



Tube formulas for valuations in complex space forms

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Abstract

Given an isometry invariant valuation on a complex space form we compute its value on the tubes of sufficiently small radii around a set of positive reach. This generalizes classical formulas of Weyl, Gray and others about the volume of tubes. We also develop a general framework on tube formulas for valuations in Riemannian manifolds.

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1 Introduction

For a compact convex set $A \subset \mathbb{R}^m$, the *Steiner formula* computes the volume of the set A_t consisting of points at distance smaller than t from A as follows

$$\text{vol}(A_t) = \sum_{i=0}^m \omega_{m-i} \mu_i(A) t^{m-i}. \quad (1)$$

Here the functionals μ_i are the so-called *intrinsic volumes*, and the normalizing constant ω_k is the volume of the k -dimensional unit ball. By Hadwiger's characterization theorem, the intrinsic volumes span the space of *valuations* (finitely additive functionals on convex bodies) that are continuous and invariant under rigid motions.

The famous *tube formula* of H. Weyl [28] is the assertion that (1) holds true for $A \subset \mathbb{R}^m$ a smooth compact submanifold and $t \geq 0$ small enough, with the additional insight that the coefficients $\mu_i(A)$ depend only on the induced Riemannian structure of A . Even more generally, Federer extended the validity of (1) to the class of compact *sets of positive reach*. Later on, the same formula has been proven to hold for bigger classes of sets (see e.g. [14, 16]). As for the coefficients μ_i , the current perspective is to view them as *smooth valuations* in the sense of Alesker's theory of valuations on manifolds (see [4]).

Already in Weyl's original work, the tube formula was extended to the sphere and to hyperbolic space. In that case, instead of a polynomial on the radius t one has a polynomial in certain functions $\sin_{\lambda}(t)$, $\cos_{\lambda}(t)$ whose definition we recall in (52). Later, Gray and Vanhecke computed the volume of tubes around submanifolds of rank one symmetric spaces (cf. [19, 20]).

All these classical tube formulas are most naturally expressed in the language of valuations on manifolds. Furthermore, this theory has allowed for the determination of kinematic formulas (a far-reaching generalization of tube formulas) in isotropic spaces. These spaces are Riemannian manifolds under the action of a group of isometries that is transitive on the sphere bundle. For instance, in [10] and [11] the kinematic formulas of complex complex space forms (i.e. complex euclidean, projective and hyperbolic spaces) were obtained, and Gray's tube formulas on such spaces were recovered.

Tube formulas, however, exist also for other valuations than the volume, and these do not follow from the kinematic formulas. For instance, differentiating the Steiner formula one easily obtains

$$\mu_k(A_t) = \sum_{j=0}^k \binom{m-j}{m-k} \frac{\omega_{m-j}}{\omega_{m-k}} \mu_j(A) t^{k-j}, \quad A \subset \mathbb{R}^m. \quad (2)$$

In real space forms (i.e. the sphere and hyperbolic space), Santaló obtained similar tube formulas for all isometry invariant valuations (see [26]). For rank one symmetric spaces, the tube formulas of a certain class of valuations (integrated mean curvatures) were found in [19], still with a differential-geometric viewpoint. There are however many invariant valuations on these spaces that were not considered.

In this paper we prove the existence of tube formulas for any smooth valuation in a Riemannian manifold. Then we develop a method to determine these formulas for the invariant valuations of an isotropic space. Using this method we compute all tube formulas explicitly in the case of complex space forms. In fact, our approach also reveals some interesting aspects in the case of real space forms.

Let us briefly describe our results. First, given a Riemannian manifold M we construct a family \mathbf{T}_t of *tubular operators* on the space $\mathcal{V}(M)$ of smooth valuations of M such that for any $\mu \in \mathcal{V}(M)$ and every compact set of positive reach $A \subset M$ one has

$$\mu(A_t) = \mathbf{T}_t \mu(A),$$

for $t \geq 0$ small enough (see Definition 4.1 and Corollary 4.7). Differentiating \mathbf{T}_t at $t = 0$ yields an operator $\partial : \mathcal{V}(M) \rightarrow \mathcal{V}(M)$. If G is a group of isometries of M acting transitively on the sphere bundle SM , the subspace $\mathcal{V}(M)^G$ of G -invariant valuations is finite dimensional, and the determination of the tube operators \mathbf{T}_t reduces to the computation of the flow generated by ∂ .

Once this general framework is established we concentrate on the complex space forms $\mathbb{C}P_\lambda^n$. For $\lambda = 0$ this refers to complex euclidean space \mathbb{C}^n under the group of complex isometries, and for $\lambda \neq 0$ this is the n -dimensional complex projective or hyperbolic space of constant holomorphic curvature 4λ , under the full group of isometries G . We simply denote $\mathcal{V}_{\lambda, \mathbb{C}}^n := \mathcal{V}(\mathbb{C}P_\lambda^n)^G$.

For $\lambda = 0$, we will readily obtain the tube formulas $\mathbf{T}_t \mu$ of all translation-invariant and $U(n)$ -invariant continuous valuations μ thanks to the existence of an \mathfrak{sl}_2 -module structure on the space $\text{Val}^{U(n)}$ of such valuations. This structure, discovered by Bernig and Fu in [10], is induced by two natural operators Λ, L , the first of which is a normalization of ∂ .

Remarkably, it turns out that also for $\lambda \neq 0$ the derivation operator ∂ is closely related to the operators Λ, L of the flat space. Indeed, in Theorem 4.11 we find an isomorphism $\Phi_\lambda : \text{Val}^{U(n)} \rightarrow \mathcal{V}_{\lambda, \mathbb{C}}^n$ such that

$$\partial|_{\mathcal{V}_{\lambda, \mathbb{C}}^n} = \Phi_\lambda \circ (\Lambda - \lambda L) \circ \Phi_\lambda^{-1}. \tag{3}$$

Using the decomposition of $\text{Val}^{U(n)}$ into \mathfrak{sl}_2 -irreducible components, the computation of the tubular operator boils down to the solution of a Cauchy problem in some abstract model spaces, yielding our main result.

Theorem *There exists a basis $\{\sigma_{k,r}^\lambda\}$ of the space $\mathcal{V}_{\lambda, \mathbb{C}}^n$ of invariant valuations of $\mathbb{C}P_\lambda^n$ such that*

$$\mathbf{T}_t \sigma_{k,r}^\lambda = \sum_{j=0}^{2n-4r} \phi_{2n-4r, k-2r, j}^\lambda(t) \sigma_{j+2r, r}^\lambda, \tag{4}$$

where

$$\phi_{m,k,j}^\lambda(t) = \sum_{h \geq 0} (-\lambda)^{j-h} \binom{m-j}{k-h} \binom{j}{h} \sin_\lambda^{k+j-2h}(t) \cos_\lambda^{m-k-j+2h}(t).$$

We describe the basis $\sigma_{k,r}^\lambda$ explicitly in terms of the previously known valuations $\tau_{k,p}^\lambda$ of [11]. The tube formulas for the $\tau_{k,p}^\lambda$ can be easily obtained from the previous ones, as we also provide the expression of these valuations in terms of the $\sigma_{k,r}^\lambda$.

Curiously, the expressions (4) are extremely similar to those obtained by Santaló in the real space form \mathbb{S}_λ^m of constant curvature λ . Indeed, for a certain basis $\{\sigma_i\}_{i=0}^m$ of the space $\mathcal{V}_{\lambda,\mathbb{R}}^m$ of isometry invariant valuations of \mathbb{S}_λ^m one has

$$\mathbf{T}_t \sigma_i^\lambda = \sum_{j=0}^{m-1} \phi_{m-1,i,j}^\lambda(t) \sigma_j^\lambda, \quad 0 \leq i \leq m-1.$$

The tube formula for $\sigma_m = \text{vol}$ is however quite different. As an explanation for these similarities, we show in Theorem 4.12 the existence of a phenomenon similar (but not completely analogous) to (3).

The paper concludes with a detailed study of the spectrum and the eigenspaces of the derivative operator ∂ in $\mathcal{V}_{\lambda,\mathbb{C}}^n$ and $\mathcal{V}_{\lambda,\mathbb{R}}^m$. In particular, we compute the kernel of ∂ in $\mathcal{V}_{\lambda,\mathbb{C}}^n$; i.e. we determine the invariant valuations of $\mathbb{C}P_\lambda^n$ for which the tube formulas are constant. We also identify the images $\partial(\mathcal{V}_{\lambda,\mathbb{C}}^n)$ and $\partial(\mathcal{V}_{\lambda,\mathbb{R}}^m)$, and we compute the preimage by ∂ of any element belonging to these subspaces.

2 Background

2.1 Valuations

Let V be a finite-dimensional real vector space, and let $\mathcal{K}(V)$ be the space of convex compact subsets of V , endowed with the Hausdorff metric. A *valuation* on V is a map $\varphi: \mathcal{K}(V) \rightarrow \mathbb{C}$ such that

$$\varphi(A \cup B) = \varphi(A) + \varphi(B) - \varphi(A \cap B),$$

for $A, B, A \cup B \in \mathcal{K}(V)$. The space of translation-invariant, continuous valuations on V is denoted by $\text{Val}(V)$.

The notion of valuation was extended to smooth manifolds by Alesker (cf. [2–4, 6]). For simplicity we will focus on the case of a Riemannian manifold M^n . It is also natural to consider here the class of compact sets of positive reach in M , which we denote $\mathcal{R}(M)$. The definition and some basic properties of such sets are recalled in Sect. 4.2.

Let SM be the sphere bundle of M consisting of unit tangent vectors, and let $\pi: SM \rightarrow M$ be the canonical projection.

Definition 2.1 (*Smooth valuation*) A *smooth valuation* on M is a functional $\varphi: \mathcal{R}(M) \rightarrow \mathbb{C}$ of the form

$$\varphi(A) = \int_{N(A)} \omega + \int_A \eta,$$

where $\omega \in \Omega^{n-1}(SM)$ and $\eta \in \Omega^n(M)$, are complex-valued differential forms, and $N(A)$ is the normal cycle of A (cf. e.g. [14]). We will denote $\varphi = \llbracket \omega, \eta \rrbracket$ in this case. For any subgroup $G \leq \text{Diff}(M)$, we will denote by $\mathcal{V}^G(M)$ the space of G -invariant valuations; i.e. $\mu \in \mathcal{V}(M)$ such that $\mu(gA) = \mu(A)$ for all $A \in \mathcal{R}(M)$ and $g \in G$.

The kernel of the map $(\omega, \eta) \mapsto \llbracket \omega, \eta \rrbracket$ was determined by Bernig and Bröcker in [8] as follows. Given $\omega \in \Omega^{n-1}(SM)$, there exists $\xi \in \Omega^{n-2}(SM)$ such that

$$D\omega := d(\omega + \alpha \wedge \xi),$$

is a multiple of α , the canonical contact form on SM . The unique n -form $D\omega$ satisfying this condition is called the *Rumin differential* of ω (see [24]). Then $\llbracket \omega, \eta \rrbracket = 0$ if and only if

$$D\omega + \pi^* \eta = 0, \quad \text{and} \quad \int_{S_x M} \omega = 0, \quad \forall x \in M. \tag{5}$$

One of the most striking aspects of Alesker’s theory of valuations on manifolds is the existence of a natural product on $\mathcal{V}(M)$, which turns this space into an algebra with χ as the unit element. The realization by Fu that this product is closely tied to kinematic formulas opened the door to the recent development of integral geometry in several spaces, including the complex space forms [1, 10, 11].

Another important algebraic structure is the *convolution of valuations* found by Bernig and Fu in linear spaces (cf. [9], but also [5]). This is a product on the dense subspace $\text{Val}^\infty(V) := \text{Val}(V) \cap \mathcal{V}(V)$ characterized as follows. Given $A \in \mathcal{K}(V)$, with smooth and positively curved boundary, we have $\mu_A(\cdot) := \text{vol}(\cdot + A) \in \text{Val}^\infty(V)$. The convolution is determined by

$$\mu_A * \varphi := \varphi(\cdot + A), \quad \varphi \in \text{Val}^\infty(V), \tag{6}$$

where $+$ refers to the Minkowski sum. In particular, vol is the unit element of this operation.

2.2 Real space forms

The fundamental examples of valuations in Euclidean space \mathbb{R}^m are the *intrinsic volumes* μ_k . These are implicitly defined by the Steiner formula

$$\text{vol}_{\mathbb{R}^m}(A + t\mathbb{B}^m) = \sum_{k=0}^m t^{m-k} \omega_{m-k} \mu_k(A), \quad A \in \mathcal{K}(\mathbb{R}^m), \tag{7}$$

where \mathbb{B}^m is the unit ball and ω_i is the volume of the i -dimensional unit ball. In particular $\mu_0 = \chi$, $\mu_{m-1} = \frac{1}{2}$ perimeter, and $\mu_n = \text{vol}_m$ are intrinsic volumes.

We will denote by \mathbb{S}_λ^m the m -dimensional complete and simply connected Riemannian manifold of constant curvature λ . That is, the sphere $S^m(\sqrt{\lambda})$ for $\lambda > 0$, Euclidean space \mathbb{R}^m for $\lambda = 0$, and hyperbolic space $H^m(\sqrt{-\lambda})$ for $\lambda < 0$. Let $G_{\lambda, \mathbb{R}}$ be the group of orientation preserving isometries of \mathbb{S}_λ^m ; i.e. $G_{\lambda, \mathbb{R}} \cong SO(m + 1)$ for $\lambda > 0$, and $G_{\lambda, \mathbb{R}} \cong SO(m) \times \mathbb{R}^m$ for $\lambda = 0$, while $G_{\lambda, \mathbb{R}} \cong PSO(m, 1)$ for $\lambda < 0$. We will denote by $\mathcal{V}_{\lambda, \mathbb{R}}^m$ the space of $G_{\lambda, \mathbb{R}}$ -invariant valuations of \mathbb{S}_λ^m .

Let $\kappa_0, \dots, \kappa_{m-1} \in \Omega^{m-1}(S\mathbb{S}_\lambda^m)^{G_{\lambda, \mathbb{R}}}$ be the differential forms defined in [13, §0.4.4]. In the same paper it was shown that the \mathbb{R} -algebra of $G_{\lambda, \mathbb{R}}$ -invariant differential forms is generated by $\kappa_0, \dots, \kappa_{m-1}, \alpha, d\alpha$. It follows by [11, Prop. 2.6] that the following valuations constitute a basis of $\mathcal{V}_{\lambda, \mathbb{R}}^m$

$$\begin{aligned} \sigma_i^\lambda &:= \llbracket \kappa_i, 0 \rrbracket, & 0 \leq i \leq m - 1 \\ \sigma_m^\lambda &:= \text{vol}_{\mathbb{S}_\lambda^m}. \end{aligned}$$

In euclidean space \mathbb{R}^m these valuations are proportional to the intrinsic volumes:

$$\sigma_i^\lambda = (m - i)\omega_{m-i}\mu_i, \quad \lambda = 0.$$

For general λ , the σ_i^λ are proportional to the valuations τ_i^λ appearing in [7, 17]

$$\sigma_i^\lambda = \frac{\pi^i(m - i)\omega_{m-i}}{i!\omega_i} \tau_i^\lambda, \quad 0 \leq i \leq m - 1, \tag{8}$$

$$\sigma_m^\lambda = \frac{\pi^m}{m!\omega_m} \tau_m^\lambda. \tag{9}$$

As we will see, the normalization taken for the σ_i^λ makes the tube formulas in $\mathcal{V}_{\lambda, \mathbb{R}}^m$ specially simple. A stronger reason in favor of this normalization is Theorem 4.12.

2.3 Complex space forms

We denote by $\mathbb{C}P_\lambda^n$ the complete, simply connected n -dimensional Kähler manifold of constant holomorphic curvature 4λ ; i.e. the complex projective space (with the suitably normalized Fubini-Study metric) for $\lambda > 0$, the complex euclidean space \mathbb{C}^n for $\lambda = 0$, and the complex hyperbolic space for $\lambda < 0$. For $\lambda \neq 0$ we let $G_{\lambda, \mathbb{C}}$ be the full isometry group of $\mathbb{C}P_\lambda^n$. For $\lambda = 0$ we put $G_{\lambda, \mathbb{C}} = U(n) \times \mathbb{C}^n$. We denote by $\mathcal{V}_{\lambda, \mathbb{C}}^n$ the space of $G_{\lambda, \mathbb{C}}$ -invariant valuations on $\mathbb{C}P_\lambda^n$.

Let $\{\beta_{k,q}, \gamma_{k,q}\} \subset \Omega^{2n-1}(S\mathbb{C}P_\lambda^n)^{G_{\lambda, \mathbb{C}}}$ be the differential forms introduced in [10] for $\lambda = 0$, and extended to the curved case $\lambda \neq 0$ in [11]. Let also

$$\mu_{k,q}^\lambda := \llbracket \beta_{k,q}, 0 \rrbracket, \quad k > 2q, \tag{10}$$

$$\mu_{2q,q}^\lambda := \sum_{i=0}^{n-q-1} \left(\frac{\lambda}{\pi}\right)^i \frac{(q+i)!}{q!} \llbracket \gamma_{2q+2i,q+i}, 0 \rrbracket + \left(\frac{\lambda}{\pi}\right)^{n-q} \frac{n!}{q!} \llbracket 0, d\text{vol} \rrbracket \quad (11)$$

where $d\text{vol}$ is the Riemannian volume element. It was shown in [10, 11] that these valuations $\mu_{k,q}^\lambda$ with $\max\{0, k-n\} \leq q \leq \frac{k}{2} \leq n$ constitute a basis of $\mathcal{V}_{\lambda,\mathbb{C}}^n$. It is convenient to emphasize that the $\mu_{k,q}^\lambda$ do *not* coincide with the *hermitian intrinsic volumes* $\mu_{k,q}^M$ for $M = \mathbb{C}P_\lambda^n$ introduced in [12].

For $\lambda = 0$ we simply write $\mu_{k,q}$ instead of $\mu_{k,q}^0$. We will also use the so-called *Tasaki valuations*

$$\tau_{k,q} := \sum_{i=q}^{\lfloor k/2 \rfloor} \binom{i}{q} \mu_{k,i}, \quad 0, k-n \leq q \leq \frac{k}{2} \leq n.$$

It will be useful to consider the following linear isomorphisms:

$$\mathcal{F}_{\lambda,\mathbb{C}}: \text{Val}^{U(n)} \longrightarrow \mathcal{V}_{\lambda,\mathbb{C}}^n, \quad \mathcal{F}_{\lambda,\mathbb{C}}(\mu_{k,q}) = \mu_{k,q}^\lambda.$$

More generally, whenever we have a valuation ν in $\text{Val}^{U(n)}$ we will denote $\nu^\lambda := \mathcal{F}_{\lambda,\mathbb{C}}(\nu)$. For instance $\tau_{k,q}^\lambda = \mathcal{F}_{\lambda,\mathbb{C}}(\tau_{k,q})$.

3 Tube formulas in linear spaces

Let V be an m -dimensional euclidean vector space. Given $t \geq 0$, let $\mathbf{T}_t: \text{Val}(V) \rightarrow \text{Val}(V)$ be given by

$$(\mathbf{T}_t \mu)(A) = \mu(A + t\mathbb{B}^m) = (\mu_{t\mathbb{B}^m} * \mu)(A) \quad A \in \mathcal{K}(V), \quad (12)$$

where \mathbb{B}^m is the unit ball. We will call \mathbf{T}_t the *tubular operator*. Let also $\partial: \text{Val}(V) \rightarrow \text{Val}(V)$ be the operator given by

$$\partial \mu = \left. \frac{d}{dt} \right|_{t=0} \mathbf{T}_t \mu. \quad (13)$$

This operator has sometimes been denoted by Λ in the literature, but following [10] we reserve the symbol Λ for a certain normalization of ∂ (see (18)).

The properties of the Minkowski sum ensure that $\mathbf{T}_{t+s} = \mathbf{T}_t \circ \mathbf{T}_s = \mathbf{T}_s \circ \mathbf{T}_t$. Differentiating with respect to s at zero yields

$$\frac{d}{dt} \mathbf{T}_t \mu = \mathbf{T}_t \partial \mu = \partial \mathbf{T}_t \mu. \quad (14)$$

It follows that

$$\partial^i \mu = \left. \frac{d^i}{dt^i} \right|_{t=0} \mathbf{T}_t \mu. \tag{15}$$

For each $\mu \in \text{Val}(V)$, the map $t \mapsto \mathbf{T}_t \mu$ is a polynomial in t of degree m by (12) and the Steiner formula (7) (or by [23]). Hence

$$\mathbf{T}_t \mu = \sum_{i=0}^m \frac{t^i}{i!} \left. \frac{d^i}{dt^i} \right|_{t=0} \mathbf{T}_t \mu \tag{16}$$

$$= \sum_{i=0}^m \frac{t^i}{i!} \partial^i \mu. \tag{17}$$

Note also that, by (15) and (16), the derivative operator ∂ is $(m + 1)$ -nilpotent; i.e. $\partial^{m+1} = 0$.

Let us compute the tube formula for the intrinsic volume μ_i for each $0 \leq i \leq m$ using (17). For that purpose we first compute ∂ . Since $\mathbf{T}_{t+s} = \mathbf{T}_s \circ \mathbf{T}_t$ we have

$$\mathbf{T}_{t+s} \text{ vol} = \sum_{j=0}^m \omega_{m-j} t^{m-j} \mathbf{T}_s \mu_j.$$

On the other hand

$$\mathbf{T}_{t+s} \text{ vol} = \sum_{j=0}^m \omega_{m-j} (t + s)^{m-j} \mu_j,$$

Differentiating at $s = 0$ and comparing coefficients yields

$$\partial \mu_j = \frac{\omega_{m-j+1}}{\omega_{m-j}} (m - j + 1) \mu_{j-1}.$$

Finally, using (17), we get

$$\begin{aligned} \mathbf{T}_t \mu_k &= \sum_{i=0}^m \frac{t^i}{i!} \partial^i \mu_k = \sum_{i=0}^k \frac{t^i}{i!} \frac{\omega_{m-k+i}}{\omega_{m-k}} \frac{(m - k + i)!}{(m - k)!} \mu_{k-i} \\ &= \sum_{j=0}^k \binom{m - j}{k - j} \frac{\omega_{m-j}}{\omega_{m-k}} t^{k-j} \mu_j, \end{aligned}$$

which is (2).

In order to compute the tube formulas for invariant valuations in \mathbb{C}^n (i.e. to determine \mathbf{T}_t on $\text{Val}^{U(n)}$), it will be useful to recall the \mathfrak{sl}_2 -module structure of $\text{Val}^{U(n)}$ found in [10]. Consider the linear maps $\Lambda, L, H: \text{Val}^\infty(V) \rightarrow \text{Val}^\infty(V)$, defined as follows

$$\Lambda v := \frac{\omega_{m-k}}{\omega_{m-k+1}} \partial v, \quad Lv := \frac{2\omega_k}{\omega_{k+1}} \mu_1 \cdot v, \quad Hv = (2k - m)v, \quad v \in \text{Val}_k^\infty(V), \tag{18}$$

where \cdot refers to the Alesker product.

Proposition 3.1 *On $\text{Val}^{O(m)}$ the operators Λ, L are given by*

$$L\mu_k = (k + 1)\mu_{k+1}, \tag{19}$$

$$\Lambda\mu_k = (m - k + 1)\mu_{k-1}, \tag{20}$$

while on $\text{Val}^{U(n)}$ one has

$$L\mu_{k,p} = (k - 2q + 1)\mu_{k+1,q} + 2(q + 1)\mu_{k+1,q+1} \tag{21}$$

$$\Lambda\mu_{k,p} = (k - 2q + 1)\mu_{k-1,q-1} + 2(n - k + q + 1)\mu_{k-1,q}, \tag{22}$$

which implies

$$L\tau_{k,q} = (k - 2q + 1)\tau_{k+1,q} \tag{23}$$

$$\Lambda\tau_{k,q} = (k - 2q + 1)\tau_{k-1,q-1} + (2n - 2q - k + 1)\tau_{k-1,q} \tag{24}$$

Proof The first two equalities are [7, eqs. (2.3.12) and (2.3.13)]. The rest is [10, Lemma 5.2]. \square

Proposition 3.2 ([7, Prop. 2.3.10 (3)]) *The operators Λ, L, H define an \mathfrak{sl}_2 -module structure on both $\text{Val}^{O(m)}$ and $\text{Val}^{U(n)}$; i.e. $[L, \Lambda] = H, [H, L] = 2L, [H, \Lambda] = -2\Lambda$.*

The decomposition into irreducible components is as follows

$$\text{Val}^{O(m)} \cong V^{(m)}, \quad \text{Val}^{U(n)} \cong \bigoplus_{0 \leq 2r \leq n} V^{(2n-4r)} \tag{25}$$

where $V^{(m)}$ is the $(m + 1)$ -dimensional irreducible \mathfrak{sl}_2 -representation. In particular, for $0 \leq 2r \leq n$, there exists a unique, up to a multiplicative constant, primitive element (i.e. annihilated by Λ) in each irreducible component of $\text{Val}^{U(n)}$. By the so-called Lefschetz decomposition, the L -orbits of these primitive elements constitute a basis of $\text{Val}^{U(n)}$. This basis was explicitly computed in [10] as follows.

Proposition 3.3 ([10, eq. (76)]) *The following valuations*

$$\pi_{2r,r} := (-1)^r (2n - 4r + 1)!! \sum_{i=0}^r (-1)^i \frac{(2r - 2i - 1)!!}{(2n - 2r - 2i + 1)!!} \tau_{2r,i}, \quad 0 \leq 2r \leq n, \tag{26}$$

are Λ -primitive; i.e. $\Lambda\pi_{2r,r} = 0$. The family

$$\pi_{k,r} := L^{k-2r} \pi_{2r,r} \tag{27}$$

$$= (-1)^r (2n - 4r + 1)!! \sum_{i=0}^r (-1)^i \frac{(k - 2i)!}{(2r - 2i)!} \frac{(2r - 2i - 1)!!}{(2n - 2r - 2i + 1)!!} \tau_{k,i},$$

$$2r \leq k \leq 2n - 2r \quad (28)$$

forms a basis of $\text{Val}^{U(n)}$.

In particular the irreducible components of $\text{Val}^{U(n)}$ are the following subspaces

$$\mathcal{T}_0^{n,r} := \{\pi_{k,r} : 2r \leq k \leq 2n - 2r\}, \quad 0 \leq 2r \leq n. \quad (29)$$

We are now able to compute the tube formulas in the complex case using (17).

Theorem 3.4

$$\mathbf{T}_t \pi_{k,r} = \frac{(k - 2r)!}{\omega_{2n-k}} \sum_{j=0}^{k-2r} \binom{2n - 4r - j}{k - 2r - j} t^{k-2r-j} \frac{\omega_{2n-2r-j}}{j!} \pi_{j+2r,r}. \quad (30)$$

Proof. By [10, Lemma 5.6],

$$\Lambda \pi_{k,r} = (k - 2r)(2n - k - 2r + 1) \pi_{k-1,r}, \quad (31)$$

and then

$$\Lambda^i \pi_{k,r} = \frac{(k - 2r)!(2n - k - 2r + i)!}{(k - 2r - i)!(2n - k - 2r)!} \pi_{k-i,r}. \quad (32)$$

Using (17), we obtain the tube formula

$$\begin{aligned} \mathbf{T}_t \pi_{k,r} &= \sum_{i=0}^{2n} \frac{t^i}{i!} \frac{\omega_{2n-k+i}}{\omega_{2n-k}} \Lambda^i \pi_{k,r} \\ &= \frac{(k - 2r)!}{\omega_{2n-k}} \sum_{i=0}^{k-2r} \frac{t^i}{i!} \omega_{2n-k+i} \frac{(2n - k - 2r + i)!}{(k - 2r - i)!(2n - k - 2r)!} \pi_{k-i,r} \\ &= \frac{(k - 2r)!}{\omega_{2n-k}} \sum_{j=0}^{k-2r} \binom{2n - 4r - j}{k - 2r - j} t^{k-2r-j} \frac{\omega_{2n-2r-j}}{j!} \pi_{j+2r,r}. \quad \square \end{aligned}$$

These tube formulas can also be given in terms of the valuations $\tau_{k,q}$. To this end, we next compute their *Lefschetz decomposition*.

Proposition 3.5 *The Lefschetz decomposition of $\tau_{k,r}$ is given by*

$$\tau_{k,r} = \frac{1}{(k - 2r)!} \sum_{i=0}^r \binom{n - 2i}{r - i} \frac{(2n - 2i - 2r)!}{(2n - 4i)!} \pi_{k,i}. \quad (33)$$

Proof. Consider the linear map $\psi : \text{Val}^{U(n)} \rightarrow \text{Val}^{U(n)}$ mapping $\tau_{k,r}$ to the left hand side of (33). We need to show that $\psi = \text{id}$. Let us check that this endomorphism commutes with both Λ and L . To check commutation with Λ , we only need to verify the following

$$\begin{aligned} & (k - 2r)! \psi(\Lambda(\tau_{k,r})) \\ &= \sum_{i=0}^{r-1} \frac{(n - 2i)!(2n - 2i - 2r + 2)!}{(r - i - 1)!(n - r - i + 1)!(2n - 4i)!} \pi_{k-1,i} \\ & \quad + (k - 2r)(2n - k - 2r + 1) \sum_{i=0}^r \frac{(n - 2i)!(2n - 2i - 2r)!}{(r - i)!(n - i - r)!(2n - 4i)!} \pi_{k-1,i} \\ &= \sum_{i=0}^r \frac{(n - 2i)!(2n - 2i - 2r)!}{(r - i)!(n - i - r)!(2n - 4i)!} (k - 2i)(2n - k - 2i + 1) \pi_{k-1,i} \\ &= (k - 2r)! \Lambda \psi(\tau_{k,r}). \end{aligned}$$

Comparing term by term, the previous identities boil down to

$$2(r - i)(2n - 2i - 2r + 1) + (k - 2r)(2n - k - 2r + 1) = (k - 2i)(2n - k - 2i + 1)$$

which is trivial.

Commutation with L is straightforward using $L\pi_{k,i} = \pi_{k+1,i}$.

Given that ψ commutes with the operators Λ and L and $\text{Val}^{U(n)}$ is multiplicity-free, Schur’s lemma implies that for each $0 \leq 2r \leq n$, there exists a constant c_r such that $\psi|_{\mathcal{T}_0^{n,r}} = c_r \text{id}$.

Let $a_{2r,j}$ and $b_{2r,i}$ be the coefficients of $\pi_{2r,j}$ and $\tau_{2r,i}$ in (33) and (26) respectively, so that $\psi(\tau_{2r,i}) = \sum_{j=0}^i a_{2r,j} \pi_{2r,j}$ and $\pi_{2r,r} = \sum_{i=0}^r b_{2r,i} \tau_{2r,i}$. Then

$$c_r \pi_{2r,r} = \psi(\pi_{2r,r}) = \sum_{i=0}^r b_{2r,i} \left(\sum_{j=0}^i a_{2r,j} \pi_{2r,j} \right) = \sum_{j=0}^r \sum_{i=j}^r b_{2r,i} a_{2r,j} \pi_{2r,j}.$$

Comparing the coefficient of $\pi_{2r,r}$ on both sides we get $c_r = b_{2r,r} a_{2r,r} = 1$ for each $0 \leq 2r \leq n$. Hence $\psi = \text{id}$, which proves (33). □

By plugging (28) and (33) in (30) one gets the tube formulas $\mathbf{T}_t \tau_{k,p}$ in terms of the $\tau_{i,j}$.

4 Tube formulas in Riemannian manifolds

4.1 Tubular and derivative operators

Next we extend to any complete Riemannian manifold M the tubular operator \mathbf{T}_t introduced in the previous section on linear spaces. Let T be the *Reeb vector field*

on SM , which is characterized by $i_T\alpha = 1$ and $\mathcal{L}_T\alpha = 0$, where \mathcal{L} is the Lie derivative. The Reeb flow $\phi : SM \times \mathbb{R} \rightarrow SM$, defined as the flow of T , is a family of contactomorphisms and coincides with the geodesic flow on SM (see e.g. [18, Theorem 1.5.2]).

Definition 4.1 (*Tubular and derivative operators*) Given $t \geq 0$, we define the *tubular operator* \mathbf{T}_t by

$$\mathbf{T}_t : \mathcal{V}(M) \longrightarrow \mathcal{V}(M), \quad \llbracket \omega, \eta \rrbracket \longmapsto \llbracket \phi_t^* \omega + (p_t)_*(\pi \circ \phi)^* \eta, \eta \rrbracket,$$

where $p_t : SM \times [0, t] \rightarrow SM$ is the projection on the first factor, and $\phi_t = \phi(\cdot, t)$. We define the *derivative operator* $\partial = \partial_M$ by

$$\partial_M : \mathcal{V}(M) \longrightarrow \mathcal{V}(M), \quad \mu \longmapsto \left. \frac{d}{dt} \right|_{t=0} \mathbf{T}_t \mu.$$

To show that these definitions are consistent, suppose $\mu = \llbracket \omega, \eta \rrbracket = 0$, and let us check that $\mathbf{T}_t \mu = 0$ for all $t \geq 0$, i.e.

$$\int_{N(A)} \phi_t^* \omega + \int_{N(A)} (p_t)_*(\pi \circ \phi)^* \eta + \int_A \eta = 0, \quad \forall A \in \mathcal{R}(M).$$

By (5) we have $\pi^* \eta = -D\omega = -d(\omega + \xi \wedge \alpha)$. Hence

$$\begin{aligned} \int_{N(A)} (p_t)_*(\pi \circ \phi)^* \eta &= - \int_{N(A)} (p_t)_* \circ \phi^* D\omega = - \int_{N(A) \times [0, t]} \phi^* d(\omega + \xi \wedge \alpha) \\ &= - \int_{N(A) \times [0, t]} d\phi^*(\omega + \xi \wedge \alpha) = - \int_{N(A) \times [0, t]} \phi^* \omega + \phi^* \xi \wedge \alpha \\ &= \int_{N(A)} \phi_0^* \omega - \int_{N(A)} \phi_t^* \omega = \int_{N(A)} \omega - \int_{N(A)} \phi_t^* \omega, \end{aligned}$$

as α vanishes on $N(A)$. Since $\llbracket \omega, \eta \rrbracket = 0$, we have $\int_{N(A)} \omega = - \int_A \eta$. Therefore $\mathbf{T}_t \mu = 0$.

Let us next establish some basic properties of these operators.

Lemma 4.1

$$\frac{d}{dt} (p_t)_* \phi^* \rho = i_T \phi_t^* \rho, \quad \rho \in \Omega^*(SM)$$

Proof Given a compact smooth submanifold $N \subset SM$,

$$\int_N (p_t)_* \phi^* \rho = \int_{N \times [0, t]} \phi^* \rho = \int_{N \times [0, t]} i_{\frac{\partial}{\partial t}} \phi^* \rho \wedge dt = \int_0^t \left(\int_N \phi_t^* i_{\frac{\partial \phi}{\partial t}} \rho \right) dt,$$

Since i_T and ϕ_t^* commute, the result follows. □

Proposition 4.2 For $\mu = \llbracket \omega, \eta \rrbracket$,

- (i) $\partial\mu = \llbracket i_T(d\omega + \pi^*\eta), 0 \rrbracket$
- (ii) $\mathbf{T}_{t+s}\mu = (\mathbf{T}_t \circ \mathbf{T}_s)\mu$

Proof Modulo exact forms we have

$$\frac{d}{dt}\phi_t^*\omega = \left. \frac{d}{ds}\right|_{s=0} \phi_{t+s}^*\omega = \mathcal{L}_T\phi_t^*\omega \equiv i_T\phi_t^*d\omega. \tag{34}$$

Together with Lemma 4.1, this yields

$$\frac{d}{dt}\mathbf{T}_t\mu = \llbracket i_T(d\phi_t^*\omega + \phi_t^*\pi^*\eta), 0 \rrbracket. \tag{35}$$

Evaluating at $t = 0$, this gives (i).

In order to prove (ii), it is enough to check that both sides have the same derivative with respect to s , as they clearly agree for $s = 0$. By (35), we have

$$\begin{aligned} \frac{d}{ds}\mathbf{T}_t \circ \mathbf{T}_s(\mu) &= \mathbf{T}_t \circ \frac{d}{ds}\mathbf{T}_s(\mu) \\ &= \mathbf{T}_t\llbracket i_T(d\phi_s^*\omega + \phi_s^*\pi^*\eta), 0 \rrbracket \\ &= \llbracket \phi_t^*i_T(d\phi_s^*\omega + \phi_s^*\pi^*\eta), 0 \rrbracket. \end{aligned}$$

Since ϕ_t^* and i_T commute, it follows from (35) that $\frac{d}{ds}\mathbf{T}_{t+s} = \frac{d}{ds}\mathbf{T}_t \circ \mathbf{T}_s$. □

Fix $\mu \in \mathcal{V}(M)$. It follows from Proposition 4.2 (ii) that

$$\frac{d}{dt}\mathbf{T}_t\mu = \partial\mathbf{T}_t\mu. \tag{36}$$

If $\mu \in \mathcal{V}(M)^G$ for a group G acting on M by isometries, then also $\mathbf{T}_t\mu \in \mathcal{V}(M)^G$. Hence, in case $\mathcal{V}^G(M)$ is finite-dimensional, computing $\mathbf{T}_t\mu$ boils down to solving the Cauchy problem (36) with initial condition $\mathbf{T}_0\mu = \mu$; i.e.

$$\mathbf{T}_t\mu = \exp(t\partial)\mu = \sum_{i \geq 0} \frac{t^i}{i!}\partial^i\mu. \tag{37}$$

This is the approach we will follow to obtain the tube formulas for invariant valuations in complex space forms. Note that (37) coincides with (16) except that ∂ does not need to be nilpotent for general M .

4.2 Tubes in Riemannian manifolds

Let M be a complete Riemannian manifold and let $d: M \times M \rightarrow [0, \infty)$ be the Riemannian distance on M . For $t \geq 0$, the *tube* of radius t around a subset $A \subset M$ is

defined as

$$A_t := \{p \in M : d_A(p) \leq t\},$$

where

$$d_A(p) := \inf \{d(p, q) : q \in A\}.$$

Next we review some basic facts about tubes around sets of positive reach (introduced by Federer in euclidean spaces and by Kleinjohann in Riemannian manifolds). For such sets A we will prove that $\mathbf{T}_t\mu(A) = \mu(A_t)$ for any $\mu \in \mathcal{V}(M)$ and sufficiently small t .

Definition 4.2 (*Sets of positive reach*) A set of positive reach in M is a closed subset $A \subset M$ for which there exists an open neighborhood $U_A \supset A$ such that for every $p \in U_A \setminus A$ there exists a unique point $f_A(p) \in A$ such that $d(p, f_A(p)) = d_A(p)$, and a unique minimizing geodesic joining p with $f_A(p)$. We denote by $\mathcal{R}(M)$ the class of compact sets of positive reach in M .

By the previous definition, there is a well-defined map

$$F_A : U_A \setminus A \longrightarrow SM, \quad F_A(p) = (\gamma(0), \gamma'(0)) \quad (38)$$

where γ is the unique minimizing geodesic such that $\gamma(0) = f_A(p)$ and $\gamma(d_A(p)) = p$.

It was shown by Kleinjohann [22, Satz 3.3] that $N(A) := F_A(U_A \setminus A)$ is a naturally oriented compact Lipschitz submanifold of SM . The corresponding current, also denoted by $N(A)$, is called the *normal cycle* of A . It follows from Proposition 4.6 below that $N(A)$ is legendrian (i.e. it vanishes on multiples of α).

Proposition 4.3 ([22, Satz 3.3, Korollar 2.7]) *Given a set of positive reach A in M there exists $r = r_A > 0$ such that $A_r \subset U_A$ and*

- (i) *for $0 < t < r$ the restriction $F_A|_{\partial A_t}$ gives a bilipschitz homeomorphism between ∂A_t and $N(A)$, preserving the natural orientations,*
- (ii) *the distance function d_A is of class C^1 in $A_r \setminus A$ and*

$$\phi_{d_A(p)}(F_A(p)) = (p, \nabla d_A(p)), \quad \partial A_t = d_A^{-1}(\{t\})$$

for $0 < t < r$. In particular, each level set ∂A_t with $0 < t < r$ is a C^1 -regular hypersurface with unit normal vector field ∇d_A .

The following propositions are certainly well-known.

Proposition 4.4 *For $0 < s < r = r_A$ the set A_s has positive reach and on $A_r \setminus A_s$ we have*

$$d_{A_s} = d_A - s, \quad F_{A_s} = \phi_s \circ F_A. \quad (39)$$

In particular $(A_s)_t = A_{t+s}$ for $t + s < r$.

Proof Let $p \in A_r \setminus A_s$, and put $d = d_A(p)$. Let $\gamma : [0, d] \rightarrow A_r$ be the unique minimizing geodesic with $\gamma(0) = f_A(p)$ and $\gamma(d) = p$. In particular $|\gamma'| = 1$ and thus $\gamma(s) \in A_s$.

Assume that $\gamma|_{[s,d]}$ does not minimize the distance between p and A_s , i.e., there exists a smooth curve $\alpha : [0, 1] \rightarrow M$ with $q := \alpha(0) \in A_s$, $\alpha(1) = p$ and length $\ell(\alpha) < d - s$. It follows that

$$d_A(p) \leq \ell(\alpha) + d_A(q) \leq \ell(\alpha) + s < d_A(p),$$

a contradiction. We conclude that $\gamma|_{[s,d]}$ realizes the distance $d_{A_s}(p)$. Hence $d_{A_s}(p) = d_A(p) - s$ and

$$F_{A_s}(p) = (\gamma(s), \gamma'(s)) = \phi_s(\gamma(0), \gamma'(0)) = \phi_s(F_A(p)).$$

□

Proposition 4.5 For $0 < s < r_A$, the restriction $\phi_s|_{N(A)}$ is a bilipschitz homeomorphism between $N(A)$ and $N(A_s)$.

Proof Take t with $s < t < \min(r_A, s+r_{A_s})$. By Proposition 4.3, both $F_A|_{\partial A_t} : \partial A_t \rightarrow N(A)$ and $F_{A_s}|_{\partial A_t} : \partial A_t \rightarrow N(A_s)$ are bilipschitz homeomorphisms. By (39) we have

$$\phi_s|_{N(A)} = F_{A_s}|_{\partial A_t} \circ (F_A|_{\partial A_t})^{-1}.$$

The statement follows. □

Proposition 4.6 For $0 < t < r_A$ the composition $\pi \circ \phi$ gives a bijective Lipschitz map between $N(A) \times (0, t]$ and $A_t \setminus A$.

Proof Since π, ϕ are smooth, the restriction of $\pi \circ \phi$ to the Lipschitz manifold $N(A) \times (0, t]$ is clearly Lipschitz.

Given $(\xi, s) \in N(A) \times (0, t]$, we know by the previous proposition that $\phi(\xi, s) \in N(A_s)$ and thus $\pi \circ \phi(x, s) \in \partial A_s \subset A_t \setminus A$.

To check surjectivity, given $p \in A_t \setminus A$ take $\xi = F_A(p)$, $s = d_A(p)$ and note that $\pi \circ \phi(\xi, s) = p$.

As for injectivity, suppose $\pi \circ \phi(\xi_1, t_1) = \pi \circ \phi(\xi_2, t_2) =: p$ for some $(\xi_1, t_1), (\xi_2, t_2) \in N(A) \times (0, t]$. By the previous proposition p belongs to both $\partial A_{t_1}, \partial A_{t_2}$, so $t_1 = t_2$. For $s \in [0, t_1]$, the geodesics $\gamma_1(s) = \pi \circ \phi(\xi_1, s)$, $\gamma_2(s) = \pi \circ \phi(\xi_2, s)$ realize the distance between p and A . Since $A_s \subset A_{r_A} \subset U_A$, we have $\gamma_1 = \gamma_2$ and thus $\xi_1 = \xi_2$. □

Corollary 4.7 For every $A \in \mathcal{R}(M)$ and $\mu \in \mathcal{V}(M)$ we have $\mu(A_t) = \mathbf{T}_t \mu(A)$ for $0 \leq t \leq r_A$.

Proof. Let $\mu = \llbracket \omega, \eta \rrbracket$. By Propositions 4.5 and 4.6 and the coarea formula,

$$\mu(A_t) = \int_{N(A_t)} \omega + \int_{A_t} \eta \int_{\phi_t(N(A))} \omega$$

$$+ \int_{(\pi \circ \phi)(N(A) \times (0, t))} \eta + \int_A \eta = \mathbf{T}_t \mu(A). \quad \square$$

Remark In the subclass $\mathcal{P}(M) \subset \mathcal{R}(M)$ of compact submanifolds with corners, the normal cycle is more naturally defined as follows. For $A \in \mathcal{P}(M)$ and $p \in A$, let

$$T_p A = \left\{ \gamma'(0) \in T_p M : \gamma \in C^1([0, 1], A), \gamma(0) = p \right\}$$

$$N'(A) = \{(p, v) \in SM : p \in A, \langle v, w \rangle \leq 0 \forall w \in T_p A\}.$$

Let us check that indeed $N'(A)$ equals $N(A) = F_A(U_A)$. Covering A by local charts (locally modelled on $\mathbb{R}^k \times [0, \infty)^l \subset \mathbb{R}^m$), and considering the copy of $N'(A)$ in the cosphere bundle of M , one sees that $N'(A)$ is a compact topological manifold.

It is also easy to show that $N(A) \subset N'(A)$. It then follows by the invariance of domain theorem that $N(A)$ is an open subset of $N'(A)$. Since $N'(A)$ is a Hausdorff space and $N(A)$ is compact, we also have that $N(A)$ is a closed subset of $N'(A)$. Since the number of connected components of both $N(A)$, $N'(A)$ clearly equals the number of connected components of A , we necessarily have $N(A) = N'(A)$.

4.3 Derivative operators in \mathbb{S}_λ^m and $\mathbb{C}P_\lambda^n$

Given $\lambda \in \mathbb{R}$ let $\partial_{\lambda, \mathbb{R}} : \mathcal{V}_{\lambda, \mathbb{R}}^m \rightarrow \mathcal{V}_{\lambda, \mathbb{R}}^m$ be the restriction of $\partial_{\mathbb{S}_\lambda^m}$ to $\mathcal{V}_{\lambda, \mathbb{R}}^m$, and let $\partial_{\lambda, \mathbb{C}}$ be the restriction of $\partial_{\mathbb{C}P_\lambda^m}$ to $\mathcal{V}_{\lambda, \mathbb{C}}^m$.

Proposition 4.8

$$\partial_{\lambda, \mathbb{R}} \sigma_i^\lambda = (m - i) \sigma_{i-1}^\lambda - \lambda(i + 1) \sigma_{i+1}^\lambda, \quad 0 \leq i \leq m - 2, \quad (40)$$

$$\partial_{\lambda, \mathbb{R}} \sigma_{m-1}^\lambda = \sigma_{m-2}^\lambda, \quad (41)$$

$$\partial_{\lambda, \mathbb{R}} \sigma_m^\lambda = \sigma_{m-1}^\lambda, \quad (42)$$

where it is understood that $\sigma_{-1}^\lambda = 0$.

Let us emphasize that (40) would make formal sense but does not hold for $i = m - 1$.

Proof By [15, Lemma 3.1], putting $\kappa_m = 0$, we have

$$d\kappa_i = \alpha \wedge ((m - i)\kappa_{i-1} - \lambda(i + 1)\kappa_{i+1}), \quad 0 \leq i \leq m - 1.$$

Contracting with T yields

$$i_T d\kappa_i = (m - i)\kappa_{i-1} - \lambda(i + 1)\kappa_{i+1}, \quad \text{mod } (\alpha, d\alpha).$$

By Proposition 4.2 the result follows. □

Lemma 4.9 *The following equalities hold modulo $\alpha, d\alpha$*

(i) For $k > 2q$

$$\begin{aligned} \frac{\omega_{2n-k}}{\omega_{2n-k+1}} i_T d\beta_{k,q} &\equiv 2(n-k+q+1)\gamma_{k-1,q} \\ &+ (k-2q+1)\beta_{k-1,q-1} \\ &- \frac{\lambda}{2\pi} (k-2q+1)(2n-k+1)\beta_{k+1,q}. \end{aligned} \quad (43)$$

(ii) For $n > k - q$

$$\begin{aligned} \frac{\omega_{2n-2q}}{\omega_{2n-2q+1}} i_T d\gamma_{2q,q} &\equiv \beta_{2q-1,q-1} \\ &- \frac{\lambda}{2\pi} \frac{(q+2)(2n-2q+1)}{n-q} \beta_{2q+1,q} \\ &- \frac{\lambda}{2\pi} \frac{(n-q-1)(2n-2q+1)}{n-q} \gamma_{2q+1,q}. \end{aligned} \quad (44)$$

Proof This is a straightforward computation using [1, Lemma 3.3, Lemma 3.6]. \square

Proposition 4.10 For $k > 2q$

$$\begin{aligned} \frac{\omega_{2n-k}}{\omega_{2n-k+1}} \partial_{\lambda, \mathbb{C}} \mu_{k,q}^\lambda &= (k-2q+1)\mu_{k-1,q-1}^\lambda + 2(n-k+q+1)\mu_{k-1,q}^\lambda \\ &- \frac{\lambda}{2\pi} (2n-k+1) \left((k-2q+1)\mu_{k+1,q}^\lambda + 2(q+1)\mu_{k+1,q+1}^\lambda \right) \end{aligned} \quad (45)$$

and

$$\frac{\omega_{2n-2q}}{\omega_{2n-2q+1}} \partial_{\lambda, \mathbb{C}} \mu_{2q,q}^\lambda = \mu_{2q-1,q-1} - (2n-2q+1) \frac{\lambda}{2\pi} \mu_{2q+1,q}. \quad (46)$$

Proof Equality (45) follows from Proposition 4.2, using (43) and the following (see [1, Proposition 2.7])

$$\llbracket \gamma_{k,q}, 0 \rrbracket = \mu_{k,q}^\lambda - \lambda \frac{(2n-k)(q+1)}{2\pi(n-k+q)} \mu_{k+2,q+1}^\lambda, \quad n-k+q > 0. \quad (47)$$

Let us now prove (46). Note first that from (44) and (47) we get

$$\begin{aligned}
 \llbracket i_T d\gamma_{2j,j}, 0 \rrbracket &= \frac{\omega_{2n-2j+1}}{\omega_{2n-2j}} \mu_{2j-1,j-1}^\lambda \\
 &\quad - \frac{\omega_{2n-2j+1}}{\omega_{2n-2j}} \frac{(2n-2j+1)(n+1)}{n-j} \frac{\lambda}{2\pi} \mu_{2j+1,j}^\lambda \\
 &\quad + \frac{\omega_{2n-2j+1}}{\omega_{2n-2j}} \frac{(2n-2j+1)(2n-2j-1)(j+1)}{n-j} \frac{\lambda^2}{4\pi^2} \mu_{2j+3,j+1}^\lambda \\
 &=: a_j \mu_{2j-1,j-1}^\lambda + b_j \frac{\lambda}{\pi} \mu_{2j+1,j}^\lambda + c_j \frac{\lambda^2}{\pi^2} \mu_{2j+3,j+1}^\lambda
 \end{aligned} \tag{48}$$

Then, by Proposition 4.2 and observing that $a_n = 2$

$$\begin{aligned}
 \partial_{\lambda, \mathbb{C}} \mu_{2q,q}^\lambda &= \sum_{i=0}^{n-q-1} \left(\frac{\lambda}{\pi}\right)^i \frac{(q+i)!}{q!} \llbracket i_T d\gamma_{2q+2i,q+i}, 0 \rrbracket + 2 \left(\frac{\lambda}{\pi}\right)^{n-q} \frac{n!}{q!} \mu_{2n-1,n-1}^\lambda \\
 &= \sum_{i=0}^{n-q} \left(\frac{\lambda}{\pi}\right)^i \frac{(q+i)!}{q!} a_{q+i} \mu_{2q+2i-1,q+i-1}^\lambda \\
 &\quad + \sum_{i=0}^{n-q-1} \left(\frac{\lambda}{\pi}\right)^{i+1} \frac{(q+i)!}{q!} b_{q+i} \mu_{2q+2i+1,q+i}^\lambda \\
 &\quad + \sum_{i=0}^{n-q-2} \left(\frac{\lambda}{\pi}\right)^{i+2} \frac{(q+i)!}{q!} c_{q+i} \mu_{2q+2i+3,q+i+1}^\lambda \\
 &= a_q \mu_{2q-1,q-1} + \frac{\lambda}{\pi} ((q+1)a_{q+1} + b_q) \mu_{2q+1,q}^\lambda \\
 &\quad + \sum_{j=2}^{n-q} \left(\frac{\lambda}{\pi}\right)^j \left(\frac{(q+j)!}{q!} a_{q+j} + \frac{(q+j-1)!}{q!} b_{q+j-1} \right. \\
 &\quad \left. + \frac{(q+j-2)!}{q!} c_{q+j-2} \right) \mu_{2q+2j-1,q+j-1}^\lambda
 \end{aligned}$$

A straightforward computation using $k\omega_k = 2\pi\omega_{k-2}$ shows

$$j(j-1)a_j + (j-1)b_{j-1} + c_{j-2} = 0$$

and the result follows. □

Note that by (18) the linear map $\Phi_0: \text{Val}^{U(n)} \rightarrow \text{Val}^{U(n)}$ given by $\Phi_0|_{\text{Val}_k^{U(n)}} = \omega_{2n-k}$ id satisfies

$$\partial_{0, \mathbb{C}} = \Phi_0 \circ \Lambda \circ \Phi_0^{-1}.$$

Remarkably, a similar identity holds for all λ , which will be crucial for our determination of tube formulas in $\mathbb{C}P_\lambda^n$.

Theorem 4.11 *The linear isomorphism*

$$\Phi_\lambda = \mathcal{F}_{\lambda, \mathbb{C}} \circ \Phi_0 : \text{Val}^{U(n)} \longrightarrow \mathcal{V}_{\lambda, \mathbb{C}}^n, \quad \mu_{k,q} \longmapsto \omega_{2n-k} \mu_{k,q}^\lambda.$$

fulfills

$$\partial_{\lambda, \mathbb{C}} = \Phi_\lambda \circ (\Lambda - \lambda L) \circ \Phi_\lambda^{-1}.$$

Proof. By combining Proposition 4.10, Proposition 3.1 and the fact $\frac{\omega_n}{\omega_{n-2}} = \frac{2\pi}{n}$, this is straightforward to check:

$$\begin{aligned} & \Phi_\lambda \circ (\Lambda - \lambda L)(\mu_{k,q}) \\ &= (k - 2q + 1)\omega_{2n-k+1}\mu_{k-1,q-1}^\lambda + 2(n - k + q + 1)\omega_{2n-k+1}\mu_{k-1,q}^\lambda \\ &\quad - \lambda(k - 2q + 1)\omega_{2n-k-1}\mu_{k+1,q}^\lambda - 2\lambda(q + 1)\omega_{2n-k-1}\mu_{k+1,q+1}^\lambda \\ &= \omega_{2n-k+1} \left((k - 2q + 1)\mu_{k-1,q-1}^\lambda + 2(n - k + q + 1)\mu_{k-1,q}^\lambda \right. \\ &\quad \left. - \frac{\lambda}{2\pi}(2n - k + 1) \left((k - 2q + 1)\mu_{k+1,q}^\lambda + 2(q + 1)\mu_{k+1,q+1}^\lambda \right) \right) \\ &= \omega_{2n-k} \partial_{\lambda, \mathbb{C}} \mu_{k,q}^\lambda = \partial_{\lambda, \mathbb{C}} \circ \Phi_\lambda(\mu_{k,q}). \quad \square \end{aligned}$$

A similar phenomenon holds in real space forms, but restricted to a hyperplane of $\mathcal{V}_{\lambda, \mathbb{R}}^m$.

Theorem 4.12 *The linear monomorphism*

$$\Psi_\lambda : \text{Val}^{O(m)} \longrightarrow \mathcal{V}_{\lambda, \mathbb{R}}^{m+1}, \quad \mu_k \longmapsto \sigma_k^\lambda$$

fulfills

$$\partial_{\lambda, \mathbb{R}} \circ \Psi_\lambda = \Psi_\lambda \circ (\Lambda - \lambda L).$$

Proof By Proposition 4.8 and Theorem 3.4

$$\begin{aligned} \partial_{\lambda, \mathbb{R}} \circ \Psi_\lambda(\mu_k) &= \partial_{\lambda, \mathbb{R}} \sigma_k^\lambda = (m - k + 1)\sigma_{k-1}^\lambda - \lambda(k + 1)\sigma_{k+1}^\lambda \\ &= \Psi_\lambda((m - k + 1)\mu_{k-1} - \lambda(k + 1)\mu_{k+1}) \\ &= \Psi_\lambda(\Lambda \mu_k - \lambda L \mu_k). \end{aligned}$$

□

Note the difference of dimensions between the source and the target of Ψ_λ . We will show that there is no isomorphism between $\text{Val}^{O(m)}$ and $\mathcal{V}_{\lambda, \mathbb{R}}^m$ intertwining ∂ and $\Lambda - \lambda L$. This is essentially due to the fact that (41) and (42) differ from (40).

5 A model space for tube formulas

We next perform some abstract computations that will easily lead to the tube formulas in both complex and real space forms via (62) and (64). The same approach will allow us to determine the kernel, the image, and the spectrum of the derivative operator ∂ on these spaces.

5.1 A system of differential equations

It is well-known that the operators $X = x \frac{\partial}{\partial y}$, $Y = y \frac{\partial}{\partial x}$ and $H = [X, Y]$ induce an \mathfrak{sl}_2 -structure on $\mathbb{C}[x, y]$. The decomposition into irreducible components is $\mathbb{C}[x, y] = \bigoplus_{m \geq 0} V^{(m)}$ where $V^{(m)}$ is the subspace of m -homogeneous polynomials:

$$V^{(m)} := \text{span}_{\mathbb{C}} \{x^k y^{m-k}\}_{k=0}^m.$$

One has $H(x^k y^{m-k}) = (m - 2k)x^k y^{m-k}$.

Motivated by Theorem 4.11, we consider $Y_\lambda = Y - \lambda X$, which is a derivation on $\mathbb{C}[x, y]$. It will be sometimes convenient to consider the monomials $\binom{m}{k} x^k y^{m-k}$. In these terms

$$\binom{m}{k} Y_\lambda(x^k y^{m-k}) = (m - k + 1) \binom{m}{k-1} x^{k-1} y^{m-k+1} - \lambda(k+1) \binom{m}{k+1} x^{k+1} y^{m-k-1}. \quad (49)$$

Our goal here is to solve the following Cauchy problem: find $p_k: \mathbb{R} \rightarrow V^{(m)}$ such that

$$\partial_t p_k(t) = Y_\lambda p_k(t), \quad p_k(0) = \binom{m}{k} x^k y^{m-k}, \quad 0 \leq k \leq m, \quad (50)$$

i.e. to compute

$$p_k(t) = \binom{m}{k} \exp(t Y_\lambda)(x^k y^{m-k}), \quad 0 \leq k \leq m. \quad (51)$$

We will use the standard notation

$$\sin_\lambda(t) := \begin{cases} \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} & \lambda > 0, \\ t & \lambda = 0, \\ \frac{\sinh(\sqrt{|\lambda|}t)}{\sqrt{|\lambda|}} & \lambda < 0, \end{cases} \quad (52)$$

which is an analytic function in both λ and t , and $\cos_\lambda(t) := \frac{d}{dt} \sin_\lambda(t)$.

Proposition 5.1 For any $\lambda, t \in \mathbb{R}$, we have

$$\exp(tY_\lambda)x = x \cos_\lambda(t) + y \sin_\lambda(t) =: u, \quad \exp(tY_\lambda)y = y \cos_\lambda(t) - \lambda x \sin_\lambda(t) =: v.$$

Proof Since clearly

$$Y_\lambda^{2k}x = (-\lambda)^k x, \quad Y_\lambda^{2k+1}x = (-\lambda)^k y,$$

we have

$$\begin{aligned} \exp(tY_\lambda)x &= \sum_{k \geq 0} \frac{t^k}{k!} Y_\lambda^k x \\ &= \sum_{k \geq 0} \frac{t^{2k}}{(2k)!} (-\lambda)^k x + \sum_{k \geq 0} \frac{t^{2k+1}}{(2k+1)!} (-\lambda)^k y \\ &= x \cos_\lambda(t) + y \sin_\lambda(t). \end{aligned}$$

In the same way we can compute $\exp(tY_\lambda)y$. □

The following standard and elementary fact will be useful.

Lemma 5.2 Let \mathbf{A} be a finite-dimensional algebra. A vector field on \mathbf{A} is a derivation iff its flow ϕ_t satisfies

$$\phi_t(pq) = \phi_t(p)\phi_t(q), \quad \forall p, q \in \mathbf{A}, \forall t \in \mathbb{R}.$$

In other words, each ϕ_t is an \mathbf{A} -morphism.

Theorem 5.3 The solution of the Cauchy problem (50) is

$$p_k(t) = \binom{m}{k} u^k v^{m-k} \tag{53}$$

$$= \binom{m}{k} (x \cos_\lambda(t) + y \sin_\lambda(t))^k (y \cos_\lambda(t) - \lambda x \sin_\lambda(t))^{m-k} \tag{54}$$

$$= \sum_{j=0}^m \phi_{m,k,j}^\lambda(t) \binom{m}{j} x^j y^{m-j}, \tag{55}$$

where

$$\phi_{m,k,j}^\lambda(t) = \sum_{h \geq 0} (-\lambda)^{j-h} \binom{m-j}{k-h} \binom{j}{h} \sin_\lambda^{k+j-2h}(t) \cos_\lambda^{m-k-j+2h}(t). \tag{56}$$

Proof Since Y_λ is a derivation, $\exp(tY_\lambda)$ is a $\mathbb{C}[x, y]$ -morphism by the previous lemma. Hence

$$\exp(tY_\lambda)x^k y^{m-k} = (\exp(tY_\lambda)x)^k (\exp(tY_\lambda)y)^{m-k} = u^k v^{m-k}.$$

Comparing with (51) yields (54).

It remains to prove (55). Putting $s = \sin_\lambda(t)$, $c = \cos_\lambda(t)$ we have

$$\begin{aligned} & \binom{m}{k} (xc + ys)^k (yc - \lambda xs)^{m-k} \\ &= \binom{m}{k} \sum_{a,b} \binom{k}{a} (ys)^a (xc)^{k-a} \binom{m-k}{b} (-\lambda xs)^b (yc)^{m-k-b} \\ &= \binom{m}{k} \sum_{a,b} \binom{k}{a} \binom{m-k}{b} (-\lambda)^b s^{a+b} c^{m-a-b} x^{k-a+b} y^{m-k+a-b} \\ &= \binom{m}{k} \sum_{j,h} \binom{k}{h} \binom{m-k}{j-h} (-\lambda)^{j-h} s^{j+k-2h} c^{m-j-k+2h} x^j y^{m-j} \end{aligned}$$

where we changed $a = k - h$, $b = j - h$. Using

$$\binom{m}{k} \binom{k}{h} \binom{m-k}{j-h} = \binom{m-j}{k-h} \binom{j}{h} \binom{m}{j} \tag{57}$$

yields (55). □

5.2 Eigenvalues and eigenvectors of Y_λ

Given $f: V \rightarrow V$ an endomorphism of \mathbb{C} -vector spaces, we denote by $\text{spec}(f)$ the set of eigenvalues of f and by $E_\alpha(f)$ the eigenspace associated to each $\alpha \in \text{spec}(f)$.

Lemma 5.4 *The endomorphism $Y_\lambda|_{V^{(m)}}$ is diagonalizable with simple multiplicities and*

$$\begin{aligned} \text{spec}(Y_\lambda|_{V^{(m)}}) &= \left\{ (2k - m)\sqrt{-\lambda} : 0 \leq k \leq m \right\}, \\ E_{(2k-m)\sqrt{-\lambda}}(Y_\lambda|_{V^{(m)}}) &= \text{span}\{e_1^k e_2^{m-k}\}, \end{aligned}$$

where $e_1 := \sqrt{-\lambda}x + y$ and $e_2 := -\sqrt{-\lambda}x + y$.

Proof The result is trivial to check for $m = 1$ as

$$Y_\lambda(e_1) = \sqrt{-\lambda}y - \lambda x = \sqrt{-\lambda}e_1, \quad Y_\lambda(e_2) = -\sqrt{-\lambda}y - \lambda x = -\sqrt{-\lambda}e_2.$$

Since Y_λ is a derivation

$$\begin{aligned} Y_\lambda e_1^k &= k e_1^{k-1} Y_\lambda e_1 = k\sqrt{-\lambda}e_1^k, \\ Y_\lambda e_2^{m-k} &= (m-k)e_2^{m-k-1} Y_\lambda e_2 = -\sqrt{-\lambda}(m-k)e_2^{m-k}. \end{aligned}$$

Hence

$$Y_\lambda(e_1^k e_2^{m-k}) = (2k - m)\sqrt{-\lambda}e_1^k e_2^{m-k},$$

as stated. □

Remark It is interesting to notice that the spectra of Y_λ and $\sqrt{-\lambda}H$, when restricted to each $V^{(m)}$, are identical. These two operators are thus intertwined e.g. by the linear isomorphism $x^k y^{m-k} \mapsto e_1^k e_2^{m-k}$.

5.3 Image of Y_λ

Using Lemma 5.4, we can conclude that $Y_\lambda|_{V^{(m)}}$ is bijective if and only if m is odd. If m is even, then the kernel is one-dimensional. An explicit description is the following.

Proposition 5.5 *If m is even, then*

$$\text{im}(Y_\lambda|_{V^{(m)}}) = \ker Z_{m,\lambda}, \tag{58}$$

where

$$Z_{m,\lambda} := \left(\frac{\partial^2}{\partial x^2} + \lambda \frac{\partial^2}{\partial y^2} \right)^{m/2}.$$

Proof By the binomial formula

$$\begin{aligned} Z_{m,\lambda}(x^k y^{m-k}) &= \sum_{i=0}^{m/2} \lambda^{m/2-i} \binom{m/2}{i} \frac{\partial^m}{\partial x^{2i} \partial y^{m-2i}} x^k y^{m-k} \delta_{k,2i} \\ &= \lambda^{\frac{m-k}{2}} \binom{m/2}{k/2} k!(m-k)! \end{aligned} \tag{59}$$

if k is even, and $Z_{m,\lambda}(x^k y^{m-k}) = 0$ if k is odd. Therefore

$$\begin{aligned} Z_{m,\lambda} \circ Y_\lambda(x^{2l+1} y^{m-2l-1}) &= Z_{m,\lambda}((2l+1)x^{2l} y^{m-2l} - \lambda(m-2l-1)x^{2l+2} y^{m-2l-2}) \\ &= \lambda^{\frac{m}{2}-l} \binom{m/2}{l} (2l+1)!(m-2l)! \\ &\quad - \lambda^{\frac{m}{2}-l} \binom{m/2}{l+1} (2l+2)!(m-2l-1)! = 0 \\ Z_{m,\lambda} \circ Y_\lambda(x^{2l} y^{m-2l}) &= 0. \end{aligned}$$

This shows that $\text{im}(Y_\lambda)$ is a subspace of $\ker Z_{m,\lambda}$. Given that $Z_{m,\lambda}$ is not zero, we have $\dim \ker Z_{m,\lambda} = m$, and by Lemma 5.4, we know that the image of $Y_\lambda|_{V^{(m)}}$ has the same dimension. This yields (58). \square

Next we compute, for even m and given φ in the image of $Y_\lambda|_{V^{(m)}}$, the preimage $Y_\lambda^{-1}(\{\varphi\})$.

Consider

$$P_{m,k} := \sum_{j \geq 0} \lambda^j \frac{(k+2j-1)!(m-k-2j-1)!}{(k-1)!(m-k+1)!} \binom{m}{k+2j} x^{k+2j} y^{m-k-2j} \in V^{(m)}. \tag{60}$$

A simple computation using (49) shows

$$Y_\lambda P_{m,k} = \binom{m}{k-1} x^{k-1} y^{m-k+1} - c_{m,k} x^m$$

where $c_{m,k} = 0$ if $m - k$ is even, and otherwise

$$c_{m,k} = \lambda^{\frac{m-k+1}{2}} \frac{m!!}{(k-1)!!(m-k+1)!!}.$$

With these ingredients at hand, for even m , we can now compute a preimage by Y_λ of any element in $\text{im } Y_\lambda$ as follows.

Proposition 5.6 *Let $\Pi: V^{(m)} \rightarrow V^{(m)}$ be given by $\binom{m}{k} x^k y^{m-k} \mapsto P_{m,k+1}$. If m is even then*

$$Y_\lambda \circ \Pi(\varphi) = \varphi, \quad \forall \varphi \in \text{im } Y_\lambda|_{V^{(m)}} \quad (61)$$

Proof Let $0 < k < m$. Since $(m-k+1)c_{m,k} - \lambda(k+1)c_{m,k+2} = 0$, using (49) we get

$$\begin{aligned} Y_\lambda \circ \Pi \circ Y_\lambda \binom{m}{k} x^k y^{m-k} &= Y_\lambda \circ \Pi \left((m-k+1) \binom{m}{k-1} x^{k-1} y^{m-k+1} - \lambda(k+1) \binom{m}{k+1} x^{k+1} y^{m-k-1} \right) \\ &= (m-k+1) Y_\lambda P_{m,k} - \lambda(k+1) Y_\lambda P_{m,k+2} \\ &= (m-k+1) \binom{m}{k-1} x^{k-1} y^{m-k+1} - \lambda(k+1) \binom{m}{k+1} x^{k+1} y^{m-k-1} \\ &\quad - ((m-k+1)c_{m,k} - \lambda(k+1)c_{m,k+2}) x^m \\ &= Y_\lambda \binom{m}{k} x^k y^{m-k}. \end{aligned}$$

For $k = 0$ and $k = m$,

$$\begin{aligned} Y_\lambda \circ \Pi \circ Y_\lambda (y^m) &= -\lambda Y_\lambda (P_{m,2}) = -\lambda (mxy^{m-1} - c_{m,2} x^m) = Y_\lambda (y^m) + \lambda c_{m,2} x^m, \\ Y_\lambda \circ \Pi \circ Y_\lambda (x^m) &= Y_\lambda (P_{m,m}) = mx^{m-1}y - c_{m,m} x^m = Y_\lambda (x^m) + c_{m,m} x^m. \end{aligned}$$

Since $c_{m,m} = 0$, and $c_{m,2} = 0$ if m is even, the result follows. \square

6 Tube formulas in \mathbb{S}_λ^m and $\mathbb{C}P_\lambda^n$

Here we will obtain our main result: the tube formulas for invariant valuations of $\mathbb{C}P_\lambda^n$ (i.e. the tubular operator \mathbf{T}_t on $\mathcal{V}_{\lambda,\mathbb{C}}^n$). We will also recover Santaló's tube formulas for $\mathcal{V}_{\lambda,\mathbb{R}}^m$ (cf. [25]) in a way that explains the similarities between the real and the complex space forms.

6.1 Tube formulas in complex space forms

Recalling (25) and Proposition 3.3, we get an isomorphism $I : W_n \rightarrow \text{Val}^{U(n)}$ of \mathfrak{sl}_2 -modules from

$$W_n := \bigoplus_{0 \leq 2r \leq n} V^{(2n-4r)}$$

to $\text{Val}^{U(n)}$ by putting $I(y^{2n-4r}) = \pi_{2r,r}$ (i.e. mapping Y -primitive elements to Λ -primitive elements) and

$$\begin{aligned} \binom{2n-4r}{k-2r} I(x^{k-2r} y^{2n-k-2r}) &= \frac{1}{(k-2r)!} I(X^{k-2r} (y^{2n-4r})) \\ &= \frac{1}{(k-2r)!} L^{k-2r} I(y^{2n-4r}) = \frac{1}{(k-2r)!} \pi_{k,r}. \end{aligned}$$

By Theorem 4.11, the map $J_{\lambda, \mathbb{C}} := \Phi_\lambda \circ I : W_n \rightarrow \mathcal{V}_{\lambda, \mathbb{C}}^n$ fulfills

$$\partial_{\lambda, \mathbb{C}} \circ J_{\lambda, \mathbb{C}} = J_{\lambda, \mathbb{C}} \circ Y_\lambda. \tag{62}$$

We define

$$\sigma_{k,r}^\lambda := \binom{2n-4r}{k-2r} J_\lambda(x^{k-2r} y^{2n-k-2r}) = \frac{\omega_{2n-k}}{(k-2r)!} \pi_{k,r}^\lambda \tag{63}$$

and arrive at our main theorem.

Theorem 6.1 *The tubular operator \mathbf{T}_t in $\mathcal{V}_{\lambda, \mathbb{C}}^n$ is given by*

$$\mathbf{T}_t(\sigma_{k,r}^\lambda) = \sum_{j=0}^{2n-4r} \phi_{2n-4r, k-2r, j}^\lambda(t) \sigma_{j+2r, r}^\lambda,$$

where

$$\phi_{m,i,j}^\lambda(t) = \sum_{h \geq 0} (-\lambda)^{j-h} \binom{m-j}{i-h} \binom{j}{h} \sin_\lambda^{i+j-2h}(t) \cos_\lambda^{m-i-j+2h}(t).$$

Proof By (37), using (62) and (63), and putting $m = 2n - 4r$, we get

$$\begin{aligned} \mathbf{T}_t \sigma_{k,q}^\lambda &= \exp(t \partial_{\lambda, \mathbb{C}})(\sigma_{k,q}^\lambda) \\ &= \binom{m}{k-2r} \exp(t \partial_{\lambda, \mathbb{C}}) \circ J_{\lambda, \mathbb{C}}(x^{k-2r} y^{m-k+2r}) \\ &= \binom{m}{k-2r} J_{\lambda, \mathbb{C}} \circ \exp(t Y_\lambda)(x^{k-2r} y^{m-k+2r}) \\ &= J_{\lambda, \mathbb{C}}(p_{k-2r}(t)). \end{aligned}$$

Using (55) the result follows. □

The tube formulas in terms of the $\tau_{k,i}^\lambda$ can be obtained from Theorem 6.1 using (28) and (33) which hold verbatim replacing $\pi_{k,r}^\lambda, \tau_{k,r}^\lambda$ for $\pi_{k,r}, \tau_{k,r}$.

Remark The tube formula for the volume $\sigma_{2n,0}^\lambda = \text{vol}_{\mathbb{C}P_\lambda^n}$ is given by the following simple expression

$$\text{vol}_{\mathbb{C}P_\lambda^n}(A_t) = \sum_{j=0}^{2n} \sin_\lambda^{2n-j}(t) \cos_\lambda^j(t) \sigma_{j,0}^\lambda(A),$$

which is Theorem 4.3 of [11], since $\sigma_{j,0}^\lambda = \omega_{2n-j} \tau_{j,0}^\lambda = \Phi_\lambda(\mu_j)$. The tube formulas $\mathbf{T}_t \sigma_{2n-2r,r}$ are equally simple.

Remark An interesting feature of the previous tube formulas is the following self-similarity property, which is explained by (62). Let

$$\mathbf{G}^{n,j} : \mathcal{V}_{\lambda,\mathbb{C}}^n \longrightarrow \mathcal{V}_{\lambda,\mathbb{C}}^{n+2j}, \quad \mathbf{G}^{n,j}(\sigma_{k,r}^\lambda) = \sigma_{k+2j,r+j}^\lambda.$$

Then one has $\mathbf{T}_t \circ \mathbf{G}^{n,j} = \mathbf{G}^{n,j} \circ \mathbf{T}_t$.

Remark It is also worth noting that $\mathcal{V}_{\lambda,\mathbb{C}}^n = \bigoplus_{0 \leq 2r \leq n} \mathcal{I}_\lambda^{n,r}$ where

$$\mathcal{I}_\lambda^{n,r} := J_{\lambda,\mathbb{C}}(V^{(2n-4r)}) = \{ \sigma_{k,r}^\lambda : 2r \leq k \leq 2n - 2r \},$$

and that these subspaces are $\partial_{\lambda,\mathbb{C}}$ -invariant. In particular, given $\varphi \in \mathcal{I}_\lambda^{n,r}$ one has $\mathbf{T}_t(\varphi) \in \mathcal{I}_\lambda^{n,r}$.

6.2 Tube formulas in real space forms

Let $I : V^{(m)} \rightarrow \text{Val}^{O(m)}$ be the isomorphism of irreducible \mathfrak{sl}_2 -representations determined by $I(y^m) = \chi$; i.e.

$$\begin{aligned} \binom{m}{i} I(x^i y^{m-i}) &= \frac{1}{i!} I(X^i(y^m)) = \frac{1}{i!} L^i(I(y^m)) \\ &= \frac{1}{i!} L^i(\mu_0) = \mu_i \end{aligned}$$

where we used (19). By Theorem 4.12, the map $J_{\lambda,\mathbb{R}} = \Psi_\lambda \circ I$ satisfies

$$\partial_{\lambda,\mathbb{R}} \circ J_{\lambda,\mathbb{R}} = J_{\lambda,\mathbb{R}} \circ Y_\lambda. \tag{64}$$

The map $J_{\lambda,\mathbb{R}}$ is explicitly given by

$$J_{\lambda,\mathbb{R}} : V^{(m)} \longrightarrow \mathcal{V}_{\lambda,\mathbb{R}}^{m+1}, \quad \binom{m}{i} x^i y^{m-i} \longmapsto \sigma_i^\lambda. \tag{65}$$

The image of $J_{\lambda,\mathbb{R}}$ is the hyperplane $\mathcal{H}_\lambda^{m+1} := \text{im } J_{\lambda,\mathbb{R}} = \text{span} \{ \sigma_0^\lambda, \dots, \sigma_m^\lambda \}$.

Theorem 6.2 *The tubular operator on $\mathcal{V}_{\lambda, \mathbb{R}}^{m+1}$ is given as follows. For $i = 0, \dots, m$,*

$$\mathbf{T}_t \sigma_i^\lambda = \sum_{j=0}^m \phi_{m,i,j}^\lambda(t) \sigma_j^\lambda. \tag{66}$$

In particular

$$\mathbf{T}_t \sigma_m^\lambda = \sum_{j=0}^m \sin_\lambda^{m-j}(t) \cos_\lambda^j(t) \sigma_j^\lambda, \tag{67}$$

and thus

$$\mathbf{T}_t \sigma_{m+1}^\lambda = \sum_{j=0}^m \left(\int_0^t \sin_\lambda^{m-j}(s) \cos_\lambda^j(s) ds \right) \sigma_j^\lambda + \sigma_{m+1}^\lambda. \tag{68}$$

These formulas were first obtained by Santaló [25].

Proof By (37), (64) and (65), we have for $0 \leq i \leq m$,

$$\begin{aligned} \mathbf{T}_t \sigma_i^\lambda &= \exp(t \partial_{\lambda, \mathbb{C}})(\sigma_i^\lambda) \\ &= \binom{m}{i} \exp(t \partial_{\lambda, \mathbb{C}}) \circ J_{\lambda, \mathbb{C}}(x^i y^{m-i}) \\ &= \binom{m}{i} J_{\lambda, \mathbb{C}} \circ \exp(t Y_\lambda)(x^i y^{m-i}) \\ &= J_{\lambda, \mathbb{C}}(p_i(t)). \end{aligned}$$

This proves (66) of which (67) is a particular case. Integrating with respect to t yields (68). □

Remark It is worth pointing out the similarity between tube formulas in real and complex space forms. More precisely, note that the isomorphism

$$\mathbf{F}_{n,r} : \mathcal{H}_\lambda^{2n-4r+1} \longrightarrow \mathcal{I}_\lambda^{n,r}, \quad \sigma_j^\lambda \longmapsto \sigma_{j+2r,r+j}^\lambda \tag{69}$$

between the subspaces $\mathcal{H}_\lambda^{2n-4r+1} \subset \mathcal{V}_{\lambda, \mathbb{R}}^{2n-4r+1}$ and $\mathcal{I}_\lambda^{n,r} \subset \mathcal{V}_{\lambda, \mathbb{C}}^n$ commutes with the tubular operator \mathbf{T}_t . This is explained by (62) and (64).

Remark Recently, Hofstätter and Wannerer [21] have found a map $\mathcal{V}_{\lambda, \mathbb{R}}^{2n+1} \rightarrow \mathcal{V}_{\lambda, \mathbb{C}}^n$ which also commutes with \mathbf{T}_t . Next we describe their results and how they relate to $\mathbf{F}_{n,0}^\lambda$. For $\lambda > 0$ let $\pi_\lambda : \mathbb{S}_\lambda^{2n+1} \rightarrow \mathbb{C}P_\lambda^n$ be the Hopf fibration. For a proper submersion $p : M \rightarrow N$, Alesker proved the existence of a push-forward of valuations $p_* : \mathcal{V}(M) \rightarrow \mathcal{V}(N)$ characterized by

$$p_* \varphi(A) = \varphi(p^{-1}(A)), \quad A \in \mathcal{R}(N).$$

Hofstätter and Wannerer have computed the push-forward of invariant valuations through the Hopf fibration. More precisely they have shown that $(\pi_\lambda)_*$ commutes

with \mathbf{T}_t and deduced from this fact that

$$(\pi_\lambda)_* \sigma_k^\lambda = 2\pi \left((2n - k + 1) \sigma_{k-1,0}^\lambda - \lambda(k + 1) \sigma_{k+1,0}^\lambda \right). \quad (70)$$

It follows from (40) that

$$(\pi_\lambda)_* = 2\pi \mathbf{F}_{n,0}^\lambda \circ \partial_{\lambda,\mathbb{R}}. \quad (71)$$

6.3 Spectral analysis of the derivative map

Here we compute the eigenvalues and eigenvectors of $\partial_{\lambda,\mathbb{R}}$ and $\partial_{\lambda,\mathbb{C}}$. Note that the tube formulas for such valuations are extremely simple: if $\partial\mu = a\mu$ with $a \in \mathbb{C}$, then $\mathbf{T}_t\mu = e^{at}\mu$.

Proposition 6.3 *For $0 \leq 2r \leq n$, the restriction of $\partial_{\lambda,\mathbb{C}}$ to $\mathcal{I}_\lambda^{n,r}$ has the following (simple) eigenvalues and eigenspaces:*

$$\begin{aligned} \text{spec} \left(\partial_{\lambda,\mathbb{C}} \big|_{\mathcal{I}_\lambda^{n,r}} \right) &= \left\{ 0, \pm 2\sqrt{-\lambda}, \pm 4\sqrt{-\lambda}, \dots, \pm 2(n - 2r)\sqrt{-\lambda} \right\}, \\ E_{(2k-2n+4r)\sqrt{-\lambda}} &= \text{span}_{\mathbb{C}} \left\{ J_{\lambda,\mathbb{C}}(e_1^k e_2^{2n-4r-k}) \right\}, \quad 0 \leq k \leq 2n - 4r. \end{aligned}$$

Hence $\partial_{\lambda,\mathbb{C}}$ diagonalizes on $\mathcal{V}_{\lambda,\mathbb{C}}^n$ with the following eigenspaces:

$$E_{2j\sqrt{-\lambda}}(\partial_{\lambda,\mathbb{C}}) = \text{span}_{\mathbb{C}} \left\{ J_{\lambda,\mathbb{C}}(e_1^{j+n-2r} e_2^{n-2r-j}) : 0 \leq 2r \leq \min\{n - j, n + j\} \right\},$$

for $-n \leq j \leq n$.

Proof Everything follows from Lemma 5.4 and (62). \square

Proposition 6.4

(i) In \mathbb{S}_λ^{2n} the derivative operator is diagonalizable with

$$\text{spec}(\partial_{\lambda,\mathbb{R}}) = \left\{ 0, \pm\sqrt{-\lambda}, \pm 3\sqrt{-\lambda}, \dots, \pm(2n - 1)\sqrt{-\lambda} \right\}, \quad (72)$$

$$E_0(\partial_{\lambda,\mathbb{R}}) = \text{span}_{\mathbb{C}}\{\chi\} \quad (73)$$

$$E_{(2k-2n+1)\sqrt{-\lambda}}(\partial_{\lambda,\mathbb{R}}) = \text{span}_{\mathbb{C}}\{J_{\lambda,\mathbb{R}}(e_1^k e_2^{2n-k-1})\}, \quad 0 \leq k \leq 2n - 1 \quad (74)$$

(ii) In $\mathbb{S}_\lambda^{2n+1}$ the derivative operator is not diagonalizable since

$$\text{spec}(\partial_{\lambda,\mathbb{R}}) = \left\{ 0, 0, \pm 2\sqrt{-\lambda}, \pm 4\sqrt{-\lambda}, \dots, \pm 2n\sqrt{-\lambda} \right\}, \quad (75)$$

$$E_0(\partial_{\lambda,\mathbb{R}}) = \text{span}_{\mathbb{C}}\{\chi\} \quad (76)$$

$$E_{(2k-2n)\sqrt{-\lambda}}(\partial_{\lambda,\mathbb{R}}) = \text{span}_{\mathbb{C}}\{J_{\lambda,\mathbb{R}}(e_1^k e_2^{2n-k})\}, \quad 0 \leq k \leq 2n. \quad (77)$$

Proof.

- (i) By Lemma 5.4 and (64) we have that $(2k - 2n + 1)\sqrt{-\lambda}$, $0 \leq k \leq 2n - 1$, is an eigenvalue of $\partial_{\lambda, \mathbb{R}}$ with eigenspace given by (74). The Euler characteristic is clearly an eigenvector with zero eigenvalue. We thus have at least $2n + 1$ eigenvalues. Since this is precisely the dimension of $\mathcal{V}_{\lambda, \mathbb{R}}^{2n}$, the statement follows.
- (ii) In light of Lemma 5.4 and (64), we ascertain that $(2k - 2n)\sqrt{-\lambda}$, $0 \leq k \leq 2n$, is an eigenvalue of $\partial_{\lambda, \mathbb{R}}$ and the corresponding eigenspace is described by (77). Our next objective is to prove that while the algebraic multiplicity of the zero eigenvalue is two, its geometric multiplicity is only one. This will entail finding a valuation μ that satisfies $\partial_{\lambda, \mathbb{R}}^2 \mu = 0$, while also ensuring that $\partial_{\lambda, \mathbb{R}} \mu \neq 0$. Consider $\sigma_{2n}^\lambda = J_{\lambda, \mathbb{R}}(x^{2n}) \in \mathcal{V}_{\lambda, \mathbb{R}}^{2n+1}$. In the notation of Lemma 5.4,

$$x = \frac{1}{2\sqrt{-\lambda}}(e_1 - e_2), \quad x^{2n} = (-4\lambda)^{-n} \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} e_1^i e_2^{2n-i}.$$

Hence

$$\partial_{\lambda, \mathbb{R}} \sigma_{2n+1}^\lambda = \sigma_{2n}^\lambda = (-4\lambda)^{-n} \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} J_{\lambda, \mathbb{R}}(e_1^i e_2^{2n-i}).$$

Consider

$$v := (-4\lambda)^{-n} \sum_{\substack{i=0 \\ i \neq n}}^{2n} \binom{2n}{i} \frac{(-1)^i}{(2i - 2n)\sqrt{-\lambda}} J_{\lambda, \mathbb{R}}(e_1^i e_2^{2n-i}),$$

and note that, by Lemma 5.4,

$$\partial_{\lambda, \mathbb{R}} v := (-4\lambda)^{-n} \sum_{\substack{i=0 \\ i \neq n}}^{2n} \binom{2n}{i} (-1)^i J_{\lambda, \mathbb{R}}(e_1^i e_2^{2n-i}),$$

since $e_1^n e_2^n \in \ker Y_\lambda$. Finally, we define $\mu = \sigma_{2n+1}^\lambda - v$. Then

$$\partial_{\lambda, \mathbb{R}} \mu = (-4\lambda)^{-n} \binom{2n}{n} (-1)^n J_{\lambda, \mathbb{R}}(e_1^n e_2^n) \neq 0,$$

while

$$\begin{aligned} \partial_{\lambda, \mathbb{R}}^2 \mu &= (-4\lambda)^{-n} \binom{2n}{n} (-1)^n \partial_{\lambda, \mathbb{R}} J_{\lambda, \mathbb{R}}(e_1^n e_2^n) \\ &= (-4\lambda)^{-n} \binom{2n}{n} (-1)^n J_{\lambda, \mathbb{R}}(Y_\lambda(e_1^n e_2^n)) = 0. \end{aligned}$$

It follows that $\dim \ker \partial_{\lambda, \mathbb{R}} < \dim \ker \partial_{\lambda, \mathbb{R}}^2$. Noting that $\chi \in \ker \partial_{\lambda, \mathbb{R}}$ this implies the statement. \square

Remark We conclude from Proposition 6.4 and Lemma 5.4 that there is no isomorphism between $\text{Val}^{O(m)}$ and $\mathcal{V}_{\lambda, \mathbb{R}}^m$ intertwining $\Lambda - \lambda L$ and $\partial_{\lambda, \mathbb{R}}$. Indeed, these two operators have different spectra no matter the parity of m .

6.4 Stable valuations in complex space forms

We say that a valuation $\varphi \in \mathcal{V}(M)$ on a Riemannian manifold M is *stable* if $\partial \mu = 0$, or equivalently, if $\mathbf{T}_t \mu = \mu$ for all t . By Propositions 6.4 and 6.3, up to multiplicative constants, the Euler characteristic is the unique isometry-invariant stable valuation in \mathbb{S}_{λ}^m . The complex case is more interesting.

Proposition 6.5 *The unique (up to multiplicative constants) stable valuation on $\mathcal{I}_{\lambda}^{n,r}$ is given by*

$$\psi_{2r} = \sum_{i=r}^{n-r} \binom{n-2r}{i-r} \binom{2n-4r}{2i-2r}^{-1} \lambda^{i-r} \sigma_{2i,r}^{\lambda}.$$

Proof By Lemma 5.4 the kernel of Y_{λ} on the space $V^{(m)}$ of homogeneous polynomials of degree $m = 2n - 4r$ is spanned by

$$\begin{aligned} e_1^{n-2r} e_2^{n-2r} &= (y + \sqrt{-\lambda}x)^{n-2r} (y - \sqrt{-\lambda}x)^{n-2r} \\ &= (y^2 + \lambda x^2)^{n-2r} = \sum_{j=0}^{n-2r} \binom{n-2r}{j} \lambda^j x^{2j} y^{n-2j} \\ &= \sum_{i=r}^{n-r} \binom{n-2r}{i-r} \binom{2n-4r}{2i-2r}^{-1} \lambda^{i-r} \binom{2n-4r}{2i-2r} x^{2i-2r} y^{2n-2i-2r} \end{aligned}$$

Therefore the kernel of $\partial_{\lambda, \mathbb{C}}$ in $\mathcal{I}_{\lambda}^{n,r}$ is spanned by $\psi_{2r} = J_{\lambda}(e_1^{n-2r} e_2^{n-2r})$, for each $0 \leq 2r \leq n$. \square

Next we express the Euler characteristic as a combination of the stable valuations ψ_{2r} . Note in particular that χ is *not* confined to any ∂ -invariant subspace $\mathcal{I}_{\lambda}^{n,r}$.

Proposition 6.6

$$\chi = \sum_{0 \leq 2r \leq n} \left(\frac{\lambda}{4\pi}\right)^r \binom{2r}{r} \frac{r!}{\omega_{2n-2r}} \psi_{2r}.$$

Proof Since χ is stable, it can be expressed as $\chi = \sum_j a_j \psi_{2j}$. By [11, Theorem 3.11]

$$\chi = \sum_{k,p \geq 0} \left(\frac{\lambda}{\pi}\right)^{k+p} \frac{\partial^{k+p}}{\partial \xi^k \partial \eta^p} \frac{1}{\sqrt{1-\xi} \sqrt{1-\eta}} \Bigg|_{(0,0)} \tau_{2k+2p,p}^{\lambda}.$$

The coefficient of $\tau_{2r,r}^\lambda$ in this expansion is

$$\begin{aligned} [\tau_{2r,r}^\lambda](\chi) &= \left(\frac{\lambda}{\pi}\right)^r \frac{\partial^r}{\partial \eta^r} \frac{1}{\sqrt{1-\eta}} \Big|_0 \\ &= \left(\frac{\lambda}{\pi}\right)^r \binom{2r}{r} r! 4^{-r}. \end{aligned}$$

By Proposition 3.3, we have

$$[\tau_{2r,r}^\lambda](\sigma_{k,r}^\lambda) = \frac{\omega_{2n-k}}{(k-2r)!} [\tau_{2r,r}^\lambda](\pi_{k,r}^\lambda) = \frac{\omega_{2n-k}}{(k-2r)!} \delta_{k,2r},$$

whence

$$\begin{aligned} [\tau_{2r,r}^\lambda] \left(\sum_j a_j \psi_{2j} \right) &= a_r [\tau_{2r,r}^\lambda](\sigma_{2r,r}^\lambda) \\ &= a_r \omega_{2n-2r}. \end{aligned}$$

Hence

$$a_r = \left(\frac{\lambda}{\pi}\right)^r \binom{2r}{r} \frac{r!}{4^r \omega_{2n-2r}}$$

and the result follows. □

6.5 Image of $\partial_{\lambda, \mathbb{C}}$ and $\partial_{\lambda, \mathbb{R}}$

Next we describe the image of the operators $\partial_{\lambda, \mathbb{C}}$ and $\partial_{\lambda, \mathbb{R}}$, and we compute the preimage of any element belonging to them.

Proposition 6.7 *Given any $\varphi = \sum_{k,r} a_{k,r} \sigma_{k,r}^\lambda \in \mathcal{V}_{\lambda, \mathbb{C}}^n$, we have $\varphi \in \text{im } \partial_{\lambda, \mathbb{C}}$ if and only if*

$$\sum_{l=r}^{n-2r} a_{2l,r} \binom{n-2r}{l-r} \lambda^{n-l-r} = 0, \quad \text{for } 0 \leq 2r \leq n. \tag{78}$$

Proof Note that $\varphi = \sum_r \varphi_r$ with $\varphi_r = \sum_k a_{k,r} \sigma_{k,r}^\lambda$ is the decomposition of φ corresponding to $\mathcal{V}_{\lambda, \mathbb{C}}^n = \bigoplus_{r=0}^{\lfloor n/2 \rfloor} \mathcal{T}_{\lambda}^{n,r}$. By (62) and Proposition 5.5 we have $\varphi \in \text{im } \partial_{\lambda, \mathbb{C}}$ if and only if for every r

$$\begin{aligned} 0 = Z_{2n-4r, \lambda}(\varphi_r) &= \sum_{k=2r}^{2n-4r} a_{k,r} \binom{2n-4r}{k-2r} Z_{2n-4r, \lambda}(x^{k-2r} y^{2n-k-2r}) \\ &= \sum_{l=r}^{n-2r} a_{2l,r} \binom{2n-4r}{2l-2r} \binom{n-2r}{l-r} \lambda^{n-l-r} (2l-2r)! (2n-2l-2r)! \end{aligned} \tag{79}$$

$$\tag{80}$$

$$= (2n - 4r)! \sum_{l=r}^{n-2r} a_{2l,r} \binom{n-2r}{l-r} \lambda^{n-l-r} \quad (81)$$

where we used (59). \square

Proposition 6.8 Given $\varphi = \sum_{k,r} a_{k,r} \sigma_{k,r}^\lambda \in \mathcal{V}_{\lambda,\mathbb{C}}^n$ satisfying (78) we have

$$\partial_{\lambda,\mathbb{C}}^{-1}(\{\varphi\}) = \sum_{k,r} a_{k,r} J_{\lambda,\mathbb{C}}(P_{2n-4r,k-2r+1}) + \text{span}\{\psi_{2r} : 0 \leq 2r \leq n\}$$

where $P_{m,l}$ is given by (60).

Proof This follows at once from Proposition 5.6 after decomposing $\varphi = \sum_r \varphi_r$ as in the previous proof. \square

Proposition 6.9 The image of $\partial_{\lambda,\mathbb{R}}$ in $\mathcal{V}_{\lambda,\mathbb{R}}^m$ is the hyperplane \mathcal{H}_{λ}^m generated by $\sigma_0^\lambda, \dots, \sigma_{m-1}^\lambda$. Moreover

$$\partial_{\lambda,\mathbb{R}} \phi^k = \frac{k! \omega_k}{\pi^k \omega_{m-k}} \sigma_{k-1}^\lambda, \quad 1 \leq k \leq m, \quad (82)$$

where $\phi^k = \sum_{j \geq 0} \left(\frac{\lambda}{4}\right)^j \tau_{k+2j}^\lambda$. In particular

$$\partial_{\lambda,\mathbb{R}}^{-1}(\{\sigma_{k-1}^\lambda\}) = \frac{\pi^k \omega_{m-k}}{k! \omega_k} \phi^k + \mathbb{C} \cdot \chi.$$

Recall from [7, eq. (118)] that $\phi^k = \int_{G_{\lambda,\mathbb{C}}} \chi(\cdot \cap g \mathbb{S}_{\lambda}^{m-k}) dg$ where dg is a properly normalized Haar measure on $G_{\lambda,\mathbb{C}}$, and $\mathbb{S}_{\lambda}^{m-k}$ is an $(m-k)$ -dimensional totally geodesic submanifold in \mathbb{S}_{λ}^m .

Proof By (8) and (9)

$$\phi^k = \sum_{j=0}^{\lfloor \frac{m-k-1}{2} \rfloor} \left(\frac{\lambda}{4\pi^2}\right)^j \frac{(k+2j)! \omega_{k+2j}}{\pi^k (m-k-2j) \omega_{m-k-2j}} \sigma_{k+2j}^\lambda + \left(\left(\frac{\lambda}{4}\right)^{\frac{m-k}{2}} \frac{m! \omega_m}{\pi^m} \sigma_m^\lambda\right),$$

where the term between brackets appears only if $m-k$ is even. Using Proposition 4.8, this yields (82). The rest of the statement follows. \square

Remark Equation (82) also follows from Theorem 4 in [27].

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