



Caloric functions and boundary regularity for the fractional Laplacian in Lipschitz open sets

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Abstract

We give Martin representation of nonnegative functions caloric with respect to the fractional Laplacian in Lipschitz open sets. The caloric functions are defined in terms of the mean value property for the space-time isotropic α -stable Lévy process. To derive the representation, we first establish the existence of the parabolic Martin kernel. This involves proving new boundary regularity results for both the fractional heat equation and the fractional Poisson equation with Dirichlet exterior conditions. Specifically, we demonstrate that the ratio of the solution and the Green function is Hölder continuous up to the boundary.

Mathematics Subject Classification Primary 35S16 · 60J50 · 35C15

1 Introduction

Let $0 < \alpha < 2$ and $d \geq 2$. For $u \in C_b^2(\mathbb{R}^d)$, define

$$(-\Delta)^{\alpha/2} u(x) := \lim_{\varepsilon \rightarrow 0^+} \int_{|x-y| > \varepsilon} (u(x) - u(y)) v(x, y) dy, \quad x \in \mathbb{R}^d,$$

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where $v(x, y) = c_{d,\alpha}|x - y|^{-d-\alpha}$, and denote $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. Let $D \subset \mathbb{R}^d$ be a nonempty bounded open Lipschitz set with localization radius $r_0 \in (0, \infty)$ and Lipschitz constant $\lambda \in (0, \infty)$. One of our goals is to investigate the structure of nonnegative solutions to the initial-boundary value problem for the fractional heat equation:

$$\begin{cases} \partial_t u(t, x) = \Delta^{\alpha/2} u(t, x), & t \in (0, T), x \in D, \\ u(t, x) = g(t, x), & t \in (0, T), x \in D^c, \\ u(0, x) = u_0(x), & x \in D. \end{cases} \quad (1.1)$$

Solutions to (1.1) are called *caloric functions*. They are defined in terms of the mean value property for the space-time α -stable Lévy process; we refer to Sect. 5 for details and connections with the classical notion of solution to (1.1). As shown by Bogdan [13] (see also Abatangelo [1] and Bogdan, Kulczycki, and Kwaśnicki [23]), nonnegative *harmonic functions* for the fractional Laplacian on D can be decomposed into a *regular* part, which can be recovered from the exterior values, and a *singular* part, vanishing outside of D and represented as an integral with respect to a finite measure on ∂D of the (elliptic) Martin kernel for D and the fractional Laplacian. Our ultimate goal, which we complete in Sect. 6, is to give a counterpart of this decomposition for nonnegative caloric functions. In particular, in Theorems 6.3 and 6.4, we show that every nonnegative *singular* caloric function, i.e., such that $u_0 = g = 0$, can be expressed as integral with respect to the *parabolic Martin kernel* $\eta_{t,Q}(x)$:

$$u(t, x) = \int_{[0,t)} \int_{\partial D} \eta_{t-s,Q}(x) \mu(dQ ds), \quad x \in D, t \in (0, T), \quad (1.2)$$

with a unique finite Borel measure μ on $\partial D \times [0, T)$.

Singular caloric functions were recently represented by Chan, Gómez-Castro, and Vázquez [28] for domains more regular than Lipschitz, such as $C^{1,1}$ domains. While the authors of [28] address more general operators than our Dirichlet, or *restricted*, fractional Laplacian, they do so by assuming that the (elliptic) Green function exhibits uniform power-type decay at the boundary. Since for Lipschitz open sets, the behavior of the Dirichlet Green function of the fractional Laplacian is more nuanced (see Jakubowski [48]), the results of [28] are not applicable in our setting. Another difference between [28] and our work is that we do not require any specific regularity or integrability conditions for caloric functions, except for assuming nonnegativity and finiteness of integrals in the mean value property. Note that in our paper, the *boundary data* may be a measure; for example $\mu = \delta_{Q_0} \otimes \delta_0$ represents a fixed parabolic Martin kernel $\eta_{t,Q_0}(x)$. Furthermore, in Theorem 6.5, we demonstrate that even without a prescribed initial condition, $u(\varepsilon, \cdot)$ converges to a measure μ_0 on D as $\varepsilon \rightarrow 0^+$. This measure finitely integrates the function $x \mapsto \mathbb{P}^x(\tau_D > 1)$ on D (see below), similar to the condition used in [28]. As a consequence, for general nonnegative caloric functions, we get the following representation:

$$\begin{aligned}
 u(t, x) &= P_t^D \mu_0(x) + \int_{[0,t)} \int_{\partial D} \eta_{t-s, Q}(x) \mu(dQ ds) \\
 &+ \int_0^t \int_{D^c} g(s, z) J^D(t, x, s, z) dz ds,
 \end{aligned}
 \tag{1.3}$$

which is our first main result. Here, P_t^D is the Dirichlet heat semigroup of D for $\Delta^{\alpha/2}$, and J^D is the so-called lateral Poisson kernel, see below for details. To obtain the representation (1.3), we prove several new boundary regularity results for the fractional Laplacian in Lipschitz sets, namely Theorems 1.2 and 1.4 and Corollaries 1.3 and 1.5. They are of independent interest and may be considered the second main contribution of the paper.

To prove the results, we utilize some basic probabilistic potential theory. Let $X = (X_t)_{t \geq 0}$ be the isotropic α -stable Lévy process in \mathbb{R}^d , see, e.g., Sato [60]. For $x \in \mathbb{R}^d$, we denote by \mathbb{P}^x and \mathbb{E}^x the probability and the expectation of the process starting from x , and $\mathbb{P} := \mathbb{P}^0$, $\mathbb{E} := \mathbb{E}^0$. We then consider

$$\tau_D := \inf\{s > 0 \mid X_s \notin D\},
 \tag{1.4}$$

the first exit time of the process X from D , and the survival probability:

$$\mathbb{P}^x(\tau_D > t) = \int_D p_t^D(x, y) dy,$$

where p_t^D is the Dirichlet heat kernel of $\Delta^{\alpha/2}$ in D (for details see Sect. 2). Furthermore, let G_D be the (elliptic) Green function of $\Delta^{\alpha/2}$ in D . We fix arbitrary $t_0 \in (0, \infty)$ and $x_0 \in D$, reference time and point.

There are several reasonable ways to define the parabolic Martin kernel in Lipschitz open sets. The general idea is to normalize p_t^D by constructing a ratio that converges to a nontrivial limit at the boundary of D . Each of the following expressions will be called a parabolic Martin kernel:

$$\eta_{t, Q}(x) := \lim_{D \ni y \rightarrow Q} \frac{p_t^D(x, y)}{\mathbb{P}^y(\tau_D > 1)},
 \tag{1.5}$$

$$\eta_{t, Q}^{x_0}(x) := \lim_{D \ni y \rightarrow Q} \frac{p_t^D(x, y)}{G_D(x_0, y)},
 \tag{1.6}$$

$$\tilde{\eta}_{t, Q}(x) := \lim_{D \ni y \rightarrow Q} \frac{p_t^D(x, y)}{p_{t_0}^D(x_0, y)}.
 \tag{1.7}$$

Here, $t > 0$, $x \in D$, and $Q \in \partial D$. We recall that the heat kernel plays the role of the Green function for the heat equation, see, e.g., Doob [36], Watson [63], or Bogdan and Hansen [21, Subsection 9.4]. This might indicate that $\tilde{\eta}$ is the canonical parabolic Martin kernel, however η and η^{x_0} offer a more explicit description of the boundary behavior of p_t^D and are more convenient to handle via the existing elliptic theory. If D is $C^{1,1}$, then one can also normalize p_t^D by using $\delta_D(y)^{\alpha/2}$ with

$$\delta_D(y) := \inf\{|x - y| : x \in \partial D\},$$

see Chen, Kim, and Song [30]; see also [28]. The next result may be considered as a consequence and a follow-up of the approximate factorization (2.6) of p_t^D by Bogdan, Grzywny, and Ryznar [19].

Theorem 1.1 *Recall that $D \subset \mathbb{R}^d$ is open, bounded, and Lipschitz with localization radius r_0 , Lipschitz constant λ , and reference point x_0 and time t_0 . Then, the limits in (1.5), (1.6), and (1.7) exist for all $t > 0$, $x \in D$, and $Q \in \partial D$. Furthermore, they are finite, strictly positive, continuous in t and x , and*

$$\eta_{1,Q}(x) \approx \mathbb{P}^x(\tau_D > 1), \quad x \in D, \quad (1.8)$$

$$\eta_{t+s,Q}(x) = \int_D \eta_{t,Q}(z) p_s^D(z, x) dz, \quad 0 < s, t < \infty, \quad x \in D. \quad (1.9)$$

The formula (1.8) is a sample of the more general estimates for η which we give in Corollary 3.6 below. The proofs of Theorem 1.1 and other results of this section are given later on. Here we note that the mere existence of a Martin-type kernel is a deep *boundary regularity*¹ result. In the elliptic setting, for G_D , it is usually proved using the boundary Harnack principle. For solutions of parabolic equations like (1.1), we may utilize the elliptic results after expressing the numerators and denominators in (1.5), (1.6), and (1.7) as Green potentials. This is precisely our approach—it was used before by Bogdan, Palmowski, and Wang [24] for Lipschitz cones at the vertex. We further remark that an early version of proof of Theorem 1.1 for (1.5) has appeared in the PhD thesis of the first-named author [4].

To obtain the representation of nonnegative caloric functions, we refine Theorem 1.1 to ensure a uniform rate of convergence in (1.5). To this end, we extend the spatial domain of the functions in (1.5), (1.6), (1.7), by additionally defining, for $t > 0$, $x \in D$, and $y \in D$,

$$\eta_{t,y}^{x_0}(x) := \frac{p_t^D(x, y)}{G_D(x_0, y)}, \quad \eta_{t,y}(x) := \frac{p_t^D(x, y)}{\mathbb{P}^y(\tau_D > 1)}, \quad \tilde{\eta}_{t,y}(x) := \frac{p_t^D(x, y)}{p_{t_0}^D(x_0, y)}.$$

Theorem 1.2 *Recall that $D \subset \mathbb{R}^d$ is open, bounded, and Lipschitz with localization radius r_0 , Lipschitz constant λ , and reference point x_0 and time t_0 . Fix $r_1 \in (0, \infty)$ and $0 < T_1 < T_2 < \infty$. For $x \in D$ and $t \in [T_1, T_2]$, η , η^{x_0} , and $\tilde{\eta}$ are Hölder continuous in y on \overline{D} , \overline{D} , and $\overline{D} \setminus B(x_0, r_1)$, respectively. The Hölder exponents and constants depend only on $d, \alpha, \underline{D}, T_1, T_2$ (for η^{x_0} also on x_0, r_1 ; for $\tilde{\eta}$ also on t_0, x_0).*

Here and below, we say constants depend on \underline{D} if they depend only on r_0, λ , and an upper bound for $\text{diam}(D)$. Theorem 1.2 yields the following boundary regularity for the Dirichlet heat semigroup

$$P_t^D f(y) := \int_D p_t^D(x, y) f(x) dx.$$

¹ Here and below, the term signals relative regularity, i.e., continuity or even Hölder continuity of *ratios* at the boundary.

Corollary 1.3 Fix $r_1 \in (0, \infty)$. Let $u_0 \in L^1(D)$, $0 < T_1 < T_2 < \infty$, and $t \in [T_1, T_2]$. Then, the functions

$$\frac{P_t^D u_0(y)}{G_D(x_0, y)}, \quad \frac{P_t^D u_0(y)}{\mathbb{P}^y(\tau_D > 1)}, \quad \frac{P_{t_0}^D u_0(y)}{p_{t_0}^D(x_0, y)}$$

are Hölder continuous in y on $\overline{D} \setminus B(x_0, r_1)$, \overline{D} , and \overline{D} respectively. The Hölder exponents and constants depend only on $d, \alpha, \underline{D}, T_1, T_2$ (and t_0, x_0, r_1 , where relevant).

Theorem 1.2 and Corollary 1.3 can be viewed as analogues of the boundary regularity result for $C^{1,1}$ open sets by Fernández-Real and Ros-Oton [39, Theorem 1.1 (b)], see also [40]. However, such regularity results for nonlocal equations are quite scarce for Lipschitz and less regular domains. That is, much is known about harmonic functions [12, 23, 48], but the first result for the Poisson equation ($\Delta^{\alpha/2}u = -f$) appeared only recently but in the PDE literature the first results for the Dirichlet problem for the Poisson equation ($\Delta^{\alpha/2}u = -f$) appeared only recently in the paper of Lian, Zhang, Li, and Hong [64, Theorem 3.11]; similar results were implicit in the probability literature, at least for bounded f , see [48, Theorem 2, Lemma 17] and [12, Lemma 3]. Other related works are by Ding and Zhang [65] and Borthagaray and Nochetto [27], but we note that [27, 64, 65] do not treat the relative boundary regularity, which is a stronger property. For regularity results in $C^{1,\gamma}$ domains with $\gamma \in (0, 1)$, see, e.g., Abels and Grubb [2] or Dong and Ryu [35] and the references therein.

Incidentally, our proof of Theorem 1.2 unveils the following integral estimate for the Green function.

Theorem 1.4 Recall that $D \subset \mathbb{R}^d$ is open, bounded, and Lipschitz with localization radius r_0 , Lipschitz constant λ , and reference point x_0 . Let $r > 0$. There exists $p_0 = p_0(d, \alpha, \underline{D}, r) > 1$ and constants $C \in (0, \infty)$ and $\sigma \in (0, 1]$ depending only on $d, \alpha, \underline{D}, p, r$, such that for all $p \in [1, p_0)$,

$$\left\| \frac{G_D(y, \cdot)}{G_D(x_0, y)} - \frac{G_D(y', \cdot)}{G_D(x_0, y')} \right\|_{L^p(D)} \leq C|y - y'|^\sigma, \quad y, y' \in \overline{D} \setminus B(x_0, r).$$

Recall that Green potentials $v(x) = G_D f(x) := \int_D G_D(x, y) f(y) dy$ solve the Dirichlet problem for the Poisson equation:

$$\begin{cases} (-\Delta)^{\alpha/2} v(x) = f(x), & x \in D, \\ v(x) = 0, & x \in D^c, \end{cases}$$

see [15]. Theorem 1.4 yields a boundary, or relative, Hölder estimate, as follows.

Corollary 1.5 Let $p > p_0/(p_0 - 1)$ and let $f \in L^p(D)$. Then, $G_D f(y)/G_D(x_0, y)$ is Hölder continuous in $\overline{D} \setminus B(x_0, r)$ with Hölder constant and exponent depending only on $d, \alpha, \underline{D}, p, r$ and $\|f\|_{L^p(D)}$.

A similar result for $C^{1,1}$ domains was obtained by Ros-Oton and Serra [57] with explicit and sharp Hölder exponents. Our regularity results are far from being sharp in terms of p_0 and σ , but this is to be expected for Lipschitz sets—some insight about precise boundary behavior can be gained from the results on cones [6, 33, 55] or numerical considerations [38], but we do not pursue this point here. Note that Corollary 1.5 implies that the Green potentials have the same decay rate at the boundary as harmonic functions, without restrictions on the Lipschitz constant of D . This stands in sharp contrast to the case of local operators, where such comparability is known to be false if the Lipschitz constant of D is too large, see, e.g., [3, 59].

Let us add a few general comments. The mean-value property for fractional caloric functions is important for our development. It was considered before, e.g., by Chen and Kumagai [31]. Here we focus on the mean-value property in cylinders, which seems adequate for the initial-exterior problem (1.1). The advantage of the approach is that from the Ikeda–Watanabe formula we obtain a semi-explicit formula for the Poisson kernel. We also have the following stochastic interpretation: if u satisfies the mean-value property $(0, T) \times D$, then $u(t, x)$ can be recovered from the space-time isotropic α -stable process $s \mapsto (t - s, X_s + x)$, which starts from (t, x) at time $s = 0$, by computing the expectation of $u(t - s, X_s + x)$ at the place of the first exit of the process from $(0, T) \times D$. The exit can occur when $x + X_s$ leaves D before time t —in which case the exterior conditions affect the expectation—or when the time coordinate $t - s$ reaches 0—then the initial condition comes into play. Singular caloric functions start to appear once we assume that the mean-value property is satisfied only on $(0, T) \times U$ for all open (relatively compact sets) $U \subset\subset D$. We refer to the book of Freidlin [41, Theorem 2.3] for a counterpart of this theory for local operators.

With a view toward applications in probability, we note that the existence of the limit (1.5) indicates how the isotropic α -stable process in D , conditioned on surviving at least time 1, behaves near the boundary of D . More precisely, it implies the existence of a “Yaglom limit”, see Theorem 3.7 below. Thanks to (1.9), $\eta_{t,Q}(y)$ may be understood as the *entrance law* for the killed process from Q into D , see Blumenthal [9]. This was used in [45, 53] to describe the behavior of the process started from a point on the boundary, e.g., the apex of a cone. Furthermore, the boundary behavior of the heat kernel yields a measure which represents the probability distribution of a rescaled process conditioned on non-extinction.

Let us now present an outline of the proofs and methods in this paper. In order to prove Theorem 1.1 we obtain an explicit representation of the survival probability as a Green potential and we show that it behaves like $G_D(x_0, \cdot)$ at the boundary. Then we *approximate* p_t^D by Green potentials and obtain the limit in (1.5) with the help of Prokhorov theorem. To this end, we utilize the uniform integrability of ratios of Green functions. The proof of Theorem 1.4 consists in splitting the integral into one region where the boundary Harnack principle can be applied, and another region where we use a technical interior regularity argument adapted to possible singularities of the Green function. In order to prove Theorem 1.2, we *represent* p_t^D as a Green potential and we apply Theorem 1.4. We make use of the spectral theory to show that p_t^D has regularity necessary for the proof; some ideas here were inspired by [28]. The boundary measure in the representation of singular caloric functions is obtained from an approximating sequence constructed via the lateral Poisson kernel. Our construction is quite different

than the one in [28], in particular it does not use the inhomogeneous fractional heat equation. Needless to say, our results point out directions of development for other nonlocal operators and various classes of open sets.

The structure of the rest of the paper is as follows. Section 2 contains basic definitions and facts. In Sect. 3, we prove Theorem 1.1 and its consequences. In Sect. 4, we prove Theorems 1.4 and 1.2. In Sect. 5, we introduce the caloric functions and the parabolic Poisson kernel and study their properties. Then in Sect. 6, we discuss the representation of nonnegative parabolic functions in Lipschitz cylinders.

2 Preliminaries

We assume throughout that the considered sets, measures, and functions are Borel. For nonnegative functions f and g , we write $f(x) \lesssim g(x), x \in A$, if there is a number $C \in (0, \infty)$, referred to as *constant*, such that $f(x) \leq Cg(x), x \in A$. We write $C = C(d, \alpha, \dots)$ if C is a *constant* depending only on d, α, \dots , that is, C may be considered as a function of the parameters d, α, \dots , but not of $x \in A$. We say that f and g are *comparable* and write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$ (this notation was used in Sect. 1). We often use $:=$ and occasionally employ *cursive* for definitions.

2.1 Geometry

Let $B(x, r) := \{y \in \mathbb{R}^d : |y - x| < r\}$. Recall that D is a Lipschitz open set with constant $\lambda \in (0, \infty)$ and localization radius $r_0 \in (0, \infty)$. This means that for every $Q \in \partial D$ there is a rigid motion R_Q and a Lipschitz function $f_Q: \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ with Lipschitz constant λ , such that $R_Q(Q) = 0$ and $D \cap B(Q, r_0) = R_Q^{-1}(B(0, r_0) \cap \{y_d > f_Q(y_1, \dots, y_{d-1})\})$. For $r > 0$, we let

$$D_r := \{x \in D : \delta_D(x) > 1/r\}. \tag{2.1}$$

Let $\kappa = 1/(4\sqrt{1 + \lambda^2})$. Of course, $\kappa < 1$. For $y \in \overline{D}$ and $r > 0$, we define

$$\mathcal{A}_r(y) := \begin{cases} \{A \in D : B(A, \kappa r) \subseteq D \cap B(y, r)\}, & r \leq r_0/2, \\ \{x_0\}, & r > r_0/2. \end{cases}$$

Lemma 2.1 *If D is Lipschitz, then $\mathcal{A}_r(y)$ is nonempty for every $r > 0$ and $y \in \overline{D}$.*

Proof Obviously, it suffices to consider $r \leq r_0/2$. For $y \in \partial D$ the statement is true even with κ replaced by $2\kappa = 1/(2\sqrt{1 + \lambda^2})$. Indeed, if we consider the *interior right-circular cone* with angle $\text{arccot}(\lambda)$ and vertex at y , then the point $A \in D$ on the axis of the cone such that $|A - y| = r$ satisfies $B(A, r/(2\sqrt{1 + \lambda^2})) \subseteq D \cap B(y, r)$. If $y \in D$ and $y \notin \mathcal{A}_r(y)$, then there is $Q \in \partial D$ with $|y - Q| = \delta_D(y) < \kappa r$ and $A \in D$ with

$$B(A, r/(4\sqrt{1 + \lambda^2})) \subseteq D \cap B(Q, r/2) \subseteq D \cap B(y, r).$$

□

Thus, by definition (see, e.g., [19]), D is κ -fat at each scale $r \in (0, r_0/2)$. We will denote by $A_r(y)$ an arbitrary point in $\mathcal{A}_r(y)$. The actual choice is unimportant in the sense that if $A_1, A_2 \in \mathcal{A}_r(y)$ and $u \geq 0$ is harmonic in $B(A_1, \kappa r)$ and $B(A_2, \kappa r)$ —see Definition 2.4 below—then we have the comparability $C^{-1}u(A_1) \leq u(A_2) \leq Cu(A_1)$, where $C = C(d, \alpha)$; see the Harnack inequality in [14, Lemma 1], see also [15, Lemma 4.4].

For $x, y \in D$, let $r_{x,y} := |x - y| \vee \delta_D(x) \vee \delta_D(y)$. Let $\mathcal{A}_{x,y} := \{x_0\}$ if $r_{x,y} > r_0/32$, and otherwise let

$$\mathcal{A}_{x,y} := \{A \in D : B(A, \kappa r_{x,y}) \subset D \cap B(x, 3r_{x,y}) \cap B(y, 3r_{x,y})\}.$$

Then, $\mathcal{A}_{x,y}$ is nonempty, see [48]. We denote by $A_{x,y}$ any point in $\mathcal{A}_{x,y}$. The actual choice is unimportant in the sense that under suitable assumptions on functions $u \geq 0$, there exists $C = C(d, \alpha, \underline{D})$ such that for all $A_1, A_2 \in \mathcal{A}_{x,y}$, $C^{-1}u(A_1) \leq u(A_2) \leq Cu(A_1)$. See Remark 2.2, following (2.10).

2.2 Potential theory

As stated in the introduction, we denote by (X_t, \mathbb{P}^x) the standard rotation invariant α -stable Lévy process in \mathbb{R}^d . The process is determined by the jump measure with density function

$$\nu(y) = \frac{2^\alpha \Gamma((d + \alpha)/2)}{\pi^{d/2} |\Gamma(-\alpha/2)|} |y|^{-d-\alpha} =: c_{d,\alpha} |y|^{-d-\alpha}, \quad y \in \mathbb{R}^d.$$

It is a process with independent and stationary increments and characteristic function $\mathbb{E}^x e^{i(\xi, X_t - x)} = e^{-t|\xi|^\alpha}$, $t > 0$, $x, \xi \in \mathbb{R}^d$. It is strong Markov with the following time-homogeneous transition probability

$$P_t(x, A) := \int_A p_t(x, y) dy, \quad t > 0, \quad x \in \mathbb{R}^d, \quad A \subseteq \mathbb{R}^d.$$

Here $p_t(x, y) := p_t(x - y)$ and p_t is the smooth real-valued function on \mathbb{R}^d with the Fourier transform:

$$\int_{\mathbb{R}^d} p_t(x) e^{i(x, \xi)} dx = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d. \quad (2.2)$$

The associated semigroup of operators acts on, e.g., $u \in C_0(\mathbb{R}^d)$ as follows:

$$P_t u(x) := \int_{\mathbb{R}^d} u(y) p_t(x, y) dy, \quad x \in \mathbb{R}^d, \quad t \geq 0.$$

We have the following scaling property as a consequence of (2.2):

$$p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha}x), \quad x \in \mathbb{R}^d, \quad t > 0. \tag{2.3}$$

Furthermore, there exists a constant c such that

$$c^{-1} \left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right) \leq p_t(x) \leq c \left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right), \quad x \in \mathbb{R}^d, \quad t > 0,$$

see, e.g., [11, 26]. Thus, in short,

$$p_t(x) \approx t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}, \quad x \in \mathbb{R}^d, \quad t > 0. \tag{2.4}$$

Recall that τ_D is the first exit time from D defined in (1.4). Since D is bounded, then $\tau_D < \infty$ almost surely, see, e.g., Pruitt [56]. The Dirichlet heat kernel $p_t^D(x, y)$ of D is defined by Hunt’s formula:

$$p_t^D(x, y) = p_t(x, y) - \mathbb{E}^x [p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t], \tag{2.5}$$

where $x, y \in \mathbb{R}^d$ and $t > 0$. Here, as usual,

$$\mathbb{E}^x [p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t] := \int_{\{\tau_D < t\}} p_{t-\tau_D}(X_{\tau_D}, y) d\mathbb{P}^x.$$

It is well known that $p_t^D(x, y)$ is jointly continuous, see Appendix A for more regularity properties. Since D is Lipschitz, it satisfies the exterior cone condition. Therefore, $\mathbb{P}^x(\tau_D = 0) = 1$ for all $x \in D^c$ by Blumenthal’s zero–one law. In particular $p_t^D(x, y) = 0$ when x or y are outside of D . For bounded or nonnegative functions f we have

$$P_t^D f(x) = \int_{\mathbb{R}^d} f(y) p_t^D(x, y) dy = \mathbb{E}^x [f(X_t); \tau_D > t],$$

see [32, Section 2]. We also note that

$$0 \leq p_t^D(x, y) = p_t^D(y, x) \leq p_t(y - x)$$

and p_t^D satisfies the Chapman–Kolmogorov equations:

$$\int p_s^D(x, y) p_t^D(y, z) dy = p_{t+s}^D(x, z), \quad s, t > 0, \quad x, z \in \mathbb{R}^d,$$

see [17, 30]. The following scaling property follows from (2.3),

$$p_t^D(x, y) = t^{-d/\alpha} p_1^{t^{-1/\alpha}D}(t^{-1/\alpha}x, t^{-1/\alpha}y), \quad x, y \in \mathbb{R}^d, \quad t > 0.$$

By [19, Theorem 1], for every $T > 0$ we have the *approximate factorization*:

$$p_t^D(x, y) \approx \mathbb{P}^x(\tau_D > t)p_t(x, y)\mathbb{P}^y(\tau_D > t), \quad x, y \in D, \quad t \in (0, T). \tag{2.6}$$

If D is (open, bounded, and) $C^{1,1}$, then the (2.6) takes on a more explicit form [30]:

$$p_t^D(x, y) \approx \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)p_t(x, y)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right), \quad x, y \in D, \quad t \in (0, T). \tag{2.7}$$

We also recall the large time estimates. Let $\lambda_1 = \lambda_1(D) > 0$ be the first eigenvalue and φ_1 the first eigenfunction of the Dirichlet fractional Laplacian on D , see Sect. 2.3 below for more details. By the intrinsic ultracontractivity due to Kulczycki [49], for every $T > 0$ we have

$$p_t^D(x, y) \approx e^{-\lambda_1 t}\varphi_1(x)\varphi_1(y), \quad x, y \in D, \quad t \in (T, \infty). \tag{2.8}$$

If D is (open, bounded, and) $C^{1,1}$, then we even have

$$p_t^D(x, y) \approx e^{-\lambda_1 t}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}, \quad x, y \in D, \quad t \in (T, \infty), \tag{2.9}$$

see [30, Theorem 1.1 (ii)]. We define the *killing intensity* of X on D as

$$\kappa_D(z) := \int_{D^c} v(z - y) dy, \quad z \in D.$$

By [60, Theorem 31.5], $\Delta^{\alpha/2}$ coincides with the generator of X_t for the class $C_c^2(\mathbb{R}^d)$ of real-valued twice continuously differentiable functions compactly supported in \mathbb{R}^d .

The *Green function* of D is given by the formula:

$$G_D(x, y) := \int_0^\infty p_t^D(x, y) dt, \quad x, y \in \mathbb{R}^d.$$

In particular, $G_D(x, y) = 0$ if either $x \in D^c$ or $y \in D^c$. We note that G_D is finite for all $x \neq y$ and by (2.5), $G_D(x, y) \leq G_{\mathbb{R}^d}(x, y) = c|x - y|^{\alpha-d}$. For further reference, we recall the Green function estimates of Jakubowski [48, Theorem 1]: If we let

$$\Phi(x) := G_D(x_0, x) \wedge 1,$$

then there exists $C(d, \alpha, D) > 0$ such that

$$C^{-1}|x - y|^{\alpha-d} \frac{\Phi(x)\Phi(y)}{\Phi(A_{x,y})^2} \leq G_D(x, y) \leq C|x - y|^{\alpha-d} \frac{\Phi(x)\Phi(y)}{\Phi(A_{x,y})^2}, \quad x, y \in D, \tag{2.10}$$

see Sect. 2.1 for notation and the following remark.

Remark 2.2 We note that if $A_1, A_2 \in \mathcal{A}_{x,y}$, then $\Phi(A_1) \approx \Phi(A_2)$; see [48, Lemma 13]. We also note that [48] uses an extra reference point x_1 to define $A_{x,y}$ for $r_{x,y} \geq r_0/32$, but the resulting values of $\Phi(A_{x,y})$ are trivially comparable in both settings. In particular, (2.10) remains true in the present (simplified) setting.

Remark 2.3 It is implicit in (2.6) and (2.8) that $\varphi_1(y) \approx \mathbb{P}^y(\tau_D > 1)$, $y \in D$. Furthermore, by [19, Theorem 2], $\mathbb{P}^y(\tau_D > 1) \approx \mathbb{E}^y \tau_D$, $y \in D$, and, by [48, Lemma 17], $\mathbb{E}^y \tau_D \approx \Phi(y)$, $y \in D$. Therefore,

$$\varphi_1(y) \approx \mathbb{P}^y(\tau_D > 1) \approx \mathbb{E}^y \tau_D \approx \Phi(y), \quad y \in D. \tag{2.11}$$

In our proofs, we mostly use the survival probability and Φ , but we also refer to results stated in terms of φ_1 and the expected exit time.

We define the Green operator (or Green potential)

$$(G_D f)(x) := \int_D G_D(x, y) f(y) dy, \quad x \in \mathbb{R}^d,$$

for integrable or nonnegative functions f . For $f \in L^1(D)$, the function $u := G_D f$ is a distributional solution of $(-\Delta)^\alpha u = f$ in D , see [15, Proposition 3.13].

Definition 2.4 Let $u \geq 0$ be a Borel measurable function on \mathbb{R}^d .

- We say that u is α -harmonic in an open set $D \subseteq \mathbb{R}^d$ if for every open (relatively compact) $B \subset\subset D$,

$$u(x) = \mathbb{E}^x u(X_{\tau_B}) < \infty, \quad x \in B.$$

- We say that u is regular α -harmonic in $D \subset \mathbb{R}^d$ if

$$u(x) = \mathbb{E}^x u(X_{\tau_D}) < \infty, \quad x \in D.$$

- We say that u is singular α -harmonic in $D \subset \mathbb{R}^d$, if u is α -harmonic in D and $u = 0$ on D^c .

We will often write ‘harmonic’ instead of ‘ α -harmonic’. Since $\tau_B \leq \tau_D$ for $B \subset D$, by the strong Markov property it follows that regular harmonic functions are harmonic. Also by the strong Markov property, $G_D(\cdot, y)$ is harmonic in $D \setminus \{y\}$, see [32, Theorem 2.5] or [50, (2.1)].

For $x \in \mathbb{R}^d$, the \mathbb{P}^x -distribution of X_{τ_D} is called the α -harmonic measure, denoted by ω_D^x . This measure is concentrated on D^c and for u regular harmonic in D , we have

$$u(x) = \int_{D^c} u(z) \omega_D^x(dz), \quad x \in D.$$

The α -harmonic measure of a Lipschitz open set is absolutely continuous with respect to the Lebesgue measure. Its density function is given by the *Poisson kernel*:

$$P_D(x, z) := \int_D G_D(x, y)v(y, z) dy, \quad x \in D, z \in D^c, \quad (2.12)$$

see [12, Lemma 6]. Therefore, for every regular harmonic u we have the representation

$$u(x) = \int_{D^c} P_D(x, z)u(z) dz, \quad x \in D.$$

We also recall the Ikeda–Watanabe formula from [47]:

$$\mathbb{P}^x[\tau_D \in I, X_{\tau_D-} \in A, X_{\tau_D} \in B] = \int_I \int_B \int_A v(y, z)p_u^D(x, dy) dz du, \quad (2.13)$$

where $I \subset (0, \infty)$, $A \subset D$, and $B \subset (\overline{D})^c$. See also [7, Lemma 1], [12], [25, (4.13)], or [62, Theorem 2.4].

Recall that $x_0 \in D$ is an arbitrary but fixed (reference) point. We define the *Martin kernel*, $M_D^{x_0}(y, Q)$ as follows: for every $Q \in \partial D$ and $y \in D$ we let

$$M_D^{x_0}(y, Q) = \lim_{D \ni x \rightarrow Q} \frac{G_D(x, y)}{G_D(x, x_0)}. \quad (2.14)$$

In [13, Lemma 6] it is shown that the Martin kernel exists, the mapping $(y, Q) \mapsto M_D^{x_0}(y, Q)$ is continuous on $D \times \partial D$, and for every $Q \in \partial D$ the function $M_D^{x_0}(\cdot, Q)$ is singular α -harmonic in D .

2.3 Auxiliary results on P_t^D and its spectral decomposition

We recall that the operators P_t^D are compact on $L^2(D)$, see, e.g., [16, Chapter 4]. Therefore there exist a nondecreasing sequence of nonnegative numbers λ_n diverging to infinity and an orthonormal sequence of functions $\varphi_n \in C_0(D)$ such that for every $\phi \in L^2(D)$, we have

$$P_t^D \phi(x) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \langle \phi, \varphi_n \rangle \varphi_n(x) \quad (2.15)$$

and

$$p_t^D(x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y), \quad x, y \in D, t > 0. \quad (2.16)$$

The fractional Weyl bounds [10, 42] read

$$\lambda_n \approx n^{\alpha/d}. \quad (2.17)$$

Note that $P_t^D \varphi_n(x) = e^{-\lambda_n t} \varphi_n(x)$ for all $x \in D$. Therefore,

$$G_D \varphi_n(x) = \int_0^\infty P_t^D \varphi_n(x) dt = \lambda_n^{-1} \varphi_n(x), \quad x \in D. \tag{2.18}$$

By iterating (2.18) and using the regularity results for the fractional Laplacian [44, 58], we find that φ_n are smooth in D . Furthermore, by [39, Proposition 3.1], there exist $C > 0$ and $w \geq 1$, such that

$$\|\varphi_n\|_\infty \leq C \lambda_n^{w-1}, \quad n \in \mathbb{N}. \tag{2.19}$$

We say that ϕ belongs to $D(L^D)$, the domain of the L^2 -generator of P_t^D , if the following limit exists in L^2 :

$$L^D \phi := \lim_{t \rightarrow 0^+} \frac{P_t^D \phi - \phi}{t}.$$

Furthermore, if the pointwise limit exists for a function ϕ and some $x \in D$, we denote it as $L^D \phi(x)$.

Lemma 2.5 (1) We have $\varphi_n \in D(L^D)$ and $L^D \varphi_n(x) = -\lambda_n \varphi_n(x)$ for all $x \in D$.
 (2) We have

$$F := \{ \phi \in L^2(D) : \sum_{n=1}^\infty \lambda_n^2 |\langle \phi, \varphi_n \rangle|^2 < \infty \} \subseteq D(L^D),$$

and for each $\phi \in F$,

$$L^D \phi = \sum_{n=1}^\infty \lambda_n \langle \phi, \varphi_n \rangle \varphi_n.$$

- (3) For every $y \in D$ and $t > 0$, $p_t^D(\cdot, y) \in F$.
- (4) For every $x, y \in D$, we have $L_x^D p_t^D(x, y) = \Delta_x^{\alpha/2} p_t^D(x, y)$.

Proof Statements (1) and (2) follow quite easily from (2.15) and (2.17). In order to prove (3), we first let $m \in \mathbb{N}$. Then, by (2.16) and (2.19),

$$| \langle p_t^D(\cdot, y), \varphi_m \rangle | = | e^{-\lambda_m t} \varphi_m(x) | \leq e^{-\lambda_m t} \|\varphi_m\|_\infty \leq C e^{-\lambda_m t} \lambda_m^{w-1}.$$

Using (2.17), we get (3).

We now prove (4). Let $x, y \in D$ and note that $z \mapsto p_t^D(z, y) \in C^2(D) \cap C_c(\mathbb{R}^d)$. Let $\phi \in C_c^2(B(x, \delta_D(x)/2))$ (extended by 0 to the whole of \mathbb{R}^d) and $g \in C_c(\mathbb{R}^d)$ be such that $\phi(z) + g(z) = p_t^D(z, y)$ for $z \in D$ and $g(z) = 0$ for $z \in B(x, \delta_D(x)/4)$.

Note that by (2.4),

$$\frac{p_t^D(x, z)}{t} \leq \frac{p_t(x, z)}{t} \lesssim v(x, z), \quad (2.20)$$

which for $|x - z| > \delta_D(x)/4$ is uniformly bounded. Furthermore, since by [52, (2.10)] we have $p_t(x, z)/t \rightarrow v(x, z)$ as $t \rightarrow 0^+$ for all $x, z \in \mathbb{R}^d$, $x \neq z$, by (2.5) we find that for fixed $x, z \in D$, $x \neq z$,

$$\lim_{t \rightarrow 0^+} \frac{p_t^D(x, z)}{t} = v(x, y) + \lim_{t \rightarrow 0^+} \frac{1}{t} \mathbb{E}^x [p_{t-\tau_D}(X_{\tau_D}, z); \tau_D < t].$$

Since x and z are fixed we have $p_{t-\tau_D}(X_{\tau_D}, z) \lesssim t$, so the limit on the right hand side is equal to 0, hence $p_t^D(x, z)/t \rightarrow v(x, z)$ as $t \rightarrow 0^+$ for all $x, z \in D$, $x \neq z$. By this, (2.20), and the dominated convergence theorem, we get $\Delta^{\alpha/2} g(x) = L^D g(x)$.

Let L be the $C_0(\mathbb{R}^d)$ -generator of the semigroup induced by p_t . By Sato [60, Theorem 31.5], we have $\Delta^{\alpha/2} \phi(x) = L\phi(x)$. Therefore,

$$L^D \phi(x) = \Delta^{\alpha/2} \phi(x) + \lim_{t \rightarrow 0^+} \frac{P_t^D \phi(x) - P_t \phi(x)}{t}.$$

We will show that the last limit exists and is equal to 0. By (2.5), Fubini–Tonelli, and the fact that $X_{\tau_D} \in D^c$ almost surely,

$$\begin{aligned} \frac{|P_t^D \phi(x) - P_t \phi(x)|}{t} &\leq \|\phi\|_\infty \frac{1}{t} \mathbb{E}^x \left[\int_{B(x, \delta_D(x)/2)} p_{t-\tau_D}(X_{\tau_D}, z) dz; \tau_D < t \right] \\ &\lesssim \mathbb{P}^x(\tau_D < t) \xrightarrow{t \rightarrow 0^+} 0. \end{aligned}$$

By collecting the above results we find that

$$\Delta_x^{\alpha/2} p_t^D(x, y) = \Delta^{\alpha/2} \phi(x) + \Delta^{\alpha/2} g(x) = L^D \phi(x) + L^D g(x) = L_x^D p_t^D(x, y),$$

which ends the proof. \square

Corollary 2.6 For every $t > 0$, $\Delta_x^{\alpha/2} p_t^D$ is bounded in $D \times D$.

Proof By Lemma 2.5 and (2.19), we have

$$\begin{aligned} |\Delta_x^{\alpha/2} p_t^D(x, y)| &= |L_x^D p_t^D(x, y)| = \left| \sum_{n=1}^{\infty} \lambda_n e^{-\lambda_n t} \varphi_n(x) \varphi_n(y) \right| \\ &\leq \sum_{n=1}^{\infty} C \lambda_n e^{-\lambda_n t} \lambda_n^{2w-2} \leq C_0 < \infty. \end{aligned}$$

\square

Lemma 2.7 *Let $\phi \in C_c^\infty(D)$. Then,*

$$P_t^D L\phi(y) = \sum_{n=1}^\infty e^{-\lambda_n t} \lambda_n \langle \phi, \varphi_n \rangle \varphi_n(y), \quad y \in D.$$

Proof Note that $L\phi \in L^2(D)$, hence

$$P_t^D L\phi(y) = \sum_{n=1}^\infty e^{-\lambda_n t} \langle \varphi_n, L\phi \rangle \varphi_n(y).$$

By (2.18) we have $\varphi_n = G_D[\lambda_n \varphi_n]$. Therefore, by [15, Proposition 3.13],

$$\langle \varphi_n, L\phi \rangle = \langle G_D[\lambda_n \varphi_n], L\phi \rangle = \langle \lambda_n \varphi_n, \phi \rangle,$$

which ends the proof of the lemma. □

The following result is a weighted Hausdorff–Young type inequality.

Lemma 2.8 *There exist $c = c(d, \alpha, \underline{D})$ and $w \in \mathbb{N}$ such that for any $p \in [2, \infty]$ and $u \in L^p(D)$,*

$$\|u\|_{L^p(D)} \leq c \left(\sum_{n=1}^\infty |\langle u, \varphi_n \rangle|^{p'} \lambda_n^{w-1} \right)^{1/p'},$$

where $p' = p/(p - 1)$ is the Hölder conjugate exponent of p .

Proof Let $\phi \in L^2(D)$. By (2.19), we have $\|\varphi_n\|_\infty \leq C\lambda_n^{w-1}$ for some $C > 0$ and $w \geq 1$ independent of n . Therefore for $x \in D$,

$$\|\phi\|_\infty \leq \sum_{n=1}^\infty |\langle \phi, \varphi_n \rangle| \|\varphi_n\|_\infty \leq C \sum_{n=1}^\infty |\langle \phi, \varphi_n \rangle| \lambda_n^{w-1}.$$

If we let $\hat{\phi} = (\langle \phi, \varphi_1 \rangle, \langle \phi, \varphi_2 \rangle, \dots)$ and denote by l_λ^p the space of sequences with the p -th powers summable with the weight $(\lambda_1^{w-1}, \lambda_2^{w-1}, \dots)$, then the above means that $\hat{\phi} \mapsto \phi$ is bounded from l_λ^1 to $L^\infty(D)$. By Parseval’s identity, this map is also bounded from l^2 to $L^2(D)$, hence also from l_λ^2 to $L^2(D)$. The statement of the lemma follows from the Riesz–Thorin theorem. □

3 Yaglom limits in Lipschitz open sets

In this section we prove Theorem 1.1. We first establish the asymptotics of Green potentials at the boundary points of D . This extends what is already known about the asymptotics of Green potentials at the vertex of cone [24, Lemma 3.5]; we also propose a different proof.

Lemma 3.1 *If f is a measurable function bounded on D and $Q \in \partial D$, then*

$$\lim_{x \rightarrow Q} \int_D \frac{G_D(x, y)}{G_D(x, x_0)} f(y) dy = \int_D \lim_{x \rightarrow Q} \frac{G_D(x, y)}{G_D(x, x_0)} f(y) dy < \infty, \quad x \in D.$$

Proof Fix two points $x_1, x_2 \in D$ and let

$$\rho = (\delta_D(x_1) \wedge \delta_D(x_2) \wedge |x_1 - x_2|)/3,$$

so that $B(x_1, \rho), B(x_2, \rho) \subset D$ and $B(x_1, \rho) \cap B(x_2, \rho) = \emptyset$. We know that $M_D^{x_0}(\cdot, Q)$ given by (2.14) is regular α -harmonic on $B(x_1, \rho)$ and $B(x_2, \rho)$, and for x sufficiently close to ∂D so is $G_D(x, \cdot)$. Therefore, for $i = 1, 2$,

$$\begin{aligned} \int_{B(x_i, \rho)^c} \lim_{x \rightarrow Q} \frac{G_D(x, y)}{G_D(x, x_0)} \omega_{B(x_i, \rho)}^{x_i}(dy) &= \int_{B(z_i, \rho)^c} M_D^{x_0}(y, Q) \omega_{B(z_i, \rho)}^{z_i}(dy) \\ &= M_D^{x_0}(x_i, Q) \\ &= \lim_{x \rightarrow Q} \frac{G_D(x, x_i)}{G_D(x, x_0)} \\ &= \lim_{x \rightarrow Q} \frac{\int_{B(x_i, \rho)^c} G_D(x, y) \omega_{B(x_i, \rho)}^{x_i}(dy)}{G_D(x, x_0)} \\ &= \lim_{x \rightarrow Q} \int_{B(x_i, \rho)^c} \frac{G_D(x, y)}{G_D(x, x_0)} \omega_{B(x_i, \rho)}^{x_i}(dy). \end{aligned}$$

The α -harmonic measures $\omega_{B(x_i, \rho)}^{x_i}(dy)$ are absolutely continuous and have radially decreasing densities g_i , see, e.g., [13]. Therefore there exists $C > 0$ such that $\omega_{B(x_i, \rho)}^{x_i}(dy) = g_i(y) dy$ and $g_i(y) \geq C$ for $y \in D \cap (B(x_i, \rho)^c)$. Let $g = g_1 + g_2$. Vitali’s theorem [61, Theorem 16.6 (i) and (iii)] yields the following L^1 convergence:

$$\lim_{x \rightarrow Q} \int_D \left| \frac{G_D(x, y)}{G_D(x, x_0)} g(y) - M_D^{x_0}(y, Q) g(y) \right| dy = 0.$$

Since $|f| \lesssim C \lesssim g$, the result follows. □

We can also establish the following identity, an analogue of [24, (3.16)].

Lemma 3.2 *For $x \in \mathbb{R}^d$, we have*

$$\mathbb{P}^x(\tau_D > 1) = (G_D P_1^D \kappa_D)(x). \tag{3.1}$$

Proof Let $x \in D$. Since our set D is Lipschitz, from Lemma 6 and the proof of Lemma 17 in [12],

$$\begin{aligned} \omega_D^x(\partial D) &= \mathbb{P}^x(X_{\tau_D} \in \partial D) = 0, \\ \mathbb{P}^x(X_{\tau_D-} = X_{\tau_D}) &= 0, \\ \mathbb{P}^x(X_{\tau_D-} \in D) &= 1. \end{aligned}$$

By the Ikeda–Watanabe formula (2.13) and the Chapman–Kolmogorov equations we have

$$\begin{aligned}
 \mathbb{P}^x(\tau_D > 1) &= \mathbb{P}^x[\tau_D > 1, X_{\tau_D-} \in D, X_{\tau_D} \in D^c] \\
 &= \int_1^\infty \int_{D^c} \int_D p_s^D(x, z)v(z - w) dz dw ds \\
 &= \int_{\mathbb{R}^d} \int_{D^c} \int_0^\infty p_{t+1}^D(x, z)v(z - w) dt dw dz \\
 &= \int_{\mathbb{R}^d} \int_{D^c} \int_0^\infty \int_D p_t^D(x, y)p_1^D(y, z) dy v(z - w) dt dw dz \\
 &= \int_D \int_0^\infty p_t^D(x, y) dt \int_{\mathbb{R}^d} p_1^D(y, z) \int_{D^c} v(z - w) dw dz dy \\
 &= \int_D G_D(x, y) \int_{\mathbb{R}^d} p_1^D(y, z)\kappa_D(z) dz dy \\
 &= \int_D G_D(x, y)(P_1^D \kappa_D)(y) dy \\
 &= (G_D P_1^D \kappa_D)(x).
 \end{aligned}$$

For $x \in D^c$, both sides of (3.1) are equal to 0. This ends the proof. □

We define

$$C_1 := \int_D \int_D M_D^{x_0}(y, Q) p_1^D(y, z)\kappa_D(z) dz dy.$$

Combining the two lemmas above, we obtain the following result.

Lemma 3.3 *We have $0 < C_1 < \infty$ and $\lim_{x \rightarrow Q} \frac{\mathbb{P}^x(\tau_D > 1)}{G_D(x, x_0)} = C_1$.*

Proof By Lemma 3.2, $\mathbb{P}^x(\tau_D > 1) = (G_D P_1^D \kappa_D)(x)$. Note that $(P_1^D \kappa_D)(y)$ is bounded. Indeed, by (2.6),

$$\begin{aligned}
 (P_1^D \kappa_D)(y) &= \int_D p_1^D(y, z)\kappa_D(z) dz \\
 &\approx \mathbb{P}^y(\tau_D > 1) \int_D \mathbb{P}^z(\tau_D > 1) p_1(y, z)\kappa_D(z) dz, \quad y \in D. \tag{3.2}
 \end{aligned}$$

Since D is bounded, by (2.4),

$$p_1(y, z) \approx 1, \quad y, z \in D. \tag{3.3}$$

Hence (3.2) becomes

$$(P_1^D \kappa_D)(y) \approx \mathbb{P}^y(\tau_D > 1) \int_D \mathbb{P}^z(\tau_D > 1)\kappa_D(z) dz, \quad y \in D. \tag{3.4}$$

Using (3.1), we see that for $x \in \mathbb{R}^d$,

$$\int_D G_D(x, y)(P_1^D \kappa_D)(y) dy = (G_D P_1^D \kappa_D)(x) = \mathbb{P}^x(\tau_D > 1) \leq 1.$$

By (2.10), $G_D(x, y)$ is strictly positive for all $x, y \in D$. Thus $P_1^D \kappa_D$ has to be finite almost everywhere. Hence the integral in (3.4) is finite and

$$(P_1^D \kappa_D)(y) \approx \mathbb{P}^y(\tau_D > 1),$$

for $y \in D$. In particular, $(P_1^D \kappa_D)(y)$ is bounded on D . By using Lemma 3.1 with $f(y) = (P_1^D \kappa_D)(y)$,

$$\begin{aligned} \lim_{x \rightarrow Q} \frac{\mathbb{P}^x(\tau_D > 1)}{G_D(x, x_0)} &= \lim_{x \rightarrow Q} \frac{(G_D P_1^D \kappa_D)(x)}{G_D(x, x_0)} \\ &= \lim_{x \rightarrow Q} \int_D \frac{G_D(x, y)}{G_D(x, x_0)} (P_1^D \kappa_D)(y) dy \\ &= \int_D M_D^{x_0}(y, Q) (P_1^D \kappa_D)(y) dy = C_1 < \infty. \end{aligned}$$

□

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Let us define

$$m_x(A) := \frac{\int_A P_1^D(x, y) dy}{\mathbb{P}^x(\tau_D > 1)}, \quad x \in D, \quad A \subseteq \mathbb{R}^d. \tag{3.5}$$

First we note that the family $\{m_x : x \in D\}$ is tight. Indeed, combining the factorization of $p_1^D(x, y)$ in (2.6) with the Eq. (3.3), we get

$$\frac{p_1^D(x, y)}{\mathbb{P}^x(\tau_D > 1)} \approx \mathbb{P}^y(\tau_D > 1), \quad x, y \in D. \tag{3.6}$$

Since the densities of the measures $m_x(A)$ are bounded by an integrable function, the tightness follows.

Next we wish to prove that the measures m_x converge weakly to a probability measure m_Q on D as $x \rightarrow Q$. To this end, consider an arbitrary sequence $\{x_n\}$ such that $x_n \rightarrow Q$. By tightness, there exists a subsequence $\{x_{n_k}\}$ such that $m_{x_{n_k}} \rightrightarrows m_Q$ for some probability measure m_Q , as $k \rightarrow \infty$. We will show that this limit is unique.

Let $\phi \in C_c^\infty(D)$ and $u_\phi = (-\Delta)^{\alpha/2} \phi$. For $x \in \mathbb{R}^d$, we claim that

$$(P_1^D \phi)(x) = (G_D P_1^D u_\phi)(x). \tag{3.7}$$

To show this, we first remark that $u_\phi \in C_0(\mathbb{R}^d)$ and that $(G_D u_\phi)(x) = \phi(x)$, see [37, Lemma 5.7] and [23, (11)]. By (2.4) it follows that

$$(P_1^D |u_\phi|)(x) = \int_D p_1^D(x, y) |u_\phi(y)| dy \leq c < \infty.$$

Therefore, since for a fixed $z \in D^c$ we have $v(y, z) \gtrsim 1$ for $y \in D$, by (2.12) we get

$$\begin{aligned} (G_D P_1^D |u_\phi|)(x) &= \int_D G_D(x, y) (P_1^D |u_\phi|)(y) dy \\ &\leq c \int_D G_D(x, y) dy < \infty. \end{aligned}$$

As a result, we can apply Fubini–Tonelli theorem and establish (3.7) as follows:

$$\begin{aligned} (G_D P_1^D u_\phi)(x) &= \int_D \int_D \int_0^\infty p_t^D(x, y) p_1^D(y, z) u_\phi(z) dt dz dy \\ &= \int_D \int_0^\infty p_{t+1}^D(x, z) u_\phi(z) dt dz \\ &= \int_D \int_0^\infty \int_D p_1^D(x, y) p_t^D(y, z) u_\phi(z) dy dt dz \\ &= \int_D \int_D \int_0^\infty p_1^D(x, y) p_t^D(y, z) u_\phi(z) dt dz dy \\ &= (P_1^D G_D u_\phi)(x) = (P_1^D \phi)(x). \end{aligned}$$

Let us denote $m_x(\phi) := \int_D \phi(y) m_x(dy)$. Using (3.7), Lemmas 3.3, and 3.1, we get

$$\begin{aligned} \lim_{x \rightarrow Q} m_x(\phi) &= \lim_{x \rightarrow Q} \frac{(P_1^D \phi)(x)}{\mathbb{P}^x(\tau_D > 1)} \\ &= \lim_{x \rightarrow Q} \frac{(P_1^D G_D u_\phi)(x)}{\mathbb{P}^x(\tau_D > 1)} \\ &= \lim_{x \rightarrow Q} \frac{(G_D P_1^D u_\phi)(x)}{G_D(x, x_0)} \frac{G_D(x, x_0)}{\mathbb{P}^x(\tau_D > 1)} \\ &= \frac{1}{C_1} \int_D M_D^{x_0}(y, Q) (P_1^D u_\phi)(y) dy. \end{aligned} \tag{3.8}$$

In particular, $m_Q(\phi) := \lim_{k \rightarrow \infty} m_{x_{n_k}}(\phi)$ does not depend on the choice of the subsequence. Thus, by the Portmanteau Theorem, $m_x \implies m_Q$ as $x \rightarrow Q$.

For $t > 1$, we consider $\phi_{t,y}(\cdot) := p_{t-1}^D(\cdot, y) \in C_0(\mathbb{R}^d)$, see [19] or [32, Proposition 1.19]. Using Chapman–Kolmogorov, we get

$$\begin{aligned} \eta_{t,Q}(y) &= \lim_{x \rightarrow Q} \frac{p_t^D(x, y)}{\mathbb{P}^x(\tau_D > 1)} \\ &= \lim_{x \rightarrow Q} \frac{\int_D p_{t-1}^D(z, y) p_1^D(x, z) dz}{\mathbb{P}^x(\tau_D > 1)} \\ &= \lim_{x \rightarrow Q} \frac{(P_1^D p_{t-1}^D(\cdot, y))(x)}{\mathbb{P}^x(\tau_D > 1)} \\ &= \lim_{x \rightarrow Q} m_x(p_{t-1}^D(\cdot, y)). \end{aligned}$$

By (3.8), the existence of $\eta_{t,Q}(y)$ for $t > 1$ follows:

$$\eta_{t,Q}(y) = m_Q(p_{t-1}^D(\cdot, y)).$$

Note that the threshold $t > 1$ is arbitrary, that is, 1 can be replaced with any $t_0 > 0$. Indeed, the results of this section can be readily reformulated with t_0 in place of 1, for instance, Lemma 3.3 may be strengthened to assert that for every $t_0 > 0$,

$$\lim_{x \rightarrow Q} \frac{\mathbb{P}^x(\tau_D > t_0)}{G_D(x, x_0)} = \int_D \int_D M_D^{x_0}(y, Q) p_{t_0}^D(y, z) \kappa_D(z) dz dy.$$

Accordingly, we get the existence of the limit

$$\lim_{x \rightarrow Q} \frac{\mathbb{P}^x(\tau_D > 1)}{\mathbb{P}^x(\tau_D > t_0)}. \tag{3.9}$$

We can also reuse the above arguments to get for all $t > t_0$, the existence of

$$\lim_{x \rightarrow Q} \frac{p_t^D(x, y)}{\mathbb{P}^x(\tau_D > t_0)}. \tag{3.10}$$

Of course, (3.9) and (3.10) give the existence of $\eta_{t,Q}(y)$ for all $t > 0$.

The Eq. (1.8) follows from Eq. (3.6), and the Eq. (1.9) follows from the Chapman–Kolmogorov equations and the dominated convergence theorem (see [19, (27)]):

$$\eta_{t+s,Q}(y) = \lim_{x \rightarrow Q} \int_D \frac{p_t^D(x, z)}{\mathbb{P}^x(\tau_D > 1)} p_s^D(z, y) dz = \int_D \eta_{t,Q}(z) p_s^D(z, y) dz.$$

The fact that $\tilde{\eta}$ and η^{x_0} exist follows from the existence of η and from Lemma 3.3. The continuity of η follows from (1.8), (1.9), continuity of $p_t^D(x, y)$, and the dominated convergence theorem. □

Corollary 3.4 *The functions $(t, y) \mapsto \eta_{t,y}(x), \tilde{\eta}_{t,y}(x), \eta_{t,y}^{x_0}(x)$ are continuous on $(0, \infty) \times \bar{D}$ for all $x \in D$.*

Proof Fix $x \in D$. By Theorem 1.1 and the fact that $p_t^D(x, y)$ and $\mathbb{P}^y(\tau_D > 1)$ are continuous for $(t, y) \in (0, \infty) \times D$, and separated from 0 in sufficiently small neighborhood of any point (t, y) , it suffices to verify that for any sequence $((t_n, y_n)) \subset (0, \infty) \times D$ such that $(t_n, y_n) \rightarrow (t, Q) \in (0, \infty) \times \partial D$, we have

$$\lim_{n \rightarrow \infty} \frac{p_{t_n}^D(x, y_n)}{\mathbb{P}^{y_n}(\tau_D > 1)} = \eta_{t, Q}(x). \tag{3.11}$$

Furthermore, by Theorem 1.1, in order to obtain (3.11) it suffices to prove that for any $t > 0$ there exists a modulus of continuity ω independent of y such that

$$\left| \frac{p_{t+\varepsilon}^D(x, y) - p_t^D(x, y)}{\mathbb{P}^y(\tau_D > 1)} \right| \leq \omega(\varepsilon), \quad \varepsilon > 0. \tag{3.12}$$

By Chapman–Kolmogorov, we have

$$\begin{aligned} & \left| \frac{p_{t+\varepsilon}^D(x, y) - p_t^D(x, y)}{\mathbb{P}^y(\tau_D > 1)} \right| \int_{D \setminus B(x, \delta_D(x)/2)} \frac{|p_t^D(z, y) - p_t^D(x, y)| p_\varepsilon^D(x, z)}{\mathbb{P}^y(\tau_D > 1)} dz \\ & + \int_{B(x, \delta_D(x)/2)} \frac{|p_t^D(z, y) - p_t^D(x, y)| p_\varepsilon^D(x, z)}{\mathbb{P}^y(\tau_D > 1)} dz =: I_1 + I_2. \end{aligned}$$

Then by (2.6),

$$I_1 \leq \int_{D \setminus B(x, \delta_D(x)/2)} p_\varepsilon^D(x, z) dz \leq \int_{D \setminus B(x, \delta_D(x)/2)} p_\varepsilon(x, z) dz \leq \omega(\varepsilon).$$

For I_2 , we use the gradient bounds of Kulczycki and Ryznar [51, Theorem 1.1] and (2.6):

$$\begin{aligned} I_2 & \leq \int_{B(x, \delta_D(x)/2)} \frac{|p_t^D(z, y) - p_t^D(x, y)| p_\varepsilon^D(x, z)}{\mathbb{P}^y(\tau_D > 1)} dz \\ & \leq \int_{B(x, \delta_D(x)/2)} |x - z| \frac{\|\nabla_x p_t^D(\cdot, y)\|_{L^\infty(B(x, \delta_D(x)/2))}}{\mathbb{P}^y(\tau_D > 1)} p_\varepsilon^D(x, z) dz \\ & \lesssim \int_{B(x, \delta_D(x)/2)} |x - z| \frac{\|p_t^D(\cdot, y)\|_{L^\infty(B(x, \delta_D(x)/2))}}{\mathbb{P}^y(\tau_D > 1)} p_\varepsilon^D(x, z) dz \\ & \lesssim \int_{B(x, \delta_D(x)/2)} |x - z| p_\varepsilon^D(x, z) dz \leq \int_{B(x, \delta_D(x)/2)} |x - z| p_\varepsilon(x, z) dz \leq \omega(\varepsilon). \end{aligned}$$

Thus, $I_1 + I_2 \leq \omega(\varepsilon)$, which ends the proof for η . For $\tilde{\eta}$ and η^{x_0} , we use Lemma 3.2 and (1.8). □

Here is a rough result on the behavior of $\eta_{s, Q}(x)$ away from the singularity at $(0, Q)$.

Lemma 3.5 *If $Q \in \partial D$ then $(s, x) \mapsto \eta_{s,Q}(x)$ is locally bounded on $((0, \infty) \times \mathbb{R}^d) \setminus \{(0, Q)\}$. Furthermore, if $t = 0$ or $y \in \partial D$, but $(t, y) \neq (0, Q)$, then $\eta_{s,Q}(x) \rightarrow 0$ as $(s, x) \rightarrow (t, y)$.*

Proof By (2.6) and (2.4), we have

$$\begin{aligned} \eta_{s,Q}(x) &= \lim_{D \ni \xi \rightarrow Q} \frac{p_s^D(x, \xi)}{\mathbb{P}^\xi(\tau_D > 1)} \lesssim \limsup_{D \ni \xi \rightarrow Q} \frac{\mathbb{P}^\xi(\tau_D > s)}{\mathbb{P}^\xi(\tau_D > 1)} p_s(x, \xi) \mathbb{P}^x(\tau_D > s) \\ &\lesssim |x - Q|^{-d-\alpha} \mathbb{P}^x(\tau_D > s) \limsup_{D \ni \xi \rightarrow Q} \frac{s \mathbb{P}^\xi(\tau_D > s)}{\mathbb{P}^\xi(\tau_D > 1)}. \end{aligned}$$

If $|x - Q| \geq \varepsilon$, then $\eta_{s,Q}(x)$ is bounded—it even converges to 0 as $s \rightarrow 0$ —see Lemma B.2. If $s > \varepsilon$, then we use the approximate factorization of p_t^D and the fact that $\mathbb{P}^x(\tau_D > s) \rightarrow 0$ as $x \rightarrow y \in \partial D$. \square

Let us summarize estimates of η that follow from the estimates of the Dirichlet heat kernel.

Corollary 3.6 *If D is $C^{1,1}$, then*

$$\eta_{t,Q}(x) \approx \begin{cases} \frac{1}{\sqrt{t}} \left(1 \wedge \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}} \right) p_t(x, Q), & t \in (0, 1), x \in D, Q \in \partial D, \\ e^{-\lambda_1 t} \delta_D(x)^{\alpha/2}, & t \in [1, \infty), x \in D, Q \in \partial D. \end{cases} \quad (3.13)$$

If D is Lipschitz, then

$$\eta_{t,Q}(x) \approx e^{-\lambda_1 t} \mathbb{P}^x(\tau_D > t), \quad t \in [1, \infty), x \in D, Q \in \partial D, \quad (3.14)$$

and

$$\eta_{t,Q}(x) \approx \frac{\mathbb{P}^x(\tau_D > t) p_t(x, Q)}{\Phi(A_{t^{1/\alpha}}(Q))}, \quad t \in (0, 1), x \in D, Q \in \partial D. \quad (3.15)$$

Furthermore, there exist $0 < \sigma_1 \leq \sigma_2 < 1$ such that

$$t^{-\sigma_1} \lesssim \frac{\eta_{t,Q}(x)}{\mathbb{P}^x(\tau_D > t) p_t(x, Q)} \lesssim t^{-\sigma_2}, \quad t \in (0, 1), x \in D, Q \in \partial D. \quad (3.16)$$

Proof The estimate (3.13) follows from (2.7) and (2.9). By [49, Theorem 1.1] and (2.6), $\mathbb{P}^y(\tau_D > 1) \approx \varphi_1(y)$, so (3.14) is a consequence of (2.8). It remains to prove (3.15) and (3.16). By (2.6),

$$\eta_{t,Q}(x) \approx \mathbb{P}^x(\tau_D > t) p_t(x, Q) \lim_{y \rightarrow Q} \frac{\mathbb{P}^y(\tau_D > t)}{\mathbb{P}^y(\tau_D > 1)}. \quad (3.17)$$

By [19, Theorem 2] and (2.11),

$$\frac{\mathbb{P}^y(\tau_D > t)}{\mathbb{P}^y(\tau_D > 1)} \approx \frac{1}{\Phi(A_{t^{1/\alpha}}(y))}.$$

By geometrical considerations, we can choose points $A_{t^{1/\alpha}}(y)$ converging to a point in $\mathcal{A}_{t^{1/\alpha}}(Q)$. This proves (3.15). By (3.17) and Lemma B.2, we get the upper bound in (3.16). The lower bound follows from (3.15) and [12, Lemma 3] with some $\sigma_1 > 0$. Of course, we must have $\sigma_1 \leq \sigma_2$ in (3.16). \square

A consequence of Theorem 1.1 is the Yaglom-type limit, obtained in the thesis of the first author [4].

Theorem 3.7 *Suppose that D is a bounded Lipschitz open set such that $0 \in \partial D$ and $D \cup \{0\}$ is star-shaped at 0. If $x \in D$ then for every Borel $A \subseteq \mathbb{R}^d$,*

$$\lim_{t \rightarrow \infty} \mathbb{P}^x \left(\frac{X_t}{t^{1/\alpha}} \in A \mid \left(\frac{X_s}{t^{1/\alpha}} \right)_{0 \leq s \leq t} \subset D \right) = m_0(A),$$

where $\mathbb{P}^x(A_1|A_2) := \mathbb{P}^x(A_1 \cap A_2) / \mathbb{P}^x(A_2)$ is the conditional probability and $m_0(A) := \int_A \eta_{1,0}(y) dy$.

Proof Let $x \in D$, $t \geq 1$, and let $A \subset \mathbb{R}^d$ be Borel. Then we have

$$\begin{aligned} \mathbb{P}^x \left(\frac{X_t}{t^{1/\alpha}} \in A \mid \left(\frac{X_s}{t^{1/\alpha}} \right)_{0 \leq s \leq t} \subset D \right) &= \frac{\mathbb{P}^x(X_t \in t^{1/\alpha} A, (X_s)_{0 \leq s \leq t} \subset t^{1/\alpha} D)}{\mathbb{P}^x((X_s)_{0 \leq s \leq t} \subset t^{1/\alpha} D)} \\ &= \frac{\int_{t^{1/\alpha} A} p_t^{1/\alpha D}(x, y) dy}{\int_{t^{1/\alpha} D} p_t^{1/\alpha D}(x, y) dy} \\ &= \frac{\int_{t^{1/\alpha} A} t^{-d/\alpha} p_1^D(t^{-1/\alpha} x, t^{-1/\alpha} y) dy}{\int_{t^{1/\alpha} D} t^{-d/\alpha} p_1^D(t^{-1/\alpha} x, t^{-1/\alpha} y) dy} \\ &= \frac{\int_A p_1^D(t^{-1/\alpha} x, y) dy}{\int_D p_1^D(t^{-1/\alpha} x, y) dy} = m_{t^{-1/\alpha} x}(A), \end{aligned}$$

where $m_{t^{-1/\alpha} x}$ is the measure defined in (3.5) above (note that $t^{-1/\alpha} x \in D$). Therefore, by Theorem 1.1, this probability approaches $m_0(A)$ as $t \rightarrow \infty$. \square

4 Hölder regularity

This section is devoted to proving Theorems 1.2 and 1.4. The proof of Theorem 1.4 uses a mix of the boundary Harnack principle and interior Hölder regularity. Then Theorem 1.2 follows by using the formulas of Sect. 3, which enable us to relate the heat kernel regularity to the elliptic regularity.

Fix $n_0 \geq 2$ such that the reference points x_0 belongs to $D_{n_0/2}$.

Lemma 4.1 *There exists $p_0 = p_0(d, \alpha, \underline{D}) > 1$ such that the family $\{(G_D(y, \cdot)/G_D(x_0, y))^p : y \in D\}$ is uniformly integrable in D for all $p \in [1, p_0)$.*

Proof For $y \in D_{n_0}$ we have a crude bound:

$$G_D(y, z)/G_D(x_0, y) \leq C(d, \alpha, \underline{D})|y - z|^{\alpha-d}, \quad z \in D.$$

Considering the functions on the right-hand side, we see that $p_0 = d/(d - \alpha)$ will do.

From now on assume that $y \in D \setminus D_{n_0}$. By (2.10), there exists $C = C(d, \alpha, \underline{D})$ such that

$$\frac{G_D(y, z)}{G_D(x_0, y)} \leq C \frac{|y - z|^{\alpha-d} \Phi(z)\Phi(A_{x_0, y})^2}{|x_0 - y|^{\alpha-d} \Phi(x_0)\Phi(A_{y, z})^2}.$$

We immediately get that

$$\frac{G_D(y, z)}{G_D(x_0, y)} \leq C'|y - z|^{\alpha-d} \frac{\Phi(z)}{\Phi(A_{y, z})^2}.$$

By the Carleson estimate [48, Lemma 13], we further find that $\Phi(z)/\Phi(A_{y, z}) \leq C(d, \alpha, \underline{D})$. If we let $U = D_{32/r_0} \cup (D \setminus B(y, r_0/32))$, then it follows that

$$\frac{G_D(y, z)}{G_D(x_0, y)} \leq \begin{cases} C(d, \alpha, \underline{D})|y - z|^{\alpha-d}, & z \in U, \\ C(d, \alpha, \underline{D})|y - z|^{\alpha-d} \Phi(A_{y, z})^{-1}, & z \in D \setminus U. \end{cases}$$

The definition of $A_{y, z}$ implies that for $z \in D \setminus U$ there exists $Q = Q(z) \in \partial D$ such that $y, z \in B(Q, 3r)$ and $B(A_{y, z}, \kappa r) \subset D \cap B(Q, 6r)$. Using [12, Lemma 5], we find that there exist $C = C(d, \alpha, \underline{D})$ and $\gamma = \gamma(d, \alpha, \underline{D}) \in (0, \alpha)$ such that

$$\Phi(A_{y, z}) \geq C|A_{y, z} - Q(z)|^\gamma \geq C\kappa^\gamma r^\gamma \geq C\kappa^\gamma |y - z|^\gamma.$$

Therefore,

$$\frac{G_D(y, z)}{G_D(x_0, y)} \leq C(|y - z|^{\alpha-d} \vee |y - z|^{\alpha-\gamma-d}), \quad y \in D \setminus D_{n_0}, \quad z \in D, \quad (4.1)$$

so the statement of the lemma holds for all $p \in [1, d/(d - \alpha + \gamma))$. We can take $p_0 = d/(d - \alpha + \gamma)$. \square

The following lemma is a specific Carleson-type estimate.

Lemma 4.2 *Let $0 < r < \delta_D(y)$, $|z - y| \geq 2r$, and $|v - y| \leq r$. There exists $C = C(d, \alpha, \underline{D})$ such that*

$$G_D(z, v) \leq CG_D(z, y).$$

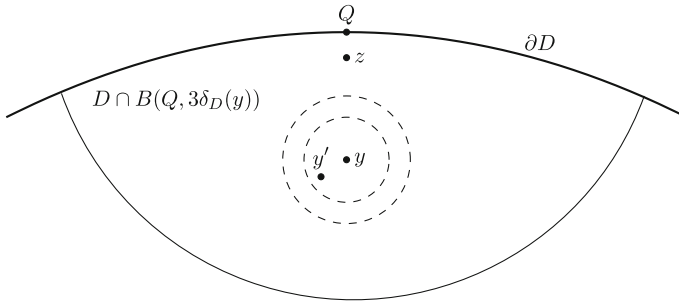


Fig. 1 Illustration for Lemma 4.3. The boundary Harnack principle cannot be used to estimate increments between y and y' because of the singularity at z . Instead we show regularity in the smaller ball using harmonicity in the larger ball

Proof Note that $2|z - v| \geq |z - y|$. By (2.10), there is $c = c(d, \alpha, \underline{D})$ such that

$$G_D(z, v) \leq c \frac{\Phi(z)\Phi(v)}{\Phi(A_{z,v})^2} |z - v|^{\alpha-d} \leq 2^{d-\alpha} c \frac{\Phi(z)\Phi(v)}{\Phi(A_{z,v})^2} |z - y|^{\alpha-d}.$$

By elementary calculations, we find that $2r_{z,v} \geq r_{z,y}$. By [48, Lemma 13] we therefore get $\Phi(A_{z,v}) \geq c(d, \alpha, \underline{D})\Phi(A_{z,y})$. Furthermore, by [12, Lemma 4 and 5] we get $\Phi(v) \leq c(d, \alpha, \underline{D})\Phi(y)$. This ends the proof. \square

The next lemma can be viewed as a more concrete, quantified version of [23, Lemma 8]. We give an interior-type Hölder regularity for ratios of Green functions, taking into account the singularity at the diagonal. The structure of the proof follows the boundary regularity approach of [12, Lemma 16], but here the singularity can occur between the boundary and the arguments of the function, see Fig. 1.

Lemma 4.3 *Let $y \in D$ and $Q \in \partial D$ satisfy $|Q - y| = \delta_D(y)$. Assume that $z \in D \cap B(Q, 3\delta_D(y))$ and let $r = |z - y|/4$, so that $\overline{B(y, r)} \subset D$. Then there exist constants $C \geq 1, k_0 \geq 4, \sigma \in (0, 1]$, and $\gamma \in (0, \alpha)$, depending only on d, α, \underline{D} , such that for every $y' \in B(y, 2^{-k_0}r)$ we have*

$$\left| \frac{G_D(y, z)}{G_D(x_0, y)} - \frac{G_D(y', z)}{G_D(x_0, y')} \right| \leq C \left(\frac{|y - y'|}{r} \right)^\sigma r^{\alpha-d-\gamma}.$$

Proof Let $k_0 \geq 4$ (further constraints on k_0 stem from the proof) and let

$$\begin{aligned} B_k &= B(y, (2^{k_0})^{-k}r), \quad k = 0, 1, \dots, \\ \Pi_k &= B_k \setminus B_{k+1}, \quad k = 0, 1, \dots, \quad \Pi_{-1} = D \setminus B_0, \\ u(y) &= G_D(y, z), \quad v(y) = G_D(x_0, y). \end{aligned}$$

We will show that there exist $c = c(d, \alpha, \underline{D})$ and $\zeta = \zeta(d, \alpha, \underline{D}) \in (0, 1]$, such that for $k = 0, 1, \dots$,

$$\sup_{B_k} \frac{u}{v} \leq (1 + c\zeta^k) \inf_{B_k} \frac{u}{v}. \tag{4.2}$$

By virtue of (4.1), this implies the statement of the theorem.

In order to obtain (4.2), for $-1 \leq l < k$ we define

$$u_k^l(x) = \mathbb{E}^x[u(X_{\tau_{B_k}})]; X_{\tau_{B_k}} \in \Pi_l, \quad v_k^l(x) = \mathbb{E}^x[v(X_{\tau_{B_k}})]; X_{\tau_{B_k}} \in \Pi_l, \quad x \in \mathbb{R}^d,$$

and we will prove the following two claims.

Claim 1 *There exist $C = C(d, \alpha, \underline{D})$ and $q = q(d, \alpha, \underline{D}) \in (0, 1)$ such that for $-1 \leq l \leq k - 2$ and $x \in B_k$,*

$$\begin{aligned} u_k^l(x) &\leq C(q^{k_0})^{k-l-1} u(x), \\ v_k^l(x) &\leq C(q^{k_0})^{k-l-1} v(x). \end{aligned}$$

We define the oscillation of function f as $\text{Osc}_A f = \sup_A f - \inf_A f$.

Claim 2 *Let $g(x) = u_{k+1}^k(x)/v_{k+1}^k(x)$. Then there is $\delta = \delta(d, \alpha, \underline{D})$ such that $\text{Osc}_{B_{k+2}} g \leq \delta \text{Osc}_{B_k} g$.*

Using Claim 1 with k_0 large enough (see [12, (5.23)–(5.25)] for details) and Claim 2 we may repeat the final part of the proof in [12, Lemma 16] to get (4.2)—we skip those details.

We will now prove Claim 1 for u , the proof for v is identical. First let $0 \leq l \leq k - 2$. By Lemma 4.2,

$$u_k^l(x) = \int_{\Pi_l} G_D(z, w) P_{B_k}(x, w) dw \leq cG_D(z, y) \mathbb{P}^x(X_{\tau_{B_k}} \in \Pi_l).$$

Furthermore, since $k \geq 1$, Lemma 4.2 yields $G_D(z, y) \leq cG_D(z, x)$. Therefore,

$$u_k^l(x) \leq cG_D(z, x) \mathbb{P}^x(X_{\tau_{B_k}} \in \Pi_l). \tag{4.3}$$

Recall the explicit formula for the Poisson kernel of the ball—see, e.g., Landkof [54]:

$$P_{B(0,r)}(x, w) = c_{d,\alpha} \frac{(r^2 - |x|^2)^{\alpha/2}}{(|w|^2 - r^2)^{\alpha/2}} |x - w|^{-d}, \quad x \in B(0, r), \quad w \in B(0, r)^c. \tag{4.4}$$

Using the formula, we find that

$$\begin{aligned} \mathbb{P}^x(X_{\tau_{B_k}} \in \Pi_l) &= \int_{\Pi_l} P_{B_k}(x, w) dw \leq c_{d,\alpha} (r(2^{k_0})^{-k})^\alpha \times \int_{\Pi_l} (|w-y|^2 - (r(2^{k_0})^{-k})^2)^{-\alpha/2} |x-w|^{-d} dw \\ &\leq \tilde{c}_{d,\alpha} \frac{(r(2^{k_0})^{-k})^\alpha}{(r(2^{k_0})^{-l-1})^\alpha} = \tilde{c}_{d,\alpha} (2^{-k_0\alpha})^{k-l-1}. \end{aligned}$$

Coming back to (4.3) we get Claim 1 for $0 \leq l \leq k - 2$.

Now, let $l = -1$. Using (4.4), we get

$$\begin{aligned} u_k^{-1}(x) &\leq C(d, \alpha, \underline{D}) \int_{D \setminus B(y,r)} G_D(z, w) \frac{((r(2^{k_0})^{-k})^2 - |x - y|^2)^{\alpha/2}}{(|w - y|^2 - (r(2^{k_0})^{-k})^2)^{\alpha/2}} |x - w|^{-d} dw \\ &\leq C(d, \alpha, \underline{D}) (2^{k_0})^{-k} \int_{D \setminus B(y,r)} G_D(z, w) \frac{r^\alpha}{(|w - y|^2 - (r(2^{k_0})^{-k})^2)^{\alpha/2}} |x - w|^{-d} dw \\ &\leq c(d, \alpha) C(d, \alpha, \underline{D}) (2^{-\alpha k_0})^k \int_{D \setminus B(y,r)} G_D(z, w) \frac{(r^2 - |x - y|^2)^{\alpha/2}}{(|w - y|^2 - r^2)^{\alpha/2}} |x - w|^{-d} dw \\ &\leq \tilde{c}(d, \alpha) C(d, \alpha, \underline{D}) (2^{-\alpha k_0})^k \int_{D \setminus B(y,r)} G_D(z, w) P_{B(y,r)}(x, w) dw. \end{aligned}$$

Since $G_D(z, \cdot)$ is harmonic in $D \setminus \{z\}$, the last integral is equal to $G_D(z, x)$, which yields Claim 1 for $l = -1$. Thus, Claim 1 is proved.

It remains to prove Claim 2, which we do now. Let $a_1 = \inf_{B_k} g$ and $a_2 = \sup_{B_k} g$. Without any loss of generality we may assume $a_1 \neq a_2$. Then, we let

$$g'(x) = \frac{g(x) - a_1}{a_2 - a_1}, \quad x \in B_k.$$

We have $0 \leq g' \leq 1$, $\text{Osc}_{B_k} g' = 1$, and $\text{Osc}_{B_{k+2}} g = \text{Osc}_{B_{k+2}} g' \text{Osc}_{B_k} g$. If $\sup_{B_{k+2}} g' \leq \frac{1}{2}$, then we are done, so assume otherwise. Note that

$$g'(x) = \frac{u_{k+1}^k(x) - a_1 v_{k+1}^k(x)}{a_2 - a_1} =: \frac{g_1(x)}{g_2(x)}, \quad x \in B_{k+2}.$$

By (4.4), we have

$$1 \leq \frac{\sup_{B_{k+2}} g_2}{\inf_{B_{k+2}} g_2} = \frac{\sup_{B_{k+2}} v_{k+1}^k}{\inf_{B_{k+2}} v_{k+1}^k} \leq C(d, \alpha). \tag{4.5}$$

Furthermore, since $v_{k+1}^k(x) \leq \sup_{B_0} v \leq C(d, \alpha, \underline{D})$ for all $x \in \mathbb{R}^d$, we get

$$g_1(x) = v_{k+1}^k(x) g'(x) \leq C(d, \alpha, \underline{D}), \quad x \in B_k.$$

If we extend g_1 to be equal to 0 on $\mathbb{R}^d \setminus B_k$, then g_1 is regular harmonic on B_{k+1} , nonnegative and bounded. Therefore, by the Harnack inequality in an explicit scale invariant formulation [14, Lemma 1]; see also Bass and Levin [8, Theorem 3.6] or Grzywny [43],

$$1 \leq \frac{\sup_{B_{k+2}} g_1}{\inf_{B_{k+2}} g_1} \leq C(d, \alpha, \underline{D}). \tag{4.6}$$

By (4.5) and (4.6), we get

$$\inf_{B_{k+2}} g' \geq C^{-2} \sup_{B_{k+2}} g' \geq \frac{1}{2} C^{-2}.$$

Therefore,

$$\text{Osc}_{B_{k+2}} g' \leq \max\left(\frac{1}{2}, 1 - \frac{1}{2} C^{-2}\right) = 1 - \frac{1}{2} C^{-2},$$

which ends the proof of Claim 2, and thus the lemma is proved. □

Proof of Theorem 1.4 By Lemma 4.1, we can assume without loss of generality that $|y - y'| \leq 1/16$.

We first consider the case $2^{k_0}|y' - y|^{1/2} \geq \delta_D(y)$, with k_0 from Lemma 4.3, and let $Q \in \partial D$ be such that $|y - Q| = \delta_D(y)$. Note that $y, y' \in B(Q, 2^{k_0+1}|y - y'|^{1/2})$, because $|y - y'| > 1$ implies $|y - y'| < |y - y'|^{1/2}$. We split the integral as follows:

$$\int_D \left| \frac{G_D(y, z)}{G_D(x_0, y)} - \frac{G_D(y', z)}{G_D(x_0, y')} \right|^p dz = \int_{D \cap B(Q, 2^{k_0+2}|y - y'|^{1/2})} + \int_{D \setminus B(Q, 2^{k_0+2}|y - y'|^{1/2})} \tag{4.7}$$

By (4.1), there exist $c = c(d, \alpha, \underline{D})$ and $\gamma = \gamma(d, \alpha, \underline{D}) \in (0, \alpha)$ such that

$$\begin{aligned} & \int_{D \cap B(Q, 2^{k_0+2}|y - y'|^{1/2})} \left| \frac{G_D(y, z)}{G_D(x_0, y)} - \frac{G_D(y', z)}{G_D(x_0, y')} \right|^p dz \\ & \leq 2^p \int_{D \cap B(Q, 2^{k_0+2}|y - y'|^{1/2})} \left(\left| \frac{G_D(y, z)}{G_D(x_0, y)} \right|^p + \left| \frac{G_D(y', z)}{G_D(x_0, y')} \right|^p \right) dz \\ & \leq c \int_{B(0, 2^{k_0+2}|y - y'|^{1/2})} |z|^{p(\alpha - \gamma - d)} dz \\ & = cC(d, \alpha, p)|y - y'|^{(d + p(\alpha - \gamma - d))/2}. \end{aligned}$$

In the second integral of (4.7) we use the boundary Harnack principle given in [13, Lemma 3]: we let $u(y) = G_D(y, z)$, $v(y) = G_D(x_0, y)$ and $r = 2^{k_0+1}|y - y'|^{1/2}$ there. By the Green function estimates (2.10) and arguments similar to the proof of Lemma 4.1 we find that for $z \in D \cap (B(Q, 2^{k+k_0+3}|y - y'|^{1/2}) \setminus B(Q, 2^{k+k_0+2}|y - y'|^{1/2}))$ we have $u(A_r(Q))/v(A_r(Q)) \leq C(d, \alpha, \underline{D})(2^k|y - y'|^{1/2})^{\alpha - \gamma - d}$, for all $k \in \{0, 1, \dots, N_0\}$, where $N_0 = \lceil \log_2(\text{diam}(D)/2^{k_0+2}|y - y'|^{1/2}) \rceil$ and we define u/v to be 0 outside D . Therefore, by [13, Lemma 3], there exist c and $\sigma > 0$ depending only on d, α, \underline{D} such that

$$\left| \frac{G_D(y, z)}{G_D(x_0, y)} - \frac{G_D(y', z)}{G_D(x_0, y')} \right|^p \leq c(2^k|y - y'|^{1/2})^{p(\alpha - \gamma - d)}|y - y'|^{\sigma p/2}$$

holds for all $z \in D \cap (B(Q, 2^{k+k_0+3}|y - y'|^{1/2}) \setminus B(Q, 2^{k+k_0+2}|y - y'|^{1/2}))$. Hence,

$$\begin{aligned} & \int_{D \setminus B(Q, 2^{k_0+2}|y - y'|^{1/2})} \left| \frac{G_D(y, z)}{G_D(x_0, y)} - \frac{G_D(y', z)}{G_D(x_0, y')} \right|^p dz \\ &= \sum_{k=0}^{N_0} \int_{D \cap (B(Q, 2^{k+k_0+3}|y - y'|^{1/2}) \setminus B(Q, 2^{k+k_0+2}|y - y'|^{1/2}))} \left| \frac{G_D(y, z)}{G_D(x_0, y)} - \frac{G_D(y', z)}{G_D(x_0, y')} \right|^p dz \\ &\leq c|y - y'|^{\sigma p/2} \sum_{k=0}^{N_0} (2^k|y - y'|^{1/2})^{p(\alpha - \gamma - d)} (2^k|y - y'|^{1/2})^d \\ &= c|y - y'|^{\sigma p/2} \sum_{k=0}^{N_0} (2^k|y - y'|^{1/2})^{d + p(\alpha - \gamma - d)}. \end{aligned}$$

The last sum is comparable to $\text{diam}(D)^{d + p(\alpha - \gamma - d)}$, so the proof is complete when $2^{k_0}|y - y'|^{1/2} \geq \delta_D(y)$.

Now assume that $2^{k_0}|y - y'|^{1/2} < \delta_D(y)$. We split the integral as follows:

$$\begin{aligned} & \int_D \left| \frac{G_D(y, z)}{G_D(x_0, y)} - \frac{G_D(y', z)}{G_D(x_0, y')} \right|^p dz \\ &= \int_{D \cap B(y, 2^{k_0}|y - y'|^{1/2})} + \int_{D \cap B(y, 2^{k_0}|y - y'|^{1/2})^c \cap B(Q, 3\delta_D(y))^c} \\ &+ \int_{D \cap B(y, 2^{k_0}|y - y'|^{1/2})^c \cap B(Q, 3\delta_D(y))}. \end{aligned} \tag{4.8}$$

The first two integrals are handled as the ones in (4.7). In particular, in the second one we can use the boundary Harnack principle. In the last integral on the right-hand side of (4.8) we will apply Lemma 4.3. To this end, we split once more:

$$\begin{aligned} \int_{D \cap B(y, 2^{k_0}|y - y'|^{1/2})^c \cap B(Q, 3\delta_D(y))} &\leq \sum_{k=0}^{M_0} \int_{D \cap B(Q, 3\delta_D(y)) \cap (B(y, 2^{k+k_0+1}|y - y'|^{1/2}) \setminus B(y, 2^{k+k_0}|y - y'|^{1/2}))} \\ &=: \sum_{k=0}^{M_0} I_k, \end{aligned}$$

where $M_0 = \lceil \log_2(3\delta_D(y)/(2^{k_0}|y - y'|^{1/2})) \rceil$. We then use Lemma 4.3 with $r = r_k = 2^{k_0+k}|y - y'|^{1/2}/4$:

$$\begin{aligned} I_k &\leq C(d, \alpha, \underline{D})|y - y'|^{\sigma p/2} \int_{B(y, 2^{k+k_0+1}|y - y'|^{1/2}) \setminus B(y, 2^{k+k_0}|y - y'|^{1/2})} r_k^{p(\alpha - d - \gamma)} dz \\ &\leq \tilde{C}(d, \alpha, \underline{D})|y - y'|^{\sigma p/2} (2^{k+k_0}|y - y'|^{1/2})^{d - p(\alpha - \gamma - d)}, \quad k = 0, \dots, M_0, \end{aligned}$$

since for $|y - y'| \leq 1/16$, we have $|y - y'| \leq |y - y'|^{1/2}/4$, so $y' \in B(y, 2^{-k_0}r)$. Therefore we get

$$\sum_{k=0}^{M_0} I_k \leq C(d, \alpha, \underline{D})|y - y'|^{\sigma p/2} \delta_D(y)^{d-p(\alpha-\gamma-d)},$$

which ends the proof. □

Proof of Theorem 1.2 Fix $x \in D$ and $t \in [T_1, T_2]$. First, we investigate η^{x_0} . By the results of Sect. 2.3,

$$p_t^D(x, y) = G_D \Delta_y^{\alpha/2} p_t^D(x, y).$$

Furthermore, by Corollary 2.6, the function $f(y) = \Delta_y^{\alpha/2} p_t^D(x, y)$ is bounded and the bound does not depend on $x \in D$. Therefore, by Theorem 1.4, for $y, y' \in D \setminus B(x_0, r_1)$,

$$\begin{aligned} \left| \frac{p_t^D(x, y)}{G_D(x_0, y)} - \frac{p_t^D(x, y')}{G_D(x_0, y')} \right| &\leq \int_D \left| \frac{G_D(y, z)}{G_D(x_0, y)} - \frac{G_D(y', z)}{G_D(x_0, y')} \right| |f(z)| dz \\ &\leq \left\| \frac{G_D(y, \cdot)}{G_D(x_0, y)} - \frac{G_D(y', \cdot)}{G_D(x_0, y')} \right\|_{L^1(D)} \|f\|_\infty \leq C|y - y'|^\sigma, \end{aligned}$$

where the constants C, σ depend only on $d, \alpha, \underline{D}, T_1, T_2, x_0$, and r_1 .

We now proceed to $\tilde{\eta}$. Note that there exist $x_1 \in D$ and $r = r(\underline{D})$ such that $B(x_1, 2r) \subset D$. Without loss of generality, we can assume that $|y - y'| < r/4$. Then, for any fixed y, y' , there exists x_2 such that $B(x_2, r/4) \subset D$ and $y, y' \notin B(x_2, r/4)$. This means that $G_D(x_2, y), G_D(x_2, y') \leq C$, where $C \geq 1$ depends only on d, α , and \underline{D} . We then split as follows:

$$\begin{aligned} \left| \frac{p_t^D(x, y)}{p_{t_0}^D(x_0, y)} - \frac{p_t^D(x, y')}{p_{t_0}^D(x_0, y')} \right| &= \left| \frac{p_t^D(x, y)}{G_D(x_2, y)} \frac{G_D(x_2, y)}{p_{t_0}^D(x_0, y)} - \frac{p_t^D(x, y')}{G_D(x_2, y')} \frac{G_D(x_2, y')}{p_{t_0}^D(x_0, y')} \right| \\ &\leq \frac{p_t^D(x, y)}{G_D(x_2, y)} \left| \frac{G_D(x_2, y)}{p_{t_0}^D(x_0, y)} - \frac{G_D(x_2, y')}{p_{t_0}^D(x_0, y')} \right| \\ &\quad + \frac{G_D(x_2, y')}{p_{t_0}^D(x_0, y')} \left| \frac{p_t^D(x, y')}{G_D(x_2, y')} - \frac{p_t^D(x, y)}{G_D(x_2, y)} \right|. \end{aligned} \tag{4.9}$$

By using Lemma 3.2 and (2.6), we find that

$$\frac{p_t^D(x, y)}{G_D(x_2, y)} \lesssim \frac{\mathbb{P}^y(\tau_D > t)}{G_D(x_2, y)} = \frac{G_D P_t^D \kappa_D(y)}{G_D(x_2, y)} \leq C(d, \alpha, \underline{D}, T_1, T_2) < \infty. \tag{4.10}$$

By similar arguments,

$$\frac{p_t^D(x_0, y)}{G_D(x_2, y)} \geq c(d, \alpha, \underline{D}, T_1, T_2, x_0) > 0. \tag{4.11}$$

From (4.9), (4.10), (4.11), and the Hölder regularity of η^{x_0} obtained above, we arrive at

$$\left| \frac{p_t^D(x, y)}{p_{t_0}^D(x_0, y)} - \frac{p_t^D(x, y')}{p_{t_0}^D(x_0, y')} \right| \leq C|y - y'|^\sigma,$$

with C and σ depending on $d, \alpha, \underline{D}, T_1, T_2, x_0$.

The arguments for η are similar to the ones for $\tilde{\eta}$, with no dependence on x_0 . The proof is complete. □

5 Space-time stable processes and caloric functions

5.1 Preliminaries

Recall that $(X_s)_{s \geq 0}$ is the isotropic α -stable Lévy process. Like for the space-time Brownian motion [36], we define the *space-time α -stable process* as the following Lévy process on \mathbb{R}^{d+1} :

$$\dot{X}_s := (-s, X_s), \quad s \geq 0.$$

Since \dot{X} is a Lévy process, it has the strong Markov property. Many properties of the space-time process are inherited from the α -stable process. Thus, for a (Borel) set $A \subseteq \mathbb{R}^{d+1}$, we let

$$\mathbb{P}^{(t,x)}(\dot{X}_s \in A) := \mathbb{P}((t-s, X_s + x) \in A),$$

and for a (Borel) function $f: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$, we have

$$\mathbb{E}^{(t,x)}[f(\dot{X}_s)] = \mathbb{E}[f(t-s, X_s + x)].$$

It can be easily verified that the transition probability of \dot{X} takes on the following form

$$\tilde{p}_s(t, x, du, dy) = p_s(x, y) dy \otimes \delta_{[t-s]}(du), \quad s \geq 0, (t, x), (u, y) \in \mathbb{R} \times \mathbb{R}^d.$$

The corresponding semigroup will be denoted by \tilde{P} . Let

$$C_b^{1,2}([0, \infty) \times \mathbb{R}^d) = \{u \in C_b([0, \infty) \times \mathbb{R}^d) : \partial_t u, \nabla_x u, D_x^2 u \in C_b([0, \infty) \times \mathbb{R}^d)\},$$

with the norm $\|u\|_{C^{1,2}} = \|u\|_\infty + \|\partial_t u\|_\infty + \|\nabla_x u\|_\infty + \|D_x^2 u\|_\infty$.

Lemma 5.1 *The pointwise generator of the semigroup of the space-time α -stable process coincides with the fractional heat operator $\Delta^{\alpha/2} - \partial_t$ for functions $u \in C_b^{1,2}([0, \infty) \times \mathbb{R}^d)$.*

Proof Let $u \in C_b^{1,2}([0, \infty) \times \mathbb{R}^d)$. For all $(t, x) \in [0, \infty) \times \mathbb{R}^d$ and $s \in (0, t)$, we have

$$\begin{aligned} \frac{1}{s}(\tilde{P}_s u(t, x) - u(t, x)) &= \frac{1}{s} \int_{\mathbb{R}^d} \int_{[0, \infty)} (u(r, y) - u(t, x)) \tilde{p}_s(t, x, dy, dr) \\ &= \frac{1}{s} \int_{\mathbb{R}^d} (u(t-s, y) - u(t, x)) p_s(x, y) dy \\ &= \frac{1}{s} \int_{\mathbb{R}^d} (u(t-s, y) - u(t-s, x)) p_s(x, y) dy \quad (5.1) \\ &\quad + \frac{1}{s}(u(t-s, x) - u(t, x)). \quad (5.2) \end{aligned}$$

Clearly, (5.2) converges to $-\partial_t u(t, x)$ as $s \rightarrow 0^+$, so it suffices to show that (5.1) converges to $\Delta_x^{\alpha/2} u(t, x)$. To this end, we will prove that

$$\frac{1}{s} \int_{\mathbb{R}^d} ((u(t, y) - u(t, x)) - (u(t-s, y) - u(t-s, x))) p_s(x, y) dy$$

converges to 0 as $s \rightarrow 0^+$. Let $\varepsilon > 0$ and let $\delta > 0$ be so small that $p_s(x, B(x, \delta)^c) < \varepsilon$. Then we also have $p_{s'}(x, B(x, \delta)^c) < \varepsilon$ for $s' \in (0, s)$. By Lagrange’s mean value theorem, we get

$$\left| \frac{1}{s} \int_{B(0, \delta)^c} ((u(t, y) - u(t, x)) - (u(t-s, y) - u(t-s, x))) p_s(x, y) dy \right| < 2\varepsilon \|u\|_{C^{1,2}}.$$

By Taylor’s expansion, $u(t-s, x) = u(t, x) - s\partial_t u(t, x) + o(s)$ as $s \rightarrow 0$, and similarly for y , so

$$\begin{aligned} &\left| \frac{1}{s} \int_{B(0, \delta)} ((u(t, y) - u(t, x)) - (u(t-s, y) - u(t-s, x))) p_s(x, y) dy \right| \\ &= \left| \int_{B(0, \delta)} (\partial_t u(t, x) - \partial_t u(t, y) + \frac{o(s)}{s}) p_s(x, y) dy \right| \\ &\leq \delta \|u\|_{C^{1,2}} + o(1). \end{aligned}$$

This ends the proof. □

In the next result we exhibit a space-time Poisson kernel for cylindrical domains. As usual, for arbitrary (open) $G \subseteq \mathbb{R} \times \mathbb{R}^d$, we let

$$\tau_G := \inf\{t > 0 : \dot{X}_t \notin G\}.$$

Lemma 5.2 *Recall that $D \subseteq \mathbb{R}^d$ is a Lipschitz open set and let $\dot{D} = (r, t) \times D$ for some (arbitrary) $-\infty \leq r < t$. Then the distribution of \dot{X}_{τ_D} —the first exit place of \dot{X}*

from \dot{D} —is given by the formula

$$\mathbb{P}^{(t,x)}(\dot{X}_{\tau_{\dot{D}}} \in (ds, dy)) = \begin{cases} \mathbf{I}_{[r,t)}(s) ds \otimes J^D(t, x, s, y) dy + \delta_{t-r}(ds) \otimes p_{t-s}^D(x, y) dy, & r > -\infty, \\ \mathbf{I}_{(-\infty,t)}(s) ds \otimes J^D(t, x, s, y) dy, & r = -\infty, \end{cases}$$

where

$$J^D(t, x, s, y) := \int_D p_{t-s}^D(x, \xi) v(\xi, y) d\xi, \quad s < t, \quad x \in D, \quad y \in D^c.$$

We call J^D the *lateral Poisson kernel*.

Remark 5.3 For the cylinder $\dot{D} = (r, t) \times D$, if the process \dot{X} starts at (t, x) with some $x \in D$, then it immediately enters \dot{D} , so $\tau_{\dot{D}} > 0$ almost surely, although $(t, x) \notin \dot{D}$. In the language of potential theory, the points on the *top* of the cylinder are *irregular*.

Proof of Lemma 5.2 Let $r > -\infty$. We have

$$\mathbb{P}^{(t,x)}(\dot{X}_{\tau_{\dot{D}}} \in (ds, dy)) = \mathbb{P}^{(t,x)}(\dot{X}_{\tau_{\dot{D}}} \in (ds, dy), \tau_{\dot{D}} > \tau_D) \tag{5.3}$$

$$+ \mathbb{P}^{(t,x)}(\dot{X}_{\tau_{\dot{D}}} \in (ds, dy), \tau_{\dot{D}} = \tau_D) \tag{5.4}$$

$$+ \mathbb{P}^{(t,x)}(\dot{X}_{\tau_{\dot{D}}} \in (ds, dy), \tau_{\dot{D}} < \tau_D). \tag{5.5}$$

Note that (5.3) vanishes, because $\mathbb{P}^{(t,x)}(\tau_{\dot{D}} > \tau_D) = 0$.

By the Ikeda–Watanabe formula (2.13), the term (5.4) is equal to

$$\mathbb{P}^{(t,x)}(X_{\tau_D} \in A, \tau_D \leq t - r, \tau_D \in ds) = \mathbf{1}_{[r,t)}(s) ds \otimes J^D(t, x, s, y) dy.$$

In (5.5) we have $\tau_D > \tau_{\dot{D}} = t - r$, so by the definition of the Dirichlet heat kernel, this term is equal to

$$\delta_{t-r}(ds) \otimes p_{t-r}^D(x, y),$$

see [32, Chapter 2]. The case of $r = -\infty$ is left to the reader. □

We see that $J^D(t, x, s, y)$ represents the scenario of \dot{X} starting at (t, x) and leaving to (s, y) , where, recall, $x \in D, y \in D^c$, and $s < t$. Another way to express the result in Lemma 5.2, is as follows:

$$\mathbb{E}^{(t,x)} u(\dot{X}_{\tau_{\dot{D}}}) = \int_r^t \int_{D^c} J^D(t, x, s, z) u(s, z) dz ds + \int_D p_{t-r}^D(x, y) u(r, y) dy, \tag{5.6}$$

whenever this integral makes sense, e.g., for nonnegative u . By analogy to the elliptic equations, we call the right-hand side of (5.6) the *Poisson integral*, and the first term on the right-hand side of (5.6)—the *lateral Poisson integral*.

Remark 5.4 Another motivation for calling $J^D(t, x, s, z)$ the lateral Poisson kernel comes from the fact that it is the *nonlocal normal derivative* of p_{t-s}^D , whereas p_{t-s}^D serves as the Green function for the fractional heat equation. Indeed, using the definition of the nonlocal normal derivative from [34]:

$$[\partial_{\bar{n}}u](x) := \int_D (u(y) - u(x))v(x, y) dy, \quad x \in D^c,$$

we see that for every $z \in D^c$,

$$\partial_{\bar{n}} p_{t-s}^D(x, \cdot)(z) = \int_D p_{t-s}^D(x, y)v(y, z) dy = J^D(t, x, s, z), \quad x \in D.$$

5.2 Caloric functions

We define the caloric functions in terms of the mean value property. We stress that we only consider finite nonnegative functions.

Definition 5.5 Let $-\infty < T_1 < T_2 < \infty$. We say that $u: (T_1, T_2) \times \mathbb{R}^d \rightarrow [0, \infty)$ is *caloric* in $(T_1, T_2) \times D$, if the *mean value property*:

$$u(t, x) = \mathbb{E}^{(t,x)} u(\dot{X}_{\tau_G}), \quad (t, x) \in (T_1, T_2) \times D, \quad (5.7)$$

holds for every open set $G \subset\subset (T_1, T_2) \times D$.

We say that $u: [T_1, T_2) \times \mathbb{R}^d \rightarrow [0, \infty)$ is *caloric* in $[T_1, T_2) \times D$ if (5.7) holds for every open $G \subset\subset [T_1, T_2) \times D$. If u is caloric in $[T_1, T_2) \times D$ and satisfies (5.7) for $G = (T_1, T_2) \times D$, then we say that u is *regular caloric*. If u is caloric in $[T_1, T_2) \times D$ and $u \equiv 0$ on the *parabolic boundary*

$$D^p := (\{T_1\} \times D) \cup ((T_1, T_2) \times D^c),$$

then we say that u is *singular caloric*.

Remark 5.6 (a) Our caloric functions are just harmonic functions of the space-time isotropic stable Lévy process.

- (b) We may also consider $T_1 = -\infty$ or $T_2 = \infty$, where appropriate, in particular when defining functions caloric on $(T_1, T_2) \times D$.
- (c) The condition $G \subset\subset [T_1, T_2) \times D$ allows G to *touch* $\{T_1\} \times D$. Caloricity in $[T_1, T_2) \times D$ may be considered as a (new) relaxation of regular caloricity, *localized* near the part $\{T_1\} \times D$ of the boundary of $(T_1, T_2) \times D$, see also Lemma 5.7. Both notions are meant to facilitate discussion of boundary conditions (they generalize to harmonic functions of other strong Markov processes).
- (d) The caloricity in $[T_1, T_2) \times D$ helps to handle initial conditions which are functions, but also rules out some interesting cases, e.g., $(t, y) \mapsto p_t^D(x, y)$. See also [23]. Remarkably, every (nonnegative) function caloric in $(T_1, T_2) \times D$ has a certain initial condition which is a measure, see Sect. 6.

- (e) A caloric function need not satisfy the fractional heat equation pointwise, due to lack of time regularity. This can be seen using the counterexample given by Chang-Lara and Dávila [29, Section 2.4.1] for viscosity solutions. See also Remark 5.12 below.

Lemma 5.7 *Regular caloricity implies caloricity in $[T_1, T_2] \times D$, which in turn implies caloricity in $(T_1, T_2) \times D$. Furthermore, (5.7) only needs to be verified for cylinders G .*

Proof Assume that (5.7) holds for G . By the strong Markov property of \dot{X} , (5.7) then holds for every open $G' \subset G$:

$$u(t, x) = \mathbb{E}^{(t,x)} u(\dot{X}_{\tau_G}) = \mathbb{E}^{(t,x)} \mathbb{E}^{\dot{X}_{\tau_{G'}}} u(\dot{X}_{\tau_G}) = \mathbb{E}^{(t,x)} u(\dot{X}_{\tau_{G'}}).$$

This first two assertions follow immediately. To clarify the third one, note that every open $G' \subset\subset [T_1, T_2] \times D$ is contained in an open cylinder, relatively compact in $[T_1, T_2] \times D$. Similarly for $(T_1, T_2) \times D$. □

We continue with several examples of caloric functions.

Example 5.8 For every fixed $x \in \mathbb{R}^d$, the function $(t, y) \mapsto p_t^D(x, y)$ satisfies the mean value property on every $(\varepsilon, T) \times D$ for $0 < \varepsilon < T < \infty$, hence it is caloric in $(0, \infty) \times D$.

Example 5.9 If we let

$$\eta_{t,Q}(x) := 0, \quad (t, x) \in (-\infty, 0] \times \mathbb{R}^d \cup (0, \infty) \times D^c, \quad Q \in \partial D, \quad (5.8)$$

then for every fixed $Q \in \partial D$, the function $(t, x) \mapsto \eta_{t,Q}(x)$ is caloric in $(-\infty, \infty) \times D$. Indeed, the mean value property in $(\varepsilon, T) \times D$, with $0 < \varepsilon < T < \infty$ is a consequence of (1.9). Then, by Lemma 3.5,

$$\begin{aligned} \eta_{t,Q}(x) &= \int_0^t \int_{U^c} J^D(t, x, s, z) \eta_{s,Q}(z) dz ds \\ &= \int_{-R}^t \int_{U^c} J^D(t, x, s, z) \eta_{s,Q}(z) dz ds, \end{aligned}$$

for any $R \geq 0$.

Example 5.10 If $f : \mathbb{R}^d \rightarrow [0, \infty)$ is a nonnegative measurable function and $P_1^D f(x)$ is finite for all $x \in D$, then $(t, x) \mapsto P_t^D f(x)$ is caloric in $[0, \infty) \times D$, with the usual convention $P_0^D f := f$.

The following class of functions is of particular interest for us. We will show in the next section that it coincides with the class of all singular caloric functions.

Lemma 5.11 *If $\mu(dQ ds)$ is a locally finite nonnegative Borel measure on $\partial D \times [0, \infty)$, then*

$$h(t, x) := \begin{cases} \int_{[0,t)} \int_{\partial D} \eta_{t-\tau, Q}(x) \mu(dQ d\tau), & t > 0, x \in D, \\ 0, & \text{elsewhere,} \end{cases}$$

is singular caloric in $[0, \infty) \times D$.

Proof By Lemma 3.5, h is finite for all $t > 0$ and $x \in D$, and by (5.8), we have

$$\int_{[0,t)} \int_{\partial D} \eta_{t-\tau, Q}(x) \mu(dQ d\tau) = \int_{[0,\infty)} \int_{\partial D} \eta_{t-\tau, Q}(x) \mu(dQ d\tau), \quad t \geq 0, x \in D.$$

Therefore, the mean value property for h follows from Fubini–Tonelli and caloricity of η . □

Remark 5.12 We note that the viscosity solution considered in [29, Section 2.4.1], although non-differentiable, is Lipschitz in time. The function $n_{t, Q}$ is not even Lipschitz in t because for $t \in (0, 1)$ and fixed $x \in D$,

$$\begin{aligned} \frac{\eta_{t, Q}(x)}{t} &= \frac{1}{t} \lim_{y \rightarrow Q} \frac{p_t^D(x, y)}{\mathbb{P}^y(\tau_D > t)} \frac{\mathbb{P}^y(\tau_D > t)}{\mathbb{P}^y(\tau_D > 1)} \gtrsim \frac{p_t(x, Q)}{t} \lim_{y \rightarrow Q} \frac{\mathbb{P}^y(\tau_D > t)}{\mathbb{P}^y(\tau_D > 1)} \\ &\gtrsim |x - Q|^{-d-\alpha} \lim_{y \rightarrow Q} \frac{\mathbb{P}^y(\tau_D > t)}{\mathbb{P}^y(\tau_D > 1)}. \end{aligned}$$

We see, indeed, that the last limit is comparable to $t^{-1/2}$ if D is $C^{1,1}$ by (2.7). Furthermore, for Lipschitz D it also explodes as $t \rightarrow 0^+$ because of the proof of Lemma B.2 and [12, Lemma 3].

Lemma 5.13 *If u is caloric in $(T_1, T_2) \times D$ for $T_1 < T_2$, then $u \in L^1_{\text{loc}}((T_1, T_2) \times \mathbb{R}^d)$.*

Proof The proof is similar to the one of [18, Lemma 4.5]. First note that for any fixed $x \in D, r > 0$, and $B = B(x, r)$, by (2.6) we have

$$\begin{aligned} J^B(t, x, s, z) &= \int_B p_{t-s}^B(x, y) v(y, z) dy \approx \int_B p_{t-s}(x, y) \mathbb{P}^y(\tau_B > t - s) v(y, z) dy \\ &\geq c \int_{B(x, r/2)} p_{t-s}(x, y) dy \geq C > 0, \end{aligned}$$

with C depending only on r and R , where $\delta_B(z), t - s \leq R$. Thus, $J^B(t, x, \cdot, \cdot)$ is locally bounded from below. Now, take two disjoint balls $B_1, B_2 \subseteq D$, centered at some points $x_1, x_2 \in D$ respectively, and let $T_1 < t_0 < t < T_2$ and $R > 0$. Since u is nonnegative and caloric, for $i = 1, 2$ we get

$$\infty > u(t, x) \geq \int_{t_0}^t \int_{B_i^c} u(s, z) J^{B_i}(t, x, s, z) dz ds \geq C \int_{t_0}^t \int_{B(0, R) \setminus B_i} u(s, z) dz ds.$$

Therefore $u \in L^1((t_0, t) \times (B(0, R) \setminus B_i))$ for $i = 1, 2$. But $B_1 \cap B_2 = \emptyset$, so $u \in L^1((t_0, t) \times B(0, R))$. Since R can be chosen arbitrarily large, the proof is complete. \square

The following result shows that the so-called ancient solutions, i.e., functions caloric in a time interval of the form $(-\infty, T)$, can be conveniently studied by considering only the lateral Poisson integrals.

Lemma 5.14 *If u is caloric in $(-\infty, T) \times D$ for some $T \in \mathbb{R}$, then for all $x \in U \subset\subset D$ and $t < T$ we have*

$$u(t, x) = \mathbb{E}^{(t,x)}[u(\tau_{(-\infty,t) \times U}, X_{\tau_{(-\infty,t) \times U}})] = \int_{-\infty}^t \int_{U^c} J^U(t, x, s, z) u(s, z) dz ds. \tag{5.9}$$

In particular, the integral on the right-hand side of (5.9) is finite.

Proof Let t, x, U be as in the statement. By the definition of caloricity, for $v < t$ we have

$$u(t, x) = \int_v^t \int_{U^c} J^U(t, x, s, z) u(s, z) dz ds + \int_U p_{t-v}^U(x, y) u(v, y) dy.$$

The first integral on the right-hand side increases to the right-hand side of (5.9) by the monotone convergence theorem and the second integral decreases. It suffices to prove that

$$a := \lim_{v \rightarrow -\infty} \int_U p_{t-v}^U(x, y) u(v, y) dy = 0.$$

To this end note that for every $v < t$,

$$\int_U p_{t-v}^U(x, y) u(v, y) dy \geq a.$$

Let $n > 0$ be so large that $U \subset\subset D_n$ (see (2.1)). Recall that $\lambda_1(V)$ is the first eigenvalue of the Dirichlet fractional Laplacian for an open set V . We claim that

$$\lambda_1(D_n) < \lambda_1(U). \tag{5.10}$$

A weak inequality is well known as the domain monotonicity. In order to prove the strict inequality, assume without loss of generality that $0 \in U$. Then there exists $q > 1$ such that $qU \subset\subset D_n$, so, by domain monotonicity, $\lambda_1(D_n) \leq \lambda_1(qU) = q^{-\alpha} \lambda_1(U)$, which yields (5.10).

By (2.9), (2.8), and the fact that each eigenfunction is bounded from above and bounded from below away from the boundary, for $s < t$, $s \rightarrow -\infty$, we get

$$\begin{aligned} \infty > u(t, x) &\geq \int_{D_n} u(s, y) p_{t-s}^{D_n}(x, y) dy \geq \int_U u(s, y) p_{t-s}^{D_n}(x, y) dy \\ &\approx \int_U u(s, y) e^{-\lambda_1(D_n)(t-s)} dy \\ &= e^{(-\lambda_1(D_n) + \lambda_1(U))(t-s)} \int_U u(s, y) e^{-\lambda_1(U)(t-s)} dy \\ &\gtrsim e^{(-\lambda_1(D_n) + \lambda_1(U))(t-s)} \int_U u(s, y) p_{t-s}^U(x, y) dy. \end{aligned}$$

By (5.10), we must have $a = 0$. □

5.3 Caloric functions are continuous

This subsection is devoted to proving that caloric functions are continuous, hence locally bounded.

The proof is based on certain estimates for the kernel J^D , which may be of independent interest. Let us note in passing that *bounded* caloric functions are known to be locally Hölder continuous [31, Theorem 4.14].

Proposition 5.15 *Assume that u is a nonnegative caloric function in $(T_0, T_1) \times D$ for some $T_0 < T_1$. Then, u is continuous and locally bounded therein.*

We fix arbitrary $(t_0, x_0) \in (T_0, T_1) \times D$, $r \in (0, \delta_D(x_0)/2)$, and let $B_\rho = B(x_0, \rho)$ for $\rho > 0$. We first establish some basic facts about the lateral Poisson kernel. With a slight conflict of notation, we introduce the Euclidean distance between $A, B \subseteq \mathbb{R}^d$,

$$d(A, B) := \inf\{|b - a| : a \in A, b \in B\}.$$

Lemma 5.16 *Let D be a Lipschitz open set, $U \subset\subset D$, and $0 < T < \infty$. Then,*

$$J^D(t, x, s, z) \approx J^D(t, x_0, s, z), \quad x \in U, z \in D^c, \quad 0 < t - s < T, \quad (5.11)$$

and

$$J^D(t, x, s, z) \lesssim J^D(t', x, s, z), \quad x \in U, z \in D^c, \quad 0 < t - s \leq t' - s < T, \quad (5.12)$$

with the comparability constants depending only on $d, \alpha, \underline{D}, d(U, D^c)$, and T .

Proof Let U' be such that $U \subset\subset U' \subset\subset D$. We pick U' so that the constants below depend only on D and U , e.g., by assuming $d(U, D^c)/2 \geq d(U', D^c) \geq d(U, D^c)/3$. We first prove (5.11). By (2.6),

$$\begin{aligned}
 J^D(t, x, s, z) &= \int_D p_{t-s}^D(x, y)v(y, z) dy \\
 &\approx \mathbb{P}^x(\tau_D > t - s) \int_D p_{t-s}(x, y)\mathbb{P}^y(\tau_D > t - s)v(y, z) dy \\
 &\approx \mathbb{P}^{x_0}(\tau_D > t - s) \left(\int_{D \setminus U'} + \int_{U'} \right) p_{t-s}(x, y)\mathbb{P}^y(\tau_D > t - s)v(y, z) dy,
 \end{aligned}
 \tag{5.13}$$

with constants depending on $d, \alpha, \underline{D}, d(U, D^c)$, and T . For $y \in D \setminus U'$, we have $|x - y| \approx |x_0 - y|$, so by (2.4),

$$\begin{aligned}
 &\int_{D \setminus U'} p_{t-s}(x, y)\mathbb{P}^y(\tau_D > t - s)v(y, z) dy \\
 &\approx \int_{D \setminus U'} p_{t-s}(x_0, y)\mathbb{P}^y(\tau_D > t - s)v(y, z) dy.
 \end{aligned}$$

For $y \in U'$, $\mathbb{P}^y(\tau_D > t - s) \approx 1$ and $v(y, z) \approx v(x_0, z)$. Using this and the fact that $U \subset\subset U'$, we find that

$$\begin{aligned}
 \int_{U'} p_{t-s}(x, y)\mathbb{P}^y(\tau_D > t - s)v(y, z) dy &\approx v(x_0, z) \int_{U'} p_{t-s}(x, y) dy \\
 &\approx v(x_0, z) \int_{U'} p_{t-s}(x_0, y) dy \\
 &\approx \int_{U'} p_{t-s}(x_0, y)\mathbb{P}^y(\tau_D > t - s)v(y, z) dy.
 \end{aligned}$$

Coming back to (5.13), we obtain (5.11). We now proceed to proving (5.12). We split in a similar way:

$$J^D(t, x, s, z) = \left(\int_{U'} + \int_{D \setminus U'} \right) p_{t-s}^D(x, y)v(y, z) dy.$$

By Lemma B.1,

$$\int_{D \setminus U'} p_{t-s}^D(x, y)v(y, z) dy \lesssim \int_{D \setminus U'} p_{t-s}^D(x, y)v(y, z) dy.$$

For the integral over U' we use (2.6):

$$\begin{aligned} & \int_{U'} p_{t-s}^D(x, y) v(y, z) dy \\ & \approx v(x_0, z) \int_{U'} p_{t-s}^D(x, y) dy \\ & \approx v(x_0, z) \int_{U'} p_{t-s}(x, y) \mathbb{P}^x(\tau_D > t-s) \mathbb{P}^y(\tau_D > t-s) dy. \end{aligned}$$

For $w \in U'$ and $0 < t-s < T$, we have $\mathbb{P}^w(\tau_D > t-s) \approx 1$ and by (2.4), $\int_{U'} p_{t-s}(x, y) dy \approx 1$, with comparability constants depending only on T , U' , and D . It follows that

$$\begin{aligned} & v(x_0, z) \int_{U'} p_{t-s}(x, y) \mathbb{P}^x(\tau_D > t-s) \mathbb{P}^y(\tau_D > t-s) dy \\ & \approx v(x_0, z) \int_{U'} p_{t'-s}(x, y) \mathbb{P}^x(\tau_D > t'-s) \mathbb{P}^y(\tau_D > t'-s) dy \\ & \approx \int_{U'} p_{t'-s}^D(x, y) v(y, z) dy, \end{aligned}$$

which ends the proof. \square

Proof of Proposition 5.15 We will show continuity at the fixed point (t_0, x_0) . Let $x \in B_{r/2}$, $t_1 \in (T_0, t_0)$ and $t \in (t_1, T_1)$, so that $T_1 < t_1 < t < T_0$. We have

$$\begin{aligned} u(t, x) &= \int_{B_r} u(t_1, y) p_{t-t_1}^{B_r}(x, y) dy + \int_{t_1}^t \int_{B_r} u(\tau, z) J^{B_r}(t, x, \tau, z) dz d\tau, \\ u(t_0, x_0) &= \int_{B_r} u(t_1, y) p_{t_0-t_1}^{B_r}(x_0, y) dy + \int_{t_1}^{t_0} \int_{B_r} u(\tau, z) J^{B_r}(t_0, x_0, \tau, z) dz d\tau. \end{aligned}$$

Since u is nonnegative and caloric, all integrals above are finite. For (t, x) sufficiently close to (t_0, x_0) , we have $p_{t-t_1}^{B_r}(x, y) \approx p_{t_0-t_1}^{B_r}(x_0, y)$ uniformly in y . Therefore, by the dominated convergence theorem,

$$\int_{B_r} u(t_1, y) p_{t-t_1}^{B_r}(x, y) dy \xrightarrow{(t,x) \rightarrow (t_0,x_0)} \int_{B_r} u(t_1, y) p_{t_0-t_1}^{B_r}(x_0, y) dy.$$

Therefore it remains to show that

$$\int_{t_1}^t \int_{B_r} u(\tau, z) J^{B_r}(t, x, \tau, z) dz d\tau \xrightarrow{(t,x) \rightarrow (t_0,x_0)} \int_{t_1}^{t_0} \int_{B_r} u(\tau, z) J^{B_r}(t_0, x_0, \tau, z) dz d\tau.$$

Assume that $t > t_0$ (we skip the other case, as it is similar). Then,

$$\begin{aligned} & \left| \int_{t_1}^t \int_{B_r} u(\tau, z) J^{B_r}(t, x, \tau, z) dz d\tau - \int_{t_1}^{t_0} \int_{B_r} u(\tau, z) J^{B_r}(t_0, x_0, \tau, z) dz d\tau \right| \\ & \leq \int_{t_1}^{t_0} \int_{B_r} u(\tau, z) |J^{B_r}(t, x, \tau, z) - J^{B_r}(t_0, x_0, \tau, z)| dz d\tau \\ & \quad + \int_{t_0}^t \int_{B_r} u(\tau, z) J^{B_r}(t, x, \tau, z) dz d\tau =: I_1 + I_2. \end{aligned}$$

By Lemma 5.16, we have $J^{B_r}(t, x, \tau, z) \lesssim J^{B_r}(t_0 + \varepsilon, x_0, \tau, z)$ for $t_1 \leq \tau \leq t \leq t_0 + \varepsilon$, $x \in B_{r/2}$, and $z \in B_r^c$. Therefore by the dominated convergence theorem, $I_2 \rightarrow 0$. Furthermore, by the properties of $p_t^{B_r}$ and the dominated convergence theorem, it is easy to see that $J^{B_r}(\cdot, \cdot, \tau, z)$ is continuous on $(\tau, \infty) \times B_r$ for all $\tau \in \mathbb{R}$ and $z \in D^c$. Therefore, using the bounds of Lemma 5.16 and the dominated convergence theorem once again, we find that $I_1 \rightarrow 0$ as well. This ends the proof. \square

6 Representation of caloric functions in Lipschitz open sets

We first discuss the representation for functions caloric on $[0, T) \times D$, where the meaning of the initial condition is clearer. We then use this case to resolve the situation of functions caloric in $(0, T) \times D$.

6.1 Functions caloric up to time 0

Lemma 6.1 *Assume that u is a nonnegative caloric function in $\dot{D} := [0, T) \times D$. Then there exists a unique decomposition $u = r + s$, where r is regular caloric in \dot{D} and s is singular caloric in \dot{D} .*

Proof Let $t < T$. Since u has the mean value property in every $\dot{D}_n = (0, t) \times D_n$ (see (2.1)), we have

$$u(t, x) = \mathbb{E}^{(t,x)} u(\dot{X}_{\tau_{\dot{D}_n}}) =: i_n(t, x) + l_n(t, x) + s_n(t, x),$$

where

$$\begin{aligned} i_n(t, x) &= \mathbb{E}^{(t,x)} [u(\dot{X}_{\tau_{\dot{D}_n}}); \tau_{\dot{D}_n} > t], \\ l_n(t, x) &= \mathbb{E}^{(t,x)} [u(\dot{X}_{\tau_{\dot{D}_n}}); \tau_{\dot{D}_n} < t, \tau_{\dot{D}_n} = \tau_D], \\ s_n(t, x) &= \mathbb{E}^{(t,x)} [u(\dot{X}_{\tau_{\dot{D}_n}}); \tau_{\dot{D}_n} < t, \tau_{\dot{D}_n} < \tau_D]. \end{aligned}$$

We let $n \rightarrow \infty$. By the monotone convergence, we get

$$i_n(t, x) = \mathbb{E}^{(t,x)} (u(\dot{X}_t); \tau_{D_n} > t) \nearrow \mathbb{E}^{(t,x)} [u(\dot{X}_t); \tau_D > t] =: i(t, x),$$

and by [12, (5.40)],

$$l_n(t, x) = \mathbb{E}^{(t,x)}[u(\dot{X}_{\tau_D}); \tau_D < t, \tau_{D_n} = \tau_D] \nearrow \mathbb{E}^{(t,x)}[u(\dot{X}_{\tau_D}); \tau_D < t] =: l(t, x),$$

the limits being finite because all $i_n, l_n,$ and s_n are nonnegative. So, $s_n(t, x)$ converges to some $s(t, x)$. Since $r(t, x) := i(t, x) + l(t, x) = \mathbb{E}^{(t,x)}u(\dot{X}_{\tau_D})$, r is regular caloric. By inspecting the definition of s_n , we find that s is singular caloric: indeed, if X_t starts from $x \in D^c$, then the event $\tau_{D_n} < \tau_D$ has probability 0, so $s_n(t, x) = 0$ for $x \in D^c$, and if \dot{X} starts from $(0, x), x \in D$, then $s_n(0, x) = 0$ because $\tau_{D_n} \geq 0$.

Assume that there is another decomposition $u = r' + s'$. Since $s' = s = 0$ on D^p , we have that $r - r' = 0$ on D^p as well and therefore $r - r' = 0$ in \dot{D} , because $r - r'$ is regular caloric on \dot{D} . □

We next give an integral representation for the singular caloric part, with the use of the parabolic Martin kernel. We first prove the following technical result.

Lemma 6.2 *Let $x \in D$ and $0 < \varepsilon < T$ be fixed. Then there exists a modulus of continuity ω , independent of y and $t \in [\varepsilon, T]$, such that for n large we have*

$$\left| \frac{p_t^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} - \frac{p_t^D(x, y)}{\mathbb{P}^y(\tau_D > 1)} \right| \leq \omega\left(\frac{1}{n}\right), \quad y \in D_n, \quad t \in [\varepsilon, T]. \tag{6.1}$$

Proof First note that the expression on the left-hand side of (6.1) converges to 0 as $n \rightarrow \infty$ for every fixed $y \in D$ (the expression is considered only when $1/n < \delta_D(y)$). In order to get (6.1) we will show that the convergence is uniform by using the Arzelà–Ascoli theorem. Indeed, by Theorem 1.2, we find that $\overline{D_n} \ni y \mapsto p_t^{D_n}(x, y)/\mathbb{P}^y(\tau_{D_n} > 1)$ are uniformly Hölder continuous for n large and $t \in [\varepsilon, T]$. Furthermore, it is well-known that a Hölder continuous function in $\overline{D_n}$ can be extended to a function on \overline{D} with the same Hölder regularity, see, e.g., Banach [5, IV (7.5)]. If we denote the corresponding extensions by f_n , then by the Arzelà–Ascoli theorem, we find that

$$\left| f_n(t, y) - \frac{p_t^D(x, y)}{\mathbb{P}^y(\tau_D > 1)} \right| \leq \omega\left(\frac{1}{n}\right), \quad y \in D, \quad t \in [\varepsilon, T].$$

In particular, (6.1) follows. □

Theorem 6.3 *Assume that u is singular caloric in $[0, T) \times D$. Then there exists a nonnegative Borel measure μ on $\partial D \times [0, T)$ such that representation (1.2) holds.*

Proof Let D_n be as in Lemma 6.1 and let N be large enough, so that $x, x_0 \in D_N$. Since u is singular caloric, for natural $n > N$ we have

$$\begin{aligned} u(t, x) &= \mathbb{E}^{(t,x)}[u(\dot{X}_{\tau_{D_n}}); \tau_{D_n} < t, X_{\tau_{D_n}} \in D \setminus D_n] \\ &= \int_0^t \int_{D \setminus D_n} u(s, z) \int_D p_{t-s}^{D_n}(x, y)v(y, z) dy dz ds \\ &= \int_0^t \int_D \frac{p_{t-s}^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} \int_{D \setminus D_n} \mathbb{P}^y(\tau_{D_n} > 1)u(s, z)v(y, z) dz dy ds. \end{aligned}$$

We define

$$\mu_n(dy ds) = \int_{D \setminus D_n} \mathbb{P}^y(\tau_{D_n} > 1)u(s, z)v(y, z) dz dy ds.$$

Note that by (2.6), if we fix $\theta > 0$, then we have $\mathbb{P}^y(\tau_{D_n} > 1) \lesssim p_{s+\theta}^{D_n}(x_0, y)$ uniformly in $s \in (0, t)$. Therefore, since u is caloric, for θ sufficiently small we have

$$\int_0^t \int_{\mathbb{R}^d} \mu_n(dy ds) \lesssim \int_0^t \int_{\mathbb{R}^d} \int_{D \setminus D_n} p_{t+\theta-s}^{D_n}(x_0, y)u(s, z)v(y, z) dz dy ds \leq u(x_0, t + \theta),$$

which means that the masses of μ_n are uniformly bounded. With this notation we have

$$u(t, x) = \int_0^t \int_D \frac{p_{t-s}^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} \mu_n(dy ds).$$

The goal is then to show that the right-hand side converges to the right-hand side of (1.2). To this end we will isolate small times and look separately at D_N and $D \setminus D_N$.

Note that all μ_n are supported in $D \times [0, T]$, so the sequence (μ_n) is tight and we can extract a subsequence μ_{n_k} converging weakly to μ . Furthermore, for every $U \subset\subset D$ and $0 < t < T$, we have that $\mu_n(U \times [0, t]) \rightarrow 0$ as $n \rightarrow \infty$, so $\mu|_{\bar{D} \times [0, T]}$ must be concentrated on $\partial D \times [0, T]$.

Since for $y \in D_N$ we have $p_{t-s}^{D_n}(x, y) \approx p_{t-s}^D(x, y)$ for $n > N + 1$, we find that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_0^t \int_{D_N} \frac{p_{t-s}^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} \mu_n(dy ds) \\ &\lesssim \lim_{n \rightarrow \infty} \int_0^t \int_{D \setminus D_n} u(s, z) \int_{D_N} p_{t-s}^D(x, y)v(y, z) dy dz ds = 0. \end{aligned} \tag{6.2}$$

We will now show that there exists a modulus of continuity ω independent of n such that

$$\int_{t-\epsilon}^t \int_D \frac{p_{t-s}^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} \mu_n(dy ds) < \omega(\epsilon). \tag{6.3}$$

To this end we will show that the left-hand side converges to 0 as $\epsilon \rightarrow 0^+$ for each $n > N$, and that it is nonincreasing with respect to n for each (small) ϵ . By the definition of μ_n and the fact that u is caloric,

$$\begin{aligned} \int_{t-\epsilon}^t \int_D \frac{p_{t-s}^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} \mu_n(dy ds) &= \int_{t-\epsilon}^t \int_{D \setminus D_n} J^{D_n}(t, x, s, z) u(s, z) dz ds \\ &= u(t, x) - \int_{D_n} p_\epsilon^{D_n}(x, y) u(t - \epsilon, y) dy. \end{aligned}$$

The last expression converges to 0 for $\epsilon \rightarrow 0^+$ for all fixed n , because u is continuous in both variables, and it is nonincreasing with respect to n because of the domain monotonicity. This proves (6.3).

Note also that the right-hand side of (1.2) is finite because μ is a finite measure and $\eta_{s,Q}(x)$ is bounded in s and Q for fixed x . Therefore,

$$\lim_{\epsilon \rightarrow 0^+} \int_{[t-\epsilon, t]} \int_{\partial D} \eta_{t-s,Q}(x) \mu(dQ ds) = 0. \tag{6.4}$$

By (6.2), (6.3), and (6.4), for any $\delta > 0$ there exist ϵ (small) and N_0 (large) such that for $n > N_0$,

$$\begin{aligned} &\left| \int_0^t \int_D \frac{p_{t-s}^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} \mu_n(dy ds) - \int_{[0,t]} \int_{\partial D} \eta_{t-s,Q}(x) \mu(dQ ds) \right| \\ &\leq \left| \int_{[t-\epsilon, t]} \int_{\partial D} \eta_{t-s,Q}(x) \mu(dQ ds) \right| + \left| \int_{t-\epsilon}^t \int_{D \setminus D_N} \frac{p_{t-s}^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} \mu_n(dy ds) \right| \\ &\quad + \left| \int_0^t \int_{D_N} \frac{p_{t-s}^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} \mu_n(dy ds) \right| \\ &\quad + \left| \int_0^{t-\epsilon} \int_{D \setminus D_N} \frac{p_{t-s}^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} \mu_n(dy ds) - \int_{[0,t-\epsilon]} \int_{\partial D} \eta_{t-s,Q}(x) \mu(dQ ds) \right| \\ &\leq 3\delta + \left| \int_0^{t-\epsilon} \int_{D \setminus D_N} \frac{p_{t-s}^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} \mu_n(dy ds) - \int_{[0,t-\epsilon]} \int_{\partial D} \eta_{t-s,Q}(x) \mu(dQ ds) \right|. \end{aligned}$$

Furthermore, if N_0 is large enough, then by Lemma 6.2,

$$\begin{aligned} &\left| \int_0^{t-\epsilon} \int_{D \setminus D_N} \frac{p_{t-s}^{D_n}(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)} \mu_n(dy ds) - \int_{[0,t-\epsilon]} \int_{\partial D} \eta_{t-s,Q}(x) \mu(dQ ds) \right| \\ &\leq \delta + \left| \int_0^{t-\epsilon} \int_{D \setminus D_N} \frac{p_{t-s}^D(x, y)}{\mathbb{P}^y(\tau_D > 1)} \mu_n(dy ds) - \int_{[0,t-\epsilon]} \int_{\partial D} \eta_{t-s,Q}(x) \mu(dQ ds) \right|. \end{aligned}$$

By Lemma C.1, $\mu_n \cdot \mathbf{1}_{D \times [0,t-\epsilon]} \rightarrow \mu \mathbf{1}_{D \times [0,t-\epsilon]}$ weakly. By Corollary 3.4, $(s, y) \mapsto \frac{p_{t-s}^D(x, y)}{\mathbb{P}^y(\tau_{D_n} > 1)}$ is in $C([0, t - \epsilon] \times \bar{D})$. So, the last expression is smaller than 2δ for n large enough, which ends the proof. \square

Theorem 6.4 *The measure μ obtained in Theorem 6.3 is unique.*

Proof Following [12, 23], we start by showing that the measures μ_n^Q corresponding to $\eta_{t,Q}$ converge to $\delta_Q \otimes \delta_0$ for $t > 0, Q \in \partial D$. To this end, fix $Q \in \partial D$ and let

$$\mu_n^Q(y, s) = \mathbb{P}^y(\tau_{D_n} > 1) \int_{D \setminus D_n} \eta_{s,Q}(z) v(y, z) dz, \quad s > 0, y \in \mathbb{R}^d.$$

By Lemma 3.5, $\mu_n^Q((B(Q, \varepsilon) \times [0, \varepsilon])^c) \rightarrow 0$ as $n \rightarrow \infty$, for any $\varepsilon > 0$. So, μ_n converges weakly to $\delta_Q \otimes \delta_0$.

Now, let u be a singular caloric function and assume that

$$u(t, x) = \int_{[0,t)} \int_{\partial D} \eta_{t-s,Q}(x) \mu(dQ ds).$$

Let $\mu_n(y, s) = \int_{D \setminus D_n} \mathbb{P}^y(\tau_D > 1) u(s, z) v(y, z) dz$. By Fubini–Tonelli,

$$\begin{aligned} \mu_n(y, s) &= \int_{D \setminus D_n} \mathbb{P}^y(\tau_D > 1) v(y, z) \int_{[0,s)} \int_{\partial D} \eta_{s-\tau,Q}(z) \mu(dQ d\tau) dz \\ &= \int_{[0,s)} \int_{\partial D} \mu_n^Q(y, s - \tau) \mu(dQ d\tau). \end{aligned}$$

Let $f \in C_b(\overline{D} \times [0, T])$. Then,

$$\begin{aligned} &\int_0^t \int_D f(y, s) \mu_n(y, s) dy ds \\ &= \int_0^t \int_D f(y, s) \int_{[0,s)} \int_{\partial D} \mu_n^Q(y, s - \tau) \mu(dQ d\tau) dy ds \\ &= \int_{[0,t)} \int_{\partial D} \int_0^{t-\tau} \int_D f(y, s + \tau) \mu_n^Q(y, s) dy ds \mu(dQ d\tau). \end{aligned}$$

Since $\mu_n^Q \implies \delta_Q \otimes \delta_0$, the above integral with respect to $dy ds$ converges to $f(Q, \tau)$. Therefore, by the dominated convergence theorem,

$$\int_0^t \int_D f(y, s) \mu_n(y, s) dy ds \xrightarrow{n \rightarrow \infty} \int_{[0,t)} \int_{\partial D} f(Q, s) \mu(dQ ds),$$

which means that $\mu_n \implies \mu \cdot \mathbf{1}_{\overline{D} \times [0,t)}$. Thus, μ is uniquely determined by u . □

6.2 Functions caloric on $(0, T) \times D$

Theorem 6.5 *Assume that u is caloric on $(0, T) \times D$ and let $g = u|_{D^c}$. Then there exist unique bounded nonnegative measures μ on $[0, T) \times \partial D$ and μ_0 on D such that for all $0 < t < T$ and $x \in D$, (1.3) holds.*

Proof By the results of the previous subsection, there is a nonnegative measure μ on $\partial D \times (0, T)$ such that for all $0 < \varepsilon < t < T$ and $x \in D$,

$$u(t, x) = P_{t-\varepsilon}^D u(\varepsilon, \cdot)(x) + \int_{[\varepsilon, t]} \int_{\partial D} \eta_{t-s, Q}(x) \mu(dQ ds) + \int_{\varepsilon}^t \int_{D^c} g(s, z) J^D(t, x, s, z) dz ds.$$

By nonnegativity and the monotone convergence theorem, the last two integrals increase and converge as $\varepsilon \rightarrow 0^+$, so that

$$u(t, x) = \lim_{\varepsilon \rightarrow 0^+} P_{t-\varepsilon}^D u(\varepsilon, \cdot)(x) + \int_{(0, t)} \int_{\partial D} \eta_{t-s, Q}(x) \mu(dQ ds) + \int_0^t \int_{D^c} g(s, z) J^D(t, x, s, z) dz ds,$$

where the remaining limit exists and the expression under it decreases. Since $p_{t-\varepsilon}^D(x, y) \approx p_t^D(x, y)$ and $p_t^D(x, \cdot) \approx 1$ for any $U \subset\subset D$ we find that $u(\varepsilon, \cdot)$ have bounded integral on U . Therefore, by the Prokhorov theorem, there is a sequence (ε_n) such that $u(\varepsilon_n, \cdot)$ converge weakly on compact subsets of D to a measure μ_0 , locally finite on D . Furthermore, we have

$$P_{t-\varepsilon}^D u(\varepsilon, \cdot)(x) = \int_D p_{t-\varepsilon}^D(x, y) u(\varepsilon, y) dy = \int_D \frac{p_{t-\varepsilon}^D(x, y)}{\mathbb{P}^y(\tau_D > 1)} \mathbb{P}^y(\tau_D > 1) u(\varepsilon, y) dy.$$

Since $\frac{p_{t-\varepsilon}^D(x, y)}{\mathbb{P}^y(\tau_D > 1)} \approx \frac{p_t^D(x, y)}{\mathbb{P}^y(\tau_D > 1)} \approx 1$ we find that the functions $y \mapsto \mathbb{P}^y(\tau_D > 1) u(\varepsilon, y)$ have bounded mass. By Prokhorov theorem, we can infer without loss of generality that $\mathbb{P}^y(\tau_D > 1) u(\varepsilon_n, y)$ converge weakly to a finite measure $\tilde{\mu}$ on \bar{D} . We have $\tilde{\mu}(dy) = \mathbb{P}^y(\tau_D > 1) \mu_0(dy)$ on D . By (3.12),

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_D \frac{p_{t-\varepsilon}^D(x, y)}{\mathbb{P}^y(\tau_D > 1)} \mathbb{P}^y(\tau_D > 1) u(\varepsilon, y) dy &= \int_{\bar{D}} \frac{p_t^D(x, y)}{\mathbb{P}^y(\tau_D > 1)} \tilde{\mu}(dy) \\ &= \int_D p_t^D(x, y) \mu_0(dy) + \int_{\partial D} \eta_{t, Q}(x) \tilde{\mu}(dQ). \end{aligned}$$

We end the proof by defining μ on $\partial D \times [0, T)$ as $\mu \mathbf{1}_{\partial D \times (0, T)} + \tilde{\mu} \otimes \delta_0(dt)$. □

Appendix A: Weak and classical formulations for caloric functions

The following result seems to be well-known, but we were unable to locate a proof. The arguments are very similar to the case of the Laplacian discussed by Hunt [46].

Lemma A.1 For any $x \in D$ the function $(t, y) \mapsto p_t^D(x, y)$ is a classical solution to the fractional heat equation with the Dirichlet condition:

$$\begin{cases} (\partial_t - \Delta_y^{\alpha/2})p_t^D(x, y) = 0 & t > 0, y \in D, \\ p_t^D(x, y) = 0 & t > 0, y \in D^c. \end{cases} \tag{A.1}$$

It is also a weak solution in the sense that for $\phi \in C_c^\infty([0, \infty) \times \mathbb{R}^d)$ and $0 < t_1 < t_2 < \infty$ we have

$$\begin{aligned} \int_{t_1}^{t_2} \int_D (\partial_t + \Delta_y^{\alpha/2})\phi(t, y)p_t^D(x, y) dy dt &= \int_D \phi(t_2, y)p_{t_2}^D(x, y) dy \\ &\quad - \int_D \phi(t_1, y)p_{t_1}^D(x, y) dy. \end{aligned}$$

Proof By definition, the exterior condition is satisfied, so it suffices to verify that $(\partial_t - \Delta_y^{\alpha/2})p_t^D(x, y) = 0$. To this end we will differentiate the Hunt formula. Using the subordination and Fourier inversion formulas (see, e.g., Bogdan and Jakubowski [22, Lemma 5]) it is easy to see that p_t is smooth in x for $t > 0$ and $\partial_y^\beta p_t(x, y)$ is bounded whenever $|x - y|$ is separated from 0 for any $\beta \in \mathbb{N}_0$. Note that this is the case for $|X_{\tau_D} - y|$. Therefore, for fixed (t, y) , by the dominated convergence theorem we find

$$\begin{aligned} \partial_y^\beta p_t^D(x, y) &= \partial_y^\beta p_t(x, y) - \partial_y^\beta \mathbb{E}^x[p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t] \\ &= \partial_y^\beta p_t(x, y) - \mathbb{E}^x[\partial_y^\beta p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t]. \end{aligned}$$

Furthermore,

$$\|p_t^D(x, \cdot)\|_{C^2(B(y, \delta_D(y)/2))} < \infty, \tag{6.2}$$

hence $\Delta_y^{\alpha/2} p_t^D(x, y)$ is well defined for $(t, y) \in D \times (0, \infty)$ and we have

$$\Delta_y^{\alpha/2} p_t^D(x, y) = \Delta_y^{\alpha/2} p_t(x, y) - \Delta_y^{\alpha/2} \mathbb{E}^x[p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t].$$

We can also interchange $\Delta_y^{\alpha/2}$ with the expectation. The easiest way to see that is by using Fubini–Tonelli, (6.2) and the Taylor expansion in the following (symmetrized) representation of the fractional Laplacian:

$$\Delta_y^{\alpha/2} u(x) = \int_{\mathbb{R}^d} (u(x + y) - 2u(x) + u(x - y))v(y) dy.$$

Thus, we obtain

$$\Delta_y^{\alpha/2} p_t^D(x, y) = \Delta_y^{\alpha/2} p_t(x, y) - \mathbb{E}^x[\Delta_y^{\alpha/2} p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t].$$

We now compute the time derivative. Note that $\partial_t p_t(x, y)$ exists and is equal to $\Delta_y^{\alpha/2} p_t(x, y)$, so it is bounded for $|x - y|$ separated from 0. We have

$$\partial_t p_t^D(x, y) = \partial_t p_t(x, y) - \partial_t \mathbb{E}^x [p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t],$$

provided the last expression exists, which we now prove. Without loss of generality, let $h > 0$. We have

$$\begin{aligned} & \frac{1}{h} \left[\mathbb{E}^x [p_{t+h-\tau_D}(X_{\tau_D}, y); \tau_D < t+h] - \mathbb{E}^x [p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t] \right] \\ &= \frac{1}{h} \mathbb{E}^x [p_{t+h-\tau_D}(X_{\tau_D}, y) - p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t] \\ & \quad + \frac{1}{h} \mathbb{E}^x [p_{t+h-\tau_D}(X_{\tau_D}, y); t \leq \tau_D < t+h]. \end{aligned}$$

By the dominated convergence theorem, we get that

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \frac{1}{h} \mathbb{E}^x [p_{t+h-\tau_D}(X_{\tau_D}, y) - p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t] \\ &= \mathbb{E}^x [\partial_t p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t]. \end{aligned}$$

Furthermore, by (2.4),

$$\frac{1}{h} \mathbb{E}^x [p_{t+h-\tau_D}(X_{\tau_D}, y); t \leq \tau_D < t+h] \leq C \mathbb{P}^x(\tau_D \in [t, t+h]),$$

and the last expression converges to 0 by the dominated convergence theorem. Therefore we get

$$\begin{aligned} \partial_t p_t^D(x, y) &= \partial_t p_t(x, y) - \mathbb{E}^x [\partial_t p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t] \\ &= \Delta_y^{\alpha/2} p_t(x, y) - \mathbb{E}^x [\Delta_y^{\alpha/2} p_{t-\tau_D}(X_{\tau_D}, y); \tau_D < t] = \Delta_y^{\alpha/2} p_t^D(x, y), \end{aligned}$$

so p_t^D is a classical solution to the fractional heat equation (A.1). It is also a weak solution, as follows from integration by parts and the fact that the support of the test function ϕ is separated from ∂D . \square

Appendix B: Almost-increasingness

The following result is used in the proof of Lemma 5.16.

Lemma B.1 *For open $U \subset\subset U' \subset\subset D$ and $T > 0$, there exists $C = C(d, \alpha, \underline{D}, U, d(U, (U')^c), T)$ such that*

$$p_s^D(x, y) \leq C p_t^D(x, y), \quad x \in U, \quad y \in D \setminus U', \quad 0 < s < t < T.$$

The proof (given below) relies on approximate factorization of p_t^D and the next lemma on the survival probability $\mathbb{P}^y(\tau_D > s)$. Of course, the latter is nonincreasing in $s \in (0, \infty)$. The following *relative* upper bound is a partial converse and may be of independent interest.

Lemma B.2 *Let $T > 0$. There exists $C = C(d, \alpha, \underline{D}, T)$ and $\sigma \in (0, 1)$ such that*

$$\frac{\mathbb{P}^y(\tau_D > s)}{\mathbb{P}^y(\tau_D > t)} \leq C \left(\frac{s}{t}\right)^{-\sigma}, \quad y \in D, \quad 0 < s < t < T. \tag{B.1}$$

Proof By [19, Remark 3] and scaling, $\mathbb{P}^y(\tau_D > t_1) \approx \mathbb{P}^y(\tau_D > t_2)$, uniformly in $y \in D$ and t_1, t_2 in each compact subset of $(0, \infty)$. Therefore we may assume that s and t in (B.1) are small. Then we can also assume that y is close to the boundary, otherwise the terms on the left-hand side of (B.1) are bounded from below by the survival probability of a sufficiently small ball (and above by 1). In this setting, recalling the notation of Sect. 2.1, by [19, Theorem 2] and [48, Lemma 17] we get

$$\frac{\mathbb{P}^y(\tau_D > s)}{\mathbb{P}^y(\tau_D > t)} \approx \frac{\mathbb{E}^{A_{t^{1/\alpha}}(y)} \tau_D}{\mathbb{E}^{A_{s^{1/\alpha}}(y)} \tau_D} \approx \frac{\Phi(A_{t^{1/\alpha}}(y))}{\Phi(A_{s^{1/\alpha}}(y))}, \tag{6.2}$$

uniformly for the considered point y and times s, t .

Let $Q \in \partial D$ be closest to y . To estimate the rightmost ratio in (6.2), we consider three geometric situations:

Case 1. If $y \in \mathcal{A}_{t^{1/\alpha}}(y) \cap \mathcal{A}_{s^{1/\alpha}}(y)$, then we can take $A_{t^{1/\alpha}}(y) = A_{s^{1/\alpha}}(y) = y$, proving (B.1).

Case 2. If $y \in \mathcal{A}_{s^{1/\alpha}}(y)$, but $y \notin \mathcal{A}_{t^{1/\alpha}}(y)$, then $\kappa s^{1/\alpha} \leq \delta_D(y) = |y - Q| < \kappa t^{1/\alpha}$, so

$$|A_{t^{1/\alpha}}(y) - A_{t^{1/\alpha}}(Q)| \leq |A_{t^{1/\alpha}}(y) - y| + |y - Q| + |Q - A_{t^{1/\alpha}}(Q)| \leq (2 + \kappa)t^{1/\alpha}.$$

By definition, $\delta_D(A_{t^{1/\alpha}}(y)) \wedge \delta_D(A_{t^{1/\alpha}}(Q)) \geq \kappa t^{1/\alpha}$. Therefore, by the Harnack inequality [14, Lemma 1], we find that $\Phi(A_{t^{1/\alpha}}(y)) \approx \Phi(A_{t^{1/\alpha}}(Q))$.

On the other hand, since $\kappa \leq 1/2$, we have $y \in \mathcal{A}_{\delta_D(y)/\kappa}(Q)$. In particular, we can take $A_{s^{1/\alpha}}(y) = A_{\delta_D(y)/\kappa}(Q) = y$. Then, by [12, Lemma 5], we get

$$\frac{\Phi(A_{t^{1/\alpha}}(y))}{\Phi(A_{s^{1/\alpha}}(y))} \approx \frac{\Phi(A_{t^{1/\alpha}}(Q))}{\Phi(A_{\delta_D(y)/\kappa}(Q))} \lesssim \left(\frac{t^{1/\alpha}}{\delta_D(y)/\kappa}\right)^\gamma \leq \left(\frac{s}{t}\right)^{-\gamma/\alpha},$$

where $\gamma = \gamma(d, \alpha, \underline{D}) \in (0, \alpha)$. This ends the proof in this case.

Case 3. If $y \notin \mathcal{A}_{s^{1/\alpha}}(y) \cup \mathcal{A}_{t^{1/\alpha}}(y)$, then $\delta_D(y) < \kappa s^{1/\alpha} < \kappa t^{1/\alpha}$. By the same argument as in the previous case, $\Phi(A_{t^{1/\alpha}}(y)) \approx \Phi(A_{t^{1/\alpha}}(Q))$, $\Phi(A_{s^{1/\alpha}}(y)) \approx \Phi(A_{s^{1/\alpha}}(Q))$, and

$$\frac{\Phi(A_{t^{1/\alpha}}(y))}{\Phi(A_{s^{1/\alpha}}(y))} \approx \frac{\Phi(A_{t^{1/\alpha}}(Q))}{\Phi(A_{s^{1/\alpha}}(Q))} \lesssim \left(\frac{s}{t}\right)^{-\gamma/\alpha}.$$

The proof is complete. □

Let us explain why (B.1) is a partial converse to nonincreasingness of the survival probability. We may interpret (B.1) as *weak lower scaling* (with exponent $-\sigma$) near $s = 0$ of the survival probability $f(s) := \mathbb{P}^x(\tau_D > s)$, uniform in $x \in D$. Such scaling is defined as almost-increasingness $f(s)/s^{-\sigma} \leq Cf(t)/t^{-\sigma}$ for (bounded arguments) $0 < s \leq t$; see, e.g., [20].

Proof of Lemma B.1 We use (2.6):

$$p_s^D(x, y) \approx \mathbb{P}^x(\tau_D > s)p_s(x, y)\mathbb{P}^y(\tau_D > s).$$

Since $x \in U \subset\subset D$, we have $\mathbb{P}^x(\tau_D > s) \lesssim \mathbb{P}^x(\tau_D > t)$ because the latter quantity is bounded from below by a constant depending only on $U, d, \alpha, \underline{D}$, and T . Furthermore, since $y \in D \setminus U'$, by (2.4) we have $p_s(x, y) \approx s$. Therefore the statement of the lemma follows from Lemma B.2. □

Appendix C: No mass concentration forward in time

Let μ_n be the sequence of measures converging to μ constructed in the proof of Theorem 6.3. We will show that $\mu_n \cdot \mathbf{1}_{[0,t] \times D}$ do not accumulate mass at time t . Here is the precise formulation.

Lemma C.1 *Let $0 < t < T$ and let $f \in C([0, t] \times \overline{D})$. Then*

$$\lim_{n \rightarrow \infty} \int_0^t \int_D f(s, y) \mu_n(dy ds) = \int_{[0,t]} \int_{\partial D} f(s, Q) \mu(dQ ds).$$

Proof It suffices to show that

$$\lim_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \int_{t-\varepsilon}^t \int_D \mu_n(dy ds) = 0. \tag{6.3}$$

Fix $\theta \in (0, T - t)$ and $x \in D$. By (2.6) we have

$$\begin{aligned} \int_{t-\varepsilon}^t \int_D \mu_n(dy ds) &= \int_{t-\varepsilon}^t \int_{D \setminus D_n} \int_{D_n} u(z, s) \mathbb{P}^y(\tau_{D_n} > 1) v(y, z) dy dz ds \\ &\approx \int_{t-\varepsilon}^t \int_{D \setminus D_n} u(z, s) \int_{D_n} p_{t+\theta-s}^{D_n}(x, y) v(y, z) dy dz ds \\ &= \int_{t-\varepsilon}^t \int_{D \setminus D_n} u(z, s) J^{D_n}(t + \theta, x, s, z) dz ds. \end{aligned}$$

Note that

$$\begin{aligned} & \int_{t-\varepsilon}^t \int_{D \setminus D_n} u(z, s) J^{D_n}(t + \theta, x, s, z) dz ds \\ &= u(t + \theta, x) - \int_t^{t+\theta} \int_{D \setminus D_n} u(z, s) J^{D_n}(t + \theta, x, s, z) dz ds - P_{\theta+\varepsilon}^{D_n} u(t - \varepsilon)(x) \\ &= P_\theta^{D_n} u(t)(x) - P_{\theta+\varepsilon}^{D_n} u(t - \varepsilon)(x) \\ &= P_\theta^{D_n} (u(t) - P_\varepsilon^{D_n} u(t - \varepsilon))(x). \end{aligned}$$

We note that by the caloricity of u , the monotone convergence theorem, and the fact that $p_t^{D_n} \nearrow p_t^D$ pointwise, we have $P_\tau^{D_n} u(t)(y) \nearrow P_\tau^D u(t)(y) \leq u(\tau + t, y) < \infty$, for $\tau = \theta, \varepsilon$. Furthermore, we have $0 \leq u(t, y) - P_\varepsilon^{D_n} u(t - \varepsilon)(y) \leq u(t, y)$ for all $y \in D$, so by the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} P_\theta^{D_n} (u(t) - P_\varepsilon^{D_n} u(t - \varepsilon))(x) = P_\theta^D (u(t) - P_\varepsilon^D u(t - \varepsilon))(x).$$

Now, it suffices to show that $u(t) - P_\varepsilon^D u(t - \varepsilon)$ converges to 0 pointwise in D . To this end, we let $y \in D$ and take n such that $y \in D_n$. Then,

$$\begin{aligned} u(t, y) - P_\varepsilon^D u(t - \varepsilon)(y) &\leq u(t, y) - P_\varepsilon^{D_n} u(t - \varepsilon)(y) \\ &= \int_{t-\varepsilon}^t \int_{D \setminus D_n} u(s, z) J^{D_n}(t, y, s, z) dz ds, \end{aligned}$$

and the last expression converges to 0 as $\varepsilon \rightarrow 0^+$ by the dominated convergence theorem. This proves (6.3). □

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Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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