$ does not follow a pure power law makes $F_4(N) \simeq \log N$. For $p > 4$, we have $\eta(p) - p\eta(2)/2 = 1 - p/4 < 0$, so R_{x_0} is intermittent in small scales when $x_0 \in \mathbb{Q}$.

13 [34, p.86, Theorem 2] shows that almost surely $\alpha_S(t) \ge 3/4$ for all t, and and [34, p. 104, Theorem 5] shows that almost surely $\alpha_S(t) \le 3/4$ for all t.

(b) The upper bound in (11) in Theorem 1.6 holds for all $x_0 \in [0, 1]$. The theorem shows that this is optimal when $x_0 \in \mathbb{Q}$, but we do not expect it to be optimal when $x_0 \notin \mathbb{Q}$. We suspect that the exact behavior, and hence $\eta_{x_0}(p)$, depends on the irrationality of x_0. We aim to study this question in a future work.

1.5 Related literature on the analytic study of Riemann's non-differentiable function

Beyond the literature for the original Riemann's function R_0, the closest work to the study of R_{x_0} is by Oskolkov and Chakhkiev [40]. They studied the regularity of $R_{x_0}(t)$ almost everywhere as a function of two variables (x_0, t), which is not fine enough to capture multifractal properties.

Alternatively, there are many works studying $R_{x_0}(t)$ as a function of x_0 with t fixed, motivated by the fact that R_{x_0} is the solution to an initial value problem for the periodic free Schrödinger equation. From this perspective, Kapitanski and Rodnianski [35] studied the Besov regularity of the fundamental solution[14] as a function of x with t fixed. This approach is also intimately related to the Talbot effect in optics which, as proposed by Berry and Klein [7], is approximated by the fundamental solution to the periodic free Schrödinger equation. Pursuing the related phenomenon of *quantization*,[15] the geometry of the profiles of Schrödinger solutions have been studied for fixed t by Berry [6] and Rodnianski [43]. Following the numeric works of Chen and Olver [16, 17], this perspective has also been extended to the nonlinear setting and other dispersive relations by Chousonis et al. [18, 24] and Boulton, Farmakis and Pelloni [8, 9].

There is a literature for other natural generalizations of Riemann's function, like

$$F(t) = \sum_{n=1}^{\infty} \frac{e^{2\pi i P(n) t}}{n^\alpha}, \qquad P \text{ a polynomial}, \quad \alpha > 1,$$

For $P(n) = n^2$, Jaffard [31] gave his results for all $\alpha > 1$. Chamizo and Córdoba [13] studied the Minkowski dimension of their graphs. Seuret and Ubis [44] studied the non-convergent case $\alpha < 1$, using a local L^2 exponent. Chamizo and Ubis [14, 15] studied the spectrum of singularities for general polynomials P. Further generalizations concerning fractional integrals of modular forms were studied by Pastor [41].

1.6 Structure of the article

In Sect. 2 we discuss the general strategy we follow to prove our theorems, stressing the new ideas related to Diophantine sets with restrictions, the Duffin–Schaeffer theorem and the Mass Transference Principle. In Sect. 3 we prove preliminary results for the local Hölder regularity of R_{x_0}, in particular the behavior around rational points t.

[14] Which, up to constants, is either $\partial_t R_{x_0}(t)$ or $\partial_{x_0}^2 R_{x_0}(t)$.

[15] See the article by Olver [39] for an instructive account of quantization.

In Sect. 4 we compute the spectrum of singularities of R_{x_0} when $x_0 \in \mathbb{Q}$ and prove Theorem 1.1. In Sect. 5 we prove Theorem 1.3. In Sect. 6 we prove Theorem 1.6 by computing the L^p norms of the high-pass filters of R_{x_0}. The proofs of some auxiliary results are postponed to Appendices A and B to avoid breaking the continuity of the main arguments.

2 An overview on the general arguments and on Diophantine approximation

2.1 General argument

An important part of the arguments in this article relies on Diophantine approximation. We will work with both the exponent of irrationality

$$\mu(x) = \sup \left\{ \mu > 0 \, : \left| x - \frac{p}{q} \right| \leq \frac{1}{q^\mu} \text{ for infinitely many coprime pairs } (p,q) \in \mathbb{N} \times \mathbb{N} \right\}, \tag{12}$$

and the Lebesgue and Hausdorff measure properties of the related sets

$$A_\mu = \left\{ x \in [0,1] \mid \left| x - \frac{p}{q} \right| \leq \frac{1}{q^\mu} \text{ for infinitely many coprime pairs } (p,q) \in \mathbb{N} \times \mathbb{N} \right\}, \tag{13}$$

where the case $\mu = \infty$ is understood as $A_\infty = \bigcap_{\mu \geq 2} A_\mu$. In a somewhat hand-waving way, $\mu(x) = \mu$ means that $|x - p/q| \simeq 1/q^\mu$ infinitely often, which ceases to be true for any larger μ.

With these concepts in hand, the classic way to study the regularity of R_{x_0} (used by Duistermaat, Jaffard and subsequent authors) is to first compute the asymptotic behavior of R_{x_0} around rationals. Using the Poisson summation formula we will get a leading order expression of the form

$$R_{x_0}\Big(\frac{p}{q} + h\Big) - R_{x_0}\Big(\frac{p}{q}\Big) \sim \frac{\sqrt{h}}{q} G_q \sim \frac{\sqrt{h}}{\sqrt{q}}, \tag{14}$$

where G_q includes a quadratic Gauss sum of period q, hence $|G_q| \sim \sqrt{q}$ whenever it does not cancel. This shows that in most rationals the regularity of R_{x_0} is 1/2. Let now $t \notin \mathbb{Q}$ with irrationality exponent $\mu(t) = \mu$. Then, essentially $|t - p/q| \simeq 1/q^\mu$, so choosing $h = t - p/q$ we get

$$R_{x_0}\Big(t\Big) - R_{x_0}\Big(t - h\Big) \sim \frac{\sqrt{h}}{\sqrt{q}} \sim h^{\frac{1}{2} + \frac{1}{2\mu}}.$$

This suggests that $\alpha_{x_0}(t) = \frac{1}{2} + \frac{1}{2\mu}$. Combining this with the Jarnik–Besicovitch theorem, which says that $\dim_{\mathcal{H}} A_\mu = 2/\mu$, we get the desired $d(\alpha) = 4\alpha - 2$ in the range $1/2 \le \alpha \le 3/4$.

This argument is essentially valid up to assuming $G_q \neq 0$ in (14). This, however, does not always hold. Apart from a parity condition on q coming from the Gauss sums (present already in previous works), an additional condition arises that depends on x_0. For example, if $x_0 = P/Q \in \mathbb{Q}$, this condition has the form of $Q \mid q$. In terms of the sets A_μ, this means that we need to restrict the denominators of the approximations to a subset of the natural numbers. So let $\mathcal{Q} \subset \mathbb{N}$, and define

$$A_{\mu,\mathcal{Q}} = \left\{ x \in [0,1] \,:\, \left| x - \frac{p}{q} \right| \le \frac{1}{q^\mu} \text{ for infinitely many coprime pairs } (p,q) \in \mathbb{N} \times \mathcal{Q} \right\}. \tag{15}$$

Clearly $A_{\mu,\mathcal{Q}} \subset A_\mu$, but a priori it could be much smaller. Does $A_{\mu,\mathcal{Q}}$ preserve the measure of A_μ? Previous works need to work with situations analogue to $Q = 2$, but here we need to argue for all $Q \in \mathbb{N}$. For that, at the level of the Lebesgue measure we will use the Duffin–Schaeffer theorem, while we will compute Hausdorff measures and dimensions via the Mass Transference Principle.

2.2 Lebesgue measure: Dirichlet approximation and the Duffin–Schaeffer theorem

Both the Dirichlet approximation theorem and the theory of continued fractions imply $A_2 = [0,1] \setminus \mathbb{Q}$. However, neither of them give enough information about the sequence of denominators they produce, so they cannot be used to determine the size of the set $A_{2,\mathcal{Q}} \subset A_2$. The recently proved Duffin–Schaeffer conjecture gives an answer to this kind of questions.

Theorem 2.1 (Duffin–Schaeffer theorem [37]) *Let* $\psi : \mathbb{N} \to [0,\infty)$ *be a function. Define*

$$A_\psi = \left\{ x \in [0,1] \,:\, \left| x - \frac{p}{q} \right| \le \psi(q) \textit{ for infinitely many coprime pairs } (p,q) \in \mathbb{N} \times \mathbb{N} \right\}.$$

Let φ *denote the Euler totient function.*[16] *Then, we have the following dichotomy:*

(a) *If* $\sum_{q=1}^\infty \varphi(q)\psi(q) = \infty$, *then* $|A_\psi| = 1$.
(b) *If* $\sum_{q=1}^\infty \varphi(q)\psi(q) < \infty$, *then* $|A_\psi| = 0$.

The relevant part of this theorem is (a), since (b) follows from the canonical limsup covering

$$A_\psi \subset \bigcup_{q=Q}^\infty \bigcup_{\substack{1 \le p \le q \\ (p,q)=1}} \left(\frac{p}{q} - \psi(q), \, \frac{p}{q} + \psi(q) \right), \quad \forall\, Q \in \mathbb{N}$$

[16] The Euler totient function: for $q \in \mathbb{N}$, $\varphi(q)$ is the number of natural numbers $i \le q$ such that $\gcd(q,i) = 1$.

$$\Longrightarrow \quad |A_\psi| \leq \sum_{q=Q}^{\infty} \varphi(q)\psi(q), \quad \forall\, Q \in \mathbb{N}. \tag{16}$$

On the other hand, as opposed to the classic theorem by Khinchin[17] [36, Theorem 32], the arbitrariness of ψ allows to restrict the denominators to a set $\mathcal{Q} \subset \mathbb{N}$ just by setting $\psi(q) = 0$ when $q \notin \mathcal{Q}$. In particular, $A_{\mu,\mathcal{Q}} = A_\psi$ if we define $\psi(q) = \mathbb{1}_{\mathcal{Q}}(q)/q^\mu$, where $\mathbb{1}_{\mathcal{Q}}$ is the indicator function of the set $\mathcal{Q}$. Hence, the relevant sum for the sets $A_{\mu,\mathcal{Q}}$ is

$$\sum_{q=1}^{\infty} \varphi(q)\psi(q) = \sum_{q\in\mathcal{Q}} \frac{\varphi(q)}{q^\mu}.$$

In particular, it is fundamental to understand the behavior of the Euler totient function φ on $\mathcal{Q}$.

The complete proof of the Duffin–Schaeffer theorem was given recently by Koukoulopoulos and Maynard [37, Theorem 1], but Duffin and Schaeffer [21] proved back in 1941 that the result holds under the additional assumption that there exists $c > 0$ such that

$$\sum_{q=1}^{N} \varphi(q)\,\psi(q) \geq c \sum_{q=1}^{N} q\,\psi(q), \quad \text{for infinitely many } N \in \mathbb{N}. \tag{17}$$

In the setting of $A_{\mu,\mathcal{Q}}$, this condition is immediately satisfied by sets $\mathcal{Q}$ for which there is a $c > 0$ such that $\varphi(q) > c\,q$ for all $q \in \mathcal{Q}$. Examples of this are:

- $\mathcal{Q} = \mathbb{P}$ the set of prime numbers, and
- $\mathcal{Q} = \{\, M^n : n \in \mathbb{N} \,\}$ where $M \in \mathbb{N}$, that is, the set of power of a given number M.

It follows from our computations in Appendix A that the condition (17) is also satisfied by

- $\mathcal{Q} = \{\, Mn : n \in \mathbb{N} \,\}$ where $M \in \mathbb{N}$, that is, the set of multiples of a given number M.

To prove Theorem 1.1 for $x_0 = P/Q$, we restrict the denominators to the latter set with $M = 4Q$; in particular, the 1941 result by Duffin and Schaeffer [21] suffices. However, in the case of $x_0 \notin \mathbb{Q}$ we need to restrict the denominators to an exponentially growing sequence q_n for which we do not know if (17) holds. Hence, in this case we need the full power of the result by Koukoulopoulos and Maynard [37]. This might give an indication of the difficulty to settle the case $x_0 \notin \mathbb{Q}$.

17 Khinchin's theorem states that if $\psi : \mathbb{N} \to [0, \infty)$ is a function such that $q^2\psi(q)$ is decreasing and $\sum_{q=1}^{\infty} q\,\psi(q) = \infty$, then the set $\{\, x \in [0, 1] : |x - p/q| \leq \psi(q)$ for infinitely many pairs $(p, q) \in \mathbb{N}\times\mathbb{N} \,\}$ has Lebesgue measure 1.

2.3 Hausdorff dimension: the Jarník–Besicovitch theorem and the Mass Transference Principle

We mentioned that $A_2 = [0,1]\setminus\mathbb{Q}$, and it follows from the argument in (16) that $|A_\mu| = 0$ for $\mu > 2$. But how small is A_μ is when $\mu > 2$? A measure theoretic answer to that is the Jarník and Besicovitch theorem from the 1930s (see [25, Section 10.3] for a modern version).

Theorem 2.2 (Jarník–Besicovitch theorem) *Let $\mu > 2$ and let A_μ be defined as in (13). Then,* $\dim_{\mathcal{H}} A_\mu = 2/\mu$ *and* $\mathcal{H}^{2/\mu}(A_\mu) = \infty$.

In this article we need to adapt this result to $A_{\mu,\mathcal{Q}}$. First, using the Duffin–Schaeffer Theorem 2.1 we will be able to find the largest $\mu_0 \geq 1$ such that $|A_{\mu_0,\mathcal{Q}}| = 1$, so that $|A_{\mu,\mathcal{Q}}| = 0$ for all $\mu > \mu_0$. To compute the Hausdorff dimension of those zero-measure sets, we will use a theorem by Beresnevich and Velani, called the Mass Transference Principle [5, Theorem 2]. We state here its application to the unit cube and to Hausdorff measures.

Theorem 2.3 (Mass Transference Principle [5]) *Let $B_n = B_n(x_n, r_n)$ be a sequence of balls in* $[0,1]^d$ *such that* $\lim_{n\to\infty} r_n = 0$. *Let $\alpha < d$ and let $B_n^\alpha = B_n(x_n, r_n^{\alpha/d})$ be the dilation of B_n centered at x_n by the exponent α. Suppose that $X^\alpha := \limsup_{n\to\infty} B_n^\alpha$ is of full Lebesgue measure, that is, $|X^\alpha| = 1$. Then, calling $X := \limsup_{n\to\infty} B_n$, we have* $\dim_{\mathcal{H}} X \geq \alpha$ *and* $\mathcal{H}^\alpha(X) = \infty$.

To illustrate the power of the Mass Transference Principle, let us explain how the Jarnik–Besicovitch Theorem 2.2 follows as a simple corollary of the Dirichlet approximation theorem. From the definition of A_μ we can write[18]

$$A_\mu = \limsup_{q\to\infty} \bigcup_{1\leq p\leq q,\,(p,q)=1} B\Big(\frac{p}{q}, \frac{1}{q^\mu}\Big). \tag{18}$$

Choose $\alpha = 2/\mu$ so that $(A_\mu)^\alpha = A_{\mu\alpha} = A_2$, which by the Dirichlet approximation theorem has full measure. Then, the Mass Transference Principle implies $\dim_{\mathcal{H}} A_\mu \geq 2/\mu$ and $\mathcal{H}^{2/\mu}(A_\mu) = \infty$. The upper bound follows from the canonical cover of A_μ in (18), proceeding like in (16).

For $A_{\mu,\mathcal{Q}}$, once we find the largest μ_0 for which $|A_{\mu_0,\mathcal{Q}}| = 1$ using the Duffin–Schaeffer theorem, we will choose $\alpha = \mu_0/\mu$ so that the property $(A_{\mu,\mathcal{Q}})^\alpha = A_{\mu\alpha,\mathcal{Q}} = A_{\mu_0,\mathcal{Q}}$ has full measure, and the Mass Transference Principle will then imply $\dim_{\mathcal{H}} A_{\mu,\mathcal{Q}} \geq \mu_0/\mu$.

3 Preliminary results on the local regularity of R_{x_0}

In this section we carry over to R_{x_0} regularity results that are by now classical for R_0. In Sect. 3.1 we prove that R_{x_0} is globally $C^{1/2}$. In Sect. 3.2 we compute the asymptotic

[18] The expression in (18) is not in the form of a limsup of balls. It follows, however, that the limsup of any enumeration whatsoever of the balls considered in the construction gives the same set.

behavior of R_{x_0} around rationals. In Sect. 3.3 we give a lower bound for $\alpha_{x_0}(t)$ that is independent of x_0.

3.1 A global Hölder regularity result

Duistermaat [22, Lemma 4.1.] proved that R_0 is globally $C^{1/2}(t)$. The same holds for all $x_0 \in \mathbb{R}$. We include the proof for completeness.

Proposition 3.1 *Let $x_0 \in \mathbb{R}$. Then, $\alpha_{x_0}(t) \geq 1/2$ for all $t \in \mathbb{R}$. That is, R_{x_0} is globally $C^{1/2}$.*

Proof For $h \neq 0$, let $N \in \mathbb{N}$ such that $\frac{1}{(N+1)^2} \leq |h| < \frac{1}{N^2}$, and write

$$
\begin{aligned}
R_{x_0}(t+h) - R_{x_0}(t) &= \sum_{|n| \leq N} \frac{e^{2\pi i n^2 t}\, e^{2\pi i n x_0}}{n^2} \left(e^{2\pi i n^2 h} - 1 \right) \\
&\quad + \sum_{|n| > N} \frac{e^{2\pi i n^2 t}\, e^{2\pi i n x_0}}{n^2} \left(e^{2\pi i n^2 h} - 1 \right).
\end{aligned}
$$

Since $|e^{ix} - 1| \leq |x|$ for all $x \in \mathbb{R}$, we bound

$$
\begin{aligned}
\left| \sum_{|n| \leq N} \frac{e^{2\pi i n^2 t}\, e^{2\pi i n x_0}}{n^2} \left(e^{2\pi i n^2 h} - 1 \right) \right| &\leq \sum_{|n| \leq N} \frac{\left| e^{2\pi i n^2 h} - 1 \right|}{n^2} \\
&\leq 2|h| N < 2|h| \frac{1}{\sqrt{|h|}} = 2\sqrt{|h|}.
\end{aligned}
$$

For the other sum, we trivially bound $\left| e^{2\pi i n^2 h} - 1 \right| \leq 2$ to get

$$
\left| \sum_{|n| > N} \frac{e^{2\pi i n^2 t}\, e^{2\pi i n x_0}}{n^2} \left(e^{2\pi i n^2 h} - 1 \right) \right| \leq 2 \sum_{n=N+1}^{\infty} \frac{2}{n^2} \leq \frac{4}{N} \leq \frac{8}{N+1} \leq 8\sqrt{|h|}.
$$

Hence $\left| R_{x_0}(t+h) - R_{x_0}(t) \right| \leq 10|h|^{1/2}$. This holds for all t, so $R_{x_0} \in C^{1/2}(t)$ for all $t \in \mathbb{R}$. □

3.2 Asymptotic behavior of R_{x_0} around rational t

The building block for all results in this article is the behavior of R_{x_0} around rationals, which we compute explicitly.

Proposition 3.2 *Let $x_0 \in \mathbb{R}$. Let $p, q \in \mathbb{N}$ be such that $(p, q) = 1$. Then,*

$$
R_{x_0}\left(\frac{p}{q} + h \right) - R_{x_0}\left(\frac{p}{q} \right) = -2\pi i h
$$

$$+\frac{\sqrt{|h|}}{q}\sum_{m\in\mathbb{Z}} G(p,m,q)\, F_\pm\left(\frac{x_0-m/q}{\sqrt{h}}\right), \quad \textit{for } h\neq 0,$$

where $F_\pm = F_+$ *if* $h>0$ *and* $F_\pm = F_-$ *if* $h<0$, *and*

$$G(p,m,q)=\sum_{r=0}^{q-1} e^{2\pi i\frac{pr^2+mr}{q}}, \quad F_\pm(\xi)=\int_{\mathbb{R}}\frac{e^{\pm 2\pi i x^2}-1}{x^2}\,e^{2\pi i x\xi}\,dx.$$

The function $F_\pm$ *is bounded and continuous,* $F_\pm(0)=2\pi(-1\pm i)$, *and*

$$F_\pm(\xi)=(1\pm i)\,\frac{e^{\mp\pi i\xi^2/2}}{\xi^2}+O\left(\frac{1}{\xi^4}\right)=O\left(\frac{1}{\xi^2}\right), \quad as \quad \xi\to\infty.$$

Proof We follow the classical approach, which can be traced back to Smith [45], of using the Poisson summation formula. From the definition of R_{x_0}, complete first the sum to $n\in\mathbb{Z}$ to write

$$R_{x_0}\left(\frac{p}{q}+h\right)-R_{x_0}\left(\frac{p}{q}\right)=-2\pi i h+\sum_{n\in\mathbb{Z}}\frac{e^{2\pi i n^2 h}-1}{n^2}\,e^{2\pi i\frac{pn^2}{q}}\,e^{2\pi i n x_0},$$

where we must interpret the term $n=0$ as the value of $\frac{e^{2\pi i n^2 h}-1}{n^2}\simeq 2\pi i h$ as $n\to 0$. Split the sum modulo q by writing $n=mq+r$ and

$$\sum_{n\in\mathbb{Z}}\frac{e^{2\pi i n^2 h}-1}{n^2}\,e^{2\pi i\frac{pn^2}{q}}\,e^{2\pi i n x_0}=\sum_{r=0}^{q-1}e^{2\pi i\frac{pr^2}{q}}\sum_{m\in\mathbb{Z}}\frac{e^{2\pi i(mq+r)^2 h}-1}{(mq+r)^2}\,e^{2\pi i(mq+r)x_0}. \tag{19}$$

Use the Poisson summation formula for the function

$$f(y)=\frac{e^{2\pi i(yq+r)^2 h}-1}{(yq+r)^2}\,e^{2\pi i(yq+r)x_0},$$

for which, changing variables $(yq+r)\sqrt{|h|}=z$, we have

$$\begin{aligned}\widehat{f}(\xi)&=\frac{\sqrt{|h|}}{q}\,e^{2\pi i r\xi/q}\int\frac{e^{2\pi i\,\mathrm{sgn}(h)z^2}-1}{z^2}\,e^{2\pi i\frac{z}{\sqrt{|h|}}(x_0-\xi/q)}\,dz\\&=\frac{\sqrt{|h|}}{q}\,e^{2\pi i r\xi/q}\,F_\pm\left(\frac{x_0-\xi/q}{\sqrt{|h|}}\right).\end{aligned}$$

Therefore,

$$(19)=\sum_{r=0}^{q-1}e^{2\pi i\frac{pr^2}{q}}\sum_{m\in\mathbb{Z}}\frac{\sqrt{|h|}}{q}\,e^{2\pi i r m/q}\,F_\pm\left(\frac{x_0-m/q}{\sqrt{|h|}}\right)$$

$$= \frac{\sqrt{|h|}}{q} \sum_{m \in \mathbb{Z}} G(p, m, q) \, F_\pm\left(\frac{x_0 - m/q}{\sqrt{|h|}} \right).$$

The properties for $F_\pm$ follow by integration by parts and the value of the Fresnel integral. □

The main term in Proposition 3.2 corresponds to $m \in \mathbb{Z}$ such that $x_0 - m/q$ is closest to 0. Define

$$\begin{cases} m_q = \operatorname{argmin}_{m \in \mathbb{Z}} \left| x_0 - \frac{m}{q} \right|, \\ x_q = x_0 - \frac{m_q}{q}, \end{cases} \qquad \text{so that} \quad |x_q| = \left| x_0 - \frac{m_q}{q} \right| = \operatorname{dist}\left(x_0, \frac{\mathbb{Z}}{q} \right) \leq \frac{1}{2q}. \tag{20}$$

Then, shifting the sum,

$$\begin{aligned} R_{x_0}\left(\frac{p}{q} + h\right) - R_{x_0}\left(\frac{p}{q}\right) = -2\pi i h + \frac{\sqrt{|h|}}{q} \, G(p, m_q, q) F_\pm\left(\frac{x_q}{\sqrt{|h|}}\right) \\ + \frac{\sqrt{|h|}}{q} \sum_{m \neq 0} G(p, m_q + m, q) \, F_\pm\left(\frac{x_q - m/q}{\sqrt{|h|}} \right). \end{aligned}$$

Let us now bound the sum as an error term. As long as $(p, q) = 1$, it is a well-known property of Gauss sums that $|G(p, m, q)| \leq \sqrt{2q}$ for all $m \in \mathbb{N}$, so

$$\frac{\sqrt{|h|}}{q} \left| \sum_{m \neq 0} G(p, m_q + m, q) \, F_\pm\left(\frac{x_q - m/q}{\sqrt{|h|}} \right) \right| \leq 2 \, \frac{\sqrt{|h|}}{\sqrt{q}} \sum_{m \neq 0} \left| F_\pm\left(\frac{x_q - m/q}{\sqrt{|h|}} \right) \right|.$$

Since $|x_q| \leq 1/(2q)$ and $m \neq 0$, we have $|x_q - m/q| \simeq |m|/q$. This suggests separating two cases:

- If $q\sqrt{|h|} < 1$, we use the property $F_\pm(x) = O(x^{-2})$ to bound

$$\sum_{m \neq 0} \left| F_\pm\left(\frac{x_q - m/q}{\sqrt{|h|}} \right) \right| \lesssim \sum_{m \neq 0} \frac{|h|}{|x_q - m/q|^2} \simeq q^2 \, |h| \sum_{m \neq 0} \frac{1}{m^2} \simeq q^2 |h|.$$

- If $q\sqrt{|h|} \geq 1$, we split the sum as

$$
\begin{aligned}
\sum_{m\neq 0} \left| F_{\pm}\left(\frac{x_q - m/q}{\sqrt{|h|}}\right)\right| &= \sum_{|m|\leq q\sqrt{|h|}} \left| F_{\pm}\left(\frac{x_q - m/q}{\sqrt{|h|}}\right)\right| \\
&\quad + \sum_{|m|\geq q\sqrt{|h|}} \left| F_{\pm}\left(\frac{x_q - m/q}{\sqrt{|h|}}\right)\right| \\
&\leq \sum_{|m|\leq q\sqrt{|h|}} C + \sum_{|m|\geq q\sqrt{|h|}} \frac{|h|}{|x_q - m/q|^2} \\
&\simeq q\sqrt{|h|} + q^2|h| \sum_{|m|\geq q\sqrt{|h|}} \frac{1}{m^2} \simeq q\sqrt{|h|}.
\end{aligned}
$$

These two bounds can be written simultaneously as

$$
\sum_{m\neq 0} \left| F_{\pm}\left(\frac{x_q - m/q}{\sqrt{|h|}}\right)\right| \lesssim \min\left(q\sqrt{|h|}, q^2|h|\right),
$$

where the underlying constant is universal. Multiply by $\sqrt{|h|}/\sqrt{q}$ to get the following corollary.

Corollary 3.3 *Let $x_0 \in \mathbb{R}$. Let $p, q \in \mathbb{N}$ be such that $(p, q) = 1$. Then,*

$$
\begin{aligned}
&R_{x_0}\left(\frac{p}{q} + h\right) - R_{x_0}\left(\frac{p}{q}\right) = -2\pi i h + \frac{\sqrt{|h|}}{q} G(p, m_q, q) F_{\pm}\left(\frac{x_q}{\sqrt{|h|}}\right) \\
&+ O\left(\min\left(\sqrt{q}\, h, q^{3/2} h^{3/2}\right)\right),
\end{aligned}
$$

where the underlying constant of the O is independent of p, q and x_0.

Remark 3.4 The difference between $x_0 = 0$ and $x_0 \neq 0$ is clear from Corollary 3.3.

- If $x_0 = 0$, we have $x_q = 0 = m_q$ for all q. The main term is $|h|^{1/2}q^{-1}\, G(p, 0, q)$ $F_{\pm}(0)$, so there is a clear dichotomy: R_0 is differentiable at p/q if and only if $G(p, 0, q) = 0$, which happens if and only if $q \equiv 2 \pmod 4$; in all other rationals, R_{x_0} is $C^{1/2}$.
- If $x_0 \neq 0$, it is in general false that $x_q = 0$, so to determine the differentiability of R_{x_0} we need to control the magnitude of $F_{\pm}(x_q/\sqrt{|h|})$.

3.3 Lower bounds for the local Hölder regularity

We now give lower bounds for $\alpha_{x_0}(t)$ that do not depend on x_0. In Sect. 3.3.1 we work with $t \in \mathbb{Q}$, and in Sect. 3.3.2 with $t \notin \mathbb{Q}$.

3.3.1 At rational points

There is a dichotomy in the Hölder regularity of R_{x_0} at rational points.

Proposition 3.5 *Let $x_0 \in \mathbb{R}$ and $t \in \mathbb{Q}$. Then, either $\alpha_{x_0}(t) = 1/2$ or $\alpha_{x_0}(t) = 3/2$.*

Proof Let $t = p/q$ with $(p, q) = 1$. If q is fixed, we get $\min\left(\sqrt{q}\,|h|, q^{3/2}\,|h|^{3/2}\right) = q^{3/2}|h|^{3/2}$ for small enough $|h|$, so from Corollary 3.3 we get

$$R_{x_0}\left(\frac{p}{q} + h\right) - R_{x_0}\left(\frac{p}{q}\right) = -2\pi i h + \frac{\sqrt{|h|}}{q}\, G(p, m_q, q) F_\pm\left(\frac{x_q}{\sqrt{|h|}}\right) + O\left(q^{3/2} h^{3/2}\right). \tag{21}$$

Then, differentiability completely depends on the Gauss sum $G(p, m_q, q)$ and on x_q.

Case 1 If $G(p, m_q, q) = 0$, then $\left|R_{x_0}\left(\frac{p}{q} + h\right) - R_{x_0}\left(\frac{p}{q}\right) + 2\pi i h\right| \lesssim_q h^{3/2}$, so $\alpha_{x_0}(p/q) \geq 3/2$.

Case 2 If $G(p, m_q, q) \neq 0$ and $x_q \neq 0$. Then, $|G(p, m_q, q)| \simeq \sqrt{q}$ and $\lim_{h\to 0} x_q/\sqrt{|h|} = \infty$, so $\left|F_\pm\left(x_q/\sqrt{|h|}\right)\right| \lesssim h/x_q^2$. Hence, $\alpha_{x_0}(p/q) \geq 3/2$ because

$$\begin{aligned} R_{x_0}\left(\frac{p}{q} + h\right) - R_{x_0}\left(\frac{p}{q}\right) &= -2\pi i h + O\left(\frac{\sqrt{h}}{\sqrt{q}}\frac{h}{x_q^2} + q^{3/2} h^{3/2}\right) \\ &= -2\pi i h + O_q\left(h^{3/2}\right). \end{aligned}$$

Case 3 If $G(p, m_q, q) \neq 0$ and $x_q = 0$, we have $|G(p, m_q, q)| \simeq \sqrt{q}$, so from (21) we get

$$\begin{aligned} \left|R_{x_0}\left(\frac{p}{q} + h\right) - R_{x_0}\left(\frac{p}{q}\right)\right| &\geq \frac{\sqrt{|h|}}{q}|G(p, m_q, q)||F_\pm(0)| \\ &\quad + O_q(h) \simeq \frac{\sqrt{h}}{\sqrt{q}} + O_q(h) \gtrsim_q h^{1/2} \end{aligned}$$

for $h \ll_q 1$. Together with Proposition 3.1, this implies $\alpha_{x_0}(p/q) = 1/2$.

That Cases 1 and 2 actually imply $\alpha_{x_0}(t) = 3/2$ is a bit more technical; we postpone the proof to Proposition B.6 in Appendix B. □

3.3.2 At irrational points

We give a lower bound $\alpha_{x_0}(t)$ that depends on the exponent of irrationality of t, but not on x_0.

Proposition 3.6 *Let $x_0 \in \mathbb{R}$ and $t \in \mathbb{R}\setminus\mathbb{Q}$. Let $\mu(t)$ be the exponent of irrationality of t. Then, $\alpha_{x_0}(t) \geq \frac{1}{2} + \frac{1}{2\mu(t)}$.*

The proof of this result, which we include for completeness, closely follows the procedure by Chamizo and Ubis [15, Proof of Theorem 2.3].

Remark 3.7 Similar to what happens for $x_0 = 0$, where $\alpha_0(t) = 1/2 + 1/2\widetilde{\mu}(t) \geq 1/2 + 1/2\mu(t)$ (see (7)), we do not expect the bound in Proposition 3.6 to be optimal for all $t \notin \mathbb{Q}$. However, it will be enough to compute the spectrum of singularities.

Proof In view of Proposition 3.1, there is nothing to prove if $\mu(t) = \infty$, so assume $\mu(t) < \infty$. Let p_n/q_n be the n-th approximation by continued fractions of t. Center the asymptotic behavior in Corollary 3.3 at p_n/q_n, and bound it from above by

$$\left| R_{x_0}\left(\frac{p_n}{q_n} + h\right) - R_{x_0}\left(\frac{p_n}{q_n}\right)\right| \lesssim \frac{\sqrt{|h|}}{\sqrt{q_n}} + |h| + \min\left(\sqrt{q_n}\, h, q_n^{3/2}\, h^{3/2}\right), \quad (22)$$

where we used that $|G(p_n, m_{q_n}, q_n)| \leq \sqrt{2q_n}$ for all $n \in \mathbb{N}$ and $|F(x)| \lesssim 1$ for all $x \in \mathbb{R}$.

Let $h \neq 0$ be small enough. The sequence $|t - p_n/q_n|$ is strictly decreasing, so choose n such that

$$\left|t - \frac{p_n}{q_n}\right| \leq |h| < \left|t - \frac{p_{n-1}}{q_{n-1}}\right|. \quad (23)$$

Then, from (22), (23) and $|t - p_n/q_n + h| \leq 2|h|$, we get

$$\begin{aligned}
&\left|R_{x_0}(t+h) - R_{x_0}(t)\right| \\
&\leq \left|R_{x_0}\left(\frac{p_n}{q_n} + t - \frac{p_n}{q_n} + h\right) - R_{x_0}\left(\frac{p_n}{q_n}\right)\right| \\
&\quad + \left|R_{x_0}\left(\frac{p_n}{q_n} + t - \frac{p_n}{q_n}\right) - R_{x_0}\left(\frac{p_n}{q_n}\right)\right| \\
&\lesssim \frac{\sqrt{|h|}}{\sqrt{q_n}} + |h| + \min\left(\sqrt{q_n}\,|h|, q_n^{3/2}\,|h|^{3/2}\right).
\end{aligned} \quad (24)$$

Next we compute the dependence between q_n and h. By the property of continued fractions

$$\frac{1}{q_n^{\mu_n}} = \left|t - \frac{p_n}{q_n}\right| \leq \frac{1}{q_{n+1}q_n},$$

we get $1/q_n \leq 1/q_{n+1}^{1/(\mu_n-1)}$ for all $n \in \mathbb{N}$. Then, from (23) we get

$$\frac{1}{q_n^{\mu_n}} \leq |h| < \frac{1}{q_{n-1}^{\mu_{n-1}}} \leq \frac{1}{q_n^{\mu_{n-1}/(\mu_{n-1}-1)}}. \quad (25)$$

We now bound each term in (24) using (25).

- For the first term, by (25), $\sqrt{|h|}/\sqrt{q_n} \leq |h|^{\frac{1}{2}+\frac{1}{2\mu_n}}$.

- The fact that $\mu_n \geq 2$ implies $\frac{1}{2} + \frac{1}{2\mu_n} \leq \frac{3}{4}$, so $|h| \leq |h|^{3/4} \leq |h|^{\frac{1}{2}+\frac{1}{2\mu_n}}$ and the second term is absorbed by the first one.
- For the third term, we write the minimum as

$$\min\left(\sqrt{q_n}\,|h|, q_n^{3/2}\,|h|^{3/2}\right) = \begin{cases} \sqrt{q_n}\,|h|, & \text{when } |h| \geq 1/q_n^2, \\ q_n^{3/2}\,|h|^{3/2} & \text{when } |h| \leq 1/q_n^2. \end{cases}$$

So we have two regions:

– When $|h| \geq 1/q_n^2$, use (25) to bound

$$\sqrt{q_n}\,|h| \leq \frac{|h|}{|h|^{(\mu_{n-1}-1)/2\mu_{n-1}}} = |h|^{\frac{1}{2}+\frac{1}{2\mu_{n-1}}}.$$

– When $|h| \leq 1/q_n^2$, we directly have $q_n \leq |h|^{-1/2}$, so

$$q_n^{3/2}\,|h|^{3/2} = |h|^{3/2-3/4} = |h|^{3/4} \leq |h|^{\frac{1}{2}+\frac{1}{2\mu_{n-1}}},$$

where in the last inequality we used $\frac{1}{2} + \frac{1}{2\mu_{n-1}} \leq \frac{3}{4}$ as before.

Gathering all cases, we get

$$|R_{x_0}(t+h) - R_{x_0}(t)| \leq |h|^{\frac{1}{2}+\frac{1}{2\mu_n}} + |h|^{\frac{1}{2}+\frac{1}{2\mu_{n-1}}}.$$

From the definition of the exponent of irrationality $\mu(t) = \limsup_{n\to\infty} \mu_n$, for any $\delta > 0$ there exists $N_\delta \in \mathbb{N}$ such that $\mu_n \leq \mu(t)+\delta$ for all $n \geq N_\delta$. Then, since $|h| < 1$, we have $|h|^{\frac{1}{2}+\frac{1}{2\mu_n}} \leq |h|^{\frac{1}{2}+\frac{1}{2\mu(t)+2\delta}}$ for all $n \geq N_\delta$. Renaming δ, we get $N_\delta \in \mathbb{N}$ such that

$$|R_{x_0}(t+h) - R_{x_0}(t)| \leq |h|^{\frac{1}{2}+\frac{1}{2\mu(t)}-\delta}, \quad \text{for all} \quad |h| \leq \left|t - \frac{p_{N_\delta}}{q_{N_\delta}}\right|,$$

so $\alpha_{x_0}(t) \geq \frac{1}{2} + \frac{1}{2\mu(t)} - \delta$. Since this holds for all $\delta > 0$, we conclude that $\alpha_{x_0}(t) \geq \frac{1}{2} + \frac{1}{2\mu(t)}$. □

4 Proof of Theorem 1.1: spectrum of singularities when $x_0 \in \mathbb{Q}$

In this section we prove Theorem 1.1. Let us fix $x_0 = P/Q$ such that $(P, Q) = 1$. To compute the spectrum of singularities d_{x_0}, we first characterize the rational points t where R_{x_0} is not differentiable, and then we give an upper bound for the regularity $\alpha_{x_0}(t)$ at irrational t.

4.1 At rational points *t*

In the proof of Proposition 3.5 we established that R_{x_0} is not differentiable at $t = p/q$ if and only if $G(p, m_q, q) \neq 0$ and $x_q = \mathrm{dist}(x_0, \mathbb{Z}/q) = 0$. We characterize this in the following proposition.

Proposition 4.1 *Let $x_0 = P/Q$ with $\gcd(P, Q) = 1$, and let $p, q \in \mathbb{N}$ such that $\gcd(p, q) = 1$. Then, R_{x_0} is non-differentiable at $t = p/q$ if and only if*

- $q = kQ$ *with* $k \equiv 0, 1, 3 \pmod 4$*, in the case* $Q \equiv 1 \pmod 2$.
- $q = kQ$ *with* $k \equiv 0 \pmod 2$*, in the case* $Q \equiv 0 \pmod 4$.
- $q = kQ$ *with* $k \in \mathbb{Z}$*, in the case* $Q \equiv 2 \pmod 4$.

In all such cases, the asymptotic behavior is

$$R_{x_0}\left(\frac{p}{q} + h\right) - R_{x_0}\left(\frac{p}{q}\right) = c\, e^{2\pi i \phi_{p,q,x_0}}\, F_{\pm}(0)\, \frac{\sqrt{|h|}}{\sqrt{q}} - 2\pi i h + O\left(\min\left(\sqrt{q}\, h, q^{3/2}\, h^{3/2}\right)\right). \tag{26}$$

where $c = 1$ or $c = \sqrt{2}$ depending on parity conditions of Q and q. In particular, $\alpha_{x_0}(t) = 1/2$.

Proof In view of the proof of Proposition 3.5, we must identify the conditions for $G(p, m_q, q) \neq 0$ and $x_q = 0$. Since $x_q = \mathrm{dist}(P/Q, \mathbb{Z}/q)$, we have $x_q = 0$ when there exists $m_q \in \mathbb{Z}$ such that

$$\frac{P}{Q} = \frac{m_q}{q} \quad \Longleftrightarrow \quad Pq = m_q Q.$$

Since $\gcd(P, Q) = 1$, then necessarily $Q|q$. Reversely, if $q = kQ$, then picking $m_q = kP$ we have $m_q/q = P/Q$. In short,

$$x_q = 0 \quad \Longleftrightarrow \quad q \text{ is a multiple of } Q.$$

So let $q = kQ$ for some $k \in \mathbb{N}$. Then, $m_q = kP$. Let us characterize the second condition $G(p, m_q, q) = G(p, kP, kQ) \neq 0$. It is well-known that

$$G(a, b, c) \neq 0 \quad \Longleftrightarrow \quad \text{either} \begin{cases} c \text{ is odd, or} \\ c \text{ is even and } \frac{c}{2} \equiv b \pmod 2. \end{cases} \tag{27}$$

We separate cases:

- Suppose Q is odd. Then, according to (27), we need either
  - kQ odd, which holds if and only if k is odd, or
  - kQ even, which holds if and only if k is even, and $kQ/2 \equiv kP \pmod 2$. Since Q is odd and k is even, this is equivalent to $k/2 \equiv 0 \pmod 2$, which means $k \equiv 0 \pmod 4$.

Therefore, if $q = kQ$, the Gauss sum $G(p, m_q, q) \neq 0$ if and only if $k \equiv 0, 1, 3 \pmod 4$.

- Suppose $Q \equiv 0 \pmod 4$. Since $q = kQ$ is even, by (27) we need $kQ/2 \equiv kP \pmod 2$. Since Q is a multiple of 4, this is equivalent to $kP \equiv 0 \pmod 2$. But since Q is even, then P must be odd. Therefore, k must be even. In short, if $q = kQ$, we have $G(p, m_q, q) \neq 0$ if and only if k is even.
- Suppose $Q \equiv 2 \pmod 4$. Since $q = kQ$ is even, by (27) we need $kQ/2 \equiv kP \pmod 2$. Now both $Q/2$ and P are odd, so this is equivalent to $k \equiv k \pmod 2$, which is of course true. Therefore, if $q = kQ$, we have $G(p, m_q, q) \neq 0$ for all $k \in \mathbb{Z}$.

Once all cases have been identified, (26) follows from Corollary 3.3 and from the fact that if $G(p, m_q, q) \neq 0$ we have $|G(p, m_q, q)| = c\sqrt{q}$ with $c = 1$ or $c = \sqrt{2}$. □

4.2 A general upper bound for irrational *t*

We begin the study of $t \notin \mathbb{Q}$ by giving a general upper bound for $\alpha_{x_0}(t)$ for $t \notin \mathbb{Q}$. The proof uses an alternative asymptotic expression around rationals that we postpone to Appendix B.

Proposition 4.2 *Let $x_0 \in \mathbb{Q}$ and $t \notin \mathbb{Q}$. Then, $\alpha_{x_0}(t) \leq 3/4$.*

Proof See Appendix B, Proposition B.3. □

4.3 Upper bounds depending on the irrationality of *t*

We now aim at an upper bound for $\alpha_{x_0}(t)$ that depends on the irrationality of t at the level of Proposition 3.6. The idea is to approximate t by rationals p/q where R_{x_0} is non-differentiable, which we characterized in Proposition 4.1. To avoid treating different cases depending on the parity of Q, let us restrict[19] $q \in 4Q\mathbb{N}$, such that the three conditions in Proposition 4.1 are simultaneously satisfied and (26) holds.

Let $\mu \in [2, \infty)$. Define the classic Diophantine set

$$A_\mu = \left\{ t \in (0,1)\backslash\mathbb{Q} \, : \, \left|t - \frac{p}{q}\right| \leq \frac{1}{q^\mu} \quad \text{for i. m. coprime pairs } (p,q) \in \mathbb{N} \times \mathbb{N} \right\}$$

and for $0 < a < 1$ small enough define the restricted Diophantine set

$$A_{\mu,Q} = \left\{ t \in (0,1)\backslash\mathbb{Q} \, : \, \left|t - \frac{p}{q}\right| \leq \frac{a}{q^\mu} \quad \text{for i. m. coprime pairs } (p,q) \in \mathbb{N} \times 4Q\mathbb{N} \right\}.$$

For $\mu = \infty$ we define $A_\infty = \bigcap_{\mu \geq 2} A_\mu$ and $A_{\infty,Q} = \bigcap_{\mu\geq 2} A_{\mu,Q}$. Clearly, $A_{\mu,Q} \subset A_\mu$. Our first step is to give an upper bound for $\alpha_{x_0}(t)$ for $t \in A_{\mu,Q}$.

[19] We lose nothing with this reduction when computing the spectrum of singularities, but it may be problematic if we aim to compute the Hölder regularity $\alpha_{x_0}(t)$ for all t.

Proposition 4.3 *Let $\mu \geq 2$ and $t \in A_{\mu,Q}$. Then, $\alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu}$.*

Proof We begin with the case $\mu < \infty$. If $t \in A_{\mu,Q}$, there is a sequence of irreducible fractions p_n/q_n with $q_n \in 4Q\mathbb{N}$, for which we can use (26) and write

$$R_{x_0}(t) - R_{x_0}\Big(\frac{p_n}{q_n}\Big) = c\, e^{2\pi i \phi_{n,x_0}} \frac{\sqrt{|h_n|}}{\sqrt{q_n}} - 2\pi i h_n + O\left(\min\left(\sqrt{q_n}\, h_n, q_n^{3/2}\, h_n^{3/2}\right)\right), \tag{28}$$

where we absorbed $F(0)$ into c and we defined h_n and μ_n as

$$h_n = t - \frac{p_n}{q_n}, \quad |h_n| = \frac{1}{q_n^{\mu_n}} \leq \frac{a}{q_n^{\mu}} < \frac{1}{q_n^{\mu}}. \tag{29}$$

We now absorb the second and third terms in (28) in the first term. First, $\mu \geq 2$ implies $q_n^2|h_n| \leq 1$, so $\min(\sqrt{q_n}\,|h_n|, q_n^{3/2}\,|h_n|^{3/2}) = q_n^{3/2}\,|h_n|^{3/2}$. Letting C be the universal constant in the O in (28),

$$C\, q_n^{3/2}|h_n|^{3/2} \leq \frac{c}{4}\frac{\sqrt{|h_n|}}{\sqrt{q_n}} \quad \Longleftrightarrow \quad q_n^2|h_n| \leq \frac{c}{4C},$$

and since $q_n^2|h_n| \leq a q_n^{2-\mu} \leq a$, it suffices to ask $a \leq c/(4C)$. Regarding the second term, we have

$$2\pi\,|h_n| \leq \frac{c}{4}\frac{\sqrt{|h_n|}}{\sqrt{q_n}} \quad \Longleftrightarrow \quad q_n\,|h_n| \leq \left(\frac{c}{8\pi}\right)^2$$

This holds for large n because $q_n^2|h_n| \leq 1$ implies $q_n\,|h_n| \leq 1/q_n$, and because $\limsup_{n\to\infty} q_n = \infty$ (otherwise q_n would be bounded and hence the sequence p_n/q_n would be finite). All together, using the reverse triangle inequality in (28) and the bound for h_n in (29)

$$\left| R_{x_0}(t) - R_{x_0}\left(\frac{p_n}{q_n}\right)\right| \geq \frac{c}{2}\frac{\sqrt{|h_n|}}{\sqrt{q_n}} \geq \frac{c}{2}\,|h_n|^{\frac{1}{2}+\frac{1}{2\mu}}, \quad \forall n \gg 1.$$

This means that R_{x_0} cannot be better than $\mathcal{C}^{\frac{1}{2}+\frac{1}{2\mu}}$ at t, thus concluding the proof for $\mu < \infty$.

If $t \in A_{\infty,Q}$, by definition $t \in A_{\mu,Q}$ for all $\mu \geq 2$, hence we just proved that $\alpha_{x_0}(t) \leq 1/2 + 1/(2\mu)$ for all $\mu \geq 2$. Taking the limit $\mu \to \infty$ we get $\alpha_{x_0}(t) \leq 1/2$. □

To prove Theorem 1.1, we need to compute $\dim_{\mathcal{H}}\{\, t : \alpha_{x_0}(t) = \alpha \,\}$ with prescribed α. For that, we need to complement Proposition 4.3 by proving that for $t \in A_{\mu,Q}$ we also have $\alpha_{x_0}(t) \geq \frac{1}{2} + \frac{1}{2\mu}$. By Proposition 3.6, it would suffice to prove that $t \in A_{\mu,Q}$

has irrationality $\mu(t) = \mu$. Unfortunately, when $\mu < \infty$ this need not be true. To fix this, for $2 \leq \mu < \infty$ define the companion sets

$$\begin{aligned} B_\mu &= A_\mu \setminus \bigcup_{\epsilon > 0} A_{\mu+\epsilon} \\ &= \left\{ t \in A_\mu \mid \forall \epsilon > 0,\ \left|t - \frac{p}{q}\right| \leq \frac{1}{q^{\mu+\epsilon}} \text{ only for finitely many } \frac{p}{q} \right\}, \end{aligned}$$

and

$$\begin{aligned} B_{\mu,Q} &= A_{\mu,Q} \setminus \bigcup_{\epsilon > 0} A_{\mu+\epsilon} \\ &= \left\{ t \in A_{\mu,Q} \mid \forall \epsilon > 0,\ \left|t - \frac{p}{q}\right| \leq \frac{1}{q^{\mu+\epsilon}} \text{ only for finitely many } \frac{p}{q} \right\}, \end{aligned} \tag{30}$$

which have the properties we need.

Proposition 4.4 *Let* $2 \leq \mu < \infty$. *Then,*

(i) $B_{\mu,Q} \subset B_\mu \subset \{ t \in \mathbb{R} \setminus \mathbb{Q} : \mu(t) = \mu \}$.
(ii) *If* $t \in B_{\mu,Q}$, *then* $\alpha_{x_0}(t) = \frac{1}{2} + \frac{1}{2\mu}$.
(iii) *If* $t \in A_{\infty,Q}$, *then* $\alpha_{x_0}(t) = 1/2$.

Proof (i) First, $B_{\mu,Q} \subset B_\mu$ because $A_{\mu,Q} \subset A_\mu$. The second inclusion is a consequence of the definition of the irrationality exponent in (12). Indeed, $t \in B_\mu \subset A_\mu$ directly implies that $\mu(t) \geq \mu$. On the other hand, for all $\epsilon > 0$, $t \in B_\mu$ implies $t \notin A_{\mu+\epsilon}$, so t can be approximated with the exponent $\mu + \epsilon$ only with finitely many fractions, and thus $\mu(t) \leq \mu + \epsilon$. Consequently, $\mu(t) \leq \mu$.

(ii) By (i), $t \in B_{\mu,Q}$ implies $\mu(t) = \mu$, so by Proposition 3.6 we get $\alpha_{x_0}(t) \geq \frac{1}{2} + \frac{1}{2\mu}$. At the same time, $t \in B_{\mu,Q} \subset A_{\mu,Q}$, so Proposition 4.3 implies $\alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu}$.
(iii) It follows directly from Propositions 3.1 and 4.3. □

Corollary 4.5 *Let* $2 < \mu < \infty$. *Then, for all* $\epsilon > 0$,

$$B_{\mu,Q} \subset \left\{ t \in (0,1) \ :\ \alpha_{x_0}(t) = \frac{1}{2} + \frac{1}{2\mu} \right\} \subset A_{\mu-\epsilon}.$$

For $\mu = 2$ *we have the slightly more precise*

$$B_{2,Q} \subset \{ t \in (0,1) \ :\ \alpha_{x_0}(t) = 3/4 \} \subset A_2.$$

For $\mu = \infty$,

$$A_{\infty,Q} \subset \{ t \in (0,1) \ :\ \alpha_{x_0}(t) = 1/2 \} \subset A_\infty \cup \mathbb{Q}.$$

Proof Left inclusions follow from Proposition 4.4 for all $\mu \geq 2$, so we only need to prove the right inclusions. When $\mu = 2$, it follows from the Dirichlet approximation theorem, which states that $\mathbb{R}\setminus\mathbb{Q} \subset A_2$, and Proposition 3.5, in which we proved that if t is rational, then either $\alpha_{x_0}(t) = 1/2$ or $\alpha_{x_0}(t) \geq 3/2$. Thus, $\{ t \in (0,1) : \alpha_{x_0}(t) = 3/4 \} \subset (0,1)\setminus\mathbb{Q} \subset A_2$. Suppose now that $2 < \mu < \infty$ and that $\alpha_{x_0}(t) = \frac{1}{2} + \frac{1}{2\mu}$. By Proposition 3.6, $\alpha_{x_0}(t) \geq \frac{1}{2} + \frac{1}{2\mu(t)}$, so we get $\mu \leq \mu(t)$. In particular, given any $\epsilon > 0$, we have $\mu - \epsilon < \mu(t)$, so $\left| t - \frac{p}{q} \right| \leq 1/q^{\mu-\epsilon}$ for infinitely many coprime pairs $(p,q) \in \mathbb{N}\times\mathbb{N}$, which means that $t \in A_{\mu-\epsilon}$. Finally, for $\mu = \infty$, if $t \notin \mathbb{Q}$ is such that $\alpha_{x_0}(t) = 1/2$, then by Proposition 3.6 we get $\mu(t) = \infty$, which implies that $t \in A_\mu$ for all $\mu \geq 2$, hence $t \in A_\infty$. □

Now, to prove Theorem 1.1 it suffices to compute $\dim_{\mathcal{H}} A_\mu$ and $\dim_{\mathcal{H}} B_{\mu,Q}$.

Proposition 4.6 *For* $2 \leq \mu < \infty$, $\dim_{\mathcal{H}} A_\mu = \dim_{\mathcal{H}} B_{\mu,Q} = 2/\mu$. *Also,* $\dim_{\mathcal{H}} A_\infty = 0$.

Form this result, whose proof we postpone, we can prove Theorem 1.1 as a corollary.

Theorem 4.7 *Let* $x_0 \in \mathbb{Q}$. *Then, the spectrum of singularities of* R_{x_0} *is*

$$d_{x_0}(\alpha) = \begin{cases} 4\alpha - 2, & 1/2 \leq \alpha \leq 3/4, \\ 0, & \alpha = 3/2, \\ -\infty, & \textit{otherwise}. \end{cases}$$

Proof Proposition 3.1 implies $d(\alpha) = -\infty$ when $\alpha < 1/2$, while Propositions 3.5 and 4.2 imply that $d_{x_0}(3/2) = 0$ and $d_{x_0}(\alpha) = -\infty$ if $\alpha > 3/4$ and $\alpha \neq 3/2$. When $1/2 \leq \alpha \leq 3/4$, it follows from Corollary 4.5, Proposition 4.6 and the periodicity of R_{x_0}. First, $d_{x_0}(1/2) \leq \dim_{\mathcal{H}}(A_\infty \cup \mathbb{Q}) = 0$ because $\dim_{\mathcal{H}} \mathbb{Q} = \dim_{\mathcal{H}} A_\infty = 0$. On the other hand, for $2 \leq \mu < \infty$ we get

$$\frac{2}{\mu} \leq d_{x_0}\left(\frac{1}{2} + \frac{1}{2\mu}\right) \leq \frac{2}{\mu - \epsilon}, \quad \forall \epsilon > 0 \quad \Longrightarrow \quad d_{x_0}\left(\frac{1}{2} + \frac{1}{2\mu}\right) = \frac{2}{\mu}.$$

which gives the result for $1/2 < \alpha \leq 3/4$ by renaming $\alpha = 1/2 + 1/(2\mu)$. □

Let us now prove Proposition 4.6.

Proof of Proposition 4.6 We have $A_2 = (0,1)\setminus\mathbb{Q}$ by Dirichlet approximation, so $\dim_{\mathcal{H}} A_2 = 1$. For $\mu > 2$ we have $\dim_{\mathcal{H}} A_\mu = 2/\mu$ by the Jarnik–Besicovitch Theorem 2.2. Also, $A_\infty \subset A_\mu$ for all $\mu \geq 2$, so $\dim_{\mathcal{H}} A_\infty \leq 2/\mu$ for all $\mu \geq 2$, hence $\dim_{\mathcal{H}} A_\infty = 0$. So we only need to prove that $\dim_{\mathcal{H}} B_{\mu,Q} = 2/\mu$ for $2 \leq \mu < \infty$. Moreover,

$$B_{\mu,Q} = A_{\mu,Q} \setminus \bigcup_{\epsilon > 0} A_{\mu+\epsilon} \subset A_{\mu,Q} \subset A_\mu,$$

which implies $\dim_{\mathcal{H}} B_{\mu,Q} \leq \dim_{\mathcal{H}} A_\mu = 2/\mu$. Hence it suffices to prove that $\dim_{\mathcal{H}} B_{\mu,Q} \geq 2/\mu$. This claim follows from $\mathcal{H}^{2/\mu}(A_{\mu,Q}) > 0$. Indeed, we first remark that the sets A_μ are nested, in the sense that $A_\sigma \subset A_\mu$ when $\sigma > \mu$. We can therefore write

$$\bigcup_{\epsilon>0} A_{\mu+\epsilon} = \bigcup_{n\in\mathbb{N}} A_{\mu+\frac{1}{n}}.$$

By the Jarnik–Besicovitch Theorem 2.2, $\dim_{\mathcal{H}} A_{\mu+1/n} = 2/(\mu+1/n) < 2/\mu$, so $\mathcal{H}^{2/\mu}(A_{\mu+1/n}) = 0$ for all $n \in \mathbb{N}$, hence

$$\mathcal{H}^{2/\mu}\left(\bigcup_{\epsilon>0} A_{\mu+\epsilon}\right) = \mathcal{H}^{2/\mu}\left(\bigcup_{n\in\mathbb{N}} A_{\mu+\frac{1}{n}}\right) = \lim_{n\to\infty} \mathcal{H}^{2/\mu}\left(A_{\mu+\frac{1}{n}}\right) = 0.$$

Therefore,

$$\begin{aligned}\mathcal{H}^{2/\mu}\left(B_{\mu,Q}\right) &= \mathcal{H}^{2/\mu}\left(A_{\mu,Q}\setminus\bigcup_{\epsilon>0} A_{\mu+\epsilon}\right) = \mathcal{H}^{2/\mu}(A_{\mu,Q}) - \mathcal{H}^{2/\mu}\left(\bigcup_{\epsilon>0} A_{\mu+\epsilon}\right)\\ &= \mathcal{H}^{2/\mu}\left(A_{\mu,Q}\right),\end{aligned}$$

so $\mathcal{H}^{2/\mu}(A_{\mu,Q}) > 0$ implies $\mathcal{H}^{2/\mu}(B_{\mu,Q}) > 0$, hence $\dim_{\mathcal{H}} B_{\mu,Q} \geq 2/\mu$.

Let us thus prove $\mathcal{H}^{2/\mu}(A_{\mu,Q}) > 0$, for which we follow the procedure outlined in Sect. 2 with the set of denominators $\mathcal{Q} = 4Q\mathbb{N}$. We first detect the largest μ such that $A_{\mu,Q}$ has full Lebesgue measure using the Duffin–Schaeffer Theorem 2.1. Define

$$\psi_{\mu,Q}(q) = a\,\frac{\mathbb{1}_{4Q\mathbb{N}}(q)}{q^\mu},$$

where $a > 0$ comes from the definition of $A_{\mu,Q}$ and $\mathbb{1}_{4Q\mathbb{N}}(q)$ is the indicator function of $4Q\mathbb{N}$,

$$\mathbb{1}_{4Q\mathbb{N}}(q) = \begin{cases} 1, & \text{if } 4Q \mid q,\\ 0, & \text{otherwise.}\end{cases}$$

Then, we have $A_{\mu,Q} = A_{\psi_{\mu,Q}}$, where

$$A_{\psi_{\mu,Q}} = \left\{ t \in [0,1] \,:\, \left|t - \frac{p}{q}\right| \leq \psi_{\mu,Q}(q) \ \text{ for i. m. coprime pairs } (p,q) \in \mathbb{N}\times\mathbb{N} \right\}$$

has the form needed for the Duffin–Schaeffer Theorem 2.1. Indeed, the inclusion $\subset$ follows directly from the definition of $\psi_{\mu,Q}$. For the inclusion $\supset$, observe first that if $t \in A_{\psi_{\mu,Q}}$ with $\mu > 1$, then $t \notin \mathbb{Q}$. Now, if a coprime pair $(p,q) \in \mathbb{N}^2$ satisfies $|t - p/q| \leq \psi_{\mu,Q}(q)$, then $q \in 4Q\mathbb{N}$ because otherwise we get the contradiction

$$0 < \left|t - \frac{p}{q}\right| \leq \psi_{\mu,Q}(q) = a\,\frac{\mathbb{1}_{4Q\mathbb{N}}(q)}{q^\mu} = 0.$$

In this setting, the Duffin–Schaeffer theorem says that $A_{\mu,Q}$ has Lebesgue measure 1 if and only if

$$\sum_{q=1}^{\infty} \varphi(q)\, \psi_{\mu,Q}(q) = \frac{a}{(4Q)^{\mu}} \sum_{n=1}^{\infty} \frac{\varphi(4Qn)}{n^{\mu}} = \infty,$$

and has zero measure otherwise. Using this characterization, we prove now

$$|A_{\mu,Q}| = \begin{cases} 1, \ \mu \leq 2, \\ 0, \ \mu > 2, \end{cases} \tag{31}$$

independently of a. To detect the critical $\mu = 2$, trivially bound $\varphi(n) < n$ so that

$$\sum_{n=1}^{\infty} \frac{\varphi(4Qn)}{n^{\mu}} < \sum_{n=1}^{\infty} \frac{4Qn}{n^{\mu}} = 4Q \sum_{n=1}^{\infty} \frac{1}{n^{\mu-1}} < \infty, \quad \text{if } \ \mu > 2.$$

However, this argument fails when $\mu = 2$. What is more, denote by $\mathbb{P}$ the set of primes so that

$$\sum_{n=1}^{\infty} \frac{\varphi(4Qn)}{n^{2}} > \sum_{p\in\mathbb{P},\, p>4Q} \frac{\varphi(4Qp)}{p^{2}}$$

If $p \in \mathbb{P}$ and $p > 4Q$, then $\gcd(p, 4Q) = 1$ because $p \nmid 4Q$ (for if $p \mid 4Q$ then $p \leq 4Q$). Therefore, $\varphi(4Qp) = \varphi(4Q)\,\varphi(p) = \varphi(4Q)\,(p-1) > \varphi(4Q)\,p/2$, so

$$\sum_{n=1}^{\infty} \frac{\varphi(4Qn)}{n^{2}} > \frac{\varphi(4Q)}{2} \sum_{p\in\mathbb{P},\, p>4Q} \frac{1}{p} = \infty,$$

because the sum of the reciprocals of the prime numbers diverges.[20] The Duffin–Schaeffer Theorem 2.1 thus implies that $|A_{2,Q}| = 1$ and, in particular, $\dim_{\mathcal{H}} A_{2,Q} = 1$. From this we immediately get $|A_{\mu,Q}| = 1$ when $\mu < 2$ because $A_{2,Q} \subset A_{\mu,Q}$.

Once we know (31), we use the Mass Transference Principle Theorem 2.3 to compute the dimension of $A_{\mu,Q}$ for $\mu > 2$. Write first

$$A_{\mu,Q} = \limsup_{q\to\infty} \bigcup_{p\leq q,\,(p,q)=1} B\left(\frac{p}{q}, \psi_{\mu,Q}(q)\right).$$

Let $\beta = 2/\mu$ so that

$$\psi_{\mu,Q}(q)^{\beta} = \left(a\, \frac{\mathbb{1}_{4Q\mathbb{N}}(q)}{q^{\mu}}\right)^{\beta} = a^{\beta}\, \frac{\mathbb{1}_{4Q\mathbb{N}}(q)}{q^{\mu\beta}} = a^{2/\mu}\, \frac{\mathbb{1}_{4Q\mathbb{N}}(q)}{q^{2}} = \psi_{2,Q}(q),$$

[20] This argument shows that the strategy used here to compute the dimension of $A_{\mu,Q}$ also works if we restrict the denominators to the primes $\mathcal{Q} = \mathbb{P}$ in the first place. This situation arises when computing the spectrum of singularities of trajectories of polygonal lines with non-zero rational torsion, studied in [4].

with a new underlying constant $a^{2/\mu}$. Therefore,

$$(A_{\mu,Q})^\beta := \limsup_{q\to\infty} \bigcup_{p\leq q,\,(p,q)=1} B\left(\frac{p}{q}, \psi_{\mu,Q}(q)^\beta\right)$$
$$= \limsup_{q\to\infty} \bigcup_{p\leq q,\,(p,q)=1} B\left(\frac{p}{q}, \psi_{2,Q}(q)\right) = A_{2,Q}.$$

Observe that β is chosen to be the largest possible exponent that gives $|(A_{\mu,Q})^\beta| = |(A_{\mu\beta,Q})| = 1$. Since (31) is independent of a, we get $|(A_{\mu,Q})^{2/\mu}| = |A_{2,Q}| = 1$, and the Mass Transference Principle Theorem 2.3 implies that $\mathcal{H}^{2/\mu}\big(A_{\mu,Q}\big) = \infty$. The proof is complete. ◻

5 Proof of Theorem 1.3: spectrum of singularities when $x_0 \notin \mathbb{Q}$

In this section we work with $x_0 \notin \mathbb{Q}$ and prove Theorem 1.3. Following the strategy for $x_0 \in \mathbb{Q}$, we first study the Hölder regularity at rational t in Sect. 5.1, and at irrational t in Sect. 5.2

5.1 Regularity at rational *t*

Let $t = p/q$ an irreducible fraction. With Corollary 3.3 in mind, we now have $x_q = \operatorname{dist}(x_0, \mathbb{Z}/q) \neq 0$. Since q is fixed, $\lim_{h\to 0} x_q/|h|^{1/2} = \infty$, so $F_\pm(x) = O(x^{-2})$ implies $F_\pm(x_q/\sqrt{|h|}) \lesssim |h|/x_q^2$ when $h \to 0$. Also $|G(p, m_q, q)| \leq \sqrt{2q}$ for all m_q. Hence,

$$\left| R_{x_0}\left(\frac{p}{q} + h\right) - R_{x_0}\left(\frac{p}{q}\right) + 2\pi i h \right| \lesssim \left(\frac{1}{\sqrt{q}\, x_q^2} + q^{3/2}\right) h^{3/2}.$$

This regularity is actually the best we can get.

Proposition 5.1 *Let $x_0 \in \mathbb{R}\setminus\mathbb{Q}$ and let $t \in \mathbb{Q}$. Then, $\alpha_{x_0}(t) = 3/2$.*

We postpone the proof of $\alpha_{x_0}(t) \leq 3/2$ to Proposition B.6. In any case, this means that when $x_0 \notin \mathbb{Q}$, R_{x_0} is more regular at rational points than when $x_0 \in \mathbb{Q}$.

5.2 Regularity at irrational *t*

Let now $t \notin \mathbb{Q}$. Again, we aim at an upper bound for $\alpha_{x_0}(t)$ that complements the lower bound in Proposition 3.6. by approximating $t \notin \mathbb{Q}$ by rationals p_n/q_n and using the asymptotic behavior in Corollary 3.3. However, now $x_0 \notin \mathbb{Q}$ implies $x_{q_n} \neq 0$, so we cannot directly assume $F_\pm(x_{q_n}/\sqrt{|h_{q_n}|}) \simeq F_\pm(0) \simeq 1$ anymore. Therefore, it is fundamental to understand the behavior of the quotient $x_{q_n}/\sqrt{|h_{q_n}|}$.

5.2.1 Heuristics

Let $q \in \mathbb{N}$ and define the exponents μ_q and σ_q as usual,

$$x_q = \text{dist}\left(x_0, \frac{\mathbb{Z}}{q}\right) = \frac{1}{q^{\sigma_q}}, \quad |h_q| = \text{dist}\left(t, \frac{\mathbb{Z}}{q}\right) = \frac{1}{q^{\mu_q}}, \quad \Longrightarrow \quad \frac{x_q}{\sqrt{|h_q|}} = \frac{1}{q^{\sigma_q - \mu_q/2}}.$$

If $\sigma_q - \mu_q/2 > c > 0$ holds for a sequence q_n, we should recover the behavior when $x_0 \in \mathbb{Q}$ because

$$\lim_{n\to\infty}\left(\sigma_{q_n} - \frac{\mu_{q_n}}{2}\right) \geq c > 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{x_{q_n}}{\sqrt{|h_{q_n}|}} = 0$$
$$\Longrightarrow \quad F_\pm\left(\frac{x_{q_n}}{\sqrt{|h_{q_n}|}}\right) \simeq F_\pm(0), \quad n \gg 1. \tag{32}$$

The main term in the asymptotic behavior for $R_{x_0}(t) - R_{x_0}(p_n/q_n)$ in Corollary 3.3 would then be

$$\text{Main Term} = \frac{\sqrt{|h_{q_n}|}}{q_n} G(p_n, m_{q_n}, q_n) F_\pm(0) \simeq \frac{\sqrt{|h_{q_n}|}}{\sqrt{q_n}} \simeq h_{q_n}^{\frac{1}{2}+\frac{1}{2\mu_{q_n}}}$$

if we assume the necessary parity conditions so that $|G(p_n, m_{q_n}, q_n)| \simeq \sqrt{q_n}$. Recalling the definition of the exponent of irrationality $\mu(\cdot)$ in (12), we may think of $\sigma_{q_n} \to \mu(x_0)$ and $\mu_{q_n} \to \mu(t)$, so these heuristic computations suggest that $\alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu(t)}$ for t such that $\mu(t) \leq 2\mu(x_0)$. Since Proposition 3.6 gives $\alpha_{x_0}(t) \geq \frac{1}{2} + \frac{1}{2\mu(t)}$, we may expect that

$$\alpha_{x_0}(t) = \frac{1}{2} + \frac{1}{2\mu(t)}, \quad \text{if} \quad 2 \leq \mu(t) \leq 2\mu(x_0), \tag{33}$$

or at least for a big subset of such t. It is less clear what to expect when $\mu(t) > 2\mu(x_0)$, since (32) need not hold. Actually, if $\sigma_{q_n} - \mu_{q_n}/2 < c < 0$ for all sequences, then since $F_\pm(x) = x^{-2} + O(x^{-4})$,

$$\lim_{n\to\infty} \frac{x_{q_n}}{\sqrt{|h_{q_n}|}} = \lim_{n\to\infty} q_n^{\mu_{q_n}/2 - \sigma_{q_n}} = \infty \quad \Longrightarrow \quad F_\pm\left(\frac{x_{q_n}}{\sqrt{|h_{q_n}|}}\right) \simeq \frac{1}{q_n^{\mu_{q_n} - 2\sigma_{q_n}}}$$
$$= |h_{q_n}|^{1 - \frac{2\sigma_{q_n}}{\mu_{q_n}}},$$

which in turn would make the main term in $R_{x_0}(t) - R_{x_0}(p_n/q_n)$ be

$$\text{Main Term} = \frac{\sqrt{h_{q_n}}}{q_n} G(p_n, m_{q_n}, q_n) F_\pm\left(\frac{x_{q_n}}{\sqrt{|h_{q_n}|}}\right) \simeq h_{q_n}^{\frac{1}{2}+\frac{1}{2\mu_{q_n}}} h_{q_n}^{1-\frac{2\sigma_{q_n}}{\mu_{q_n}}} \simeq h_{q_n}^{\frac{3}{2}-\frac{4\sigma_{q_n}-1}{2\mu_{q_n}}},$$

which corresponds to an exponent $\frac{3}{2}-\frac{4\mu(x_0)-1}{2\mu(t)}$. Together with lower bound in Proposition 3.6, we would get $\frac{1}{2}+\frac{1}{2\mu(t)}\leq\alpha_{x_0}(t)\leq\frac{3}{2}-\frac{4\mu(x_0)-1}{2\mu(t)}$, which leaves an open interval for $\alpha_{x_0}(t)$.

The main difficulty to materialize the ideas leading to (33) is that we need the sequence q_n to generate good approximations of both x_0 and t simultaneously, which a priori may be not possible. In the following lines we show how we can partially dodge this problem to prove Theorem 1.3.

5.2.2 Proof of Theorem 1.3

Let $\sigma\geq 2$. Recalling the definition of the sets $A_{\mu,\mathcal{Q}}$ in (15), define

$$A_{\sigma,\,\mathbb{N}\backslash 4\mathbb{N}}=\Big\{x\in[0,1]\,:\,\Big|x-\frac{b}{q}\Big|<\frac{1}{q^\sigma}\quad\text{for infinitely many coprime pairs } (b,q)\in\mathbb{N}\times(\mathbb{N}\backslash 4\mathbb{N})\Big\}.$$

We first prove that the restriction in the denominators[21] does not affect the Hausdorff dimension.

Proposition 5.2 *Let* $\sigma\geq 2$. *Then,* $\dim_{\mathcal{H}}A_{\sigma,\,\mathbb{N}\backslash 4\mathbb{N}}=2/\sigma$. *Moreover,* $A_{2,\,\mathbb{N}\backslash 4\mathbb{N}}=(0,1)\backslash\mathbb{Q}$, *hence* $|A_{2,\,\mathbb{N}\backslash 4\mathbb{N}}|=1$. *If* $\sigma>2$, *then* $\mathcal{H}^{2/\sigma}(A_{\sigma,\,\mathbb{N}\backslash 4\mathbb{N}})=\infty$.

Proof The proof for the upper bound for the Hausdorff dimension is standard. Writing

$$\begin{aligned}A_{\sigma,\,\mathbb{N}\backslash 4\mathbb{N}}&=\limsup_{q\to\infty\ (q\notin 4\mathbb{N})}\ \bigcup_{1\leq b<q,\ (b.q)=1}B\Big(\frac{b}{q},\frac{1}{q^\sigma}\Big)\\&=\bigcap_{Q=1}^{\infty}\ \bigcup_{q\geq Q,\ q\notin 4\mathbb{N}}\Big(\bigcup_{1\leq b<q,\ (b.q)=1}B\Big(\frac{b}{q},\frac{1}{q^\sigma}\Big)\Big),\end{aligned}$$

we get an upper bound for the Hausdorff measures using the canonical cover

$$\begin{aligned}&A_{\sigma,\,\mathbb{N}\backslash 4\mathbb{N}}\subset\bigcup_{q\geq Q,\ q\notin 4\mathbb{N}}\Big(\bigcup_{1\leq b<q}B\Big(\frac{b}{q},\frac{1}{q^\sigma}\Big)\Big),\quad\forall Q\in\mathbb{N}\\&\implies\quad\mathcal{H}^\beta(A_{\sigma,\,\mathbb{N}\backslash 4\mathbb{N}})\leq\lim_{Q\to\infty}\sum_{q\geq Q}\frac{1}{q^{\sigma\beta-1}}.\end{aligned}\tag{34}$$

Thus, $\mathcal{H}^\beta(A_{\sigma,\,\mathbb{N}\backslash 4\mathbb{N}})=0$ when $\sigma\beta-1>1$, and consequently $\dim_{\mathcal{H}}A_{\sigma,\,\mathbb{N}\backslash 4\mathbb{N}}\leq 2/\sigma$.

For the lower bound we follow the procedure discussed in Sect. 2, though unlike in the proof of Proposition 4.6 we do not need the Duffin–Schaeffer theorem here. We first study the Lebesgue measure of $A_{\sigma,\,\mathbb{N}\backslash 4\mathbb{N}}$. From (34) with $\beta=1$, we directly get

[21] This condition, which will be apparent later, comes from parity the conditions for the Gauss sums not to vanish.

$|A_{\sigma,\,\mathbb{N}\setminus 4\mathbb{N}}| = 0$ when $\sigma > 2$. When $\sigma = 2$, we get $A_{2,\,\mathbb{N}\setminus 4\mathbb{N}} = A_2 = (0,1)\setminus\mathbb{Q}$. Indeed, if b_n/q_n is the sequence of approximations by continued fractions of $x \in (0,1)\setminus\mathbb{Q}$, two consecutive denominators q_n and q_{n+1} are never both even.[22] This means that there is a subsequence b_{n_k}/q_{n_k} such that $|x - b_{n_k}/q_{n_k}| < 1/q_{n_k}^2$ and q_{n_k} is odd for all $k \in \mathbb{N}$. In particular, $q_{n_k} \notin 4\mathbb{N}$, so $(0,1)\setminus\mathbb{Q} \subset A_{2,\,\mathbb{N}\setminus 4\mathbb{N}}$. Hence,

$$|A_{\sigma,\,\mathbb{N}\setminus 4\mathbb{N}}| = \begin{cases} 1, & \sigma \le 2, \\ 0, & \sigma > 2, \end{cases} \tag{35}$$

With this in hand, we use the Mass Transference Principle Theorem 2.3. For $\beta > 0$,

$$\begin{aligned}(A_{\sigma,\,\mathbb{N}\setminus 4\mathbb{N}})^\beta &= \limsup_{\substack{q\to\infty \\ q\notin 4\mathbb{N}}} \bigcup_{1\le b<q,\,(b,q)=1} B\left(\frac{b}{q}, \left(\frac{1}{q^\sigma}\right)^\beta\right) \\ &= \limsup_{\substack{q\to\infty \\ q\notin 4\mathbb{N}}} \bigcup_{1\le b<q,\,(b,q)=1} B\left(\frac{b}{q}, \frac{1}{q^{\sigma\beta}}\right) = A_{\sigma\beta,\,\mathbb{N}\setminus 4\mathbb{N}}.\end{aligned}$$

Thus, choosing $\beta = 2/\sigma$ we get $(A_{\sigma,\,\mathbb{N}\setminus 4\mathbb{N}})^{2/\sigma} = A_{2,\,\mathbb{N}\setminus 4\mathbb{N}}$, hence by (35) we get $|(A_{\sigma,\,\mathbb{N}\setminus 4\mathbb{N}})^{2/\sigma}| = 1$. The Mass Transference Principle implies $\dim_{\mathcal{H}} A_{\sigma,\,\mathbb{N}\setminus 4\mathbb{N}} \ge 2/\sigma$ and $\mathcal{H}^{2/\sigma}(A_{\sigma,\,\mathbb{N}\setminus 4\mathbb{N}}) = \infty$. □

Let $x_0 \in A_{\sigma,\,\mathbb{N}\setminus 4\mathbb{N}}$. Then there exists a sequence of pairs $(b_n, q_n) \in \mathbb{N}\times(\mathbb{N}\setminus 4\mathbb{N})$ such that $|x_0 - b_n/q_n| < 1/q_n^\sigma$ and moreover b_n/q_n are all approximations by continued fractions. Define

$$\mathcal{Q}_{x_0} = \{\, q_n \;:\; n \in \mathbb{N} \,\}$$

to be the set of such denominators. This sequence exists because:

- if $\sigma = 2$, there is a subsequence of continued fraction approximations with odd denominator, in particular with $q_n \notin 4\mathbb{N}$.
- if $\sigma > 2$, by definition there exist a sequence of pairs $(b_n, q_n) \in \mathbb{N}\times(\mathbb{N}\setminus 4\mathbb{N})$ such that

$$\left|x_0 - \frac{b_n}{q_n}\right| < \frac{1}{q_n^\mu} \le \frac{1}{2q_n^2}, \qquad \text{for large enough } n \in \mathbb{N}.$$

  By a theorem of Khinchin [36, Theorem 19], all such b_n/q_n are continued fraction approximations of x_0.

[22] If $x = [a_0; a_1, a_2, \ldots]$ is a continued fraction, then $q_0 = 1$, $q_1 = a_1$ and $q_n = a_n q_{n-1} + q_{n-2}$ for $n \ge 2$. If q_N and q_{N+1} were both even for some N, then q_{N-1} would also be, and by induction $q_0 = 1$ would be even.

Since all such q_n are the denominators of continued fraction approximations, the sequence q_n grows exponentially.[23] Following again the notation in (15) in Sect. 2, for $\mu \geq 1$ and $0 < c < 1/2$, let[24]

$$A_{\mu,\mathcal{Q}_{x_0}} = \left\{ t \in [0,1] : \left| t - \frac{p}{q} \right| < \frac{c}{q^\mu} \text{ for infinitely many coprime pairs } (p,q) \in \mathbb{N} \times \mathcal{Q}_{x_0} \right\}.$$

Proposition 5.3 *For $\mu \geq 1$, $\dim_{\mathcal{H}}(A_{\mu,\mathcal{Q}_{x_0}}) = 1/\mu$.*

Proof As in the proof of Proposition 5.2, the upper bound follows from the limsup expression $A_{\mu,\mathcal{Q}_{x_0}} = \limsup_{n\to\infty} \bigcup_{1\leq p\leq q_n,\,(p,q_n)=1} B(p/q_n, c/q_n^\mu)$ and its canonical covering

$$A_{\mu,\mathcal{Q}_{x_0}} \subset \bigcup_{n\geq N} \bigcup_{1\leq p\leq q_n} B\left(\frac{p}{q_n}, \frac{c}{q_n^\mu}\right), \quad \forall N \in \mathbb{N} \implies \mathcal{H}^\beta\left(A_{\mu,\mathcal{Q}_{x_0}}\right) \leq c^\beta \lim_{N\to\infty} \sum_{n=N}^\infty \frac{1}{q_n^{\mu\beta-1}}. \quad (36)$$

Since $q_n \geq 2^{n/2}$, the series converges if and only if $\mu\beta - 1 > 0$. Thus, $\mathcal{H}^\beta(A_{\mu,\mathcal{Q}_{x_0}}) = 0$ for all $\beta > 1/\mu$, hence $\dim_{\mathcal{H}}(A_{\mu,\mathcal{Q}_{x_0}}) \leq 1/\mu$.

For the lower bound we follow again the procedure in Sect. 2. First we compute the Lebesgue measure of $A_{\mu,\mathcal{Q}_{x_0}}$. From (36) with $\beta = 1$ we get $|A_{\mu,\mathcal{Q}_{x_0}}| = 0$ if $\mu > 1$. When $\mu \leq 1$, we need the full strength of the Duffin–Schaeffer theorem proved by Koukoulopoulos and Maynard [37] (see Theorem 2.1 in this paper). Indeed, we have $|A_{\mu,\mathcal{Q}_{x_0}}| = 1$ if and only if $\sum_{n=1}^\infty \varphi(q_n)/q_n^\mu = \infty$, and otherwise $|A_{\mu,\mathcal{Q}_{x_0}}| = 0$. If $\mu < 1$, we use one of the classic properties of Euler's totient function, namely that for $\epsilon = (1-\mu)/2 > 0$ there exists $N \in \mathbb{N}$ such that $\varphi(n) \geq n^{1-\epsilon}$ for all $n \geq N$. In particular, there exists $K \in \mathbb{N}$ such that

$$\sum_{n=1}^\infty \frac{\varphi(q_n)}{q_n^\mu} \geq \sum_{n=K}^\infty \frac{\varphi(q_n)}{q_n^\mu} \geq \sum_{n=K}^\infty q_n^{1-\mu-\epsilon} \geq \sum_{n=K}^\infty 1 = \infty,$$

so $|A_{\mu,\mathcal{Q}_{x_0}}| = 1$ if $\mu < 1$. For $\mu = 1$, none of these arguments work, and we need to know the behavior of $\varphi(q_n)$ for $q_n \in \mathcal{Q}_{x_0}$, of which we have little control. So independently of $c > 0$,

$$|A_{\mu,\mathcal{Q}_{x_0}}| = \begin{cases} 1, & \mu < 1, \\ ?, & \mu = 1, \\ 0, & \mu > 1. \end{cases} \quad (37)$$

23 We actually have $q_n \geq 2^{n/2}$. To see this, rename this sequence as a subsequence $(b_{n_k}/q_{n_k})_k$ of the continued fraction convergents of x_0. By the properties of the continued fractions, $q_{n_k} \geq 2^{n_k/2}$. Since $n_k \geq k$, we get $q_{n_k} \geq 2^{k/2}$.

24 When $\mu = \infty$ the definition is adapted as usual as $A_{\infty,\mathcal{Q}_{x_0}} = \cap_\mu A_{\mu,\mathcal{Q}_{x_0}}$. Proofs for forthcoming results are written for $\mu < \infty$, but the simpler $\mu = \infty$ case is proved the same way we did in Sect. 4.3.

Even not knowing $|A_{1,\mathcal{Q}_{x_0}}|$, the Mass Transference Principle Theorem 2.3 allows us to compute the Hausdorff dimension of $A_{\mu,\mathcal{Q}_{x_0}}$ from (37). As usual, dilate the set with an exponent $\beta > 0$:

$$(A_{\mu,\mathcal{Q}_{x_0}})^\beta = \limsup_{n\to\infty} \bigcup_{1\le p\le q_n} B\left(\frac{p}{q_n}, \left(\frac{c}{q_n^\mu}\right)^\beta\right)$$
$$= \limsup_{n\to\infty} \bigcup_{1\le p\le q_n} B\left(\frac{p}{q_n}, \frac{c^\beta}{q_n^{\mu\beta}}\right) = A_{\mu\beta,\mathcal{Q}_{x_0}},$$

with a new constant c^β. Since (37) is independent of c, we have $|(A_{\mu,\mathcal{Q}_{x_0}})^\beta| = |A_{\mu\beta,\mathcal{Q}_{x_0}}| = 1$ if $\mu\beta < 1$, and the Mass Transference Principle implies $\dim_{\mathcal{H}} A_{\mu,\mathcal{Q}_{x_0}} \ge \beta$. Taking $\beta \to 1/\mu$, we deduce $\dim_{\mathcal{H}} A_{\mu,\mathcal{Q}_{x_0}} \ge 1/\mu$. □

As in Proposition 4.4 and in the definition of $B_{\mu,\mathcal{Q}}$ in (30), to get information about $\alpha_{x_0}(t)$ for $t \in A_{\mu,\mathcal{Q}_{x_0}}$ we need to restrict their exponent of irrationality. We do this by removing sets $A_{\mu+\epsilon}$ defined in (13). However, compared to Proposition 4.4 we have two fundamental difficulties:

(a) The dimensions $\dim_{\mathcal{H}} A_\mu = 2/\mu > 1/\mu = \dim_{\mathcal{H}} A_{\mu,\mathcal{Q}_{x_0}}$ do not match anymore.
(b) Because do not know the Lebesgue measure of $A_{1,\mathcal{Q}_{x_0}}$ in (37), we cannot conclude that $\mathcal{H}^{1/\mu}(A_{\mu,\mathcal{Q}_{x_0}}) = \infty$ if $\mu > 1$.

To overcome these difficulties, let $\delta_1, \delta_2 > 0$ and define the set

$$B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}} = \left(A_{\mu,\mathcal{Q}_{x_0}} \setminus A_{\mu+\delta_1,\mathcal{Q}_{x_0}}\right) \setminus \left(\bigcup_{\epsilon>0} A_{2\mu+\delta_2+\epsilon}\right).$$

Remark 5.4 (*Explanation of the definition of* $B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}}$) The role of δ_2 is to avoid the problem (b) above, while δ_1 has a technical role when controlling the behavior of $F_\pm(x_{q_n}/\sqrt{h_{q_n}})$ in (40). Last, we remove $A_{2\mu+\epsilon}$ instead of $A_{\mu+\epsilon}$ to avoid problem (a) and to ensure that $B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}}$ is not too small. The downside of this is that we can only get $\mu(t) \in [\mu, 2\mu + \delta_2]$ for the exponent of irrationality of $t \in B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}}$. If instead we worked with the set

$$\widetilde{B}^{\delta_1}_{\mu,\mathcal{Q}_{x_0}} = \left(A_{\mu,\mathcal{Q}_{x_0}} \setminus A_{\mu+\delta_1,\mathcal{Q}_{x_0}}\right) \setminus \left(\bigcup_{\epsilon>0} A_{\mu+\epsilon}\right)$$

we would deduce $\mu(t) = \mu$ and therefore $\alpha_{x_0}(t) = 1/2 + 1/(2\mu)$. However, we do not know how to compute the dimension of $\widetilde{B}^{\delta_1}_{\mu,\mathcal{Q}_{x_0}}$.

Proposition 5.5 *Let $\mu \geq 1$. Then,*

(a) $\dim_{\mathcal{H}} B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}} = 1/\mu$.

(b) *If $t \in B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}}$, then $\alpha_{x_0}(t) \geq \frac{1}{2} + \frac{1}{4\mu+2\delta_2}$.*

(c) *If $2 \leq \mu < 2\sigma - \delta_1$ and $t \in B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}}$, then $\alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu}$.*

Proof of Proposition 5.5 (a) The inclusion $B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}} \subset A_{\mu,\mathcal{Q}_{x_0}}$ directly implies $\dim_{\mathcal{H}} B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}} \leq 1/\mu$. We prove the lower bound following the proof of Proposition 4.6 in a few steps:

(a.1) Since $\dim_{\mathcal{H}} A_{\mu+\delta_1,\mathcal{Q}_{x_0}} = 1/(\mu+\delta_1) < 1/\mu$, we have $\dim_{\mathcal{H}}(A_{\mu,\mathcal{Q}_{x_0}} \setminus A_{\mu+\delta_1,\mathcal{Q}_{x_0}}) = 1/\mu$.

(a.2) The sets A_μ are nested, so by the Jarnik–Besicovitch Theorem 2.2

$$\dim_{\mathcal{H}}\left(\bigcup_{\epsilon>0} A_{2\mu+\delta_2+\epsilon}\right) = \sup_{n\in\mathbb{N}}\left\{\dim_{\mathcal{H}}\left(A_{2\mu+\delta_2+\frac{1}{n}}\right)\right\}$$
$$= \sup_{n\in\mathbb{N}} \frac{2}{2\mu+\delta_2+\frac{1}{n}} = \frac{1}{\mu+\delta_2/2}.$$

Moreover, $\mathcal{H}^\gamma\left(\bigcup_{\epsilon>0} A_{2\mu+\delta_2+\epsilon}\right) = \lim_{n\to\infty} \mathcal{H}^\gamma\left(A_{2\mu+\delta_2+1/n}\right) = 0$ for all $\gamma \geq 1/(\mu+\delta_2/2)$.

Take γ such that $1/(\mu+\delta_2/2) < \gamma < 1/\mu$. From (a.1) we get $\mathcal{H}^\gamma(A_{\mu,\mathcal{Q}_{x_0}} \setminus A_{\mu+\delta_1,\mathcal{Q}_{x_0}}) = \infty$, and from (a.2) we hav e $\mathcal{H}^\gamma\left(\bigcup_{\epsilon>0} A_{2\mu+\delta_2+\epsilon}\right) = 0$, so

$$\mathcal{H}^\gamma(B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}}) = \mathcal{H}^\gamma(A_{\mu,\mathcal{Q}_{x_0}} \setminus A_{\mu+\delta_1,\mathcal{Q}_{x_0}}) - \mathcal{H}^\gamma\left(\bigcup_{\epsilon>0} A_{2\mu+\delta+\epsilon}\right) > 0.$$

Consequently $\dim_{\mathcal{H}} B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}} \geq \gamma$, and taking $\gamma \to 1/\mu$ we conclude $\dim_{\mathcal{H}} B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}} \geq 1/\mu$.

(b) Let $t \in B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}}$. Then, $t \notin \bigcup_{\epsilon>0} A_{2\mu+\delta_2+\epsilon}$ implies $\mu(t) \leq 2\mu+\delta_2$, where $\mu(t)$ is the exponent of irrationality of t. Combining this with Proposition 3.6 we get $\alpha_{x_0}(t) \geq \frac{1}{2} + \frac{1}{2\mu(t)} \geq \frac{1}{2} + \frac{1}{4\mu+2\delta_2}$.

(c) Let $t \in B^{\delta_1,\delta_2}_{\mu,\mathcal{Q}_{x_0}}$. Since $t \in A_{\mu,\mathcal{Q}_{x_0}} \setminus A_{\mu+\delta_1,\mathcal{Q}_{x_0}}$, there is a subsequence of denominators $(q_{n_k})_k \subset \mathcal{Q}_{x_0}$ such that $c/q_{n_k}^{\mu+\delta_1} \leq \left|t - p_{n_k}/q_{n_k}\right| < c/q_{n_k}^{\mu}$ for $k \in \mathbb{N}$. Define the errors h_{n_k} and x_{n_k}, and the exponent μ_{n_k} as

$$h_{n_k} = t - \frac{p_{n_k}}{q_{n_k}}, \quad |h_{n_k}| = \frac{1}{q_{n_k}^{\mu_{n_k}}} \quad \text{and} \quad x_{n_k} = \left|x_0 - \frac{b_{n_k}}{q_{n_k}}\right| < \frac{1}{q_{n_k}^{\sigma}}. \tag{38}$$

From the condition above, since $c < 1$, we immediately get that for any $\epsilon > 0$,

$$\mu < \mu_{n_k} \leq \mu + \delta_1 + \epsilon, \quad \forall k \gg_\epsilon 1. \tag{39}$$

By the asymptotic expansion in Corollary 3.3, we have

$$R_{x_0}(t) - R_{x_0}\left(\frac{p_{n_k}}{q_{n_k}}\right) = \frac{|h_{n_k}|^{1/2}}{q_{n_k}} G(p_{n_k}, b_{n_k}, q_{n_k})\, F_{\pm}\left(\frac{x_{n_k}}{\sqrt{h_{n_k}}}\right) - 2\pi i h_{n_k} + \text{Error},$$

where $\text{Error} = O\Big(\min\big(q_{n_k}^{3/2}\, h_{n_k}^{3/2}, q_{n_k}^{1/2}\, h_{n_k}\big)\Big)$. Let us treat the elements in this expression separately.

- Since $q_{n_k} \notin 4\mathbb{N}$, we have $|G(p_{n_k}, b_{n_k}, q_{n_k})| \geq \sqrt{q_{n_k}}$ for $k \in \mathbb{N}$. Indeed, if q_{n_k} is odd, then $|G(p_{n_k}, b_{n_k}, q_{n_k})| = \sqrt{q_{n_k}}$. If $q_{n_k} \equiv 2 \pmod 4$, then b_{n_k} is odd, so $q_{n_k}/2 \equiv b_{n_k} \pmod 2$ and hence $|G(p_{n_k}, b_{n_k}, q_{n_k})| = \sqrt{2q_{n_k}}$. Also, by (38) and (39),

$$\frac{x_{n_k}}{\sqrt{|h_{n_k}|}} = x_{n_k}\, q_{n_k}^{\mu_{n_k}/2} < \frac{q_{n_k}^{\mu_{n_k}/2}}{q_{n_k}^{\sigma}} \leq \frac{q_{n_k}^{\frac{\mu}{2}+\frac{\delta_1}{2}+\frac{\epsilon}{2}}}{q_{n_k}^{\sigma}} = \frac{1}{q_{n_k}^{\sigma-\frac{\mu}{2}-\frac{\delta_1}{2}-\frac{\epsilon}{2}}}. \tag{40}$$

  Hence, if $2\sigma > \mu + \delta_1$, we can choose $\epsilon = \sigma - \mu/2 - \delta_1/2 > 0$ and we get

$$\lim_{k\to\infty} \frac{x_{n_k}}{\sqrt{|h_{n_k}|}} \leq \lim_{k\to\infty} \frac{1}{q_{n_k}^{\sigma-\mu/2-\delta_1/2-\epsilon/2}} = \lim_{k\to\infty} \frac{1}{q_{n_k}^{(\sigma-\mu/2-\delta_1/2)/2}} = 0.$$

  Since $F_{\pm}$ is continuous, we get $|F_{\pm}(x_{n_k}/|h_{n_k}|^{1/2})| \geq |F_{\pm}(0)|/2 \simeq 1$ for all $k \gg 1$. Therefore,

$$\text{Main term} = \left| \frac{\sqrt{|h_{n_k}|}}{q_{n_k}} G(p_{n_k}, b_{n_k}, q_{n_k})\, F\left(\frac{x_{n_k}}{|h_{n_k}|^{1/2}}\right)\right| \simeq \frac{\sqrt{|h_{n_k}|}}{\sqrt{q_{n_k}}}, \quad \forall k \gg 1.$$

- The term $2\pi i h_{n_k}$ is absorbed by the Main Term if $|h_{n_k}| \ll \sqrt{|h_{n_k}|}/\sqrt{q_{n_k}}$, which is equivalent to $|h_{n_k}| \ll 1/q_{n_k}$. If $\mu > 1$, we get precisely $|h_{n_k}| < c/q_{n_k}^{\mu} \ll 1/q_{n_k}$.
- Regarding the error term, we can write

$$q_{n_k}^{1/2}|h_{n_k}| = \frac{\sqrt{|h_{n_k}|}}{\sqrt{q_{n_k}}}\,(q_{n_k}^2|h_{n_k}|)^{1/2}, \quad q_{n_k}^{3/2}|h_{n_k}|^{3/2} = \frac{\sqrt{|h_{n_k}|}}{\sqrt{q_{n_k}}}\, q_{n_k}^2|h_{n_k}|.$$

  Since $\text{Error} \leq C \min\big(q_{n_k}^{3/2}\,|h_{n_k}|^{3/2}, q_{n_k}^{1/2}\,|h_{n_k}|\big)$ for some $C > 0$, the error is absorbed by the Main Term if $q_{n_k}^2\,|h_{n_k}| \leq c$ for a small enough, but universal constant c. Choosing $c > 0$ in the definition of $A_{\mu,\mathcal{Q}_{x_0}}$, the condition $|h_{n_k}| \leq c/q_{n_k}^{\mu} \leq c/q_{n_k}^2$ is satisfied if $\mu \geq 2$.

Hence, if $2 \leq \mu < 2\sigma - \delta_1$ and $t \in B_{\mu,\mathcal{Q}_{x_0}}^{\delta_1,\delta_2}$, then $|R_{x_0}(t) - R_{x_0}(p_{n_k}/q_{n_k})| \gtrsim \sqrt{|h_{n_k}|}/\sqrt{q_{n_k}}$ for all $k \gg 1$. From (39) we have $1/\sqrt{q_{n_k}} = |h_{n_k}|^{1/(2\mu_{n_k})} > |h_{n_k}|^{1/(2\mu)}$, so $|R_{x_0}(t) - R_{x_0}(p_{n_k}/q_{n_k})| \gtrsim |h_{n_k}|^{\frac{1}{2}+\frac{1}{2\mu}}$ for all $k \gg 1$, which implies $\alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu}$. □

From Proposition 5.5 we can deduce the main part of Theorem 1.3.

Theorem 5.6 *Let $\sigma \geq 2$ and let $x_0 \in A_{\sigma, \mathbb{N}\setminus 4\mathbb{N}}$. Let $2 \leq \mu < 2\sigma$. Then, for all $\delta > 0$,*

$$\frac{1}{\mu} \leq \dim_{\mathcal{H}} \left\{ t : \frac{1}{2} + \frac{1}{4\mu} - \delta \leq \alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu} \right\} \leq \frac{2}{\mu}.$$

Proof Choose $\delta_2 > 0$ and any $\delta_1 < 2\sigma - \mu$. Hence, $2 \leq \mu < 2\sigma - \delta_1$ and Proposition 5.5 implies

$$B^{\delta_1,\delta_2}_{\mu, \mathcal{Q}_{x_0}} \subset \left\{ t : \frac{1}{2} + \frac{1}{4\mu + 2\delta_2} \leq \alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu} \right\}.$$

Since $\dim_{\mathcal{H}} B^{\delta_1,\delta_2}_{\mu, \mathcal{Q}_{x_0}} = 1/\mu$ and δ_2 is arbitrary, we get the lower bound. Let us now prove the upper bound. If $\alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu}$, by Proposition 3.6 we get $\frac{1}{2} + \frac{1}{2\mu(t)} \leq \alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu}$, hence $\mu(t) \geq \mu$. This implies $t \in A_{\mu-\epsilon}$ for all $\epsilon > 0$, so by the Jarnik–Besicovitch Theorem 2.2 we get

$$\dim_{\mathcal{H}} \left\{ t : \frac{1}{2} + \frac{1}{4\mu} - \delta \leq \alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu} \right\} \leq \dim_{\mathcal{H}} A_{\mu-\epsilon} = \frac{2}{\mu - \epsilon}$$

for all $\delta \geq 0$. We conclude by taking the limit $\epsilon \to 0$. □

To get the precise statement of Theorem 1.3, we only need to relate the sets $A_{\sigma, \mathbb{N}\setminus 4\mathbb{N}}$ with the exponent $\sigma(x_0) = \limsup_{n\to\infty}\{ \mu_n : q_n \notin 4\mathbb{N} \}$ defined in (8). We proceed as follows. Since $\{A_{\sigma, \mathbb{N}\setminus 4\mathbb{N}}\}_{\sigma \geq 2}$ is a nested family and $A_{2, \mathbb{N}\setminus 4\mathbb{N}} = (0, 1)\setminus\mathbb{Q}$, for every $x_0 \in (0, 1)\setminus\mathbb{Q}$ there exists $\widetilde{\sigma}(x_0) = \sup\{ \sigma : x_0 \in A_{\sigma, \mathbb{N}\setminus 4\mathbb{N}} \}$. Let us check that $\sigma(x_0) = \widetilde{\sigma}(x_0)$. Indeed, call $\widetilde{\sigma}(x_0) = \widetilde{\sigma}$.

- If $\widetilde{\sigma} > 2$. Then for $\epsilon > 0$ small enough there exists a sequence b_k/q_k such that $q_k \notin 4\mathbb{N}$ and $|x_0 - b_k/q_k| < 1/q_k^{\widetilde{\sigma}-\epsilon} < 1/(2q_k^2)$. By Khinchin's theorem [36, Theorem 19], b_k/q_k is an approximation by continued fraction, for which $|x_0 - b_k/q_k| = 1/q_k^{\mu_k} < 1/q_k^{\widetilde{\sigma}-\epsilon}$, and therefore $\mu_k \geq \widetilde{\sigma} - \epsilon$. This implies $\sigma(x_0) \geq \widetilde{\sigma} - \epsilon$ for all $\epsilon > 0$, hence $\sigma(x_0) \geq \widetilde{\sigma}$. On the other hand, for all approximations by continued fractions with $q_n \notin 4\mathbb{N}$ with large enough n we have $|x_0 - b_n/q_n| = 1/q_n^{\mu_n} > 1/q_n^{\widetilde{\sigma}+\epsilon}$, hence $\mu_n \leq \widetilde{\sigma} + \epsilon$. This holds for all $\epsilon > 0$, so $\sigma(x_0) \leq \widetilde{\sigma}$.
- If $\widetilde{\sigma} = 2$, then $|x_0 - b_n/q_n| = 1/q_n^{\mu_n} > 1/q_n^{2+\epsilon}$, hence $\mu_n \leq 2 + \epsilon$, for all approximations by continued fractions with $q_n \notin 4\mathbb{N}$. Therefore, $\sigma(x_0) \leq 2$. Since $\sigma(x_0) \geq 2$ always holds, we conclude.
  Therefore, let $x_0 \in (0, 1)\setminus\mathbb{Q}$. Then, $x_0 \in A_{\sigma, \mathbb{N}\setminus 4\mathbb{N}}$ for all $\sigma < \sigma(x_0)$, so the conclusion of Theorem 5.6 holds for $2 \leq \mu < 2\sigma$, for all $\sigma < \sigma(x_0)$. That implies that for every $\delta > 0$,

$$\frac{1}{\mu} \leq \dim_{\mathcal{H}} \left\{ t : \frac{1}{2} + \frac{1}{4\mu} - \delta \leq \alpha_{x_0}(t) \leq \frac{1}{2} + \frac{1}{2\mu} \right\} \leq \frac{2}{\mu}, \quad \text{for all} \quad 2 \leq \mu < 2\sigma(x_0).$$

6 Proof of Theorem 1.6—the high-pass filters when $x_0 \in \mathbb{Q}$

In this section we prove Theorem 1.6. For that, we compute the L^p norms of the high-pass filters of R_{x_0} when $x_0 \in \mathbb{Q}$. In Sect. 6.1 we define Fourier high-pass filters using smooth cutoffs, reduce the computation of their L^p norms to the study of Fourier localized L^p estimates, state such localized estimates and deduce Theorem 1.6 from them. We prove such localized estimates in Sect. 6.2.

6.1 High-pass filters and frequency localization

We begin with the definition of high-pass filters we use in the proofs. Let $\phi \in C^\infty$ a positive and even cutoff with support on $[-1, 1]$ and such that $\phi(x) = 1$ on $x \in [-1/2, 1/2]$. Let $\psi(x) = \phi(x/2) - \phi(x)$, and

$$\psi_{-1}(x) = \frac{\phi(x)}{\phi(x) + \sum_{i\in\mathbb{N}} \psi(x/2^i)}, \quad \psi_k(x) = \frac{\psi(x/2^k)}{\phi(x) + \sum_{i\in\mathbb{N}} \psi(x/2^i)}, \quad \text{for } k \geq 0,$$

so that we have the partition of unity $\sum_{k=-1}^{\infty} \psi_k(x) = 1$. For $k \geq 0$, ψ_k is supported on $[-2^{k+1}, -2^{k-1}] \cup [2^{k-1}, 2^{k+1}]$. Let f be a periodic function with Fourier series $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n t}$. With the partition of unity above, we perform a Littlewood–Paley decomposition

$$f(t) = \sum_{k=-1}^{\infty} P_k f(t), \quad \text{where} \quad P_k f(t) = \sum_{n\in\mathbb{Z}} \psi_k(n) a_n e^{2\pi i n t}.$$

The Fourier high-pass filter at frequency $N \in \mathbb{N}$ is roughly $P_{\geq N} f(t) = \sum_{k\geq \log N} P_k f(t)$. Let us be more precise working directly with R_{x_0}, whose frequencies in t are squared. Let $N \in \mathbb{N}$ be large, and define k_N to be the unique $k_N \in \mathbb{N}$ such that $2^{k_N} \leq \sqrt{N} < 2^{k_N+1}$. We define the high-pass filter of R_{x_0} at frequency N as

$$P_{\geq N} R_{x_0}(t) = \sum_{k\geq k_N} P_k R_{x_0}(t), \quad \text{where} \quad P_k R_{x_0}(t) = \sum_{n\in\mathbb{N}} \psi_k(n) \frac{e^{2\pi i (n^2 t + n x_0)}}{n^2}. \tag{41}$$

We first estimate $\| P_k R_{x_0} \|_p$ and then extend the result to estimate $\| P_{\geq N} R_{x_0} \|_p$.

Remark 6.1 At a first glance, using pure Littlewood–Paley blocks in the definition for high-pass filters in (41) may seem restrictive, since it is analogue to estimating high-frequency cutoffs only for a sequence $N_k \simeq 2^k \to \infty$. However, the estimates we give depend only on the L^1 norm of the cutoff ψ, so slightly varying the definition and support of ψ does not affect the estimates. This is analogous to having a cutoff $\Phi(x/N)$ for a fixed Φ as we state in the introduction.

We now state the estimates for the frequency localized L^p estimates. For the sake of generality, let $\Psi \in C^\infty$ be compactly supported outside the origin and bounded below in an interval of its support (for instance, ψ defined above).

Theorem 6.2 *Let $x_0 \in \mathbb{R}$. Then, for $N \gg 1$,*

$$\Big\| \sum_{n\in\mathbb{Z}} \Psi\big(\frac{n}{N}\big)\, e^{2\pi i (n^2 t + n x_0)} \Big\|_{L^p(0,1)}^p \lesssim \begin{cases} N^{p-2}, & \text{when } p > 4, \\ N^2 \log N, & \text{when } p = 4, \\ N^{p/2}, & \text{when } p < 4. \end{cases} \tag{42}$$

When $p = 2$, the upper bound is sharp, that is, $\| \sum_{n\in\mathbb{Z}} \Psi(n/N)\, e^{2\pi i (n^2 t + n x_0)} \|_{L^2(0,1)}^2 \simeq N$.

If $x_0 \in \mathbb{Q}$, then the upper bound is sharp. That is, if $x_0 = P/Q$ with $(P, Q) = 1$, then

$$\Big\| \sum_{n\in\mathbb{Z}} \Psi\big(\frac{n}{N}\big)\, e^{2\pi i (n^2 t + n x_0)} \Big\|_{L^p(0,1)}^p \simeq_Q \begin{cases} N^{p-2}, & \text{when } p > 4, \\ N^2 \log N, & \text{when } p = 4, \\ N^{p/2}, & \text{when } p < 4. \end{cases} \tag{43}$$

Remark 6.3 All estimates in Theorem 6.2 depend on $\|\Psi\|_1$ due to Lemma 6.4.

We postpone the proof of Theorem 6.2 to Sect. 6.2. and use it now to compute the L^p norms of the high-pass filters $\|P_{\geq N} R_{x_0}\|_p$ and therefore to prove Theorem 1.6.

Proof of Theorem 1.6 Denote the estimate for $x_0 \in \mathbb{Q}$ on (43) in Theorem 6.2 by

$$\Big\| \sum_{n\in\mathbb{Z}} \Psi(n/N)\, e^{2\pi i (n^2 t + n x_0)} \Big\|_{L^p(0,1)}^p \simeq G_p(N). \tag{44}$$

First, use the triangle inequality in (41) to bound

$$\|P_{\geq N} R_{x_0}\|_p \leq \sum_{k\geq k_N} \|P_k R_{x_0}\|_p = \sum_{k\geq k_N} \Big\| \sum_{n\in\mathbb{Z}} \psi_k(n)\, \frac{e^{2\pi i (n^2 t + n x_0)}}{n^2} \Big\|_p.$$

Since ψ_k is supported on $[2^{k-1}, 2^{k+1}]$, we can take the denominator n^2 out of the L^p norm to get

$$\|P_{\geq N} R_{x_0}\|_p \lesssim \sum_{k\geq k_N} \frac{1}{2^{2k}} \Big\| \sum_{n\in\mathbb{Z}} \psi_k(n)\, e^{2\pi i (n^2 t + n x_0)} \Big\|_p,$$

for example using [23, Lemma 3.1, Corollary 3.2]. We can now use (44) to get[25]

$$\|P_{\geq N} R_{x_0}\|_p \lesssim \sum_{k \geq k_N} \frac{G_p(2^k)^{1/p}}{2^{2k}} \simeq \frac{G_p(2^{k_N})^{1/p}}{2^{2k_N}}, \tag{45}$$

where the last equality follows by direct calculation because the defintion of G_p makes the series be geometric. For the lower bound, as long as $1 < p < \infty$, the Mihklin multiplier theorem.[26] combined again with [23, Lemma 3.1, Corollary 3.2] and (44) gives

$$\|P_{\geq N} R_{x_0}\|_p \gtrsim \|P_{k_N} R_{x_0}\|_p \simeq \frac{1}{2^{2k_N}} \left\| \sum_n \psi_{k_N}(n)\, e^{2\pi i (n^2 t + n x_0)} \right\|_p \simeq \frac{G_p(2^{k_N})^{1/p}}{2^{2k_N}}. \tag{46}$$

Joining (45) and (46) and recalling that $2^{k_N} \simeq \sqrt{N}$, we conclude that

$$\|P_{\geq N} R_{x_0}\|_p \simeq \frac{G_p(2^{k_N})^{1/p}}{2^{2k_N}} \simeq \begin{cases} N^{-1/2-1/p}, & p > 4, \\ N^{-3/4} (\log N)^{1/4}, & p = 4, \\ N^{-3/4}, & p < 4, \end{cases}$$

from which we immediately get

$$\eta(p) = \lim_{N \to \infty} \frac{\log(\|P_{\geq N} R_{x_0}\|_p^p)}{\log(1/N)} = \begin{cases} p/2 + 1, & p > 4, \\ 3p/4, & p \leq 4. \end{cases}$$

□

6.2 Frequency localized L^p norms

In this section we prove Theorem 6.2. The L^2 estimate, which holds for all x_0, follows from Plancherel's theorem. For $p \neq 2$, we use the following well-known lemma, whose proof can be found in [10, Lemma 3.18] (see also [4, Lemma 4.4]).

Lemma 6.4 *Let $\Psi \in C_0^\infty(\mathbb{R})$. Let $N \in \mathbb{N}$ and $q \in \mathbb{N}$ such that $q \leq N$. Let also $a \in \mathbb{Z}$ such that $(a, q) = 1$. Then,*

$$\left| t - \frac{a}{q} \right| \leq \frac{1}{qN} \quad \Longrightarrow \quad \left| \sum_{n \in \mathbb{Z}} \Psi\left(\frac{n}{N}\right) e^{2\pi i (n^2 t + n x)} \right|$$

[25] The estimates in Theorem 6.2 depend on $\|\Psi\|_1$, so strictly speaking we need to check that for large enough $k \gg 1$, the norm $\|\psi_k(2^k \cdot)\|_1$ does not depend on k. This is the case, since

$$\int \psi_k(2^k x)\, dx = \int_{1/2}^{2} \frac{\psi(x)}{\phi(2^k x) + \sum_{i=0}^{\infty} \psi(2^k x/2^i)}\, dx = \int_{1/2}^{2} \frac{\psi(x)}{\psi(x/2) + \psi(x) + \psi(2x)}\, dx = C_\psi.$$

[26] Apply Mihklin's theorem in $\mathbb{R}$ to the operator P_{k_N} in (41) to get $\|P_{k_N} f\|_p \simeq \|P_{k_N} P_{\geq N} f\|_p \lesssim \|P_{\geq N} f\|_p$, and then periodize the result using a theorem by Stein and Weiss [46, Chapter 7, Theorem 3.8]

$$\lesssim_{\|\Psi\|_1} \frac{N}{\sqrt{q}\,\left(1+N\sqrt{|t-a/q|}\right)}. \tag{47}$$

Moreover, there exist $\delta, \epsilon \leq 1$ *only depending on* Ψ *such that if*

$$q \leq \epsilon N, \quad \left|t-\frac{a}{q}\right| \leq \frac{\delta}{N^2}, \quad \left|x-\frac{b}{q}\right| \leq \frac{\delta}{N}$$

for some $b \in \mathbb{Z}$*, then*

$$\left|\sum_{n\in\mathbb{Z}} \Psi\left(\frac{n}{N}\right) e^{2\pi i(n^2 t+nx)}\right| \simeq_{\|\Psi\|_1} \frac{N}{\sqrt{q}}.$$

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2 Let $x_0 \in \mathbb{R}$. For simplicity, we prove the L^2 estimate for a symmetric Ψ. Considering f as a Fourier series in t, by Plancherel's theorem we write

$$\left\|\sum_{n\in\mathbb{Z}} \Psi\Big(\frac{n}{N}\Big) e^{2\pi i(n^2 t+n x_0)}\right\|^2_{L^2(0,1)} = \sum_{n=1}^{\infty} \left|\Psi\Big(\frac{n}{N}\Big) e^{2\pi in x_0} + \Psi\Big(-\frac{n}{N}\Big) e^{-2\pi in x_0}\right|^2$$
$$= \sum_{n=1}^{\infty} \Psi\Big(\frac{n}{N}\Big)^2 \left|e^{2\pi inx_0}+e^{-2\pi inx_0}\right|^2 \simeq \sum_{n=1}^{\infty} \Psi\Big(\frac{n}{N}\Big)^2 \cos^2(2\pi n x_0)$$

This sum is upper bounded by N by the triangle inequality. If x_0 is rational, say $x_0 = P/Q$, the bound from below follows[27] by summing only over multiples of Q in $[N, 2N]$, so that

$$\left\|\sum_{n\in\mathbb{Z}} \Psi\Big(\frac{n}{N}\Big) e^{2\pi i(n^2 t+n x_0)}\right\|^2_{L^2(0,1)} \gtrsim \sum_{k=N/Q}^{2N/Q} \cos^2(2\pi k Q x_0) = \frac{N}{Q} \simeq_Q N.$$

If x_0 is irrational, it is known that the sequence $(nx_0)_n$ is equidistributed in the torus, which means that for any continuous p-periodic function

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(nx_0) = \int_0^p f.$$

[27] Without loss of generality assume that $\Psi(x) \simeq 1$ for $x \in (1,2)$.

In particular, since for $f(y) = \cos(4\pi y)$ we have $\int_0^{1/2} f(y)\,dy = 0$, we get[28] for large N that

$$\Big\| \sum_{n\in\mathbb{Z}} \Psi\big(\frac{n}{N}\big)\, e^{2\pi i(n^2 t + n x_0)} \Big\|^2_{L^2(0,1)} \gtrsim \sum_{n=N}^{2N} \cos^2(2\pi n x_0) \simeq N + \sum_{n=N}^{2N} \cos(4\pi n x_0) \simeq N.$$

We now prove the upper bound (42) for any $x_0 \in \mathbb{R}$. The Dirichlet approximation theorem implies that any $t \in \mathbb{R}\backslash\mathbb{Q}$ can be approximated as follows:

$$\forall N \in \mathbb{N}, \quad \exists q \le N, \quad 1 \le a \le q \quad \text{such that} \quad \Big| t - \frac{a}{q} \Big| \le \frac{1}{qN},$$

which can be rewritten as $\mathbb{R}\backslash\mathbb{Q} \subset \bigcup_{q=1}^{N} \bigcup_{a=1}^{q} B\big(\frac{a}{q}, \frac{1}{qN}\big)$ for all $N \in \mathbb{N}$. Therefore, for any $N \in \mathbb{N}$,

$$\begin{aligned} &\Big\| \sum_{n\in\mathbb{Z}} \Psi(n/N)\, e^{2\pi i(n^2 t + n x_0)} \Big\|^p_{L^p(0,1)} \\ &\quad \le \sum_{q=1}^{N} \sum_{a=1}^{q} \int_{B(\frac{a}{q}, \frac{1}{qN})} \Big| \sum_{n\in\mathbb{Z}} \Psi(n/N)\, e^{2\pi i(n^2 t + n x_0)} \Big|^p dt. \end{aligned} \tag{48}$$

We split each integral according to the two situations in (47) in Lemma 6.4:

$$\begin{aligned} &\int_{|t-\frac{a}{q}|<\frac{1}{N^2}} \Big| \sum_{n\in\mathbb{Z}} \Psi(n/N)\, e^{2\pi i(n^2 t + n x_0)} \Big|^p dt \\ &\quad + \int_{\frac{1}{N^2}<|t-\frac{a}{q}|<\frac{1}{qN}} \Big| \sum_{n\in\mathbb{Z}} \Psi(n/N)\, e^{2\pi i(n^2 t + n x_0)} \Big|^p dt \\ &\le \int_{|t-\frac{a}{q}|<\frac{1}{N^2}} \Big(\frac{N}{\sqrt{q}} \Big)^p dt + \int_{\frac{1}{N^2}<|t-\frac{a}{q}|<\frac{1}{qN}} \Big(\frac{1}{\sqrt{q}\,|t-\frac{a}{q}|^{1/2}} \Big)^p dt \\ &\simeq \frac{N^{p-2}}{q^{p/2}} + \frac{1}{q^{p/2}} \int_{\frac{1}{N^2}}^{\frac{1}{qN}} \frac{1}{h^{p/2}}\, dh. \end{aligned} \tag{49}$$

The behavior of that last integral changes depending on p being greater or smaller than 2.

- If $p < 2$,

$$(49) \simeq \frac{N^{p-2}}{q^{p/2}} + \frac{1}{q^{p/2}} \left(\Big(\frac{1}{qN}\Big)^{1-p/2} - \Big(\frac{1}{N^2}\Big)^{1-p/2} \right) \le \frac{N^{p-2}}{q^{p/2}} + \frac{1}{q\,N^{1-p/2}},$$

[28] Using the trigonometric identity $\cos^2(x) = (1 + \cos(2x))/2$.

so

$$(48) \leq N^{p-2} \sum_{q=1}^{N} \sum_{a=1}^{q} \frac{1}{q^{p/2}} + \frac{1}{N^{1-p/2}} \sum_{q=1}^{N} \sum_{a=1}^{q} \frac{1}{q} \lesssim N^{p/2}.$$

- If $p = 2$,

$$(49) \simeq \frac{1}{q}\Big(1 + \int_{\frac{1}{N^2}}^{\frac{1}{qN}} \frac{dh}{h}\Big) \lesssim \frac{1}{q}\Big(1 + \log(N^2) - \log(qN)\Big) = \frac{1 + \log(N/q)}{q},$$

hence

$$\begin{aligned}(48) &\lesssim \sum_{q=1}^{N} \Big(1 - \log(q/N)\Big) \simeq N - \int_1^N \log(x/N)\, dx \simeq N\Big(1 - \int_{\frac{1}{N}}^{1} \log(y)\, dy\Big) \\ &\simeq N.\end{aligned}$$

- If $p > 2$,

$$\begin{aligned}(49) &\simeq \frac{N^{p-2}}{q^{p/2}} + \frac{\big(N^2\big)^{p/2-1} - (qN)^{p/2-1}}{q^{p/2}} \lesssim \frac{N^{p-2}}{q^{p/2}} \\ \implies \quad (48) &\lesssim N^{p-2} \sum_{q=1}^{N} \frac{1}{q^{p/2-1}}.\end{aligned}$$

This series converges if and only if $p > 4$, and more precisely,

$$(48) \lesssim \begin{cases} N^{p-2}, & p > 4, \\ N^2 \log N, & p = 4, \\ N^{p-2} N^{2-p/2} = N^{p/2}, & p < 4. \end{cases}$$

This concludes the proof of (42).

We now prove the lower bound in (43) for $x_0 \in \mathbb{Q}$. Let $x_0 = P/Q$ with $(P, Q) = 1$. Let $\delta, \epsilon > 0$ as given in Lemma 6.4, and let $N \in \mathbb{N}$ be such that $Q \leq \epsilon N$. Bound the L^p norm from below by

$$\Big\| \sum_{n \in \mathbb{Z}} \Psi(n/N)\, e^{2\pi i (n^2 t + n x_0)} \Big\|_{L^p(0,1)}^p \geq \int_{B\left(\frac{a}{Q}, \frac{\delta}{N^2}\right)} \Big| \sum_{n \in \mathbb{Z}} \Psi(n/N)\, e^{2\pi i (n^2 t + n x_0)} \Big|^p dt, \tag{50}$$

where a is any $1 \le a \le Q$ such that $(a, Q) = 1$. Use Lemma 6.4 with $q = Q$ and $b = P$, for which the condition $0 = |x_0 - P/Q| < \delta/N$ is satisfied trivially, and $|t - a/Q| < \delta/N^2$, which is valid on the domain of integration. Then, for $N \ge Q/\epsilon$,

$$\Big\| \sum_{n\in\mathbb{Z}} \Psi(n/N)\, e^{2\pi i (n^2 t + n x_0)} \Big\|_{L^p(0,1)}^p \gtrsim \int_{B\left(\frac{a}{Q}, \frac{\delta}{N^2}\right)} \left(\frac{N}{\sqrt{Q}}\right)^p dt \simeq \frac{N^p}{Q^{p/2}} \frac{\delta}{N^2} \simeq_Q N^{p-2}.$$

In view of the upper bound in (42), this is optimal when $p > 4$. When $p \le 4$, we refine the bound in (50) as follows. Define the set

$$\mathcal{Q}_N = \{ q \in \mathbb{N} \,:\, Q \mid q \;\text{ and }\; q \le \epsilon N \},$$

whose cardinality $\simeq \epsilon N/Q$ is as large as needed if $N \gg 1$. Observe that

$$B\left(\frac{a}{q}, \frac{\delta}{N^2}\right) \cap B\left(\frac{a'}{q'}, \frac{\delta}{N^2}\right) = \emptyset, \quad \forall q, q' \in \mathcal{Q}_N, \quad (a, q) = 1 = (a', q'),$$

as long as $a/q \ne a'/q'$. Indeed, the distance from the centers is $\frac{|aq' - a'q|}{q\,q'} \ge \frac{1}{q\,q'} \ge \frac{1}{\epsilon^2 N^2}$, while the radius is $\frac{\delta}{N^2} < \frac{1}{\epsilon^2 N^2}$ (choosing a smaller $\delta > 0$ if needed). Hence the balls in the family $\{B(a/q, \delta/N^2) \,:\, q \in \mathcal{Q}_N,\ (a, q) = 1\}$ are pairwise disjoint, and we can bound

$$\begin{aligned} &\Big\| \sum_{n\in\mathbb{Z}} \Psi(n/N)\, e^{2\pi i (n^2 t + n x_0)} \Big\|_{L^p(0,1)}^p \\ &\quad \gtrsim \sum_{q\in\mathcal{Q}_N} \sum_{a:(a,q)=1} \int_{B\left(\frac{a}{q}, \frac{\delta}{N^2}\right)} \Big| \sum_{n\in\mathbb{Z}} \Psi(n/N)\, e^{2\pi i (n^2 t + n x_0)} \Big|^p dt. \end{aligned} \tag{51}$$

For each of those integrals we have $q = Qn$ for some $n \in \mathbb{N}$. To use Lemma 6.4 we chose $b = Pn$ so that $0 = |x_0 - b/q| < \delta/N$, hence

$$\begin{aligned} (51) &\gtrsim \sum_{q\in\mathcal{Q}_N} \sum_{a:(a,q)=1} \int_{B\left(\frac{a}{q}, \frac{\delta}{N^2}\right)} \left(\frac{N}{\sqrt{q}}\right)^p dt \\ &\simeq \delta\, N^{p-2} \sum_{q\in\mathcal{Q}_N} \frac{\varphi(q)}{q^{p/2}} \simeq \frac{N^{p-2}}{Q^{p/2}} \sum_{n=1}^{\epsilon N/Q} \frac{\varphi(Qn)}{n^{p/2}}. \end{aligned} \tag{52}$$

We estimate this sum in the following lemma, which we prove in Appendix A, Corollary A.5.

Lemma 6.5 *Let $Q \in \mathbb{N}$. Then, for $N \gg 1$,*

$$\sum_{n=1}^{N} \frac{\varphi(Qn)}{n^2} \simeq \log N, \quad \textit{and} \quad \sum_{n=1}^{N} \frac{\varphi(Qn)}{n^\alpha} \simeq N^{2-\alpha}, \quad \textit{for} \quad \alpha < 2,$$

where the implicit constants depend on Q and α.

Using this lemma in (52), when $p < 4$ we get

$$\left\| \sum_{n\in\mathbb{Z}} \Psi(n/N)\, e^{2\pi i(n^2 t+n x_0)} \right\|_{L^p(0,1)}^p \simeq_{p,Q} \frac{N^{p-2}}{Q^{p/2}} \left(\frac{\epsilon N}{Q}\right)^{2-\frac{p}{2}} \simeq_{p,Q} N^{p/2}.$$

Similarly, when $p = 4$ we get

$$\left\| \sum_{n\in\mathbb{Z}} \Psi(n/N)\, e^{2\pi i(n^2 t+n x_0)} \right\|_{L^4(0,1)}^4 \simeq_Q \frac{N^2}{Q^2} \log\left(\frac{\epsilon N}{Q}\right) \simeq_Q N^2 \log N.$$

Together with the upper bounds in (42), this completes the proof. □

Appendix A. Sums of Euler's totient function

Sums of the Euler totient function play a relevant role in this article, especially in Lemma 6.5. In Sect. A.1 we state the classical results and briefly prove them for completeness. In Sect. A.2 we adapt these classical proofs to sums modulo Q that we need in this article. Throughout this appendix, φ denotes the Euler totient function and μ denotes the Möbius function.[29]

A.1 Sums of Euler's totient function

Define the sum function

$$\Phi(N) = \sum_{n=1}^{N} \varphi(n), \quad N \in \mathbb{N}.$$

Proposition A.1 *For $N \gg 1$,*

$$\Phi(N) = CN^2 + O\Big(N \log N\Big), \quad \textit{where} \quad C = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{3}{\pi^2}$$

Proof By the Möbius inversion formula,

$$\Phi(N) = \sum_{n=1}^{N} \varphi(n) = \sum_{n=1}^{N} n\left(\sum_{d|n} \frac{\mu(d)}{d}\right) = \sum_{n=1}^{N} \sum_{d|n} \frac{n}{d}\, \mu(d).$$

[29] For $n \in \mathbb{N}$, $\mu(n) = 1$ if n is has no squared prime factor and if it has an even number of prime factors; $\mu(n) = -1$ if n is has no squared prime factor and if it has an odd number of prime factors; and $\mu(n) = 0$ if it has a squared prime factor.

Calling $n/d = d'$, the sum is in all natural numbers d and d' such that $dd' \leq N$. Therefore,

$$\Phi(N) = \sum_{d,d' : dd' \leq N} d'\mu(d) = \sum_{d=1}^{N} \mu(d) \sum_{d'=1}^{\lfloor N/d \rfloor} d' = \sum_{d=1}^{N} \mu(d) \frac{\lfloor N/d \rfloor (\lfloor N/d \rfloor + 1)}{2}.$$

For $x \in \mathbb{R}$, write $x = \lfloor x \rfloor + \{x\}$, where $0 \leq \{x\} < 1$ is the fractional part of x. Then, direct computation shows that $\lfloor x \rfloor (\lfloor x \rfloor + 1) = x^2 + O(x)$ when $x \geq 1$, so

$$\Phi(N) = \frac{1}{2} \sum_{d=1}^{N} \mu(d) \left(\left(\frac{N}{d} \right)^2 + O\left(\frac{N}{d} \right) \right) = \frac{N^2}{2} \sum_{d=1}^{N} \frac{\mu(d)}{d^2} + O\left(N \sum_{d=1}^{N} \frac{1}{d} \right).$$

The series $\sum_{d=1}^{\infty} \mu(d)/d^2$ is absolutely convergent, and its value is known to be $2C = 6/\pi^2$, so write

$$\sum_{d=1}^{N} \frac{\mu(d)}{d^2} = 2C - \sum_{d=N+1}^{\infty} \frac{\mu(d)}{d^2} = 2C + O\left(\sum_{d=N+1}^{\infty} \frac{1}{d^2} \right) = 2C + O\left(\frac{1}{N} \right).$$

Since $\sum_{d=1}^{N} 1/d \simeq \log N$, we get $\Phi(N) = C\,N^2 + O(N) + O(N \log N) = CN^2 + O(N \log N)$. $\square$

As a Corollary of Lemma A.1 we obtain the analogue result for the sums weighted by $n^{-\alpha}$. Observe that when $\alpha > 2$ the sum is convergent.

Corollary A.2 *Let $\alpha \leq 2$. For $N \gg 1$,*

$$\sum_{n=1}^{N} \frac{\varphi(n)}{n^2} \simeq \log N, \quad \textit{and} \quad \sum_{n=1}^{N} \frac{\varphi(n)}{n^\alpha} \simeq N^{2-\alpha}, \quad \textit{if } \alpha < 2.$$

Proof Upper bounds immediately follow from $\varphi(n) \leq n$. For lower bounds, assume first that $\alpha \geq 0$. From Proposition A.1 we directly get

$$\sum_{n=1}^{N} \frac{\varphi(n)}{n^\alpha} \geq \frac{1}{N^\alpha} \sum_{n=1}^{N} \varphi(n) = \frac{1}{N^\alpha} \Phi(N) \simeq N^{2-\alpha},$$

which is optimal when $\alpha < 2$. For the case $\alpha = 2$ we use the summation by parts formula[30] to get

$$\sum_{n=1}^{N} \frac{\varphi(n)}{n^2} = \frac{\Phi(N)}{N^2} - \sum_{n=1}^{N-1} \Phi(n) \left(\frac{1}{(n+1)^2} - \frac{1}{n^2} \right)$$

[30] Let a_n and b_n be two sequences, and let $B_N = \sum_{n=1}^{N} b_n$. Then, $\sum_{n=1}^{N} a_n b_n = a_N B_N - \sum_{n=1}^{N-1} B_n (a_{n+1} - a_n)$.

$$= \frac{\Phi(N)}{N^2} + \sum_{n=1}^{N-1} \Phi(n) \frac{2n+1}{n^2\,(n+1)^2}. \tag{53}$$

Restrict the sum to $\log N \leq n \leq N-1$, and combine it with $\Phi(n) \simeq n^2$ for $n \gg 1$ from Proposition A.1 to get

$$\sum_{n=1}^{N} \frac{\varphi(n)}{n^2} \gtrsim 1 + \sum_{n \geq \log N}^{N-1} \frac{1}{n} \simeq \log N - \log\log N \simeq \log N, \quad \text{for } N \gg 1.$$

When $\alpha < 0$, restrict the sum to $n \in [N/2, N]$ and use $\Phi(N) = CN^2 + O(N \log N)$ in Proposition A.1 to get

$$\sum_{n=1}^{N} \frac{\varphi(n)}{n^\alpha} = \sum_{n=1}^{N} \varphi(n)\, n^{|\alpha|} \geq \left(\frac{N}{2}\right)^{|\alpha|} \sum_{n \geq N/2}^{N} \varphi(n) \simeq_{|\alpha|} \frac{\Phi(N) - \Phi(N/2)}{N^\alpha} \simeq N^{2-\alpha}.$$

□

A.2 Sums of Euler's totient function modulo Q

For $Q \in \mathbb{N}$, let

$$\Phi_Q(N) = \sum_{n=1}^{N} \varphi(Qn) \quad \text{when } N \gg 1,$$

To estimate the behavior when $N \to \infty$ we adapt the proofs of Proposition A.1 and Corollary A.2.

Proposition A.3 *Let $Q \in \mathbb{N}$. Then, $\Phi_Q(N) \leq QN^2$, and there exists a constant $c_Q > 0$ such that*

$$\Phi_Q(N) \geq c_Q N^2 + O_Q(N \log N).$$

Consequently, $\Phi_Q(N) \simeq_Q N^2$ when $N \gg 1$.

Proof The upper bound follows directly from $\varphi(n) < n$ for all $n \in \mathbb{N}$, so it suffices to prove the lower bound. For that, first restrict the sum to $n \leq N$ such that $(Q, n) = 1$. By the multiplicative property of the Euler function, we get

$$\Phi_Q(N) \geq \sum_{\substack{n=1 \\ (Q,n)=1}}^{N} \varphi(Qn) = \varphi(Q) \sum_{\substack{n=1 \\ (Q,n)=1}}^{N} \varphi(n). \tag{54}$$

The proof now follows the same strategy as in Proposition A.1. Use Möbius inversion to write

$$\sum_{\substack{n=1\\(Q,n)=1}}^{N} \varphi(n) = \sum_{\substack{n=1\\(Q,n)=1}}^{N} \left(n \sum_{d|n} \frac{\mu(d)}{d} \right) = \sum_{\substack{n=1\\(Q,n)=1}}^{N} \sum_{d|n} \frac{n}{d}\, \mu(d).$$

Observe that if $(Q, n) = 1$ and if we decompose $n = d\,d'$, then both d and d' are coprime with Q. Conversely, if d and d' are coprime with Q, then so is $n = d\,d'$. Thus,

$$\sum_{\substack{n=1\\(Q,n)=1}}^{N} \varphi(n) = \sum_{\substack{d,d' \,:\, d\,d' \leq N\\(Q,d)=1=(Q,d')}} d'\,\mu(d) = \sum_{\substack{d=1\\(Q,d)=1}}^{N} \mu(d) \left(\sum_{\substack{d'=1\\(Q,d')=1}}^{\lfloor N/d \rfloor} d' \right). \tag{55}$$

In the following lemma we give a closed formula for the inner sum. We postpone its proof.

Lemma A.4 *Let $Q \in \mathbb{N}$, $Q \geq 2$. Then,*

$$S_Q = \sum_{\substack{n=1\\(Q,n)=1}}^{Q-1} n = \frac{Q\,\varphi(Q)}{2}, \quad \textit{and} \quad S_{Q,k} = \sum_{\substack{n=1\\(Q,n)=1}}^{kQ-1} n = \frac{Q\,\varphi(Q)}{2}\,k^2, \quad \forall k \in \mathbb{N}.$$

Now, for every $d \leq N$, find $k_d \in \mathbb{N} \cup \{0\}$ such that $k_d Q \leq \lfloor N/d \rfloor < (k_d + 1)Q$, and write

$$\begin{aligned} \sum_{\substack{d'=1\\(Q,d')=1}}^{\lfloor N/d \rfloor} d' &= \sum_{\substack{d'=1\\(Q,d')=1}}^{k_d Q-1} d' + \sum_{\substack{d'=k_d Q+1\\(Q,d')=1}}^{\lfloor N/d \rfloor} d' = S_{Q,k_d} + O\Big((k_d + 1) Q^2 \Big) \\ &= \frac{Q\,\varphi(Q)}{2}\,k_d^2 + O\Big((k_d + 1) Q^2 \Big). \end{aligned} \tag{56}$$

Since the definition of k_d is equivalent to $\frac{1}{Q}\lfloor N/d \rfloor - 1 < k_d \leq \frac{1}{Q}\lfloor N/d \rfloor$, we deduce that $k_d = \lfloor \frac{1}{Q} \lfloor N/d \rfloor \rfloor$. Consequently, since $\lfloor x \rfloor = x + O(1)$ and $\lfloor x \rfloor^2 = x^2 + O(x)$, we get

$$k_d = \frac{N}{Qd} + O(1) \quad \text{and} \quad k_d^2 = \frac{N^2}{Q^2 d^2} + \frac{1}{Q}\, O\Big(\frac{N}{d}\Big). \tag{57}$$

Hence, from (56) and (57) we get

$$\sum_{\substack{d'=1\\(Q,d')=1}}^{\lfloor N/d \rfloor} d' = \frac{\varphi(Q)}{2Q}\,\frac{N^2}{d^2} + O\left(\varphi(Q)\,\frac{N}{d} + Q\frac{N}{d} + Q^2\right) = \frac{\varphi(Q)}{2Q}\,\frac{N^2}{d^2} + Q^2\, O\Big(\frac{N}{d}\Big).$$

We plug this in (55) to get

$$\sum_{\substack{n=1\\(Q,n)=1}}^{N} \varphi(n) = \frac{\varphi(Q)}{2Q} N^2 \sum_{\substack{d=1\\(Q,d)=1}}^{N} \frac{\mu(d)}{d^2} + O\bigg(Q^2 N \sum_{\substack{d=1\\(Q,d)=1}}^{N} \frac{\mu(d)}{d}\bigg).$$

The sum $\sum_{n=1}^{\infty} \mu(d)/d^2$ is absolutely convergent, and $c_Q := \sum_{d=1,\,(Q,d)=1}^{\infty} \mu(d)/d^2 > 0$ because

$$c_Q = 1 + \sum_{\substack{d=2\\(Q,d)=1}}^{\infty} \frac{\mu(d)}{d^2} \quad \text{and} \quad \Bigg| \sum_{\substack{d=2\\(Q,d)=1}}^{\infty} \frac{\mu(d)}{d^2} \Bigg| \le \frac{\pi^2}{6} - 1 < 1.$$

Hence,

$$\sum_{\substack{d=1\\(Q,d)=1}}^{N} \frac{\mu(d)}{d^2} = c_Q - \sum_{\substack{d=N+1\\(Q,d)=1}}^{\infty} \frac{\mu(d)}{d^2} = c_Q + O\bigg(\sum_{d=N+1}^{\infty} \frac{1}{d^2}\bigg) = c_Q + O(1/N).$$

Together with $|\sum_{d=1,\,(Q,d)=1}^{N} \mu(d)/d| \lesssim \log N$, this implies

$$\begin{aligned}\sum_{\substack{n=1\\(Q,n)=1}}^{N} \varphi(n) &= c_Q\,\frac{\varphi(Q)}{2Q} N^2 + O\Big(\frac{\varphi(Q)}{Q} N\Big) + O(Q^2 N \log N)\\ &= c_Q\,\frac{\varphi(Q)}{2Q} N^2 + O_Q(N \log N).\end{aligned}$$

Together with (54) we conclude $\Phi_Q(N) \ge c_Q\, \frac{\varphi(Q)^2}{2Q} N^2 + O_Q(N \log N)$. □

Proof of Lemma A.4 We begin with $k = 1$. When $Q = 2$, we have $S_{2,1} = 1 = 2\,\varphi(2)/2$, so we may assume $Q \ge 3$. We first observe that $\varphi(Q)$ is even, because if Q has an odd prime factor p, then $\varphi(p) = p - 1$, which is even, is a factor of $\varphi(Q)$. Otherwise, $Q = 2^r$ with $r \ge 2$, so $\varphi(Q) = 2^{r-1}$ is even. Now, the observation that $(Q, n) = 1 \iff (Q, Q - n) = 1$ implies

$$S_{Q,1} = \sum_{\substack{n=1 \\ (Q,n)=1}}^{\lfloor Q/2 \rfloor} n + \sum_{\substack{n=\lfloor Q/2 \rfloor+1 \\ (Q,n)=1}}^{Q-1} n = \sum_{\substack{n=1 \\ (Q,n)=1}}^{\lfloor Q/2 \rfloor} \big(n + (Q-n)\big) = Q\,\frac{\varphi(Q)}{2}.$$

Let now $k \geq 2$, so that

$$\sum_{\substack{n=(k-1)Q+1 \\ (Q,n)=1}}^{kQ-1} n = \sum_{\substack{n=1 \\ (Q,n)=1}}^{Q-1} \Big(n + (k-1)Q\Big)$$
$$= S_{Q,1} + (k-1)Q\varphi(Q) = Q\varphi(Q)\Big(k - \frac{1}{2}\Big).$$

Consequently,

$$S_{Q,k} = \sum_{\ell=1}^{k} \left(\sum_{\substack{n=(\ell-1)Q+1 \\ (Q,n)=1}}^{\ell Q} n \right) = \sum_{\ell=1}^{k} Q\varphi(Q)\Big(\ell - \frac{1}{2}\Big) = \frac{Q\varphi(Q)}{2} k^2.$$

□

To conclude, we prove the estimates for the weighted sums that we needed in Lemma 6.5 as a corollary of Proposition A.3. As before, when $\alpha > 2$ the sums are absolutely convergent.

Corollary A.5 (Lemma 6.5) *Let $Q \in \mathbb{N}$ and $\alpha \leq 2$. For $N \gg 1$,*

$$\sum_{n=1}^{N} \frac{\varphi(Qn)}{n^2} \simeq \log N, \quad \textit{and} \quad \sum_{n=1}^{N} \frac{\varphi(Qn)}{n^\alpha} \simeq N^{2-\alpha} \quad \textit{for} \quad \alpha < 2.$$

The implicit constants depend on Q, and also on α when $\alpha < 0$.

Proof Upper bounds follow directly from $\varphi(n) \leq n$. Lower bounds follow from Proposition A.3 with the same strategy as in the proof of Corollary A.2. If $\alpha \geq 0$, by Proposition A.3 we get

$$\sum_{n=1}^{N} \frac{\varphi(Qn)}{n^\alpha} \geq \frac{1}{N^\alpha}\,\Phi_Q(N) \simeq_Q N^{2-\alpha}, \quad \text{when } N \gg 1.$$

When $\alpha = 2$, combine Proposition A.3 with summing by parts as in (53) to get

$$\sum_{n=1}^{N} \frac{\varphi(Qn)}{n^2} = \frac{\Phi_Q(N)}{N^2} + \sum_{n=1}^{N-1} \Phi_Q(n)\frac{2n+1}{n^2\,(n+1)^2} \gtrsim 1 + \sum_{n=\log N}^{N-1} \frac{1}{n} \simeq \log N.$$

When $\alpha < 0$, choosing $\delta > 0$ small enough depending on Q, Proposition A.3 implies

$$\sum_{n=1}^{N} \frac{\varphi(Qn)}{n^\alpha} \geq_\alpha N^{|\alpha|} \sum_{n=\delta N}^{N} \varphi(Qn) = N^{|\alpha|}\Big(\Phi_Q(N) - \Phi_Q(\delta N)\Big) \simeq_{Q,\alpha} N^{|\alpha|} N^2 = N^{2-\alpha}.$$

□

Appendix B. Alternative asymptotic behavior of R_{x_0} around rational t

Following Duistermaat [22], we give an alternative asymptotic behavior of R_{x_0} around rationals that complements Corollary 3.3 and allows us to prove Propositions 3.5 and 4.2.

Proposition B.1 *Let $x_0 \in \mathbb{R}$. Let $p, q \in \mathbb{N}$ be such that $(p,q) = 1$. Let $x_q = \mathrm{dist}(x_0, \mathbb{Z}/q)$. Let $h \neq 0$ and denote $\mathrm{sign}(h) = \pm$ so that $h = \pm|h|$. If $x_q = 0$,*

$$\begin{aligned} &R_{x_0}\Big(\frac{p}{q} + h\Big) - R_{x_0}\Big(\frac{p}{q}\Big) + 2\pi i h \\ &\quad = 2\pi(-1 \pm i)\frac{\sqrt{|h|}}{\sqrt{q}} \frac{G(p, m_q, q)}{\sqrt{q}} \\ &\qquad + 2(1 \pm i)\, q^{3/2}|h|^{3/2} \sum_{m \neq 0} \frac{G(p, m_q + m, q)}{\sqrt{q}} \frac{e^{-2\pi i \frac{m^2}{4q^2 h}}}{m^2} + O\left(q^{7/2} h^{5/2}\right), \end{aligned} \tag{58}$$

If $x_q \neq 0$,

$$\begin{aligned} &R_{x_0}\Big(\frac{p}{q} + h\Big) - R_{x_0}\Big(\frac{p}{q}\Big) + 2\pi i h \\ &\quad = 2(1 \pm i)\, q^{3/2}|h|^{3/2} \sum_{m \in \mathbb{Z}} \frac{G(p, m_q + m, q)}{\sqrt{q}} \frac{e^{-2\pi i \frac{(m - q x_q)^2}{4q^2 h}}}{(m - q x_q)^2} \\ &\qquad + O\left(q^{7/2} h^{5/2} \sum_{m \in \mathbb{Z}} \frac{1}{(m - q x_q)^4}\right). \end{aligned} \tag{59}$$

Proof From the definition $R_{x_0}(t) = \sum_{n \neq 0} e^{2\pi i (n^2 t + n x_0)}/n^2$, we first write

$$\begin{aligned} R_{x_0}\Big(\frac{p}{q} + h\Big) - R_{x_0}\Big(\frac{p}{q}\Big) + 2\pi i h &= 2\pi i h + \sum_{n \neq 0} \frac{e^{2\pi i n^2 h} - 1}{n^2} e^{2\pi i \frac{p n^2}{q}} e^{2\pi i n x_0} \\ &= 2\pi i h \sum_{n \in \mathbb{Z}} \Big(\int_0^1 e^{2\pi i n^2 h \tau}\, d\tau\Big) e^{2\pi i \frac{p n^2}{q}} e^{2\pi i n x_0}. \end{aligned}$$

Split the sum modulo q by writing $n = mq + r$ and use the Poisson summation formula to obtain

$$
\begin{aligned}
&R_{x_0}\Big(\frac{p}{q}+h\Big) - R_{x_0}\Big(\frac{p}{q}\Big) + 2\pi i h \\
&= 2\pi i h \sum_{r=0}^{q-1} e^{2\pi i r^2 p/q} \sum_{m\in\mathbb{Z}} \Big(\int_0^1 e^{2\pi i (mq+r)^2 h\tau}\, d\tau \Big)\, e^{2\pi i (mq+r)x_0} \\
&= 2\pi i h \sum_{r=0}^{q-1} e^{2\pi i r^2 p/q} \sum_{m\in\mathbb{Z}} \int \Big(\int_0^1 e^{\pm 2\pi i (zq+r)^2 |h|\tau}\, d\tau \Big)\, e^{2\pi i (zq+r)x_0}\, e^{-2\pi i m z}\, dz \\
&= \pm 2\pi i \frac{\sqrt{|h|}}{q} \sum_{m\in\mathbb{Z}} \sum_{r=0}^{q-1} e^{2\pi i \frac{pr^2+mr}{q}} \int_0^1 \int e^{\pm 2\pi i y^2 \tau}\, e^{2\pi i \frac{y}{\sqrt{|h|}}(x_0 - \frac{m}{q})}\, dy\, d\tau.
\end{aligned}
$$

where we changed variables $(zq + r)^2|h| = y^2$. Now complete the square to get

$$
\begin{aligned}
&R_{x_0}\Big(\frac{p}{q}+h\Big) - R_{x_0}\Big(\frac{p}{q}\Big) + 2\pi i h \\
&= \pm 2\pi i \frac{\sqrt{|h|}}{q} \sum_{m\in\mathbb{Z}} G(p,m,q) \int_0^1 \Big(\int e^{\pm 2\pi i \tau \big(y \pm \frac{x_0 - m/q}{2\tau\sqrt{|h|}} \big)^2}\, dy \Big)\, e^{\mp 2\pi i \frac{(x_0-m/q)^2}{4\tau |h|}}\, d\tau \\
&= \pm 2\pi i \frac{1\pm i}{2} \frac{\sqrt{|h|}}{q} \sum_{m\in\mathbb{Z}} G(p,m,q) \int_0^1 \frac{1}{\sqrt{\tau}}\, e^{\mp 2\pi i \frac{(x_0-m/q)^2}{4\tau |h|}}\, d\tau. \qquad (60)
\end{aligned}
$$

By changing variables, and defining $x_q = \min_{m\in\mathbb{Z}} |x_0 - m/q| = |x_0 - m_q/q|$ as in (20), we write

$$
\begin{aligned}
&R_{x_0}\Big(\frac{p}{q}+h\Big) - R_{x_0}\Big(\frac{p}{q}\Big) + 2\pi i h \\
&= \pi(-1\pm i) \frac{\sqrt{|h|}}{\sqrt{q}} \sum_{m\in\mathbb{Z}} \frac{G(p, m_q + m, q)}{\sqrt{q}} \int_1^\infty \frac{1}{\xi^{3/2}}\, e^{-2\pi i \frac{(x_q - m/q)^2}{4h}\xi}\, d\xi. \qquad (61)
\end{aligned}
$$

We now separate cases. If $x_q = 0$, the integral of the term $m = 0$ is $\int_1^\infty \xi^{-3/2} d\xi = 2$. In all other cases, that is, if either $x_q \neq 0$ or $m \neq 0$, integration by parts implies

$$
\begin{aligned}
&\int_1^\infty \frac{1}{\xi^{3/2}}\, e^{-2\pi i \frac{(x_q - m/q)^2}{4h}\xi}\, d\xi \\
&= \frac{2}{\pi i} \frac{q^2 h}{(m - q x_q)^2} \left(e^{-2\pi i \frac{(m - q x_q)^2}{4 q^2 h}} + \frac{3}{2} \int_1^\infty \frac{1}{\xi^{5/2}}\, e^{-2\pi i \frac{(x_q - m/q)^2}{4h}\xi}\, d\xi \right) \\
&= O\left(\frac{q^2 h}{(m - q x_q)^2} \right).
\end{aligned}
$$

What is more, integrating by parts again we obtain

$$\int_1^\infty \frac{1}{\xi^{3/2}} e^{-2\pi i \frac{(x_q - m/q)^2}{4h}\xi} \, d\xi = \frac{2}{\pi i} \frac{q^2 h}{(m - q x_q)^2} \left(e^{-2\pi i \frac{(m - q x_q)^2}{4q^2 h}} + O\Big(\frac{q^2 h}{(m - q x_q)^2} \Big) \right).$$

Combining these with (61) give the desired expressions. □

Remark B.2 Computations for (60) are made rigorous to avoid convergence problems by writing

$$\sum_{n\in\mathbb{Z}} \frac{e^{2\pi i n^2 h} - 1}{n^2} e^{2\pi i n^2 p/q} \, e^{2\pi i n x_0} = \lim_{\epsilon\to 0} \sum_{n\in\mathbb{Z}} \frac{e^{2\pi i n^2 h(1+i\epsilon)} - 1}{n^2} e^{2\pi i n^2 p/q} \, e^{2\pi i n x_0}.$$

Proposition B.1 will allow us to give upper bounds of $\alpha_{x_0}(t)$ for general t.

Proposition B.3 *Let $x_0 \in \mathbb{Q}$ and $t \notin \mathbb{Q}$. Then, $\alpha_{x_0}(t) \leq 3/4$.*

Proof Set $x_0 = P/Q$ with $P, Q \in \mathbb{N}$ and $(P, Q) = 1$. Let $t \notin \mathbb{Q}$ and let p_n/q_n be its approximations by continued fractions. It is well-known[31] that there is a subsequence of odd denominators q_{n_k}. Renaming that subsequence back to q_n, we may assume that all q_n are odd. Consequently, $|G(p_n, m, q_n)| = \sqrt{q}$ for all $m, n \in \mathbb{N}$. As usual, let

$$h_n = t - \frac{p_n}{q_n}, \quad |h_n| < \frac{1}{q_n^2}, \quad x_{q_n} = \min_{m\in\mathbb{Z}} \left| \frac{P}{Q} - \frac{m}{q_n} \right| = \left| \frac{P}{Q} - \frac{m_{q_n}}{q_n} \right|,$$

and we immediately deduce that either $x_{q_n} = 0$ or $1/Q \leq q_n x_{q_n} \leq 1/2$. We separate cases:

Case 1 We have $x_{q_n} = 0$ for infinitely many $n \in \mathbb{N}$. Rename that subsequence and rewrite (58) as

$$\begin{aligned} &\left| R_{x_0}\Big(\frac{p_n}{q_n} + h_n\Big) - R_{x_0}\Big(\frac{p_n}{q_n}\Big) + 2\pi i h_n \right| \\ &= 2\pi\sqrt{2} \frac{\sqrt{|h_n|}}{\sqrt{q_n}} + O\left(q_n^{3/2} h_n^{3/2}\right) \simeq \frac{\sqrt{|h_n|}}{\sqrt{q_n}} \left(1 + O\big(q_n^2 h_n\big)\right). \end{aligned} \tag{62}$$

Let $\delta > 0$ which we determine later. Separate cases again:

Case 1.1. Suppose that $\left|1 + O\big(q_n^2 h_n\big)\right| \geq \delta$ for infinitely many $n \in \mathbb{N}$. Then,

$$\left| R_{x_0}(t) - R_{x_0}(t - h_n) + 2\pi i h_n \right| \geq \delta \frac{\sqrt{|h_n|}}{\sqrt{q_n}} \geq \delta \, |h_n|^{3/4},$$

because $q_n^2 |h_n| \leq 1$. Hence $|R_{x_0}(t) - R_{x_0}(t - h_n)| \geq (\delta/2)\, |h_n|^{3/4}$ for infinitely many $n \in \mathbb{N}$, and consequently $\alpha_{x_0}(t) \leq 3/4$.

[31] Because two consecutive denominators q_n and q_{n+1} are never both even.

Case 1.2. We have $\left|1+O\left(q_n^2 h_n\right)\right| < \delta$ for all large enough n. In that case, we evaluate (62) at a point closer to p_n/q_n. Let $\epsilon > 0$ and write (58) for ϵh_n, so that instead of (62) we get

$$\left|R_{x0}\left(\frac{p_n}{q_n}+\epsilon h_n\right)-R_{x0}\left(\frac{p_n}{q_n}\right)+2\pi i\epsilon h_n\right| \simeq \sqrt{\epsilon}\,\frac{\sqrt{|h_n|}}{\sqrt{q_n}}\left(1+\epsilon O\left(q_n^2 h_n\right)\right).$$

Since $q_n^2|h_n| < 1$ and the constant underlying the big-O is universal, say C, choose $\epsilon \leq 1/(2C)$, in such a way that

$$\left|R_{x0}\left(\frac{p_n}{q_n}+\epsilon h_n\right)-R_{x0}\left(\frac{p_n}{q_n}\right)+2\pi i\epsilon h_n\right| \gtrsim \frac{\sqrt{\epsilon}}{2}\,\frac{\sqrt{|h_n|}}{\sqrt{q_n}}.$$

From this and (62), we write

$$\begin{aligned}
\frac{\sqrt{\epsilon}}{2}\,\frac{\sqrt{|h_n|}}{\sqrt{q_n}} &\lesssim \left|R_{x0}\left(\frac{p_n}{q_n}+\epsilon h_n\right)-R_{x0}\left(\frac{p_n}{q_n}\right)\right| + 2\pi\epsilon|h_n| \\
&\leq \left|R_{x0}\left(\frac{p_n}{q_n}+\epsilon h_n\right)-R_{x0}(t)\right| + \left|R_{x0}(t)-R_{x0}\left(\frac{p_n}{q_n}\right)\right| + 2\pi\epsilon|h_n| \\
&\lesssim \left|R_{x0}\left(\frac{p_n}{q_n}+\epsilon h_n\right)-R_{x0}(t)\right| \\
&\quad + \frac{\sqrt{|h_n|}}{\sqrt{q_n}}\left(1+O\left(q_n^2 h_n\right)\right) + 2\pi(1+\epsilon)|h_n| \\
&\leq \left|R_{x0}\left(\frac{p_n}{q_n}+\epsilon h_n\right)-R_{x0}(t)\right| + 2\delta\,\frac{\sqrt{|h_n|}}{\sqrt{q_n}}.
\end{aligned}$$

In the last line we used the hypothesis of Case 1.2 and $|h_n| \leq \frac{\sqrt{|h_n|}}{\sqrt{q_n}}\,\frac{1}{\sqrt{q_n}}$. Hence,

$$\left|R_{x0}\left(\frac{p_n}{q_n}+\epsilon h_n\right)-R_{x0}(t)\right| \gtrsim \left(\frac{\sqrt{\epsilon}}{2}-C\delta\right)\frac{\sqrt{|h_n|}}{\sqrt{q_n}},$$

for some $C > 0$. Fix $\sqrt{\epsilon} = 4C\delta$ small enough. Writing $p_n/q_n + \epsilon h_n = t-(1-\epsilon)h_n$ and observing that $(1-\epsilon)|h_n| \simeq |h_n|$, we conclude that

$$\begin{aligned}
&\left|R_{x0}\left(t-(1-\epsilon)h_n\right)-R_{x0}(t)\right| \\
&\quad \gtrsim \delta\,\frac{\sqrt{|h_n|}}{\sqrt{q_n}} \geq \delta\,|h_n|^{3/4} \simeq |(1-\epsilon)h_n|^{3/4}, \qquad \text{for large enough } n.
\end{aligned}$$

Hence $\alpha_{x0}(t) \leq 3/4$.

Case 2 We have $x_{q_n} \neq 0$ for all large enough $n \in \mathbb{N}$, hence $1/Q \leq q_n x_{q_n} \leq 1/2$. We now use (59) which has no leading $h^{1/2}$ term. Rewrite it,[32] assuming $1/Q \leq q x_q \leq 1/2$, as

$$R_{x_0}\left(\frac{p}{q}+h\right) - R_{x_0}\left(\frac{p}{q}\right) + 2\pi i h$$
$$= 2(1 \pm i)\, \frac{G(p,0,q)}{\sqrt{q}}\, \frac{\sqrt{|h|}}{\sqrt{q}}\, q^2 |h| \left[\sum_{m\in\mathbb{Z}} e^{2\pi i (4p)^{-1} \frac{(m_q+m)^2}{q}}\, \frac{e^{-2\pi i \frac{(m-qx_q)^2}{4q^2 h}}}{(m-qx_q)^2} + O_Q\left(q^2 h\right) \right].$$

Define the auxiliary function

$$f_q(y) = \sum_{m\in\mathbb{Z}} e^{2\pi i (4p)^{-1} \frac{m^2+2m_q m}{q}}\, \frac{e^{-2\pi i (m^2 - 2q x_q m) y}}{(m-qx_q)^2}. \tag{63}$$

Take absolute values and write

$$\left| R_{x_0}\left(\frac{p}{q}+h\right) - R_{x_0}\left(\frac{p}{q}\right) + 2\pi i h \right|$$
$$= 2\sqrt{2} \frac{\sqrt{|h|}}{\sqrt{q}}\, q^2 |h| \left| f_q\left(\frac{1}{4q^2 h}\right) + O_Q\left(q^2 h\right) \right|. \tag{64}$$

We now state the properties of this function, whose proof we postpone.

Lemma B.4 *Let $q \in \mathbb{N}$, let $p \in \mathbb{N}$ be coprime with q and f_q defined in (63). Then,*

(a) *f_q is periodic of period Q.*
(b) *there exists $y_0^q \in [0, Q]$ depending on q (and on p) such that $|f_q(y_0^q)| \geq 5$.*
(c) *The sequence defined by $y_k^q = y_0^q + kQ$ satisfies*

$$\lim_{k\to\infty} y_k^q = \infty, \quad \textit{and} \quad |f_q(y_k^q)| \geq 5, \quad \forall k \in \mathbb{N}.$$

Remark B.5 The dependence on p of the point y_0^q is irrelevant for our purposes. Indeed, once we fix $t \notin \mathbb{Q}$, we get the sequence of approximations p_n/q_n, hence each q_n comes with one and only one p_n. Hence, we can assume that the sequence f_{q_n} only depends on q_n.

We now evaluate (64) at p_n/q_n and $h_n = t - p_n/q_n$ and we separate two cases:

[32] When q is odd and coprime with p, the inverses of 2 and p modulo q exist. Therefore,

$$G(p,m,q) = \sum_{r=1}^{q} e^{2\pi i \frac{pr^2+mr}{q}} = e^{2\pi i (4p)^{-1} \frac{m^2}{q}} \sum_{r=1}^{q} e^{2\pi i p \frac{(r+(2p)^{-1}m)^2}{q}} = e^{2\pi i (4p)^{-1} \frac{m^2}{q}}\, G(p,0,q).$$

Case 2.1. Suppose $\limsup_{n\to\infty} q_n^2|h_n| > 0$, so that there exists $c > 0$ and a subsequence for which $c < q_n^2|h_n| \leq 1$. Then, from (64) we get

$$\left| R_{x_0}(t) - R_{x_0}\left(\frac{p_n}{q_n}\right) + 2\pi i h_n \right| \geq c \frac{\sqrt{|h_n|}}{\sqrt{q_n}} \left| f_{q_n}\left(\frac{1}{4q_n^2 h_n}\right) + O_Q\left(q_n^2 h_n\right) \right|.$$

Fix $\delta > 0$ which we later determine. Proceeding like in **Case 1**, we separate two cases:

Case 2.1.1. Suppose $\left| f_{q_n}\left(\frac{1}{4q_n^2 h_n}\right) + O_Q\left(q_n^2 h_n\right) \right| \geq \delta$ for infinitely many n. Then,

$$\left| R_{x_0}(t) - R_{x_0}\left(\frac{p_n}{q_n}\right) + 2\pi i h_n \right| \geq c\delta \frac{\sqrt{|h_n|}}{\sqrt{q_n}} \geq c\delta |h_n|^{3/4}$$

for infinitely many n, which implies $\alpha_{x_0}(t) \leq 3/4$.

Case 2.1.2. Suppose $\left| f_{q_n}\left(\frac{1}{4q_n^2 h_n}\right) + O_Q\left(q_n^2 h_n\right) \right| < \delta$ for all large enough n. Then, let ϵ_n be a sequence which we determine later, and define $\eta_n = \epsilon_n / q_n^2$. Observe that $\eta_n = \epsilon_n |h_n| / (q_n^2 |h_n|) \simeq \epsilon_n |h_n|$. Evaluate (64) at η_n to get

$$\begin{aligned} &\left| R_{x_0}\left(\frac{p_n}{q_n} + \eta_n\right) - R_{x_0}\left(\frac{p_n}{q_n}\right) + 2\pi i \eta_n \right| \\ &\quad = 2\sqrt{2} \frac{\sqrt{\eta_n}}{\sqrt{q_n}} q_n^2 \eta_n \left| f_q\left(\frac{1}{4q_n^2 \eta_n}\right) + O_Q\left(q_n^2 \eta_n\right) \right| \\ &\quad = 2\sqrt{2}\, \epsilon_n \frac{\sqrt{\eta_n}}{\sqrt{q_n}} \left| f_{q_n}\left(\frac{1}{4\epsilon_n}\right) + O_Q\left(\epsilon_n\right) \right|. \end{aligned}$$

Fix $k \in \mathbb{N}$ large enough and set $\epsilon_n = 1/(4y_k^{q_n})$. Then, by Lemma B.4 (c),

$$\left| f_{q_n}\left(\frac{1}{4\epsilon_n}\right) \right| = \left| f_{q_n}(y_K^{q_n}) \right| \geq 5, \quad \forall n \text{ large enough.}$$

Since $\epsilon_n \simeq 1/(kQ)$, if $k \in \mathbb{N}$ is large enough we get $O_Q\left(\epsilon_n\right) \leq C_Q \epsilon_n \leq 1$. In particular, $|h_n| \simeq_Q k\eta_n$. Therefore,

$$\left| R_{x_0}\left(\frac{p_n}{q_n} + \eta_n\right) - R_{x_0}\left(\frac{p_n}{q_n}\right) + 2\pi i \eta_n \right| \geq \epsilon_n \frac{\sqrt{\eta_n}}{\sqrt{q_n}} \simeq \epsilon_n^{3/2} \frac{\sqrt{|h_n|}}{\sqrt{q_n}}.$$

With this, and using the assumption of this case in (64), we write

$$\begin{aligned}\epsilon_n^{3/2}\frac{\sqrt{|h_n|}}{\sqrt{q_n}} &\lesssim \left|R_{x_0}\left(\frac{p_n}{q_n}+\eta_n\right)-R_{x_0}\left(\frac{p_n}{q_n}\right)\right|+2\pi\eta_n \\ &\leq \left|R_{x_0}\left(\frac{p_n}{q_n}+\eta_n\right)-R_{x_0}(t)\right|+\left|R_{x_0}(t)-R_{x_0}\left(\frac{p_n}{q_n}\right)\right|+2\pi\eta_n \\ &\lesssim \left|R_{x_0}\left(\frac{p_n}{q_n}+\eta_n\right)-R_{x_0}(t)\right|+\delta\frac{\sqrt{|h_n|}}{\sqrt{q_n}}+2\pi(|h_n|+\eta_n) \\ &\lesssim \left|R_{x_0}\left(\frac{p_n}{q_n}+\eta_n\right)-R_{x_0}(t)\right|+\delta\frac{\sqrt{|h_n|}}{\sqrt{q_n}},\end{aligned}$$

for large enough n, where in the last line we used $\eta_n \simeq |h_n|/k \leq |h_n|$ and $|h_n| \leq \frac{\sqrt{|h_n|}}{\sqrt{q_n}}\frac{1}{\sqrt{q_n}} \ll \frac{\sqrt{|h_n|}}{\sqrt{q_n}}$. Since $\epsilon_n \simeq_Q 1/k$, set $\delta = 1/(c_Q k^{3/2})$ with some small enough $c_Q > 0$ so that

$$\left|R_{x_0}\left(\frac{p_n}{q_n}+\eta_n\right)-R_{x_0}(t)\right| \gtrsim \left(\epsilon_n^{3/2}-C\delta\right)\frac{\sqrt{|h_n|}}{\sqrt{q_n}} \gtrsim \delta\frac{\sqrt{|h_n|}}{\sqrt{q_n}} \geq \delta|h_n|^{3/4}.$$

Write $p_n/q_n+\eta_n = t-(h_n-\eta_n)$. Since $|h_n-\eta_n| \leq 2|h_n|$, we get

$$\left|R_{x_0}\big(t-(h_n-\eta_n)\big)-R_{x_0}(t)\right| \geq \delta|h_n|^{3/4} \gtrsim \delta|h_n-\eta_n|^{3/4}, \quad \text{for large enough } n,$$

which implies $\alpha_{x_0}(t) \leq 3/4$.

Case 2.2. Suppose $\lim_{n\to\infty} q_n^2|h_n| = 0$. In this case, the term $q_n^2|h_n|$ in (64) tends to zero, which kills the desired $|h_n|^{3/4}$ that came from $\sqrt{h_n}/\sqrt{q_n}$. To counter that, define $\eta_n = \epsilon_n/q_n^2$ as in Case 2.1.2. By (64),

$$\left|R_{x_0}\left(\frac{p_n}{q_n}+\eta_n\right)-R_{x_0}\left(\frac{p_n}{q_n}\right)+2\pi i\eta_n\right| = 2\sqrt{2}\epsilon_n\frac{\sqrt{\eta_n}}{\sqrt{q_n}}\left|f_{q_n}\left(\frac{1}{4\epsilon_n}\right)+O_Q(\epsilon_n)\right|.$$

Fix $k \in \mathbb{N}$ large enough and set $\epsilon_n = 1/(4y_k^{q_n})$. Then,

$$\left|f_{q_n}\left(\frac{1}{4\epsilon_n}\right)\right| = \left|f_{q_n}\left(y_k^{q_n}\right)\right| \geq 5, \quad \text{and} \quad O_Q(\epsilon_n) \leq C_Q\epsilon_n = \frac{C_Q}{4y_k^{q_n}} \simeq \frac{C_Q}{kQ} \leq 1,$$

so

$$\left|R_{x_0}\left(\frac{p_n}{q_n}+\eta_n\right)-R_{x_0}\left(\frac{p_n}{q_n}\right)+2\pi i\eta_n\right| \geq \epsilon_n\frac{\sqrt{\eta_n}}{\sqrt{q_n}}.$$

With this and (64), we can write

$$\begin{aligned}\epsilon_n \frac{\sqrt{\eta_n}}{\sqrt{q_n}} &\leq \left| R_{x_0}\left(\frac{p_n}{q_n}+\eta_n\right) - R_{x_0}\left(\frac{p_n}{q_n}\right)\right| + 2\pi\eta_n \\ &\leq \left| R_{x_0}\left(\frac{p_n}{q_n}+\eta_n\right) - R_{x_0}(t)\right| + \left| R_{x_0}(t) - R_{x_0}\left(\frac{p_n}{q_n}\right)\right| + 2\pi\eta_n \\ &\lesssim \left| R_{x_0}\left(\frac{p_n}{q_n}+\eta_n\right) - R_{x_0}(t)\right| + \frac{\sqrt{|h_n|}}{\sqrt{q_n}}\, q_n^2|h_n| + 2\pi(\eta_n + |h_n|).\end{aligned}$$

Since $\lim_{n\to\infty} q_n^2|h_n| = 0$ implies $h_n = o(\eta_n)$, and $\eta_n = \frac{\sqrt{\eta_n}}{\sqrt{q_n}}\frac{\sqrt{\epsilon_n}}{\sqrt{q_n}}$, we get

$$\left| R_{x_0}\left(\frac{p_n}{q_n}+\eta_n\right) - R_{x_0}(t)\right| \gtrsim \left(\epsilon_n - q_n^2 h_n - \frac{\sqrt{\epsilon_n}}{\sqrt{q_n}}\right)\frac{\sqrt{\eta_n}}{\sqrt{q_n}} \geq \frac{\epsilon_n}{2}\,\frac{\sqrt{\eta_n}}{\sqrt{q_n}} = \frac{\epsilon_n^{3/4}}{2}\,\eta_n^{3/4}.$$

Write $p_n/q_n + \eta_n = t + (\eta_n - h_n)$. Recalling $\epsilon_n \simeq 1/(kQ)$ for all n, and since $h_n = o(\eta_n)$ implies $|\eta_n - h_n| \simeq \eta_n$, we conclude

$$\left| R_{x_0}\Big(t + (\eta_n - h_n)\Big) - R_{x_0}(t)\right| \geq \frac{\epsilon_n^{3/4}}{2}\,\eta_n^{3/4} \simeq_Q |\eta_n - h_n|^{3/4},$$

and therefore $\alpha_{x_0}(t) \leq 3/4$. □

We now prove Lemma B.4.

Proof of Lemma B.4 (*a*) Write first

$$0 \neq q x_q = q \min_{m\in\mathbb{Z}} \left| x_0 - \frac{m}{q}\right| = q\left| x_0 - \frac{m_q}{q}\right| = \frac{1}{Q}\,|Pq - Qm_q| = \frac{m'_q}{Q},$$

where $m'_q = |Pq - Qm_q| \in \mathbb{N}\backslash\{0\}$. Hence, the variable y in (63) only appears in

$$e^{2\pi i (m^2 - 2qx_q m)\, y} = e^{2\pi i (Qm^2 - 2m'_q m)\frac{y}{Q}},$$

which is Q-periodic. Hence f_q has period Q.

(*b*) Split the sum in f_p in the terms $m = 0, 1$ and the rest,

$$f_q(y) = \frac{1}{(qx_q)^2} + e^{2\pi i (4p)^{-1}\frac{1+2m_q}{q}}\,\frac{e^{-2\pi i(1-2qx_q)y}}{(1-qx_q)^2} + \text{Error}$$

where $1/Q \le qx_q \le 1/2$ implies

$$|\text{Error}| = \left| \sum_{m \neq 0,1} e^{2\pi i (4p)^{-1} \frac{m^2+2m_q m}{q}} \frac{e^{-2\pi i (m^2 - 2qx_q m) y}}{(m-qx_q)^2} \right|$$
$$\le \sum_{m=2}^{\infty} \frac{1}{(m-qx_q)^2} + \sum_{m=1}^{\infty} \frac{1}{(m+qx_q)^2} \le \sum_{m=2}^{\infty} \frac{1}{(m-1/2)^2} + \sum_{m=1}^{\infty} \frac{1}{m^2}$$
$$= \frac{\pi^2}{2} - 4 + \frac{\pi^2}{6} \le 3.$$

On the other hand, the phase in

$$e^{2\pi i (4p)^{-1} \frac{1+2m_q}{q}} e^{-2\pi i (1-2qx_q) y}.$$

is continuous, decreasing, and Q-periodic. That implies that there exists $y_0^q \in [0, Q]$ such that $e^{2\pi i (4p)^{-1} \frac{1+2m_q}{q}} e^{-2\pi i (1-2qx_q) y_0^q} = 1$, and consequently,

$$|f_q(y_0^q)| \ge \frac{1}{(qx_q)^2} + \frac{1}{(1-qx_q)^2} - 3 \ge \frac{1}{(1/2)^2} + \frac{1}{(1-1/2)^2} - 3 = 5$$

because in $(0, 1)$ the function $1/x^2 + 1/(1-x)^2$ has a minimum in $x = 1/2$.

(c) The fact that f_q is Q-periodic implies that $|f_q(y_n^q)| = |f_q(y_0^q + nQ)| = |f_q(y_0^q)| \ge 5$. □

We now complete the proof of Proposition 3.5.

Proposition B.6 *Let $x_0 \in \mathbb{R}$ and $t \in \mathbb{Q}$. If $\alpha_{x_0}(t) \neq 1/2$, then $\alpha_{x_0}(t) = 3/2$.*

Proof By Proposition B.1, $\alpha_{x_0}(t) = 1/2$ happens only if $x_q = 0$ and $G(p, m_q, q) \neq 0$.

- If $x_q = 0$ and $G(p, m_q, q) = 0$, then $x_0 \in \mathbb{Q}$ and $q \in 2\mathbb{N}$. From (58) and the fact that

$$G(p,m,q) = \begin{cases} e^{2\pi i (4p)^{-1} m^2/q}\, G(p,0,q), & q \text{ odd}, \\ e^{2\pi i p^{-1} (m/2)^2/q}\, G(p,0,q), & q \equiv 0 \pmod 4 \text{ and } m \text{ even}, \\ e^{2\pi i p^{-1} ((m-1)/2)^2/q} e^{2\pi i p^{-1} ((m-1)/2)/q}\, G(p,1,q), & q \equiv 2 \pmod 4 \text{ and } m \text{ odd}, \end{cases} \tag{65}$$

and $G(p,m,q) = 0$ otherwise, we have

$$R_{x_0}\left(\frac{p}{q} + h\right) - R_{x_0}\left(\frac{p}{q}\right) + 2\pi i h$$
$$= 2(1 \pm i)\, q^{3/2} |h|^{3/2} \sum_{m \text{ odd}} \frac{G(p, m_q + m, q)}{\sqrt{q}} \frac{e^{-2\pi i \frac{m^2}{4q^2 h}}}{m^2} + O\left(q^{7/2} h^{5/2}\right).$$

It suffices to find a sequence $y_k \to \infty$ such that $|g(y_k)| \geq c > 0$ for some $c > 0$, where

$$g(y) = \sum_{m \text{ odd}} \frac{G(p, m_q + m, q)}{\sqrt{q}} \frac{e^{-2\pi i m^2 y}}{m^2},$$

because that way, defining $h_k = 1/(4q^2 y_k)$, we get

$$\left| R_{x_0}\left(\frac{p}{q} + h_k\right) - R_{x_0}\left(\frac{p}{q}\right) + 2\pi i h_k \right| \gtrsim_q h_k^{3/2} |g(y_k)| - O(h_k^{5/2}) \gtrsim_q h_k^{3/2}$$

for all k large enough, hence $\alpha_{x_0}(t) \leq 3/2$. So let us find that sequence y_k. According to (65), if $q \equiv 0 \pmod 4$, by symmetry we can write

$$\begin{aligned} g(y) &= \frac{G(p,0,q)}{\sqrt{q}} \sum_{m \geq 0 \text{ odd}} \frac{e^{-2\pi i m^2 y}}{m^2} \left(e^{2\pi i p^{-1} \left(\frac{m_q+m}{2}\right)^2 \frac{1}{q}} + e^{2\pi i p^{-1} \left(\frac{m_q-m}{2}\right)^2 \frac{1}{q}} \right) \\ &= 2\, \frac{G(p,0,q)}{\sqrt{q}}\, e^{2\pi i p^{-1} \frac{m_q^2}{4q}} \sum_{m \geq 0 \text{ odd}} \frac{e^{-2\pi i m^2 (y - \frac{p^{-1}}{4q})}}{m^2} \cos\left(2\pi \frac{p^{-1} m_q}{2q} m\right). \end{aligned}$$

On the other hand, if $q \equiv 2 \pmod 4$, then

$$\begin{aligned} g(y) &= \frac{G(p,1,q)}{\sqrt{q}} \sum_{m \geq 0 \text{ odd}} \frac{e^{-2\pi i m^2 y}}{m^2} \\ &\quad \times \left(e^{2\pi i p^{-1} \left[\left(\frac{m_q+m-1}{2}\right)^2 + \frac{m_q+m-1}{2} \right] \frac{1}{q}} + e^{2\pi i p^{-1} \left[\left(\frac{m_q-m-1}{2}\right)^2 + \frac{m_q-m-1}{2} \right] \frac{1}{q}} \right) \\ &= 2\, \frac{G(p,1,q)}{\sqrt{q}}\, e^{2\pi i p^{-1} \frac{(m_q-1)^2 + 2(m_q-1)}{4q}} \sum_{m \geq 0 \text{ odd}} \frac{e^{-2\pi i m^2 (y - \frac{p^{-1}}{4q})}}{m^2} \cos\left(2\pi \frac{p^{-1} m_q}{2q} m\right). \end{aligned}$$

Choose the sequence $y_k = p^{-1}/(4q) + k$ for $k \in \mathbb{N}$. Then, since $x_q = |x_0 - m_q/q| = 0$ implies $x_0 = m_q/q$, but also $x_0 = P/Q$ in its reduced form, we get

$$|g(y_k)| \simeq \left| \sum_{m=0}^{\infty} \frac{\cos\left(\pi \frac{p^{-1} P}{Q} (2m+1)\right)}{(2m+1)^2} \right|, \quad \forall k \in \mathbb{N}. \tag{66}$$

Define the Fourier series

$$G(z) = \sum_{m=0}^{\infty} \frac{\cos((2m+1)\pi z)}{(2m+1)^2} = \frac{\pi^2}{8}(1 - |2z|) \quad z \in (-1, 1),$$

so that, after extending periodically to $\mathbb{R}$, in view of (66), we have $|g(y_n)| = |G(p^{-1}P/Q)|$ for all $n \in \mathbb{N}$. Observe that the only zeros of G are $(2m+1)/2$

for $m \in \mathbb{Z}$. We separate two cases again. If $q \equiv 0 \pmod 4$, by (65) m_q must be odd. Then $Qm_q = Pq$ implies $4 \mid Q$, hence both p^{-1} and P are odd. We deduce $p^{-1}P/Q \neq (2m+1)/2$ for any $m \in \mathbb{Z}$, because otherwise $p^{-1}P = (2m+1)Q/2$ for some m, so $p^{-1}P$ would be even. If $q \equiv 2 \pmod 4$, then m_q is even and $Q(m_q/2) = P(q/2)$ implies that Q is odd. Hence $p^{-1}P/Q \neq (2m+1)/2$ for any $m \in \mathbb{Z}$. In both cases, this implies that $|g(y_k)| = |G(p^{-1}P/Q)| \neq 0$ for all k, which is what we wanted to prove.

- If $x_q \neq 0$, according to (59) we get

$$\left| R_{x_0}\left(\frac{p}{q}+h\right) - R_{x_0}\left(\frac{p}{q}\right) + 2\pi i h \right| \simeq \left| (qh)^{3/2} \sum_{m\in\mathbb{Z}} \frac{G(p, m_q+m, q)}{\sqrt{q}} \frac{e^{-2\pi i \frac{(m-qx_q)^2}{4q^2h}}}{(m-qx_q)^2} + O\left(q^{7/2}h^{5/2}\right) \right| \tag{67}$$

because $0 < qx_q \leq 1/2$. If q is odd, we use (65) and the definition of f_q in (63) to write

$$\left| R_{x_0}\left(\frac{p}{q}+h\right) - R_{x_0}\left(\frac{p}{q}\right) + 2\pi i h \right| \simeq q^{3/2}h^{3/2} \left| f_q\left(\frac{1}{4q^2h}\right) + O\big(q^2h\big) \right|. \tag{68}$$

With the definition of y_k^q in Lemma B.4, choose the sequence $h_k = 1/(4q^2y_k^q)$ that tends to zero and for which $|f_q(1/(4q^2h_k^q))| = |f_q(y_k^q)| \geq 5$. This and (68) show that $\alpha_{x_0}(t) = 3/2$. When q is even, by (65), the sum in (67) only has either even or odd terms. The main term is $m = 0$ if even terms survive, and $m = 1$ if odd terms survive, and crude estimates in the error suffice to conclude. □

Acknowledgements L. Vega is thankful to S. Jaffard and S. Seuret for insightful conversations.

Funding V. Banica is partially supported by the Institut Universitaire de France, by the French ANR project SingFlows. D. Eceizabarrena is funded in part by the Simons Foundation Collaboration Grant on Wave Turbulence (Nahmod's award ID 651469) and by the American Mathematical Society and the Simons Foundation under an AMS-Simons Travel Grant for the period 2022-2024. A. Nahmod is funded in part by NSF DMS-2052740, NSF DMS-2101381 and the Simons Foundation Collaboration Grant on Wave Turbulence (Nahmod's award ID 651469). L. Vega is funded in part by MICINN (Spain) projects Severo Ochoa CEX2021-001142, and PID2021-126813NB-I00 (ERDF A way of making Europe), and by Eusko Jaurlaritza project IT1615-22 and BERC program.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

References

1. Anselmet, F., Gagne, Y., Hopfinger, E.J., Antonia, R.A.: High-order velocity structure functions in turbulent shear flows. J. Fluid Mech. **140**, 63–89 (1984)

2. Apolinário, G.B., Chevillard, L., Mourrat, J.C.: Dynamical fractional and multifractal fields. J. Stat. Phys. **186**, no.1, Paper No. 15, 35 pp. (2022)
3. Banica, V., Vega, L.: Evolution of polygonal lines by the binormal flow. Ann. PDE **6**(1), 53 (2020)
4. Banica, V., Vega, L.: Riemann's non-differentiable function and the binormal curvature flow. Arch. Ration. Mech. Anal. **244**(2), 501–540 (2022)
5. Beresnevich, V., Velani, S.: A mass transference principle and the Duffin–Schaeffer conjecture for Hausdorff measures. Ann. Math. (2) **164**(3), 971–992 (2006)
6. Berry, M.V.: Quantum fractals in boxes. J. Phys. A Math. Gen. **29**, 6617–6629 (1996)
7. Berry, M.V., Klein, S.: Integer, fractional and fractal Talbot effects. J. Mod. Opt. **43**(10), 2139–2164 (1996)
8. Boulton, L., Farmakis, G., Pelloni, B.: Beyond periodic revivals for linear dispersive PDEs. Proc. A. **477**(2251), 20210241 (2021)
9. Boulton, L., Farmakis, G., Pelloni, B.: The phenomenon of revivals on complex potential Schrödinger's equation. Z. Anal. Anwend. **43**(3), 401–416 (2024)
10. Bourgain, J.: Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. Geom. Funct. Anal. **3**(2), 107–156 (1993)
11. Broucke, F., Vindas, J.: The pointwise behavior of Riemann's function. J. Fractal Geom. **10**(3–4), 333–349 (2023)
12. Buckmaster, T., Vicol, V.: Convex integration and phenomenologies in turbulence. EMS Surv. Math. Sci. **6**(1–2), 173–263 (2019)
13. Chamizo, F., Córdoba, A.: Differentiability and dimension of some fractal Fourier series. Adv. Math. **142**(2), 335–354 (1999)
14. Chamizo, F., Ubis, A.: Some Fourier series with gaps. J. Anal. Math. **101**, 179–197 (2007)
15. Chamizo, F., Ubis, A.: Multifractal behavior of polynomial Fourier series. Adv. Math. **250**, 1–34 (2014)
16. Chen, G., Olver, P.J.: Dispersion of discontinuous periodic waves. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **469**(2149), 20120407 (2013)
17. Chen, G., Olver, P.J.: Numerical simulation of nonlinear dispersive quantization. Discrete Contin. Dyn. Syst. **34**(3), 991–1008 (2014)
18. Chousionis, V., Erdoğan, M.B., Tzirakis, N.: Fractal solutions of linear and nonlinear dispersive partial differential equations. Proc. Lond. Math. Soc. (3) **110**(3), 543–564 (2015)
19. Da Rios, L.S.: On the motion of an unbounded fluid with a vortex filament of any shape. Rend. Circ. Mat. Palermo **22**, 117–135 (1906)
20. de la Hoz, F., Vega, L.: Vortex filament equation for a regular polygon. Nonlinearity **27**(12), 3031–3057 (2014)
21. Duffin, R.J., Schaeffer, A.C.: Khintchine's problem in metric Diophantine approximation. Duke Math. J. **8**, 243–255 (1941)
22. Duistermaat, J.J.: Self-similarity of "Riemann's nondifferentiable function". Nieuw Arch. Wisk. (4) **9**(3), 303–337 (1991)
23. Eceizabarrena, D., Vilaça Da Rocha, V.: An analytical study of flatness and intermittency through Riemann's nondifferentiable functions. SIAM J. Math. Anal. **54**(3), 3575–3608 (2022)
24. Erdoğan, M.B., Tzirakis, N.: Talbot effect for the cubic non-linear Schrödinger equation on the torus. Math. Res. Lett. **20**(6), 1081–1090 (2013)
25. Falconer, K.: Fractal Geometry: Mathematical Foundations and Applications. Wiley, New York (2014)
26. Fontelos, M.A., Vega, L.: Evolution of viscous vortex filaments and desingularization of the Biot–Savart integral. arxiv: 2311.12246
27. Frisch, U.: Turbulence: The Legacy of A. N. Kolmogorov. Cambridge University Press, Cambridge (1995)
28. Gerver, J.: The differentiability of the Riemann function at certain rational multiples of π. Am. J. Math. **92**, 33–55 (1970)
29. Gerver, J.: More on the differentiability of the Riemann function. Am. J. Math. **93**, 33–41 (1971)
30. Hardy, G.H.: Weierstrass's non-differentiable function. Trans. Am. Math. Soc. **17**(3), 301–325 (1916)
31. Jaffard, S.: The spectrum of singularities of Riemann's function. Rev. Mat. Iberoamericana **12**(2), 441–460 (1996)
32. Jaffard, S.: Multifractal formalism for functions. I. Results valid for all functions. SIAM J. Math. Anal. **28**(4), 944–970 (1997)
33. Jerrard, R.L., Smets, D.: On the motion of a curve by its binormal curvature. J. Eur. Math. Soc. (JEMS) **17**(6), 1487–1515 (2015)

34. Kahane, J.-P.: Some Random Series of Functions, Second ed., Vol. 5 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1985)
35. Kapitanski, L., Rodnianski, I.: Does a quantum particle know the time? In: Emerging Applications of Number Theory (Minneapolis, MN, 1996), Vol. 109 of IMA Vol. Math. Appl. Springer, New York, pp. 355–371 (1999)
36. Khinchin, A.Y.: Continued Fractions. Dover Publications (1997). Reprint of the 1964 translation
37. Koukoulopoulos, D., Maynard, J.: On the Duffin–Schaeffer conjecture. Ann. Math. (2) **192**(1), 251–307 (2020)
38. Levi-Civita, T.: Teoremi di unicitá e di esistenza per le piccole oscillazioni di un filetto vorticoso prossimo alla forma circolare. R. C. Accad. Lincei **6**(15), 409–416 (1932)
39. Olver, P.J.: Dispersive quantization. Am. Math. Mon. **117**(7), 599–610 (2010)
40. Oskolkov, K.I., Chakhkiev, M.A.: On the “nondifferentiable” Riemann function and the Schrödinger equation. Tr. Mat. Inst. Steklova **269**, 193–203 (2010)
41. Pastor, C.: On the regularity of fractional integrals of modular forms. Trans. Am. Math. Soc. **372**(2), 829–857 (2019)
42. Ricard, G., Falcon, E.: Experimental evidence of random shock-wave intermittency. Phys. Rev. E **108**, 045106 (2023)
43. Rodnianski, I. Fractal solutions of the Schrödinger equation. In: Nonlinear PDE’s, Dynamics and Continuum Physics (South Hadley, MA, 1998: Vol. 255 of Contemp. Math. Amer. Math. Soc., Providence, pp. 181–187 (2000)
44. Seuret, S., Ubis, A.: Local L^2-regularity of Riemann’s Fourier series. Ann. Inst. Fourier (Grenoble) **67**(5), 2237–2264 (2017)
45. Smith, A.: The differentiability of Riemann’s functions. Proc. Am. Math. Soc. **34**, 463–468 (1972)
46. Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces, Vol. 32 of Princeton Mathematical Series. Princeton University Press, Princeton (1971)
47. Weierstrass, K.: Über continuirliche Functionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten Differentialquotienten besitzen. In: Mathematische Werke. II. Abhandlungen 2, pp. 71–74 (1995)