



Generalized Schauder theory and its application to degenerate/singular parabolic equations

Takwon Kim¹ · Ki-Ahm Lee^{2,3} · Hyungsung Yun⁴ 

Received: 18 April 2023 / Revised: 11 April 2024 / Accepted: 16 May 2024

© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2024

Abstract

In this paper, we study generalized Schauder theory for the degenerate/singular parabolic equations of the form

$$u_t = a^{i'j'} u_{i'j'} + 2x_n^{\gamma/2} a^{i'n} u_{i'n} + x_n^{\gamma} a^{nn} u_{nn} + b^{i'} u_{i'} + x_n^{\gamma/2} b^n u_n + cu + f \quad (\gamma \leq 1).$$

When the equation above is singular, it can be derived from Monge–Ampère equations by using the partial Legendre transform. Also, we study the fractional version of Taylor expansion for the solution u , which is called s -polynomial. To prove $C_s^{2+\alpha}$ -regularity and higher regularity of the solution u , we establish generalized Schauder theory which approximates coefficients of the operator with s -polynomials rather than constants. The generalized Schauder theory not only recovers the proof for uniformly parabolic equations but is also applicable to other operators that are difficult to apply the bootstrap argument to obtain higher regularity.

Mathematics Subject Classification 35B65 · 35K65 · 35K67

Ki-Ahm Lee was supported by the National Research Foundation of Korea (NRF), Ministry of Science and ICT, South Korea, Grant: NRF-2020R1A2C1A01006256. Hyungsung Yun was supported by a KIAS Individual Grant (no. MG097801) at Korea Institute for Advanced Study.

✉ Hyungsung Yun
hyungsung@kias.re.kr

Takwon Kim
takwonkim@kaist.ac.kr

Ki-Ahm Lee
kiahm@snu.ac.kr

- ¹ Stochastic Analysis and Application Research Center, Korea Advanced Institute of Science and Technology, Daejeon 34141, Republic of Korea
- ² Research Institute of Mathematics, Seoul National University, Seoul 08826, Republic of Korea
- ³ Department of Mathematical Sciences, Seoul National University, Seoul 08826, Republic of Korea
- ⁴ School of Mathematics, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea

1 Introduction

In this paper, we study generalized Schauder theory and fractional order expansion of solutions for the following degenerate/singular parabolic equations

$$u_t = Lu + f \quad \text{in } Q_1^+, \quad (1.1)$$

where

$$\begin{aligned} L = & a^{i'j'}(x, t)D_{i'j'} + 2x_n^{\gamma/2}a^{i'n}(x, t)D_{i'n} + x_n^\gamma a^{nn}(x, t)D_{nn} \\ & + b^{i'}(x, t)D_{i'} + x_n^{\gamma/2}b^n(x, t)D_n + c(x, t) \end{aligned}$$

with a constant $\gamma \leq 1$ and

$$Q_1^+ = \{x \in \mathbb{R}_+^n : |x_i| < 1 (1 \leq i \leq n)\} \times (-1, 0].$$

The repeated index with prime i' means the summation from 1 to $(n - 1)$; that is,

$$A^{i'}B_{i'} = \sum_{i=1}^{n-1} A^i B_i \quad \text{and} \quad A^{i'j'}B_{i'j'} = \sum_{i,j=1}^{n-1} A^{ij} B_{ij}.$$

In classical regularity theory, to obtain $C^{k,\alpha}$ -regularity of the solution u , it is shown that the following statement holds (cf. [10]): For each $Y \in \overline{\Omega}$, there exists a polynomial p^Y of degree k satisfying

$$\|u - p^Y\|_{L^\infty(B_r(Y) \cap \Omega)} \leq Cr^{k+\alpha} \quad \text{for all } r > 0. \quad (1.2)$$

Furthermore, we can obtain the higher order partial derivative of u at Y from the coefficients of the polynomial p^Y . However, the function that is not smooth enough cannot be approximated by polynomials. Breaking away from the stereotype that polynomial approximation should be used, we consider generalized polynomials represented by monomials of fractional order, and we call them s -polynomials in this study. In simple terms, s -polynomial approximation can be understood as a fractional version of Taylor expansion. The coefficients of an s -polynomial p approximating the solution can be regarded as suitable constant multiples of the weighted derivatives, and the degree of p gives the order of regularity. While conventional regularity theory so far has focused on differentiability for classical solutions, in this study, we are interested in how well s -polynomial approximates the solution instead of differentiability. In other words, a generalized concept of regularity theory can be developed through the s -polynomial.

The regularity of solutions for (1.1) can be expected to be $C^{1,\alpha}$ -regularity up to the boundary, but higher regularity up to the boundary can be obtained by considering s -polynomials and a new metric that preserves scaling. In classical Schauder theory, by showing that the derivatives satisfy the equations of the same class, $C^{2,\alpha}$ -regularity of the gradient Du is obtained, and iteratively, higher regularity can be obtained. However,

this bootstrap argument is not applicable in (1.1) since the equation that the partial derivative u_n satisfies is not of the same class as (1.1), so the generalized Schauder theory was developed to solve this issue. It is also applicable to other operators that are difficult to apply the bootstrap argument to obtain higher regularity.

The study on the regularity of solutions for (1.1) was inspired by several previous studies.

- (1) Monge–Ampère equations: Daskalopoulos and Savin [5] converted the Monge–Ampère equations

$$\det D^2u = |x|^\gamma \quad \text{in } B_1$$

into the singular equations with $\gamma < 0$

$$|x|^\gamma v_{xx} + v_{yy} = 0 \quad \text{in } B_1$$

using the partial Legendre transform. In [5], they solved the problem for Monge–Ampère equation by finding the fractional order expansion

$$v(x, y) = a_1 + a_2x + a_3y + a_4xy + a_5 \left(\frac{1}{2}y^2 - \frac{1}{(2-\gamma)(1-\gamma)}|x|^{2-\gamma} \right) + O((y^2 + |x|^{2-\gamma})^{1+\delta})$$

for some universal constant $\delta = \delta(\gamma) > 0$.

In the study of Schauder estimates up to the boundary for degenerate Monge–Ampère equations, Le and Savin [7] show that a bounded solution w of the singular equations with $\gamma < 0$

$$\begin{cases} \Delta_{x'}w + x_n^\gamma w_{nn} = 0 & \text{in } B_1^+ \\ w = 0 & \text{on } \{x_n = 0\} \end{cases}$$

satisfies

$$|w(x) - p(x')x_n| \leq C(x_1^2 + \dots + x_{n-1}^2 + x_n^{2-\gamma})^{\frac{3-\gamma}{2-\gamma}}$$

for all $x \in B_{1/2}^+$, where C is a universal constant and $p(x')$ is a standard polynomial of degree 1.

As they used a distance function that allows solutions and equations to be scaling invariant, we also defined a distance function corresponding to (1.1) in our study. It is noteworthy that such a fractional order expansion and estimates are developed to higher orders by s -polynomial and the distance function.

- (2) Gauss curvature flow: Related studies can also be found in differential geometry, for example, Daskalopoulos and Hamilton [4] showed that the regularity of the interface for the Gauss curvature flow with flat sides can be transformed into the regularity of solutions for the following degenerate equations

$$v_t = xv_{xx} + v_{yy} + vv_x + f \text{ in } \mathbb{R}_+^2 \times (0, T]. \tag{1.3}$$

They showed Schauder estimates for smooth solutions of (1.3) using a new metric that preserves scaling.

- (3) Non-local equations: In the research field of non-local equations, Caffarelli and Silvestre [1] showed that the solution u of the degenerate/singular equations

$$\begin{cases} \Delta_x u + z^{\frac{2s-1}{s}} u_{zz} = 0 & \text{in } \mathbb{R}^n \times [0, \infty) \\ u = f & \text{on } \mathbb{R}^n \times \{0\} \end{cases} \tag{1.4}$$

satisfies

$$(-\Delta)^s f(x) = -C(n, s)u_z(x, 0)$$

for some constant $C(n, s) > 0$. The regularity of solutions for the extension problem (1.4) can be applied to the regularity of solutions for the fractional Laplace equations. In fact, in [1], Hölder’s regularity of solutions for fractional Laplace equations was shown by using the Harnack inequality for the extension problem (1.4).

- (4) Mathematical Finance: The Black–Scholes equations for the constant elasticity of variance (CEV) model, introduced by Cox [2] and Cox and Ross [3], have a similar structure to (1.1). The risky asset’s price X_t of the CEV model evolves according to the following stochastic differential equations

$$\begin{cases} dX_t = \mu X_t dt + \sigma X_t^{\gamma/2} dW_t \\ X_0 = x, \end{cases}$$

where W_t is a one-dimensional Brownian motion for some positive constants μ, σ , and γ . Using the Feynman–Kac formula, we can derive the following degenerate backward equations

$$u_t + \frac{1}{2}\sigma^2 x^\gamma u_{xx} + rxu_x - ru = 0 \text{ in } \mathbb{R}_+ \times [0, T).$$

As with the aforementioned equations, many applications arise for $\gamma \leq 1$, so it is necessary to research the regularity of solutions to (1.1) for $\gamma \leq 1$.

Although there is a slight difference compared to (1.1), the methodology covered in our study is expected to be applicable to equations that have similar degenerate/singular structures of (1.1) like aforementioned equations. In addition, it is a new version of Schauder theory that can directly demonstrate higher regularity of solutions without relying on the bootstrap argument for uniformly parabolic equations.

In order to obtain the heuristic idea of s -polynomial, let us start with a simplified version of (1.1). The following equation

$$u_t = x^\gamma u_{xx} + 1 \text{ in } Q_1^+ \quad (0 < \gamma < 1) \tag{1.5}$$

admits a solution of the form

$$u(x, t) = x + \frac{t^2}{2}x - \frac{1}{(2 - \gamma)(1 - \gamma)}x^{2-\gamma} + \frac{t}{(3 - \gamma)(2 - \gamma)}x^{3-\gamma} + \frac{1}{(5 - 2\gamma)(4 - 2\gamma)(3 - \gamma)(2 - \gamma)}x^{5-2\gamma}. \tag{1.6}$$

Since u is not twice differentiable at $x = 0$ in the x -direction, the regularity of u for a spatial variable is at most $C^{1,\alpha}$, so the polynomial that satisfies (1.2) with $k = 1$ is $p(x, t) = x$. However, considering the following estimate

$$\left| u(x, t) - \left(x + \frac{t^2}{2}x - \frac{1}{(2 - \gamma)(1 - \gamma)}x^{2-\gamma} + \frac{t}{(3 - \gamma)(2 - \gamma)}x^{3-\gamma} \right) \right| \leq Cr^{\frac{2(5-2\gamma)}{2-\gamma}} \tag{1.7}$$

for all $(x, t) \in Q_r^+$, it can be expected that fractional order expansion is possible. Also, (1.7) contains more terms than p and provides a more accurate approximation.

Looking at the pattern of each monomial when it is substituted into (1.5), it is possible to find a solution with a fractional order expansion of more terms than (1.6). This analysis can be extended for (1.1), which is the motivation to consider s -polynomials. Although the forcing term f is a general function rather than a constant function, the solution of (1.1) can be approximated by such a fractional order expansion. This is why we established Schauder estimates using s -polynomials rather than standard polynomials. In addition, the distance function for uniformly parabolic equations does not have scaling invariance for (1.1) at the boundary $\{x_n = 0\}$, so it is not suitable for describing the behavior of solutions for (1.1) in the neighborhood of $\{x_n = 0\}$. Inspired by [4, 5, 7], we define a scaling-preserving distance function $s : Q_1^+ \times Q_1^+ \rightarrow [0, \infty)$ and use it to show Schauder estimates.

Since the bootstrap argument is not applicable in (1.1), it is necessary to obtain $C_s^{k,2+\alpha}$ -regularity of solutions for (1.1) directly from $C_s^{k,\alpha}$ -regularity of the coefficients of L . However, it is not enough to consider perturbative methods, by ‘‘freezing’’ the coefficients around a certain point. In order to solve this difficulty, we will use s -polynomial approximation of coefficients that fully reflects the regularity of coefficients and study equations with s -polynomial coefficients first.

The solution u of (1.1) satisfies the following estimates depending on the range of γ .

$$|u(x, t)| \leq \begin{cases} Cx_n \approx Cs[(x, t), (0, 0)]^{\frac{2}{2-\gamma}} & (\gamma < 1) \\ Cx_n^{2-\gamma} \approx Cs[(x, t), (0, 0)]^2 & (1 < \gamma < 2) \end{cases} \tag{1.8}$$

for all $(x, t) \in \overline{Q_{1/2}^+}$, where $C > 0$ is a universal constant. Assuming $f = 0$, the order of (1.9) can be improved to a higher order, but not in the case of (1.8). In the case of $1 < \gamma < 2$, $C_s^{2+\alpha}$ -regularity of solutions for (1.1) is directly obtained from the improved estimates of (1.9), and then the generalized coefficient freezing method can

be applied immediately thanks to $C_s^{2+\alpha}$ -regularity. However, for $\gamma < 1$, the regularity of solutions for (1.1) expected from (1.8) is C_s^α -regularity or $C_s^{1,\alpha}$ -regularity depending on the range on γ . Thus, a process to show $C_s^{2+\alpha}$ -regularity of solutions for (1.1) is necessary. In addition, since s -polynomials for $\gamma = 1$ include logarithmic functions, an appropriate method for dealing with them is also required.

The paper is organized as follows. In Sect. 2, we introduce notations used throughout this paper and state the main theorem. Also, the maximal principle, comparison principle, the existence of a unique solution for the degenerate/singular equations are covered in Sect. 2. It is similar to uniformly parabolic equations, so readers familiar with it can skip it. In the last part of Sect. 2, we derive global regularity from interior regularity and boundary regularity of solutions for (1.1). In Sect. 3, we prove the boundary Lipschitz estimates of solutions for (1.1) and use it to show global C_s^α -regularity of solutions for (1.1). In Sect. 4, we prove $C^{1,\alpha}$ -regularity and $C_s^{k,2+\alpha}$ -regularity of solutions for equations with constant coefficients. In Sect. 5, we deal with generalized Schauder theory, which is the essence of this study. Unlike equations with constant coefficients, $C_s^{k,2+\alpha}$ -regularity cannot be obtained immediately, so $C_s^{2,\alpha}$ -regularity for solutions of (1.1) and $C_s^{k,2+\alpha}$ -regularity of solutions for equations with s -polynomial coefficients are first shown, and then $C_s^{k,2+\alpha}$ -regularity of solutions for (1.1) can be obtained by using generalized coefficient freezing method.

2 Preliminaries

2.1 Notations

We summarize some basic notations as follows.

(1) Points: For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote

$$x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, \quad X = (x, t) \in \mathbb{R}^{n+1}, \quad \text{and} \quad O = (0, \dots, 0) \in \mathbb{R}^{n+1},$$

respectively. For $r > 0$, we denote

$$rX = (rx', r^{\frac{2}{2-\gamma}}x_n, r^2t)$$

which allows solutions and equations to be scaling invariant.

(2) Sets: We denote the open upper half-space and the set of nonnegative integers as

$$\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\} \quad \text{and} \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

respectively. We denote an intrinsic cube with side $2r$ and center $Y = (y, \tau) \in \mathbb{R}^{n+1}$ as

$$Q_r^+(Y) = \{x \in \mathbb{R}_+^n : |x_i - y_i| < r \ (1 \leq i < n), \ |x_n^{\frac{2-\gamma}{2}} - y_n^{\frac{2-\gamma}{2}}| < r\} \times (\tau - r^2, \tau]$$

and standard cube with side $2r$ and center $Y = (y, \tau) \in \mathbb{R}^{n+1}$ as

$$Q_r(Y) = \{x \in \mathbb{R}^n : |x_i - y_i| < r \ (1 \leq i \leq n)\} \times (\tau - r^2, \tau].$$

For convenience, we denote $Q_r^+ = Q_r^+(O)$. We denote the set of degrees for s -polynomial as

$$\mathcal{D} = \left\{ i + \frac{2j}{2-\gamma} : i, j \in \mathbb{N}_0 \right\}.$$

- (3) Universal constant means a constant that depends only on $n, \lambda, \Lambda, \gamma, k$, and α with $n \in \mathbb{N}, k \in \mathbb{N}_0, 0 < \alpha < 1, 0 < \lambda \leq \Lambda$, and $\gamma \leq 1$.
- (4) Distance functions: The parabolic distance function $d : \overline{Q_1^+} \times \overline{Q_1^+} \rightarrow [0, \infty)$ from $X = (x, t)$ to $Y = (y, \tau)$ is given by

$$d[X, Y] = \max \left\{ \max_{1 \leq i \leq n} |x_i - y_i|, \sqrt{|t - \tau|} \right\}.$$

In addition, the intrinsic distance function $s : \overline{Q_1^+} \times \overline{Q_1^+} \rightarrow [0, \infty)$ from $X = (x, t)$ to $Y = (y, \tau)$ is given by

$$s[X, Y] = \max \left\{ \max_{1 \leq i < n} |x_i - y_i|, |x_n^{\frac{2-\gamma}{2}} - y_n^{\frac{2-\gamma}{2}}|, \sqrt{|t - \tau|} \right\}.$$

- (5) Partial derivatives: We denote partial derivatives of u as subscriptions.

$$u_t = \partial_t u = \frac{\partial u}{\partial t}, \quad u_i = D_i u = \frac{\partial u}{\partial x_i}, \quad \text{and} \quad u_{ij} = D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

- (6) Multiindex notation: A vector of the form $\beta = (\beta_1, \beta_2, \dots, \beta_k) \in \mathbb{N}_0^k$ is called a k -dimensional multiindex of order $|\beta| = \beta_1 + \beta_2 + \dots + \beta_k$. For k -dimensional multiindices $\beta, \tilde{\beta} \in \mathbb{N}_0^k, \tilde{\beta} \leq \beta$ means $\tilde{\beta}_i \leq \beta_i \ (i = 1, 2, \dots, k)$. The factorial and binomial coefficients of a multiindex are defined as follows:

$$\beta! = \beta_1! \beta_2! \cdots \beta_k! \quad \text{and} \quad \binom{\beta}{\tilde{\beta}} = \frac{\beta!}{\tilde{\beta}!(\beta - \tilde{\beta})!}.$$

Given $\beta \in \mathbb{N}_0^k, m \in \mathbb{N}$, and $(x_1, x_2, \dots, x_k) \in \mathbb{R}^k$, define

$$(x_1, x_2, \dots, x_k)^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}, \quad D_{(x_1, x_2, \dots, x_k)}^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_k^{\beta_k}},$$

and $D_{(x_1, x_2, \dots, x_k)}^m u = \{D_{(x_1, x_2, \dots, x_k)}^\beta u : |\beta| = m\}$.

Remark 2.1 (Leibniz’s formula) Let Ω be a domain in \mathbb{R}^n and let $u, v : \Omega \rightarrow \mathbb{R}$ be smooth functions. Then

$$D_x^\beta (uv) = \sum_{\tilde{\beta} \leq \beta} \binom{\beta}{\tilde{\beta}} D_x^{\tilde{\beta}} u D_x^{\beta - \tilde{\beta}} v.$$

(7) *s*-Polynomials: Let κ be a positive real number. We say p is an *s*-polynomial of degree m corresponding to κ at $Y = (y, \tau) \in \mathbb{R}_+^n \times \mathbb{R}$ provided

$$p(X) = \begin{cases} \sum_{\beta, i, j, l} A^{\beta i j l} (x' - y')^\beta (x_n^{\frac{2-\gamma}{2}} - y_n^{\frac{2-\gamma}{2}})^i (x_n - y_n)^j (t - \tau)^l & (\gamma < 1) \\ \sum_{\beta, i, j, l} A^{\beta i j l} (x' - y')^\beta (\sqrt{x_n} - \sqrt{y_n})^i (\sqrt{x_n} \log x_n - \sqrt{y_n} \log y_n)^j (t - \tau)^l & (\gamma = 1) \end{cases}$$

for some $A^{\beta i j l} \in \mathbb{R}$, where $\beta \in \mathbb{N}_0^{n-1}$ and $i, j, l \in \mathbb{N}_0$ satisfy

$$\begin{cases} |\beta| + i + \frac{2j}{2 - \gamma} + 2l < \kappa & (\gamma < 1) \\ |\beta| + i + j + 2l < \kappa & (\gamma = 1). \end{cases}$$

The degree m corresponding to κ is given by

$$\text{deg } p = \begin{cases} \max \left\{ |\beta| + i + \frac{2j}{2 - \gamma} + 2l \in [0, \kappa) : \beta \in \mathbb{N}_0^{n-1}, i, j, l \in \mathbb{N}_0 \right\} & (\gamma < 1) \\ \max \left\{ |\beta| + i + j + 2l \in [0, \kappa) : \beta \in \mathbb{N}_0^{n-1}, i, j, l \in \mathbb{N}_0 \right\} & (\gamma = 1). \end{cases}$$

(8) Hölder norms: Let $\Omega \subset \mathbb{R}_+^n \times \mathbb{R}$ be an open set and $\alpha \in (0, 1)$. We define the α^{th} -Hölder seminorm of $u : \Omega \rightarrow \mathbb{R}$ to be

$$[u]_{C^\alpha(\overline{\Omega})} := \sup_{\substack{X \neq Y \\ X, Y \in \Omega}} \frac{|u(X) - u(Y)|}{d[X, Y]^\alpha}.$$

Also, we define the α^{th} -Hölder norm of u to be

$$\|u\|_{C^\alpha(\overline{\Omega})} := \|u\|_{C^0(\overline{\Omega})} + [u]_{C^\alpha(\overline{\Omega})}.$$

Moreover, we define for a nonnegative integer k ,

$$\|u\|_{C^{k,0}(\overline{\Omega})} := \sum_{\substack{|\beta| + 2i \leq k \\ \beta \in \mathbb{N}_0^n, i \in \mathbb{N}_0}} \|D_x^\beta \partial_t^i u\|_{C^0(\overline{\Omega})}$$

and

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} := \|u\|_{C^{k,0}(\overline{\Omega})} + \sum_{\substack{|\beta|+2i=k \\ \beta \in \mathbb{N}_0^n, i \in \mathbb{N}_0}} [D_x^\beta \partial_t^i u]_{C^\alpha(\overline{\Omega})}.$$

Next, we define the Hölder norm with respect to the distance function s to be

$$\|u\|_{C_s^{k,\alpha}(\overline{\Omega})} := \sup_{Y \in \overline{\Omega}} \sup_{r>0} \inf_p \left\{ \frac{|u(X) - p(X)|}{r^{k+\alpha}} + \sum_{\beta,i,j,l} |A^{\beta i j l}| : X \in \overline{Q_r^+(Y)} \cap \overline{\Omega} \right\},$$

where p is among s -polynomials of degree m corresponding to $\kappa = k + \alpha$. In particular, we denote the α^{th} -Hölder norm with respect to distance function s to be

$$\|u\|_{C_s^\alpha(\overline{\Omega})} := \|u\|_{C_s^{0,\alpha}(\overline{\Omega})}.$$

Finally, we define the $(2 + \alpha)^{\text{th}}$ -Hölder norm and the higher $(2 + \alpha)^{\text{th}}$ -Hölder norm of $u : \Omega \rightarrow \mathbb{R}$ to be

$$\|u\|_{C_s^{2+\alpha}(\overline{\Omega})} := \|u\|_{C_s^{2,\alpha}(\overline{\Omega})} + \|(\partial_t - L)u\|_{C_s^\alpha(\overline{\Omega})}$$

and

$$\|u\|_{C_s^{k,2+\alpha}(\overline{\Omega})} := \|u\|_{C_s^{k+2,\alpha}(\overline{\Omega})} + \|(\partial_t - L)u\|_{C_s^{k,\alpha}(\overline{\Omega})}.$$

- (9) Function spaces: For any nonnegative integer k , $0 < \alpha < 1$, and domain $\Omega \subset \mathbb{R}_+^n \times \mathbb{R}$, we define function space $C_s^{k,\alpha}(\overline{\Omega})$ as the completion of $\{u \in C^\infty(\Omega) \cap C(\overline{\Omega}) : \|u\|_{C_s^{k,\alpha}(\overline{\Omega})} < \infty\}$ for $\|\cdot\|_{C_s^{k,\alpha}(\overline{\Omega})}$ and

$$C_s^{k,2+\alpha}(\overline{\Omega}) = \{u \in C_s^{k+2,\alpha}(\overline{\Omega}) : \|u\|_{C_s^{k,2+\alpha}(\overline{\Omega})} < \infty\}$$

which is the space that allows the operator

$$\partial_t - L : C_s^{k,2+\alpha}(\overline{\Omega}) \rightarrow C_s^{k,\alpha}(\overline{\Omega})$$

to be well-defined.

All of the function spaces defined so far become Banach spaces.

Through this article, we assume that the symmetric matrix (a^{ij}) has the following property:

$$\lambda |\xi|^2 \leq a^{ij}(X) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \text{for any } X \in \overline{Q_1^+}, \quad \xi \in \mathbb{R}^n \tag{2.1}$$

and b^i , c have the boundedness as follows:

$$\sum_{i=1}^n \|b^i\|_{L^\infty(Q_1^+)} + \|c\|_{L^\infty(Q_1^+)} \leq \Lambda.$$

In addition, since the function $v = e^{-(\Lambda+1)t}u$ satisfies

$$v_t = Lv - (\Lambda + 1)v + e^{-(\Lambda+1)t}f \quad \text{in } Q_1^+,$$

we may assume that the coefficient c of L is negative.

2.2 Main result

In this subsection, we are going to state the main theorem.

Theorem 2.2 *Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$ with $k + 2 + \alpha \notin \mathcal{D}$, and assume*

$$a^{ij}, b^i, c, f \in C_s^{k,\alpha}(\overline{Q_1^+}) \quad (i, j = 1, 2, \dots, n).$$

Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ is a solution of (1.1) satisfying $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. Then $u \in C_s^{k,2+\alpha}(\overline{Q_{1/2}^+})$ and

$$\|u\|_{C_s^{k,2+\alpha}(\overline{Q_{1/2}^+})} \leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})} \right),$$

where C is a positive constant depending only on $n, \lambda, \Lambda, \gamma, k, \alpha, \|a^{ij}\|_{C_s^{k,\alpha}(\overline{Q_1^+})}, \|b^i\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$, and $\|c\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$.

2.3 Maximum principle

First, we will show the maximum principle for the following initial/boundary-value problem

$$\begin{cases} u_t = Lu + f & \text{in } Q_1^+ \\ u = g & \text{on } \partial_p Q_1^+. \end{cases} \quad (2.2)$$

Lemma 2.3 *Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ satisfies $u_t < Lu$ in Q_1^+ . If $u < 0$ on $\partial_p Q_1^+$, then*

$$u(X) < 0 \quad \text{for all } X \in \overline{Q_1^+}.$$

Proof Suppose not. Then there exists $\tau = \inf\{t : u(x, t) \geq 0 \text{ for some } (x, t) \in \overline{Q_1^+}\}$. Since $u \in C(\overline{Q_1^+})$ and $u < 0$ on $\partial_p Q_1^+$, we know that there is $Y = (y, \tau) \in \overline{Q_1^+} \setminus \partial_p Q_1^+$ such that $u(Y) = 0$. Meanwhile τ is the first time u becomes nonnegative, the function $U(x) := u(x, \tau)$ attains its maximum at y and hence $u_t(Y) \geq 0$, $Du(Y) = 0$, and $D^2u(Y) \leq 0$. Then we have

$$\mathcal{D}^2u(Y) := \begin{pmatrix} I_{n-1} & \mathbf{0} \\ \mathbf{0} & y_n^{\gamma/2} \end{pmatrix} D^2u(Y) \begin{pmatrix} I_{n-1} & \mathbf{0} \\ \mathbf{0} & y_n^{\gamma/2} \end{pmatrix} \leq 0$$

and this implies that

$$u_t(Y) - Lu(Y) = u_t(Y) - \text{tr}(a^{ij}(Y)\mathcal{D}^2u(Y)) \geq 0.$$

This yields a contradiction to $u_t < Lu$ in Q_1^+ . □

Lemma 2.4 Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ satisfies $u_t \leq Lu$ in Q_1^+ . If $u \leq 0$ on $\partial_p Q_1^+$, then $u \leq 0$ in $\overline{Q_1^+}$.

Proof For any $\varepsilon > 0$, consider $v(X) = u(X) - \varepsilon(t + 2)$. Then we have $v_t < Lv$ in Q_1^+ and $v < 0$ on $\partial_p Q_1^+$. Thus, by Lemma 2.3

$$u(X) < \varepsilon(t + 2) \text{ for all } X \in \overline{Q_1^+}.$$

We finish the proof by letting $\varepsilon \rightarrow 0$. □

Lemma 2.5 Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ is a solution of (2.2). Then

$$\|u\|_{L^\infty(Q_1^+)} \leq C \left(\|f\|_{L^\infty(Q_1^+)} + \|g\|_{L^\infty(Q_1^+)} \right),$$

where $C > 0$ is a universal constant.

Proof Let $v = u - e^{t+1}(\|f\|_{L^\infty(Q_1^+)} + \|g\|_{L^\infty(Q_1^+)})$. Then we have $v_t \leq Lv$ in Q_1^+ and $v \leq 0$ on $\partial_p Q_1^+$. By Lemma 2.4, we have

$$u(X) \leq e \left(\|f\|_{L^\infty(Q_1^+)} + \|g\|_{L^\infty(Q_1^+)} \right) \text{ for all } X \in Q_1^+.$$

Considering the function $w = -u - e^{t+1}(\|f\|_{L^\infty(Q_1^+)} + \|g\|_{L^\infty(Q_1^+)})$, we can obtain the lower bound in a similar way. □

Lemma 2.6 Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ is a solution of (2.2) with $\gamma < 0$ and $x_n^{-\gamma} f \in L^\infty(Q_1^+)$. Then

$$\|u\|_{L^\infty(Q_1^+)} \leq C \left(\|x_n^{-\gamma} f\|_{L^\infty(Q_1^+)} + \|g\|_{L^\infty(Q_1^+)} \right),$$

where $C > 0$ is a universal constant.

Proof Let $w = e^{\mu(t+1)} \|x_n^{-\gamma} f\|_{L^\infty(Q_1^+)} (1 + x_n - x_n^2)$. Then, we have

$$\begin{aligned} w_t - Lw &= e^{\mu(t+1)} \|x_n^{-\gamma} f\|_{L^\infty(Q_1^+)} \left(\mu(1 + x_n - x_n^2) \right. \\ &\quad \left. + 2x_n^\gamma a^{nn} - x_n^{\gamma/2} b^n (1 - 2x_n) - c(1 + x_n - x_n^2) \right) \\ &\geq e^{\mu(t+1)} \|x_n^{-\gamma} f\|_{L^\infty(Q_1^+)} \left(\mu + 2x_n^\gamma \lambda - 3x_n^{\gamma/2} \Lambda \right) \\ &= \lambda e^{\mu(t+1)} \|x_n^{-\gamma} f\|_{L^\infty(Q_1^+)} x_n^\gamma + e^{\mu(t+1)} \|x_n^{-\gamma} f\|_{L^\infty(Q_1^+)} \\ &\quad \times \left(\left(\sqrt{\lambda} x_n^{\gamma/2} - \frac{3\Lambda}{2\sqrt{\lambda}} \right)^2 + \mu - \frac{9\Lambda^2}{4\lambda} \right). \end{aligned}$$

For sufficiently large $\mu > \frac{9\Lambda^2}{4\lambda}$, we obtain

$$w_t - Lw \geq \lambda \|x_n^{-\gamma} f\|_{L^\infty(Q_1^+)} x_n^\gamma \quad \text{in } Q_1^+.$$

As in the proof Lemma 2.5, we can see the desired result, by considering the function $v_\pm = \pm u - (w/\lambda + \|g\|_{L^\infty(Q_1^+)})$. □

2.4 Existence of solutions

In this subsection, we assume that $a^{ij}, b^i, c,$ and f are $C^\alpha_{\text{loc}}(Q_1^+)$ for the existence of a classical solution.

Definition 2.7 A function $v \in C(\overline{Q_1^+})$ is called a subsolution (resp. supersolution) of (2.2) if

$$v \leq g \text{ (resp. } \geq g) \quad \text{on } \partial_p Q_1^+$$

and if for any $\Omega \subset Q_1^+,$ the solution $\tilde{v} \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{cases} \tilde{v}_t = L\tilde{v} + f & \text{in } \Omega \\ \tilde{v} = v & \text{on } \partial_p \Omega \end{cases}$$

is greater (resp. less) than or equal to v in Ω .

Lemma 2.8 (Comparison principle) *If $w \in C(\overline{Q_1^+})$ is a supersolution of (2.2) and $v \in C(\overline{Q_1^+})$ is a subsolution of (2.2), then*

$$w \geq v \quad \text{in } Q_1^+.$$

Proof Suppose $h = v - w$ has a positive maximum value M achieved at some point $Y = (y, \tau) \in \overline{Q_1^+} \setminus \partial_p Q_1^+$. Then, we have $h(Y) = M > 0$ and $h(X) \leq M$ for all $X \in \Omega$, where $\Omega = \{X \in Q_1^+ : x_n > y_n/2, t < \tau\}$. The operator $(\partial_t - L)$ is a uniformly parabolic in Ω since Ω is away from the boundary $\{x_n = 0\}$. So, we can take $\tilde{v} \in C^2(\Omega) \cap C(\overline{\Omega})$ and $\tilde{w} \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying

$$\begin{cases} \tilde{v}_t = L\tilde{v} + f & \text{in } \Omega \\ \tilde{v} = v & \text{on } \partial_p \Omega \end{cases} \quad \text{and} \quad \begin{cases} \tilde{w}_t = L\tilde{w} + f & \text{in } \Omega \\ \tilde{w} = w & \text{on } \partial_p \Omega. \end{cases}$$

Then, the function $\tilde{h} = \tilde{v} - \tilde{w} - M$ satisfying $\tilde{h}_t \leq L\tilde{h}$ in Ω and $\tilde{h} \leq 0$ on $\partial_p \Omega$. Furthermore, since v and w are subsolution and supersolution of (2.2), respectively, we know that $\tilde{v} \geq v$ in Ω and $\tilde{w} \leq w$ in Ω . Hence, we have $\tilde{h}(Y) \geq 0$. By the strong maximum principle for uniformly parabolic equations, we have $\tilde{h} = \tilde{v} - \tilde{w} - M = 0$ in $\overline{\Omega}$. It follows that $v - w = M$ on $\partial_p \Omega$. Since $\partial_p Q_1^+ \cap \partial_p \Omega$ is nonempty, this contradicts the assumption that $v \leq g \leq w$ on $\partial_p Q_1^+$. \square

Lemma 2.9 Let $k \in \mathbb{N}_0, 0 < \alpha < 1$, and assume

$$a^{ij}, b^i, c, f \in C_s^{k,\alpha}(\overline{Q_1^+}) \text{ (} i, j = 1, 2, \dots, n \text{) and } g \in C(\overline{Q_1^+}).$$

Then there exists a unique bounded solution $u \in C_{\text{loc}}^{k+2,\alpha}(Q_1^+) \cap C(\overline{Q_1^+})$ of (2.2) such that

$$\|u\|_{L^\infty(Q_1^+)} \leq C \left(\|f\|_{L^\infty(Q_1^+)} + \|g\|_{L^\infty(Q_1^+)} \right),$$

where $C > 0$ is a universal constant.

Proof Let \mathcal{A} be a set of bounded subsolutions of (2.2) and let $u = \sup_{w \in \mathcal{A}} w$. Consider the function $\varphi_\pm = \pm e^{t+1} (\|f\|_{L^\infty(Q_1^+)} + \|g\|_{L^\infty(Q_1^+)})$. Note first that φ_+ is a supersolution of (2.2) and φ_- is a subsolution of (2.2), so u is bounded by φ_\pm by Lemma 2.8. For any cube $Q_r(Y) \subset\subset Q_1^+$, the operator $(\partial_t - L)$ is a uniformly parabolic in $Q_r(Y)$ since $Q_r(Y)$ is away from the boundary $\{x_n = 0\}$. Thus, we can apply Perron’s method for the problem (2.2) as in [6, Lemma 3.3] and hence $u \in C_{\text{loc}}^{k+2,\alpha}(Q_1^+)$. Finally, for any $Y = (y, \tau) \in \{X \in \partial_p Q_1^+ : x_n = 0\}$, take a local barrier w near Y given by

$$w(x, t) = |x' - y'|^2 + (t - \tau)^2 + \begin{cases} \frac{2x_n}{r} - \left(\frac{x_n}{r}\right)^{2-\gamma/2} - \left(\frac{x_n}{r}\right)^{2-\gamma} & (\gamma < 0) \\ \frac{x_n}{r} - \left(\frac{x_n}{r}\right)^{2-\gamma} & (0 \leq \gamma < 1) \\ -\frac{x_n}{r} \log x_n & (\gamma = 1) \end{cases}$$

for some small $r > 0$. The continuity of u up to the boundary $\{x_n = 0\}$ is automatically guaranteed since Y is a regular boundary point. In the case $Y \in \{X \in \partial_p Q_1^+ : x_n \neq 0\}$, the continuity of u is guaranteed by using the barrier function of the uniformly parabolic equations as in [8]. \square

2.5 Global regularity

The proof of Theorem 2.2 is sufficient if we show higher regularity of solutions at the boundary. Main equation (1.1) is a uniformly parabolic equation in any $\Omega \subset\subset Q_1^+$, so if the coefficients and forcing term are in $C_s^{k,\alpha}(\overline{Q_1^+})$, the solution of (1.1) is in $C_{loc}^{k+2,\alpha}(\Omega)$ and global regularity can be obtained by combining interior regularity and boundary regularity. More strictly speaking, it is described by the following lemma and theorem.

Lemma 2.10 *Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$, and assume*

$$a^{ij}, b^i, c, f \in C_s^{k,\alpha}(\overline{Q_1^+}) \quad (i, j = 1, 2, \dots, n).$$

Suppose $u \in C_{loc}^{k+2,\alpha}(Q_1^+)$ be a solution of (1.1). Then, for each $Y \in \{X \in \overline{Q_{1/2}^+} : x_n > 0\}$, there is an s -polynomial p of degree $(k + 2)$ at Y such that

$$\|p\|_{C_s^{k+2,\alpha}(\overline{Q_{r/2}^+(Y)})} \leq C \left(\frac{\|u\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|f\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right)$$

and

$$|u(X) - p(X)| \leq C \left(\frac{\|u\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|f\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right) s[X, Y]^{k+2+\alpha}$$

for all $X \in \overline{Q_{r/2}^+(Y)}$ and fixed $\rho \in (1/2, 1)$, where $r = y_n^{\frac{2-\gamma}{2}}$ and $C > 0$ is a constant which is independent of Y . Moreover

$$|p(X)| \leq C \left(\frac{\|u\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|f\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right) s[X, Y]^{k+2+\alpha}$$

for all $X \in \overline{Q_1^+}$ with $s[X, Y] \geq r/2$.

Proof For each $Y = (y, \tau) \in \{X \in \overline{Q_{1/2}^+} : x_n > 0\}$, we consider the function v defined in a standard cube $Q_\rho(0', 1, 0)$ by

$$v(x, t) = u(y' + rx', (rx_n)^{\frac{2}{2-\gamma}}, \tau + r^2t),$$

where $\rho \in (1/2, 1)$ and $r = y_n^{\frac{2-\gamma}{2}}$. Then, v satisfies the uniformly parabolic equation

$$v_t = \tilde{a}^{ij}(X)v_{ij} + \tilde{b}^i(X)v_i - \frac{\gamma}{2-\gamma} \frac{\tilde{a}^{nn}(X)}{x_n} v_n + \tilde{c}(X)v + \tilde{f}(X) \quad \text{in } Q_\rho(0', 1, 0), \tag{2.3}$$

where

$$\begin{aligned} \tilde{a}^{ij}(X) &= \begin{cases} a^{ij}(y' + rx', (rx_n)^{\frac{2}{2-\gamma}}, \tau + r^2t) & \text{if } i, j \neq n \\ \frac{2-\gamma}{2}a^{in}(y' + rx', (rx_n)^{\frac{2}{2-\gamma}}, \tau + r^2t) & \text{if } i \neq n, j = n \\ \frac{2-\gamma}{2}a^{nj}(y' + rx', (rx_n)^{\frac{2}{2-\gamma}}, \tau + r^2t) & \text{if } i = n, j \neq n \\ \frac{(2-\gamma)^2}{4}a^{nn}(y' + rx', (rx_n)^{\frac{2}{2-\gamma}}, \tau + r^2t) & \text{if } i = j = n, \end{cases} \\ \tilde{b}^i(X) &= \begin{cases} rb^i(y' + rx', (rx_n)^{\frac{2}{2-\gamma}}, \tau + r^2t) & \text{if } i \neq n \\ \frac{2-\gamma}{2}rb^n(y' + rx', (rx_n)^{\frac{2}{2-\gamma}}, \tau + r^2t) & \text{if } i = n, \end{cases} \\ \tilde{c}(X) &= r^2c(y' + rx', (rx_n)^{\frac{2}{2-\gamma}}, \tau + r^2t), \\ \tilde{f}(X) &= r^2f(y' + rx', (rx_n)^{\frac{2}{2-\gamma}}, \tau + r^2t). \end{aligned}$$

Since $Q_{r\rho}^+(Y)$ is away from $\{X \in \partial_p Q_1^+ : x_n = 0\}$, we can show that

$$\begin{cases} \|\tilde{a}^{ij}\|_{C^{k,\alpha}(\overline{Q_\rho(0',1,0)})} \leq C \|a^{ij}\|_{C_s^{k,\alpha}(\overline{Q_1^+})} \\ \|\tilde{b}^i\|_{C^{k,\alpha}(\overline{Q_\rho(0',1,0)})} \leq C \|b^i\|_{C_s^{k,\alpha}(\overline{Q_1^+})} \\ \|\tilde{c}\|_{C^{k,\alpha}(\overline{Q_\rho(0',1,0)})} \leq C \|c\|_{C_s^{k,\alpha}(\overline{Q_1^+})} \end{cases}$$

and $\|\tilde{f}\|_{C^{k,\alpha}(\overline{Q_\rho(0',1,0)})} \leq r^2C \|f\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}$ for some constant C which is independent Y . Thus, by the interior Schauder estimate, there exists a polynomial \tilde{p} of degree $(k + 2)$ of the form

$$\tilde{p}(X) = \sum_{\substack{|\beta|+i+2j \leq k+2 \\ \beta \in \mathbb{N}_0^{n-1}, i, j \in \mathbb{N}_0}} A^{\beta ij} x'^{\beta} (x_n - 1)^i t^j$$

and

$$\|\tilde{p}\|_{C^{k+2,\alpha}(Q_\rho(0',1,0))} \leq C \left(\|v\|_{L^\infty(Q_\rho(0',1,0))} + \|\tilde{f}\|_{C^{k,\alpha}(Q_\rho(0',1,0))} \right), \tag{2.4}$$

where C depends on $n, \lambda, \Lambda, k, \alpha, \|\tilde{a}^{ij}\|_{C^{k,\alpha}(\overline{Q_\rho(0',1,0)})}, \|\tilde{b}^i\|_{C^{k,\alpha}(\overline{Q_\rho(0',1,0)})}$, and $\|\tilde{c}\|_{C^{k,\alpha}(\overline{Q_\rho(0',1,0)})}$ such that

$$|v(X) - \tilde{p}(X)| \leq C \left(\|v\|_{L^\infty(\overline{Q_\rho(0',1,0)})} + \|\tilde{f}\|_{C^{k,\alpha}(\overline{Q_\rho(0',1,0)})} \right) d[X, (0', 1, 0)]^{k+2+\alpha} \tag{2.5}$$

for all $X \in \overline{Q_{1/2}(0', 1, 0)}$. Putting

$$\begin{aligned}
 p(X) &= \tilde{p}(r^{-1}(x' - y'), r^{-1}x_n^{\frac{2-\gamma}{2}}, r^{-2}(t - \tau)) \\
 &= \sum_{\substack{|\beta|+i+2j \leq k+2 \\ \beta \in \mathbb{N}_0^{n-1}, i, j \in \mathbb{N}_0}} \frac{A^{\beta ij}}{r^{|\beta|+i+2j}} (x' - y')^\beta (x_n^{\frac{2-\gamma}{2}} - y_n^{\frac{2-\gamma}{2}})^i (t - \tau)^j,
 \end{aligned}$$

we can rewrite (2.4) and (2.5) into

$$\|p\|_{C^{k+2,\alpha}(\overline{Q_{r/2}^+(Y)})} \leq C \left(\frac{\|u\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|f\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right)$$

and

$$\begin{aligned}
 &|u(X) - p(X)| \\
 &\leq C \left(\|u\|_{L^\infty(Q_{r\rho}^+(Y))} + r^2 \|f\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})} \right) \\
 &\quad \times d[(r^{-1}(x' - y'), r^{-1}x_n^{\frac{2-\gamma}{2}}, r^{-2}(t - \tau)), (0', 1, 0)]^{k+2+\alpha} \\
 &= C \left(\frac{\|u\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|f\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right) s[X, Y]^{k+2+\alpha}
 \end{aligned}$$

for all $X \in \overline{Q_{r/2}^+(Y)}$. From (2.4), we know that

$$\sum_{\substack{|\beta|+i+2j \leq k+2 \\ \beta \in \mathbb{N}_0^{n-1}, i, j \in \mathbb{N}_0}} |A^{\beta ij}| \leq C \left(\|u\|_{L^\infty(Q_{r\rho}^+(Y))} + r^2 \|f\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})} \right)$$

and hence, we have

$$\begin{aligned}
 |p(X)| &\leq C \left(\frac{\|u\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|f\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right) \sum_{\substack{|\beta|+i+2j \leq k+2 \\ \beta \in \mathbb{N}_0^{n-1}, i, j \in \mathbb{N}_0}} \frac{r^{k+2+\alpha}}{r^{|\beta|+i+2j}} s[X, Y]^{|\beta|+i+2j} \\
 &\leq C \left(\frac{\|u\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|f\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right) s[X, Y]^{k+2+\alpha}
 \end{aligned}$$

for all $X \in \overline{Q_1^+}$ with $s[X, Y] \geq r/2$. □

Theorem 2.11 (Global Regularity up to $\{x_n = 0\}$) *Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$, and assume $u \in C(\overline{Q_1^+})$. Suppose the following statements hold:*

(1) If for each $Y \in \{X \in \overline{Q_{1/2}^+} : x_n > 0\}$, there is an s -polynomial p^{k+2} of degree $(k + 2)$ at Y such that

$$\|p^{k+2}\|_{C_s^{k+2,\alpha}(\overline{Q_{r/2}^+(Y)})} \leq A \left(\frac{\|u\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|u_t - Lu\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right)$$

and

$$|u(X) - p^{k+2}(X)| \leq A \left(\frac{\|u\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|u_t - Lu\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right) s[X, Y]^{k+2+\alpha}$$

for all $X \in \overline{Q_{r/2}^+(Y)}$ and fixed $\rho \in (1/2, 1)$, where $r = y_n^{\frac{2-\gamma}{2}}$ and $A > 1$ is a constant that is independent of Y . Moreover

$$|p^{k+2}(X)| \leq A \left(\frac{\|u\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|u_t - Lu\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right) s[X, Y]^{k+2+\alpha}$$

for all $X \in \overline{Q_1^+}$ with $s[X, Y] \geq r/2$.

(2) If for each $\tilde{Y} \in \{X \in \partial_p \overline{Q_{1/2}^+} : x_n = 0\}$, there is an s -polynomial p^m of degree m corresponding to $\kappa = k + 2 + \alpha$ at \tilde{Y} and a constant $B > 1$ such that

$$\begin{cases} |(\partial_t - L)(u - p^m)(X)| \leq Bs[X, \tilde{Y}]^{k+\alpha} \\ |u(X) - p^m(X)| \leq Bs[X, \tilde{Y}]^{k+2+\alpha} \end{cases}$$

for all $X \in \overline{Q_1^+}$ and

$$\|p^m\|_{C_s^{k+2,\alpha}(\overline{Q_1^+})} \leq B.$$

(3) For each s -polynomial p of degree m corresponding to $\kappa = k + 2 + \alpha$ with $\|p\|_{C_s^{k+2,\alpha}(\overline{Q_1^+})} \leq B$, $v = u - p$ satisfies (1) again.

Then $u \in C_s^{k+2,\alpha}(\overline{Q_{1/2}^+})$ and

$$\|u\|_{C_s^{k+2,\alpha}(\overline{Q_{1/2}^+})} \leq ABC,$$

where $C = C(k, \alpha) > 0$ is a constant.

Proof For each $Y = (y, \tau) \in \overline{Q_{1/2}^+}$, we just need to find an s -polynomial p that satisfies

$$\|p\|_{C_s^{k+2,\alpha}(\overline{Q_1^+})} \leq ABC$$

and

$$|u(X) - p(X)| \leq ABCs[X, Y]^{k+2+\alpha} \quad \text{for all } X \in \overline{Q_1^+}.$$

If $y_n = 0$, then our goal p is just p^m in (2). Let us think about $y_n > 0$ case. Let $\tilde{Y} = (y', 0, \tau)$. From (2), there is an s -polynomial p^m of degree m corresponding to $\kappa = k + 2 + \alpha$ at \tilde{Y} and a constant $B > 1$ such that

$$\begin{cases} |(\partial_t - L)(u - p^m)(X)| \leq Bs[X, \tilde{Y}]^{k+\alpha} \\ |u(X) - p^m(X)| \leq Bs[X, \tilde{Y}]^{k+2+\alpha} \end{cases} \quad (2.6)$$

for all $X \in \overline{Q_1^+}$ and $\|p^m\|_{C_s^{k+2,\alpha}(\overline{Q_1^+})} \leq B$. Now applying (1) on the function $v = u - p^m$, we have an s -polynomial p^{k+2} of degree $(k + 2)$ such that

$$\|p^{k+2}\|_{C_s^{k+2,\alpha}(\overline{Q_{r/2}(Y)})} \leq A \left(\frac{\|v\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|v_t - Lv\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right) \quad (2.7)$$

and

$$|v(X) - p^{k+2}(X)| \leq A \left(\frac{\|v\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|v_t - Lv\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right) s[X, Y]^{k+2+\alpha} \quad (2.8)$$

for all $X \in \overline{Q_{r/2}^+(Y)}$ and $\rho \in (1/2, 1)$, where $r = y_n^{\frac{2-\gamma}{2}}$ and $A > 1$ is a constant that is independent of Y . Moreover

$$|p^{k+2}(X)| \leq A \left(\frac{\|v\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^{k+2+\alpha}} + \frac{\|v_t - Lv\|_{C_s^{k,\alpha}(\overline{Q_{r\rho}^+(Y)})}}{r^{k+\alpha}} \right) s[X, Y]^{k+2+\alpha} \quad (2.9)$$

for all $X \in \overline{Q_1^+}$ with $s[X, Y] \geq r/2$. From (2.6), we see

$$\begin{aligned} |v(X)| &\leq Bs[X, \tilde{Y}]^{k+2+\alpha} \leq B(s[X, Y] + s[Y, \tilde{Y}])^{k+2+\alpha} \leq B \left(r\rho + y_n^{\frac{2-\gamma}{2}} \right)^{k+2+\alpha} \\ &\leq BCr^{k+2+\alpha} \end{aligned} \quad (2.10)$$

for all $X \in \overline{Q_{r\rho}^+(Y)}$ and we also see

$$|(v_t - Lv)(X)| \leq BCr^{k+\alpha} \quad \text{for all } X \in \overline{Q_{r\rho}^+(Y)}. \quad (2.11)$$

Thus, combining (2.7), (2.10), and (2.11) gives

$$\|p^{k+2}\|_{C_s^{k+2,\alpha}(\overline{Q_{r/2}(Y)})} \leq ABC. \quad (2.12)$$

Since p^{k+2} is an s -polynomial, we can extend the domain $\overline{Q_{r/2}^+(Y)}$ to $\overline{Q_1^+}$ in (2.12). Furthermore, combining (2.8), (2.10), and (2.11) gives

$$|u(X) - p^m(X) - p^{k+2}(X)| \leq ABCs[X, Y]^{k+2+\alpha} \quad \text{for all } X \in \overline{Q_{r/2}^+(Y)}.$$

Meanwhile, combining (2.6), (2.9), (2.10), and (2.11) gives

$$\begin{aligned} |u(X) - p^m(X) - p^{k+2}(X)| &\leq |u(X) - p^m(X)| + |p^{k+2}(X)| \\ &\leq Bs[X, \tilde{Y}]^{k+2+\alpha} + ABCs[X, Y]^{k+2+\alpha} \\ &\leq ABCs[X, Y]^{k+2+\alpha} \end{aligned}$$

for all $X \in \overline{Q_1^+}$ with $s[X, Y] \geq r/2$. □

Thanks to Theorem 2.11, we are only concerned with showing regularity at the boundary.

3 C_s^α -Regularity

The goal of this section is to show global C_s^α -regularity of solutions for (1.1). In Sect. 4, we obtain boundary $C^{1,\alpha}$ -regularity of solutions for equations with constant coefficients using the result of this section, and in Sect. 5, it is used to prove the approximation lemma.

Lemma 3.1 *Let $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ be a solution of (1.1) satisfying $\|u\|_{L^\infty(Q_1^+)} \leq 1$ and $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$ and for some nonnegative constant $\delta \leq \min\{1 - \gamma/2, 1 - \gamma\}$ and $M > 0$, if f satisfies*

$$|f(X)| \leq Mx_n^{-\delta} \quad \text{for all } X \in Q_1^+, \tag{3.1}$$

then

$$|u(X)| \leq \begin{cases} Cx_n & (\gamma + \delta < 1) \\ -Cx_n \log x_n & (\gamma + \delta = 1) \end{cases}$$

for all $X \in \overline{Q_{1/2}^+}$, where $C > 0$ is a constant depending only on $n, \lambda, \Lambda, \gamma$, and M .

Proof For $\varepsilon < \min\{1, 2/\Lambda, 2/M\}$, we consider the function \tilde{u} defined in Q_1 by $\tilde{u}(X) = u(2^{-1}\varepsilon^2 X)$, where

$$Q_\rho = \{(x, t) : |x_i| < 2\rho\varepsilon^{-1} (1 \leq i < n), 0 < x_n < \rho^{\frac{2}{2-\gamma}}, -2\varepsilon^{-2}\rho^2 < t \leq 0\}.$$

Then, \tilde{u} is a solution of

$$\tilde{u}_t = \tilde{L}\tilde{u} + \tilde{f} \quad \text{in } Q_1$$

satisfying $\|\tilde{u}\|_{L^\infty(Q_1)} \leq 1$ and $\tilde{u} = 0$ on $\{X \in \partial_p Q_1 : x_n = 0\}$, where

$$\begin{aligned} \tilde{L} = & \tilde{a}^{i'j'}(X)D_{i'j'} + 2x_n^{\gamma/2}\tilde{a}^{i'n}(X)D_{i'n} + x_n^\gamma\tilde{a}^{nn}(X)D_{nn} + \tilde{b}^{i'}(X)D_{i'} \\ & + x_n^{\gamma/2}\tilde{b}^n(X)D_n + \tilde{c}(X) \end{aligned}$$

with

$$\tilde{a}^{ij}(X) = a^{ij}(2^{-1}\varepsilon^2 X), \quad \tilde{b}^i(X) = 2^{-1}\varepsilon^2 b^i(2^{-1}\varepsilon^2 X), \quad \tilde{c}(X) = (2^{-1}\varepsilon^2)^2 c(2^{-1}\varepsilon^2 X),$$

and $\tilde{f}(X) = (2^{-1}\varepsilon^2)^2 f(2^{-1}\varepsilon^2 X)$. Since $\|b^i\|_{L^\infty(Q_1^+)} \leq \Lambda$ ($i = 1, 2, \dots, n$) and (3.1), we have $|\tilde{b}^i| \leq 2^{-1}\varepsilon^2 \Lambda < \varepsilon$ ($i = 1, 2, \dots, n$) in Q_1 and

$$|\tilde{f}(X)| \leq (2^{-1}\varepsilon^2)^{2-\frac{2\delta}{\gamma}} M x_n^{-\delta} \leq 2^{-1}\varepsilon^2 M x_n^{-\delta} \leq \varepsilon x_n^{-\delta} \quad \text{for all } X \in Q_1.$$

From these observations, we may assume without loss of generality that $u \in C^2(Q_1) \cap C(\bar{Q}_1)$ is a solution of

$$u_t = Lu + f \quad \text{in } Q_1$$

satisfying $\|u\|_{L^\infty(Q_1)} \leq 1$ and $u = 0$ on $\{X \in \partial_p Q_1 : x_n = 0\}$, where the coefficients b^i , c , and the forcing term f satisfy $|b^i| \leq \varepsilon$ ($i = 1, 2, \dots, n$), $c \leq 0$ in Q_1 , and $|f(X)| \leq \varepsilon x_n^{-\delta}$ for all $X \in Q_1$. Now, we define $\Omega_\varepsilon = \{(x, t) : |x_i| < \varepsilon^{-1} (1 \leq i < n), 0 < x_n < 1, -\varepsilon^{-2} < t \leq 0\}$. For each $Y = (y, \tau) \in \bar{Q}_{1/2}$, consider the function v defined in $\Omega_\varepsilon \subset Q_1$ by

$$v(x, t) = u(x' + y', x_n, t + \tau) + \varphi(x_n) - (|x'|^2 - t)\varepsilon^2,$$

where

$$\varphi(x_n) = \begin{cases} x_n^{2-\gamma-\delta} + x_n^{2-\gamma/2} - 3x_n & (\gamma + \delta < 0) \\ x_n^{2-\gamma-\delta} - 2x_n & (0 \leq \gamma + \delta < 1) \\ x_n \log x_n - x_n & (\gamma + \delta = 1). \end{cases}$$

Then $v \leq 0$ on $\partial_p \Omega_\varepsilon$ and for $\varepsilon < \frac{1}{3} (2 - \frac{\gamma}{2}) (1 - \frac{\gamma}{2}) \lambda$, we have

$$v_t - \hat{L}v \leq x_n^{-\delta} \left(-\lambda\eta(\gamma, \delta) + \left(5 + \frac{3}{2}|\gamma|\right)\varepsilon + [1 + 2(n-1)(\Lambda + 1)]\varepsilon^2 \right)$$

in Ω_ε , where

$$\begin{aligned} \hat{L} = & \hat{a}^{i'j'}(X)D_{i'j'} + 2x_n^{\gamma/2}\hat{a}^{i'n}(X)D_{i'n} + x_n^\gamma\hat{a}^{nn}(X)D_{nn} + \hat{b}^{i'}(X)D_{i'} \\ & + x_n^{\gamma/2}\hat{b}^n(X)D_n + \hat{c}(X) \end{aligned}$$

with

$$\hat{a}^{ij}(X) = a^{ij}(x' + y', x_n, t + \tau), \quad \hat{b}^i(X) = b^i(x' + y', x_n, t + \tau),$$

$$\hat{c}(X) = c(x' + y', x_n, t + \tau),$$

and

$$\eta(\gamma, \delta) = \begin{cases} (2 - \gamma - \delta)(1 - \gamma - \delta) & (\gamma + \delta \neq 1) \\ 1 & (\gamma + \delta = 1). \end{cases}$$

For sufficiently small $\varepsilon > 0$, we obtain $v_t - \hat{L}v < 0$ in Ω_ε . By Lemma 2.4, we have

$$u(x' + y', x_n, t + \tau) \leq (|x'|^2 - t)\varepsilon^2 - \varphi(x_n) \quad \text{for all } (x, t) \in \overline{\Omega_\varepsilon}.$$

Thus, if we take $(x', x_n, t) = (0', y_n, 0)$, then

$$u(Y) \leq -\varphi(y_n) \leq \begin{cases} 3y_n & (\gamma + \delta < 1) \\ y_n(1 - \log y_n) & (\gamma + \delta = 1) \end{cases}$$

for all $Y \in \overline{Q_{1/2}}$. Next, consider the following function to find the lower bound

$$w(x, t) = -u(x' + y', x_n, t + \tau) + \varphi(x_n) - (|x'|^2 - t)\varepsilon^2.$$

Similarly, we can obtain the lower bound. Hence, we conclude that

$$|u(X)| \leq \begin{cases} Cx_n & (\gamma + \delta < 1) \\ -Cx_n \log x_n & (\gamma + \delta = 1) \end{cases}$$

for all $X \in \overline{Q_{1/2}^+}$. □

Corollary 3.2 *Let $0 < \alpha < \min\{\frac{2}{2-\gamma}, 1\}$, and assume $f \in L^\infty(Q_1^+)$. Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ is a solution of (1.1) satisfying $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. Then $u \in C_s^\alpha(Q_{1/2}^+)$ and*

$$\|u\|_{C_s^\alpha(\overline{Q_{1/2}^+})} \leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{L^\infty(Q_1^+)} \right),$$

where $C > 0$ is a universal constant.

Proof By Lemma 3.1, we have

$$\begin{aligned}
 |u(X)| &\leq \begin{cases} C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{L^\infty(Q_1^+)} \right) x_n & (\gamma < 1) \\ -C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{L^\infty(Q_1^+)} \right) x_n \log x_n & (\gamma = 1) \end{cases} \\
 &\leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{L^\infty(Q_1^+)} \right) s[X, O]^\alpha
 \end{aligned}$$

for all $X \in \overline{Q_{1/2}^+}$, since the function $x_n^{1-\alpha/2} \log x_n$ is bounded. Thus, we know that for each $Y = (y, \tau) \in \{X \in \overline{Q_{1/2}^+} : x_n > 0\}$,

$$|u(X)| \leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{L^\infty(Q_1^+)} \right) s[X, \tilde{Y}]^\alpha \quad \text{for all } X \in \overline{Q_1^+}, \tag{3.2}$$

where $\tilde{Y} = (y', 0, \tau)$, since (1.1) is invariant for translation in the x' -direction. We now consider the function v defined in a standard cube $Q_\rho(0', 1, 0)$ by $v(x, t) = u(y' + rx', (rx_n)^{\frac{2}{2-\gamma}}, \tau + r^2t)$, where fixed $\rho \in (1/2, 1)$ and $r = y_n^{\frac{2-\gamma}{2}}$ as in the proof of Lemma 2.10. Then v is a solution of the uniformly parabolic equation (2.3). By the interior Hölder regularity,

$$|v(X) - v(0', 1, 0)| \leq C \left(\|v\|_{L^\infty(Q_\rho(0', 1, 0))} + \|\tilde{f}\|_{L^\infty(Q_\rho(0', 1, 0))} \right) d[X, (0', 1, 0)]^\alpha \tag{3.3}$$

for all $X \in \overline{Q_{1/2}(0', 1, 0)}$. From (3.2), we see

$$|u(X)| \leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{L^\infty(Q_1^+)} \right) r^\alpha \quad \text{for all } X \in \overline{Q_{r\rho}^+(Y)}. \tag{3.4}$$

Then, we can rewrite (3.3) into

$$\begin{aligned}
 &|u(X) - u(Y)| \\
 &\leq C \left(\|u\|_{L^\infty(Q_{r\rho}^+(Y))} + r^2 \|f\|_{L^\infty(Q_{r\rho}^+(Y))} \right) \\
 &\quad \times d[(r^{-1}(x' - y'), r^{-1}x_n^{\frac{2-\gamma}{2}}, r^{-2}(t - \tau)), (0', 1, 0)]^\alpha \\
 &= C \left(\frac{\|u\|_{L^\infty(Q_{r\rho}^+(Y))}}{r^\alpha} + r^{2-\alpha} \|f\|_{L^\infty(Q_{r\rho}^+(Y))} \right) d[(x', x_n^{\frac{2-\gamma}{2}}, t), (y', y_n^{\frac{2-\gamma}{2}}, \tau)]^\alpha \\
 &\leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{L^\infty(Q_1^+)} \right) s[X, Y]^\alpha
 \end{aligned}$$

for all $X \in \overline{Q_{r/2}^+(Y)}$ which is extensible throughout $\overline{Q_1^+}$ by using (3.2) and (3.4). \square

Lemma 3.3 *Let $0 < \alpha < \min\{\frac{2}{2-\gamma}, 1\}$ and let $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ be a solution of*

$$\begin{cases} u_t = Lu & \text{in } Q_1^+ \\ u = g & \text{on } \partial_p Q_1^+ \end{cases} \tag{3.5}$$

satisfying $\|u\|_{L^\infty(Q_1^+)} \leq 1$ and $g = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. If $\|g\|_{C_s^\alpha(\overline{Q_1^+})} \leq 1$, then

$$|u(X)| \leq Cx_n^{\frac{2-\gamma}{2}\alpha} \text{ for all } X \in \overline{Q_1^+},$$

where $C > 0$ is a universal constant.

Proof Let $K = \left\{ \frac{\lambda}{2\Lambda} \left(1 - \frac{(2-\gamma)\alpha}{2} \right) \right\}^{\frac{2}{2-\gamma}} \in (0, 1)$. Since $\|g\|_{C_s^\alpha(\overline{Q_1^+})} \leq 1$ and $\|u\|_{L^\infty(Q_1^+)} \leq 1$, there exists a constant $C > 0$ such that $|g(X)| \leq Cx_n^{\frac{2-\gamma}{2}\alpha}$ for all $X \in \partial_p Q_1^+$ and $|u(X)| \leq CK^{\frac{2-\gamma}{2}\alpha}$ for all $X = (x', K, t) \in Q_1^+$. Consider now the function v defined in $\Omega = \{X \in Q_1^+ : 0 < x_n < K\}$ by $v(X) = u(X) - Cx_n^{\frac{2-\gamma}{2}\alpha}$. Then $v \leq 0$ on $\partial_p \Omega$ and

$$\begin{aligned} v_t - Lv &= C \left\{ \frac{(2-\gamma)\alpha}{2} \left(\frac{(2-\gamma)\alpha}{2} - 1 \right) a^{nn} x_n^{\frac{2-\gamma}{2}(\alpha-2)} \right. \\ &\quad \left. + \frac{(2-\gamma)\alpha}{2} b^n x_n^{\frac{2-\gamma}{2}(\alpha-1)} + cx_n^{\frac{2-\gamma}{2}\alpha} \right\} \\ &\leq \frac{C(2-\gamma)\alpha}{2} \left\{ \left(\frac{(2-\gamma)\alpha}{2} - 1 \right) \lambda + \Lambda x_n^{\frac{2-\gamma}{2}} \right\} x_n^{\frac{2-\gamma}{2}(\alpha-2)} \\ &\leq \frac{C(2-\gamma)\alpha\lambda}{4} \left(\frac{(2-\gamma)\alpha}{2} - 1 \right) x_n^{\frac{2-\gamma}{2}(\alpha-2)} \\ &< 0. \end{aligned}$$

By Lemma 2.4, we have

$$u(X) \leq Cx_n^{\frac{2-\gamma}{2}\alpha} \text{ for all } X \in \overline{\Omega}.$$

Considering $v(x, t) = -u(x, t) + Cx_n^{\frac{2-\gamma}{2}\alpha}$ similarly, we have lower bound of u . \square

Corollary 3.4 *Let $0 < \alpha < \min\{\frac{2}{2-\gamma}, 1\}$. Then for each $a^{ij}, b^i, c \in C_{loc}^\alpha(Q_1^+)$, and $g \in C_s^\alpha(\overline{Q_1^+})$ satisfying $g = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$, there exists a unique solution $u \in C^2(Q_1^+) \cap C_s^\alpha(\overline{Q_1^+})$ of (3.5) and*

$$\|u\|_{C_s^\alpha(\overline{Q_1^+})} \leq C\|g\|_{C_s^\alpha(\overline{Q_1^+})},$$

where $C > 0$ is a universal constant.

4 Regularity of solutions for equations with constant coefficients

Before treating (1.1) with variable coefficients, we establish a necessary preliminary result for equations with constant coefficients given by

$$u_t = L_0 u + f, \tag{4.1}$$

where the operator L_0 is given as

$$L_0 = A^{i'j'} D_{i'j'} + 2x_n^{\gamma/2} A^{i'n} D_{i'n} + x_n^\gamma A^{nn} D_{nn} + B^i D_{i'} + x_n^{\gamma/2} B^n D_n + C^-,$$

with the constant symmetric matrix (A^{ij}) satisfies the following condition

$$\lambda |\xi|^2 \leq A^{ij} \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n$$

and the constants B^1, B^2, \dots, B^n , and C^- satisfy

$$\sum_{i=1}^n |B^i| + |C^-| \leq \Lambda \quad \text{and} \quad C^- < 0.$$

Theorem 4.1 *Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$ with $k + 1 + \alpha - \frac{2}{2-\gamma} \notin \mathbb{N}_0$, and assume $f \in C_s^{k,\alpha}(\overline{Q_1^+})$. Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ is a solution of (4.1) satisfying $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. Then $u \in C_s^{k,2+\alpha}(\overline{Q_{1/2}^+})$ and*

$$\|u\|_{C_s^{k,2+\alpha}(\overline{Q_{1/2}^+})} \leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})} \right),$$

where C is a universal constant.

Lemma 4.2 *Let $0 < \alpha < \min\{\frac{2}{2-\gamma}, 1\}$ and let $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ be a solution of (4.1) satisfying $\|u\|_{L^\infty(Q_1^+)} \leq 1$ and $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. If*

$$\|f\|_{L^\infty(Q_1^+)} \leq 1 \quad \text{and} \quad \|f_i\|_{L^\infty(Q_1^+)} \leq 1 \quad (i \neq n),$$

then

$$\|u_i\|_{C_s^\alpha(\overline{Q_{1/2}^+})} \leq C,$$

where $C > 0$ is a universal constant.

Proof It is sufficient to prove only the Hölder continuity of u_1 in $\overline{Q_{1/2}^+}$. Consider $\tilde{u}(x_1)$ as a function whose variables are fixed except for the x_1 -coordinate in the function u . By Corollary 3.2, for $h \in \mathbb{R}$ with $|h| \ll 1$, we have

$$\|u\|_{C_s^\alpha(\overline{Q_{1-|h|}^+})} \leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{L^\infty(Q_1^+)} \right).$$

This implies that $\tilde{u} \in C^\alpha(I_h)$ and the function

$$v^\alpha(x, t) = \frac{u(x_1 + h, x_2, \dots, x_n, t) - u(x, t)}{|h|^\alpha}$$

is a bounded solution of $v_t^\alpha = L_0 v^\alpha + f^\alpha$, where

$$f^\alpha(x, t) = \frac{f(x_1 + h, x_2, \dots, x_n, t) - f(x, t)}{|h|^\alpha} \quad \text{and} \quad I_h = \begin{cases} [-1, 1 - h] & (h > 0) \\ [-1 - h, 1] & (h < 0). \end{cases}$$

We can apply Corollary 3.2 to v^α again,

$$\begin{aligned} \|v^\alpha\|_{C_s^\alpha(\overline{Q_{1-2|h|}^+})} &\leq C \left(\|v^\alpha\|_{L^\infty(Q_{1-|h|}^+)} + \|f^\alpha\|_{L^\infty(Q_{1-|h|}^+)} \right) \\ &\leq C \left(\|u\|_{C_s^\alpha(\overline{Q_{1-|h|}^+})} + \|f_1\|_{L^\infty(Q_1^+)} \right) \\ &\leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{L^\infty(Q_1^+)} + \|f_1\|_{L^\infty(Q_1^+)} \right). \end{aligned}$$

This implies that $\tilde{u} \in C^{2\alpha}(I_{2h})$. We can repeat this process, we have $\tilde{u} \in C^{0,1}(I_{kh})$ for some $k \in \mathbb{N}$. Finally, v^1 is also a bounded solution of $v_t^1 = L_0 v^1 + f^1$ and v^1 is the difference quotient of u for x_1 -direction, we conclude that $\tilde{u} \in C^{1,\alpha}(I_{(k+1)h})$ and

$$\|u_1\|_{C_s^\alpha(\overline{Q_{1/2}^+})} \leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{L^\infty(Q_1^+)} + \|f_1\|_{L^\infty(Q_1^+)} \right).$$

□

Since $v = u_i$ ($i \neq n$) satisfies $v_t = L_0 v + f_i$, we can apply Lemma 4.2 again to obtain the Hölder continuity of u_{ij} on $\overline{Q_{1/2}^+}$.

Corollary 4.3 *Let $0 < \alpha < \min\{\frac{2}{2-\gamma}, 1\}$ and let $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ be a solution of (4.1) satisfying $\|u\|_{L^\infty(Q_1^+)} \leq 1$ and $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. If for each $i, j = 1, 2, \dots, n - 1$,*

$$\|f\|_{L^\infty(Q_1^+)} \leq 1, \quad \|f_i\|_{L^\infty(Q_1^+)} \leq 1, \quad \text{and} \quad \|f_{ij}\|_{L^\infty(Q_1^+)} \leq 1,$$

then

$$\|u_{ij}\|_{C_s^\alpha(\overline{Q_{1/2}^+})} \leq C,$$

where $C > 0$ is a universal constant.

Lemma 4.4 *Let $0 < \alpha < \min\{\frac{2}{2-\gamma}, 1\}$ and let $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ be a solution of (4.1) satisfying $\|u\|_{L^\infty(Q_1^+)} \leq 1$ and $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. If*

$$\|f\|_{L^\infty(Q_1^+)} \leq 1 \quad \text{and} \quad \|f_t\|_{L^\infty(Q_1^+)} \leq 1,$$

then

$$\|u_t\|_{C_s^\alpha(Q_{1/2}^+)} \leq C,$$

where $C > 0$ is a universal constant.

Proof Considering the function

$$w^\alpha(x, t) = \frac{u(x, t+h) - u(x, t)}{|h|^{\alpha/2}},$$

we know that the proof is exactly the same as Lemma 4.2. □

Lemma 4.5 *Let $u \in C^\infty(Q_1^+) \cap C(\overline{Q_1^+})$ be a solution of (4.1) satisfying $\|u\|_{L^\infty(Q_1^+)} \leq 1$ and $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. If for each $i, j, k = 1, 2, \dots, n-1$,*

$$\begin{aligned} \|f\|_{L^\infty(Q_1^+)} \leq 1, \quad \|f_i\|_{L^\infty(Q_1^+)} \leq 1, \quad \|f_{ij}\|_{L^\infty(Q_1^+)} \leq 1, \\ \|f_{ijk}\|_{L^\infty(Q_1^+)} \leq 1, \quad \|f_t\|_{L^\infty(Q_1^+)} \leq 1, \quad \|f_{kt}\|_{L^\infty(Q_1^+)} \leq 1, \end{aligned}$$

and there exist constants $\theta \geq \gamma/2$ and $M > 0$ such that

$$\begin{cases} |f(X)| \leq Mx_n^\theta & \text{for all } X \in \overline{Q_1^+} \\ |f_k(X)| \leq Mx_n^\theta & \text{for all } X \in \overline{Q_1^+}, \end{cases}$$

then u_n is well-defined on $\overline{Q_{1/2}^+}$ and

$$|u_n(x', x_n, t) - u_n(x', y_n, t)| \leq \begin{cases} C|x_n - y_n|^{1-\gamma/2} & (0 < \gamma \leq 1) \\ C|x_n - y_n| & (\gamma \leq 0) \end{cases}$$

for all $(x', x_n, t), (x', y_n, t) \in \overline{Q_{1/2}^+}$, where $C > 0$ is a constant depending only on $n, \lambda, \Lambda, \gamma, \theta$, and M .

Proof Differentiate both sides of (4.1) with respect to x_k ($k \neq n$), then

$$\begin{cases} v_t = L_0 v + f_k & \text{in } Q_1^+ \\ v = 0 & \text{on } \{X \in \partial_p Q_1^+ : x_n = 0\}, \end{cases} \tag{4.2}$$

where $v = u_k$. By Lemma 4.2, v is bounded solution of (4.2), we can apply Lemma 3.1 to v , we have

$$|u_k(X)| = |v(X)| \leq \begin{cases} Cx_n & (\gamma < 1) \\ -Cx_n \log x_n & (\gamma = 1) \end{cases}$$

for all $X \in \overline{Q_{1/2}^+}$. We can obtain the same inequality not only for u_k but also for u_i and u_{ij} ($i, j = 1, 2, \dots, n - 1$). An integration by parts yields

$$\begin{aligned} \left| \int_0^z x_n^{-\gamma/2} u_{i'n}(x, t) dx_n \right| &= \left| \left[x_n^{-\gamma/2} u_{i'}(x, t) \right]_{x_n=0}^{x_n=z} + \frac{\gamma}{2} \int_0^z x_n^{-\gamma/2-1} u_{i'}(x, t) dx_n \right| \\ &\leq \left| z^{-\gamma/2} u_{i'}(x', z, t) \right| + \frac{|\gamma|}{2} \int_0^z |x_n^{-\gamma/2-1} u_{i'}(x, t)| dx_n \end{aligned} \tag{4.3}$$

$$\leq \begin{cases} Cz^{1-\gamma/2} & (\gamma < 1) \\ -C\sqrt{z} \log z & (\gamma = 1) \end{cases} \tag{4.4}$$

for all $(x', z, t) \in \overline{Q_{1/2}^+}$. Similarly, we also have

$$\left| \int_0^z x_n^{-\gamma/2} u_n(x, t) dx_n \right| \leq \begin{cases} Cz^{1-\gamma/2} & (\gamma < 1) \\ -C\sqrt{z} \log z & (\gamma = 1) \end{cases} \tag{4.5}$$

for all $(x', z, t) \in \overline{Q_{1/2}^+}$. Combining (4.1), (4.4), and (4.5) gives

$$\begin{aligned} &\left| \int_0^z u_{nn} dx_n \right| \\ &= \left| \int_0^z \frac{u_t - A^{i'j'} u_{i'j'} - 2x_n^{\gamma/2} A^{i'n} u_{i'n} - B^{i'} u_{i'} - x_n^{\gamma/2} B^n u_n - C^- u - f}{x_n^\gamma A^{nn}} dx_n \right| \\ &\leq \begin{cases} Cz^{1-\gamma/2} & (\gamma < 1) \\ -C\sqrt{z} \log z & (\gamma = 1) \end{cases} \end{aligned} \tag{4.6}$$

for all $(x', z, t) \in \overline{Q_{1/2}^+}$ and hence for $\varepsilon < 2^{-\frac{2}{2-\gamma}}$,

$$\begin{aligned} |u_n(x', \varepsilon, t)| &= \left| u_n(x', 2^{-\frac{2}{2-\gamma}}, t) - \int_0^{2^{-\frac{2}{2-\gamma}}} u_{nn}(x, t) dx_n + \int_0^\varepsilon u_{nn}(x, t) dx_n \right| \\ &\leq \begin{cases} |u_n(x', 2^{-\frac{2}{2-\gamma}}, t)| + C(2^{-1} + \varepsilon^{1-\gamma/2}) & (\gamma < 1) \\ |u_n(x', 1/4, t)| + C(\log 2 - \sqrt{\varepsilon} \log \varepsilon) & (\gamma = 1). \end{cases} \end{aligned}$$

By interior gradient estimates for uniformly parabolic equations, u_n is bounded by a universal constant on $\{x_n = 2^{-\frac{2}{2-\gamma}}\}$. Therefore, we have $|u_n| \leq C$ in $Q_{1/2}^+$. This implies that even though $\gamma = 1$, we have the following Lipschitz estimate

$$|u(x, t)| \leq Cx_n \quad \text{for all } (x, t) \in \overline{Q_{1/2}^+}.$$

Furthermore $v = u_k$ ($k \neq n$) satisfies assumptions of Lemma 4.5 again, we have $|u_{kn}| \leq C$ in $Q_{1/2}^+$. Thus, (4.6) gives

$$\begin{aligned} |u_n(x', x_n, t) - u_n(x', y_n, t)| &= \left| \int_{y_n}^{x_n} u_{nn}(x', z, t) dz \right| \\ &\leq C \left(\left| \int_{y_n}^{x_n} z^{1-\gamma} dz \right| + \left| \int_{y_n}^{x_n} z^{-\gamma/2} dz \right| + \left| \int_{y_n}^{x_n} z^{\theta-\gamma} dz \right| \right) \\ &= C(|x_n^{2-\gamma} - y_n^{2-\gamma}| + |x_n^{1-\gamma/2} - y_n^{1-\gamma/2}| + |x_n^{\theta-\gamma+1} - y_n^{\theta-\gamma+1}|) \end{aligned} \tag{4.7}$$

$$\leq \begin{cases} C|x_n - y_n|^{1-\gamma/2} & (0 < \gamma \leq 1) \\ C|x_n - y_n| & (\gamma \leq 0) \end{cases} \tag{4.8}$$

for all $x_n, y_n \in (0, 2^{-\frac{2}{2-\gamma}})$.

In order to extend (4.8) to $\overline{Q_{1/2}^+}$, we define the function $u_n^* : [0, 2^{-\frac{2}{2-\gamma}}] \rightarrow \mathbb{R}$ as in [9]:

$$u_n^*(x_n) = \begin{cases} \inf_{0 < y_n < 2^{-\frac{2}{2-\gamma}}} \{u_n(x', y_n, t) + C|x_n - y_n|^{1-\gamma/2}\} & (0 < \gamma \leq 1) \\ \inf_{0 < y_n < 2^{-\frac{2}{2-\gamma}}} \{u_n(x', y_n, t) + C|x_n - y_n|\} & (\gamma \leq 0). \end{cases}$$

Then, u_n^* is uniformly continuous on $[0, 2^{-\frac{2}{2-\gamma}}]$ such that

$$|u_n^*(x_n) - u_n^*(y_n)| \leq \begin{cases} C|x_n - y_n|^{1-\gamma/2} & (0 < \gamma \leq 1) \\ C|x_n - y_n| & (\gamma \leq 0) \end{cases}$$

for all $x_n, y_n \in [0, 2^{-\frac{2}{2-\gamma}}]$ and $u_n^*(x_n) = u_n(x', x_n, t)$ for all $x_n \in (0, 2^{-\frac{2}{2-\gamma}})$. Therefore, the value $u_n(x', 0, t)$ can be defined as $u_n^*(0)$ and we conclude that

$$|u_n(x', x_n, t) - u_n(x', y_n, t)| \leq \begin{cases} C|x_n - y_n|^{1-\gamma/2} & (0 < \gamma \leq 1) \\ C|x_n - y_n| & (\gamma \leq 0) \end{cases}$$

for all $(x', x_n, t), (x', y_n, t) \in \overline{Q_{1/2}^+}$. □

Remark 4.6 In Lemma 4.5, we can verify that the proof can continue for u_k ($k \neq n$), u_t , or higher order differentiation of u except for x_n -direction. Therefore, on the same

assumption of Lemma 4.5 for forcing term $D_{(x',t)}^\beta f(X)$ for each $\beta \in \mathbb{N}_0^n$, the partial derivative $D_{(x',t)}^\beta u_n(X)$ is well-defined on $\overline{Q_{1/2}^+}$ and

$$|D_{(x',t)}^\beta u(X)| \leq Cx_n \quad \text{and} \quad |D_{(x',t)}^\beta u_n(X)| \leq C$$

for all $X \in \overline{Q_{1/2}^+}$, where $C > 0$ is a constant depending only on $n, \lambda, \Lambda, \gamma, \theta$, and M .

Lemma 4.7 *Let $u \in C^\infty(Q_1^+) \cap C(\overline{Q_1^+})$ be a solution of (4.1) with $f = 0$. If $\|u\|_{L^\infty(Q_1^+)} \leq 1$ and $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$, then for each $N \in \mathbb{N}_0$, there exists an s -polynomial p of degree $(N + \frac{2}{2-\gamma})$ at O such that*

$$|(p_t - L_0 p)(X)| \leq Cx_n^{1+\frac{2-\gamma}{2}(N-1)} \quad \text{for all } X \in \overline{Q_{1/2}^+}$$

and

$$|u(X) - p(X)| \leq Cs[X, O]^{N+1+\frac{2}{2-\gamma}} \quad \text{for all } X \in \overline{Q_1^+},$$

where $C > 0$ is a constant depending only on $n, \lambda, \Lambda, \gamma$, and N .

Proof By Lemma 4.5, the function $U^0(x', t) := u_n(x', 0, t)$ is well-defined. Furthermore, by Remark 4.6, U^0 is infinitely differentiable and hence we have $D_{(x',t)}^\beta U^0(x', t) = D_{(x',t)}^\beta u_n(x', 0, t)$ for all $\beta \in \mathbb{N}_0^n$. For each $N \in \mathbb{N}_0$, consider a function v^N of the form

$$v^N(X) = \sum_{i=0}^N U^i(x', t)x_n^{1+\frac{2-\gamma}{2}i}.$$

Then, the function $(v_t^N - L_0 v^N)$ is expressed as follows:

$$(v_t^N - L_0 v^N)(X) = \sum_{i=1}^{N+2} g^i(x', t)x_n^{1+\frac{2-\gamma}{2}(i-2)},$$

where each g^i is an unknown smooth function. For arbitrary fixed positive integer $l \leq N$, the function g^l in $(\partial_t - L_0)v^N$ is the sum of the coefficient functions of

$$x_n^{1+\frac{2-\gamma}{2}(l-2)} \tag{4.9}$$

in the functions obtained by expanding the following operator

$$(v_t^N - L_0 v^N)(X) = \sum_{i=0}^N (\partial_t - L_0) \left(U^i(x', t)x_n^{1+\frac{2-\gamma}{2}i} \right). \tag{4.10}$$

If we choose U^1 such that $g^1 \equiv 0$, then

$$U^1(x', t) = -\frac{4}{(2-\gamma)(4-\gamma)A^{nn}}(2A^{i'n}U_{i'}^0(x', t) + B^n U^0(x', t)).$$

In the expansion of (4.10), the coefficient function for the monomial of the form (4.9) can be obtained as follows:

$$\begin{aligned} & -x_n^\gamma A^{nn} D_n^2 \left(U^l x_n^{1+\frac{2-\gamma}{2}l} \right) - x_n^{\gamma/2} (2A^{i'n} D_{i'n} + B^n D_n) \left(U^{l-1} x_n^{1+\frac{2-\gamma}{2}(l-1)} \right) \\ & + (\partial_t - A^{i'j'} D_{i'j'} - B^{i'} D_{i'} - C^-) \left(U^{l-2} x_n^{1+\frac{2-\gamma}{2}(l-2)} \right) \\ & = -\left[\frac{(2-\gamma)l}{2} \left(1 + \frac{(2-\gamma)l}{2} \right) A^{nn} U^l + \left(1 + \frac{(2-\gamma)(l-1)}{2} \right) (2A^{i'n} D_{i'} + B^n) U^{l-1} \right. \\ & \quad \left. - (\partial_t - A^{i'j'} D_{i'j'} - B^{i'} D_{i'} - C^-) U^{l-2} \right] x_n^{1+\frac{2-\gamma}{2}(l-2)}. \end{aligned}$$

Now, taking U^l such that $g^l \equiv 0$, we can determine U^l inductively using the following formula:

$$U^l = -\frac{\left(1 + \frac{(2-\gamma)(l-1)}{2} \right) (2A^{i'n} D_{i'} + B^n) U^{l-1} - (\partial_t - A^{i'j'} D_{i'j'} - B^{i'} D_{i'} - C^-) U^{l-2}}{\frac{(2-\gamma)l}{2} \left(1 + \frac{(2-\gamma)l}{2} \right) A^{nn}}. \tag{4.11}$$

Since l was arbitrary, we found all U^i .

Using the recurrence relation (4.11), for any $N \in \mathbb{N}_0$, we can construct the function v^N such that all monomials of the form

$$g^i(x', t) x_n^{1+\frac{2-\gamma}{2}(i-2)} \quad (i \leq N)$$

are deleted in $(v_i^N - L_0 v^N)$. Thus, we have

$$(v_i^N - L_0 v^N)(X) = g^{N+1}(x', t) x_n^{1+\frac{2-\gamma}{2}(N-1)} + g^{N+2}(x', t) x_n^{1+\frac{2-\gamma}{2}N}.$$

This implies that

$$|D_{(x',t)}^\beta (v_i^N - L_0 v^N)(X)| \leq C x_n^{1+\frac{2-\gamma}{2}(N-1)} \tag{4.12}$$

for all $X \in \overline{Q_{3/4}^+}$ and $\beta \in \mathbb{N}_0^n$, where C depends on $n, \lambda, \Lambda, \gamma, \beta$, and N .

We now claim, for any $N \in \mathbb{N}_0$, the following inequality holds:

$$|D_{(x',t)}^\beta (u - v^N)(X)| \leq C x_n^{1+\frac{2-\gamma}{2}(N+1)} \tag{4.13}$$

for all $X \in \overline{Q_{1/2}^+}$ and $\beta \in \mathbb{N}_0^n$, where $C > 0$ is a constant depending only on $n, \lambda, \Lambda, \gamma, \beta$, and N . The proof is by induction on N . Suppose first $N = 0$ and let $w^0 = D_{(x',t)}^\beta(u - v^0)$, then w^0 is a solution of

$$w_t^0 = L_0 w^0 + f^0 \quad \text{in } Q_{3/4}^+,$$

where $f^0 = D_{(x',t)}^\beta(L_0 v^0 - v_t^0)$. From (4.12), f^0 satisfy assumptions of Lemma 4.5, thus inequality (4.7) yields

$$|w_n^0(X)| = |w_n^0(X) - w_n^0(x', 0, t)| \leq C x_n^{\frac{2-\gamma}{2}} \quad \text{for all } X \in \overline{Q_{1/2}^+}$$

and hence

$$|D_{(x',t)}^\beta(u - v^0)(X)| = |w^0(X) - w^0(x', 0, t)| \leq C x_n^{1+\frac{2-\gamma}{2}} \quad \text{for all } X \in \overline{Q_{1/2}^+}.$$

Next, assume (4.13) is valid for some nonnegative integer $(N - 1)$. Let $w^N = D_{(x',t)}^\beta(u - v^N)$, then

$$\begin{aligned} |w^N(X)| &\leq |D_{(x',t)}^\beta u(X) - D_{(x',t)}^\beta v^{N-1}(X)| + |D_{(x',t)}^\beta U^N(x', t) x_n^{1+\frac{2-\gamma}{2}N}| \\ &\leq C x_n^{1+\frac{2-\gamma}{2}N} \end{aligned} \tag{4.14}$$

for all $X \in \overline{Q_{1/2}^+}$ and

$$w_t^N = L_0 w^N + f^N \quad \text{in } Q_{3/4}^+,$$

where $f^N = D_{(x',t)}^\beta(L_0 v^N - v_t^N)$. Since β was arbitrary, the partial derivatives of w^N except in the x_n -direction satisfy (4.14) again. We know from (4.12) that f^N satisfies assumptions of Lemma 4.5. Combining (4.3) and (4.14) gives

$$\begin{aligned} &\left| \int_0^{x_n} z^{-\gamma/2} w_{i'n}^N(x', z, t) dz \right| \\ &\leq \left| x_n^{-\gamma/2} w_{i'n}^N(x, t) \right| + \frac{|\gamma|}{2} \int_0^{x_n} |z^{-\gamma/2-1} w_{i'n}^N(x', z, t)| dz \leq C x_n^{\frac{2-\gamma}{2}(N+1)} \end{aligned}$$

for all $X \in \overline{Q_{1/2}^+}$. Similarly, we also have

$$\left| \int_0^{x_n} z^{-\gamma/2} w_n^N(x', z, t) dz \right| \leq C x_n^{\frac{2-\gamma}{2}(N+1)} \quad \text{for all } X \in \overline{Q_{1/2}^+}.$$

Combining (4.6), (4.12), and (4.14) gives

$$\begin{aligned}
 |w_n^N(X)| &= \left| \int_0^{x_n} w_{nn}^N(x', z, t) dz \right| \\
 &\leq C \left(\int_0^{x_n} z^{1+\frac{2-\gamma}{2}N-\gamma} dz + \left| \int_0^{x_n} z^{-\gamma/2} w_{i'n}^N(x', z, t) dz \right| \right. \\
 &\quad \left. + \left| \int_0^{x_n} z^{-\gamma/2} w_n^N(x', z, t) dz \right| + \left| \int_0^{x_n} z^{-\gamma} f^N(x', z, t) dz \right| \right) \\
 &\leq Cx_n^{\frac{2-\gamma}{2}(N+1)}
 \end{aligned}$$

for all $X \in \overline{Q_{1/2}^+}$ and hence

$$|D_{(x',t)}^\beta(u - v^N)(X)| = |w^N(X)| \leq Cx_n^{1+\frac{2-\gamma}{2}(N+1)} \quad \text{for all } X \in \overline{Q_{1/2}^+}.$$

Thus, by mathematical induction, (4.13) holds for all $N \in \mathbb{N}_0$.

Since U^l ($l = 0, 1, 2, \dots, N$) is smooth, by Taylor theorem, for any $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1}) \in \mathbb{N}_0^{n-1}$ and $k \in \mathbb{N}_0$ with $|\beta| + 2k = N + 1 - l$, there exists Taylor polynomial $T^l(x', t)$ of degree $(N - l)$ such that

$$|U^l(x', t) - T^l(x', t)| \leq C \sum_{|\beta|+2k=N-l+1} |x_1|^{\beta_1} \cdots |x_{n-1}|^{\beta_{n-1}} |t|^k$$

for all $|x_i| < 1/2$ ($i = 1, 2, \dots, n - 1$) and $-1/4 < t \leq 0$. This implies that

$$\begin{aligned}
 \left| v^N(X) - \sum_{l=0}^N T^l(x', t)x_n^{1+\frac{2-\gamma}{2}l} \right| &\leq C \sum_{l=0}^N \sum_{|\beta|+2k=N-l+1} |x_1|^{\beta_1} \cdots |x_{n-1}|^{\beta_{n-1}} |t|^k x_n^{1+\frac{2-\gamma}{2}l} \\
 &\leq Cs[X, O]^{N+1+\frac{2-\gamma}{2}}
 \end{aligned} \tag{4.15}$$

for all $X \in \overline{Q_{1/2}^+}$. Now put

$$p(X) = \sum_{l=0}^N T^l(x', t)x_n^{1+\frac{2-\gamma}{2}l} \tag{4.16}$$

which is an s -polynomial of degree $(N + \frac{2}{2-\gamma})$. Then, from (4.12), we know that

$$|(p_t - L_0p)(X)| \leq Cx_n^{1+\frac{2-\gamma}{2}(N-1)} \quad \text{for all } X \in \overline{Q_{1/2}^+}.$$

Combining (4.13) and (4.15) gives

$$\begin{aligned} |u(X) - p(X)| &\leq |u(X) - v^N(X)| + |v^N(X) - p(X)| \\ &\leq Cx_n^{1+\frac{2-\gamma}{2}(N+1)} + Cs[X, O]^{N+1+\frac{2}{2-\gamma}} \\ &\leq Cs[X, O]^{N+1+\frac{2}{2-\gamma}} \end{aligned}$$

for all $X \in \overline{Q_{1/2}^+}$ which is extensible throughout Q_1^+ . □

Lemma 4.8 *Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$ and assume that the function $f \in C_s^{k,\alpha}(\overline{Q_1^+})$ is of the form*

$$f(X) = \begin{cases} \sum_{\substack{\frac{2i}{2-\gamma} + j < k + \alpha \\ i, j \in \mathbb{N}_0}} \tilde{f}^{ij}(x', t)x_n^{i+\frac{2-\gamma}{2}j} & (\gamma < 1) \\ \sum_{\substack{i \leq j < k + \alpha \\ i, j \in \mathbb{N}_0}} \tilde{f}^{ij}(x', t)(\log x_n)^i x_n^{j/2} & (\gamma = 1), \end{cases}$$

where each \tilde{f}^{ij} is a smooth function for (x', t) . Then there is a function $h \in C_s^{k+2,\alpha}(\overline{Q_1^+})$ of the form

$$h(X) = \begin{cases} \sum_{\substack{\frac{2i}{2-\gamma} + j < k + \alpha \\ i, j \in \mathbb{N}_0}} \tilde{h}^{ij}(x', t)x_n^{i+\frac{2-\gamma}{2}(j+2)} & (\gamma < 1) \\ \sum_{\substack{i \leq j < k + \alpha \\ i, j \in \mathbb{N}_0}} \tilde{h}^{ij}(x', t)(\log x_n)^{i+1} x_n^{(j+2)/2} & (\gamma = 1) \end{cases} \tag{4.17}$$

such that

$$|(h_t - L_0h - f)(X)| \leq C\|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})} x_n^{\frac{2-\gamma}{2}(k+\alpha)} \text{ for all } X \in \overline{Q_1^+}$$

and

$$\|h\|_{C_s^{k+2,\alpha}(\overline{Q_1^+})} \leq C\|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})},$$

where each \tilde{h}^{ij} is a smooth functions for (x', t) and C is a universal constant.

In fact, Lemma 4.8 holds true even when the coefficients are s -polynomial, and the general case will be proved in Lemma 5.4, so it is omitted here.

Lemma 4.9 *Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$ with $k + 1 + \alpha - \frac{2}{2-\gamma} \notin \mathbb{N}_0$, and assume $f \in C_s^{k,\alpha}(\overline{Q_1^+})$. Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ is a solution of (4.1) satisfying $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. Then there exists an s -polynomial p of degree m corresponding to $\kappa := k + 2 + \alpha$ at O such that*

$$|u(X) - p(X)| \leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})} \right) s[X, O]^{k+2+\alpha} \quad \text{for all } X \in \overline{Q_1^+},$$

where $C > 0$ is a universal constant.

Proof By considering $u/(\|u\|_{L^\infty(Q_1^+)} + \varepsilon^{-1}\|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})})$ for $\varepsilon > 0$, we may assume that $\|u\|_{L^\infty(Q_1^+)} \leq 1$ and $\|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})} \leq \varepsilon$. Since $f \in C_s^{k,\alpha}(\overline{Q_1^+})$, we also assume that

$$|f(X) - F(X)| \leq \varepsilon s[X, O]^{k+\alpha} \quad \text{for all } X \in \overline{Q_1^+}, \tag{4.18}$$

where F is an s -polynomial with degree \tilde{m} corresponding to $(k + \alpha)$ at O . By Lemma 4.8, there exists a function $h \in C_s^{k+2,\alpha}(\overline{Q_1^+})$ of the form (4.17) such that

$$|(F + L_0h - h_t)(X)| \leq C_*\varepsilon s[X, O]^{k+\alpha} \quad \text{for all } X \in \overline{Q_1^+} \tag{4.19}$$

and $\|h\|_{C_s^{k+2,\alpha}(\overline{Q_1^+})} \leq C_*\varepsilon$, where C_* is a universal constant.

Decompose $(u - h)$ into the sum of v^1 and w^1 such that

$$\begin{cases} v_t^1 = L_0v^1 & \text{in } Q_1^+ \\ v^1 = u - h & \text{on } \partial_p Q_1^+ \end{cases} \quad \text{and} \quad \begin{cases} w_t^1 = L_0w^1 + \tilde{f}^1 & \text{in } Q_1^+ \\ w^1 = 0 & \text{on } \partial_p Q_1^+, \end{cases}$$

where $\tilde{f}^1 = f + L_0h - h_t$. Then, combining (4.18) and (4.19) leads us to the estimate

$$|\tilde{f}^1(X)| \leq |f(X) - F(X)| + |(F + L_0h - h_t)(X)| \leq (1 + C_*)\varepsilon s[X, O]^{k+\alpha} \tag{4.20}$$

for all $X \in \overline{Q_1^+}$. By Lemma 2.5, for fixed $r \in (0, 1)$, (4.20) gives

$$|w^1(X)| \leq C\|\tilde{f}^1\|_{L^\infty(Q_1^+)} \leq (1 + C_*)\varepsilon \quad \text{for all } X \in \overline{Q_r^+}$$

and hence, we have

$$|u(X) - v^1(X) - h(X)| = |w^1(X)| \leq (1 + C_*)\varepsilon \quad \text{for all } X \in \overline{Q_r^+}.$$

Choose now ε small enough such that $(1 + C_*)\varepsilon < r^{k+2+\alpha}$. Then,

$$|u(X) - v^1(X) - h(X)| \leq r^{k+2+\alpha} \quad \text{for all } X \in \overline{Q_r^+}.$$

Let us now define

$$u^2(X) := \frac{(u - v^1 - h)(rX)}{r^{k+2+\alpha}} \quad \text{and} \quad \tilde{f}^2(X) := \frac{\tilde{f}^1(rX)}{r^{k+\alpha}}.$$

Then, $\|u^2\|_{L^\infty(Q_1^+)} \leq 1$ and

$$u_t^2 = L_0^2 u^2 + \tilde{f}^2 \quad \text{in } Q_1^+,$$

where

$$L_0^2 = A^{i'j'} D_{i'j'} + 2x_n^{\gamma/2} A^{i'n} D_{i'n} + x_n^\gamma A^{nn} D_{nn} + r B^{i'} D_{i'} + x_n^{\gamma/2} r B^n D_n + r^2 C^-.$$

Furthermore, (4.20) leads us to the estimate

$$\begin{aligned} |\tilde{f}^2(X)| &\leq r^{-(k+\alpha)} (|f(rX) - F(rX)| + |(F + L_0 h - h_t)(rX)|) \\ &\leq (1 + C_*) \varepsilon_S[X, O]^{k+\alpha} \end{aligned}$$

for all $X \in \overline{Q_1^+}$. That is, the same hypotheses as before are fulfilled. Repeating the same procedure, we decompose u^2 into the sum of v^2 and w^2 such that

$$\begin{cases} v_t^2 = L_0^2 v^2 & \text{in } Q_1^+ \\ v^2 = u^2 & \text{on } \partial_p Q_1^+, \end{cases} \quad \begin{cases} w_t^2 = L_0^2 w^2 + \tilde{f}^2 & \text{in } Q_1^+ \\ w^2 = 0 & \text{on } \partial_p Q_1^+, \end{cases}$$

and

$$|u^2(X) - v^2(X)| \leq r^{k+2+\alpha} \quad \text{for all } X \in \overline{Q_r^+}.$$

By substituting back, we have

$$|u(X) - h(X) - v^1(X) - r^{k+2+\alpha} v^2(r^{-1}X)| \leq r^{2(k+2+\alpha)} \quad \text{for all } X \in \overline{Q_{r^2}^+}.$$

Continuing iteratively, for each integer $l \geq 3$, let us define the sequence of functions $\{u^l\}$ inductively as follows:

$$u^l(X) := \frac{(u^{l-1} - v^{l-1})(rX)}{r^{k+2+\alpha}} \quad \text{and} \quad \tilde{f}^l(X) := \frac{\tilde{f}^{l-1}(rX)}{r^{k+\alpha}}.$$

Then, $\|u^l\|_{L^\infty(Q_1^+)} \leq 1$ and

$$u_t^l = L_0^l u^l + \tilde{f}^l \quad \text{in } Q_1^+,$$

where

$$L_0^l = A^{i'j'} D_{i'j'} + 2x_n^{\gamma/2} A^{i'n} D_{i'n} + x_n^\gamma A^{nn} D_{nn} + r^{l-1} B^{i'} D_{i'} \\ + x_n^{\gamma/2} r^{l-1} B^n D_n + r^{2(l-1)} C^-.$$

Furthermore,

$$|\tilde{f}^l(X)| \leq (1 + C_*) \varepsilon_S[X, O]^{k+\alpha} \quad \text{for all } X \in \overline{Q_1^+},$$

and hence we decompose u^l into the sum of v^l and w^l such that

$$\begin{cases} v_i^l = L_0^l v^l & \text{in } Q_1^+ \\ v^l = u^l & \text{on } \partial_p Q_1^+, \end{cases} \quad \begin{cases} w_i^l = L_0^l w^l + \tilde{f}^l & \text{in } Q_1^+ \\ w^l = 0 & \text{on } \partial_p Q_1^+, \end{cases}$$

and

$$|u^l(X) - v^l(X)| \leq r^{k+2+\alpha} \quad \text{for all } X \in \overline{Q_r^+}.$$

By substituting back, we have

$$\left| u(X) - h(X) - \sum_{i=1}^l r^{(i-1)(k+2+\alpha)} v^i(r^{-i+1}X) \right| \leq r^{l(k+2+\alpha)} \quad \text{for all } X \in \overline{Q_r^+}. \quad (4.21)$$

Since $k+1+\alpha - \frac{2}{2-\gamma} \notin \mathbb{N}_0$, so we can choose $N \in \mathbb{N}_0$ such that

$$k+1+\alpha - \frac{2}{2-\gamma} < N < k+2+\alpha - \frac{2}{2-\gamma}.$$

By Lemma 4.7 and (4.16), for each $l \in \mathbb{N}$, there exists an s -polynomial

$$p^l(X) = \sum_{|\beta|+i+2j+\frac{2}{2-\gamma} < k+2+\alpha} A_l^{\beta ij} x'^\beta x_n^{1+\frac{2-\gamma}{2}i} t^j$$

of degree $(N + \frac{2}{2-\gamma})$ at O such that

$$|v^l(X) - p^l(X)| \leq C_S[X, O]^{N+1+\frac{2}{2-\gamma}} \quad \text{for all } X \in \overline{Q_1^+}, \quad (4.22)$$

where $C > 0$ is a universal constant. Combining (4.21) and (4.22) leads us to the estimate

$$\begin{aligned}
 & \left| u(X) - h(X) - \sum_{i=1}^l r^{(i-1)(k+2+\alpha)} p^i(r^{-i+1} X) \right| \\
 & \leq r^{l(k+2+\alpha)} + C \sum_{i=1}^l r^{(i-1)(k+2+\alpha)} s[r^{-i+1} X, O]^{N+1+\frac{2}{2-\gamma}} \\
 & \leq r^{l(k+2+\alpha)} + Cr^{l(N+1+\frac{2}{2-\gamma})} \sum_{i=1}^l r^{-(i-1)(N+\frac{2}{2-\gamma}-k-1-\alpha)} \\
 & = r^{l(k+2+\alpha)} + \frac{C(r^{(l+1)(N+\frac{2}{2-\gamma}-k-1-\alpha)} - r^{N+\frac{2}{2-\gamma}-k-1-\alpha})}{r^{N+\frac{2}{2-\gamma}-k-1-\alpha} - 1} r^{l(k+2+\alpha)} \\
 & \leq Cr^{l(k+2+\alpha)} \tag{4.23}
 \end{aligned}$$

for all $X \in \overline{Q_{r^l}^+}$. Now put

$$P^l(X) = \sum_{i=1}^l r^{(i-1)(k+2+\alpha)} p^i(r^{-i+1} X).$$

Then, (4.23) gives

$$\begin{aligned}
 |P^l(X) - P^{l-1}(X)| & \leq \left| u(X) - h(X) - P^l(X) \right| + \left| u(X) - h(X) - P^{l-1}(X) \right| \\
 & \leq Cr^{l(k+2+\alpha)} \tag{4.24}
 \end{aligned}$$

for all $X \in \overline{Q_{r^l}^+}$. Consider the rescaling s -polynomial

$$\begin{aligned}
 \tilde{P}(X) & := P^l(r^l X) - P^{l-1}(r^l X) \\
 & = r^{(l-1)(k+2+\alpha)} \sum_{|\beta|+i+2j+\frac{2}{2-\gamma} < k+2+\alpha} r^{|\beta|+i+2j+\frac{2}{2-\gamma}} A_l^{\beta ij} x^{\beta} x_n^{1+\frac{2-\gamma}{2}i} t^j.
 \end{aligned}$$

From (4.24), we know that $\|\tilde{P}\|_{L^\infty(Q_1^+)} \leq Cr^{l(k+2+\alpha)}$. Since the coefficients of s -polynomial \tilde{P} on Q_1^+ are controlled by the L^∞ norm, we have

$$|A_l^{\beta ij}| \leq \frac{Cr^{k+2+\alpha}}{r^{|\beta|+i+2j+\frac{2}{2-\gamma}}}$$

for all $\beta \in \mathbb{N}_0^{n-1}$ and $i, j \in \mathbb{N}_0$ with $|\beta| + i + 2j + \frac{2}{2-\gamma} < k + 2 + \alpha$. It follows that P^l converges uniformly to an s -polynomial

$$P^\infty(X) = \sum_{|\beta|+i+2j+\frac{2}{2-\gamma} < k+2+\alpha} B^{\beta ij} x'^\beta x_n^{1+\frac{2-\gamma}{2}i} t^j$$

of degree $(N + \frac{2}{2-\gamma})$ at O such that

$$|u(X) - h(X) - P^\infty(X)| \leq C_s[X, O]^{k+2+\alpha} \quad \text{for all } X \in \overline{Q_1^+}.$$

Finally, as in Lemma 4.7, if h is approximated with an s -polynomial, the desired s -polynomial p can be obtained. □

5 Generalized Schauder theory

5.1 $C_s^{2+\alpha}$ -Regularity

In this section, we establish $C_s^{2+\alpha}$ -regularity of solutions for (1.1). Since generalized Schauder theory approximates coefficients using s -polynomials rather than constants, boundary $C^{1,\alpha}$ -regularity and higher regularity for equations with s -polynomials must first be obtained. To prove these, we first need $C_s^{2+\alpha}$ -regularity of solutions for (1.1).

For this section only, L_0 is considered the operator with the following constant coefficients

$$A^{ij} = a^{ij}(O), \quad B^i = b^i(O), \quad C^- = c(O).$$

Theorem 5.1 *Let $0 < \alpha < 1$ with $2 + \alpha \notin \mathcal{D}$, and assume*

$$a^{ij}, b^i, c, f \in C_s^\alpha(\overline{Q_1^+}) \quad (i, j = 1, 2, \dots, n).$$

Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ is a solution of (1.1) satisfying $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. Then $u \in C_s^{2+\alpha}(\overline{Q_{1/2}^+})$ and

$$\|u\|_{C_s^{2+\alpha}(\overline{Q_{1/2}^+})} \leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{C_s^\alpha(\overline{Q_1^+})} \right),$$

where C is a positive constant depending only on $n, \lambda, \Lambda, \gamma, \alpha, \|a^{ij}\|_{C_s^\alpha(\overline{Q_1^+})}, \|b^i\|_{C_s^\alpha(\overline{Q_1^+})}$, and $\|c\|_{C_s^\alpha(\overline{Q_1^+})}$.

Lemma 5.2 (Approximation lemma) *Let $0 < \alpha < \min\{\frac{2}{2-\gamma}, 1\}$, and assume $f \in L^\infty(Q_1^+)$. Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ is a solution of (1.1) satisfying $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$ with $\|u\|_{L^\infty(Q_1^+)} \leq 1$. If for any $\varepsilon > 0$,*

$$\|a^{ij} - A^{ij}\|_{L^\infty(Q_1^+)} \leq \varepsilon, \quad \|b^i - B^i\|_{L^\infty(Q_1^+)} \leq \varepsilon, \quad \|c - C^-\|_{L^\infty(Q_1^+)} \leq \varepsilon,$$

then there exists a solution $h \in C^\infty(Q_{3/4}^+) \cap C_s^\alpha(\overline{Q_{3/4}^+})$ of

$$\begin{cases} h_t = L_0 h & \text{in } Q_{3/4}^+ \\ h = u & \text{on } \partial_p Q_{3/4}^+ \end{cases}$$

such that

$$\|u - h\|_{L^\infty(Q_{1/2}^+)} \leq C \left(\varepsilon^\theta + \|f\|_{L^\infty(Q_1^+)} \right),$$

where $C > 0$ and $\theta \in (0, 1)$ are universal constants.

Proof By Corollary 3.2 and Corollary 3.4, there exists a unique bounded solution $h \in C^2(Q_{3/4}^+) \cap C_s^\alpha(\overline{Q_{3/4}^+})$ of

$$\begin{cases} h_t = L_0 h & \text{in } Q_{3/4}^+ \\ h = u & \text{on } \partial_p Q_{3/4}^+, \end{cases}$$

and

$$\|h\|_{C_s^\alpha(\overline{Q_{3/4}^+})} \leq C \left(\|h\|_{L^\infty(Q_{3/4}^+)} + \|u\|_{C_s^\alpha(\overline{Q_{3/4}^+})} \right) \leq C \left(2 + \|f\|_{L^\infty(Q_1^+)} \right).$$

For $X \in \partial_p Q_{3/4-\delta}^+$, we can choose $Y \in \partial_p Q_{3/4}^+$ such that $s[X, Y] = \delta$ and hence we have

$$|u(X) - h(X)| \leq \|u - h\|_{C_s^\alpha(\overline{Q_{3/4}^+})} s[X, Y]^\alpha \leq C \delta^\alpha \left(2 + \|f\|_{L^\infty(Q_1^+)} \right).$$

By Lemma 2.5 and Theorem 4.1, we have for any $\delta \in (0, 1/4)$

$$\|h\|_{C_s^{2,\alpha}(\overline{Q_{3/4-\delta}^+})} \leq C \delta^{-2-\alpha} \|h\|_{L^\infty(Q_{3/4}^+)} \leq C \delta^{-2-\alpha} \|u\|_{L^\infty(Q_{3/4}^+)} \leq C \delta^{-2-\alpha}.$$

Let $v = u - h$, then

$$v_t = Lv + \tilde{f} \quad \text{in } Q_{3/4-\delta}^+,$$

where

$$\begin{aligned} \tilde{f} &= (a^{ij'} - A^{ij'})h_{i'j'} + 2x_n^{\gamma/2}(a^{in} - A^{in})h_{i'n} + x_n^\gamma(a^{nn} - A^{nn})h_{nn} \\ &\quad + (b^{i'} - B^{i'})h_{i'} + x_n^{\gamma/2}(b^n - B^n)h_n + (c - C^-)h + f. \end{aligned}$$

By Lemma 2.5 and Lemma 2.6 with scaling argument,

$$\begin{aligned} \|u - h\|_{L^\infty(Q_{3/4-\delta}^+)} &\leq \begin{cases} C \left(\|u - h\|_{L^\infty(\partial_p Q_{3/4-\delta}^+)} + \|\tilde{f}\|_{L^\infty(Q_{3/4-\delta}^+)} \right) & (0 < \gamma < 1) \\ C \left(\|u - h\|_{L^\infty(\partial_p Q_{3/4-\delta}^+)} + \|x_n^{-\gamma} \tilde{f}\|_{L^\infty(Q_{3/4-\delta}^+)} \right) & (\gamma < 0) \end{cases} \\ &\leq C \left(\delta^\alpha (2 + \|f\|_{L^\infty(Q_1^+)}) + \varepsilon \delta^{-2-\alpha} + \|f\|_{L^\infty(Q_1^+)} \right). \end{aligned}$$

Take $\delta = \varepsilon^{\frac{1}{2(2+\alpha)}}$ and $\theta = \frac{\alpha}{2(2+\alpha)}$, we conclude that

$$\|u - h\|_{L^\infty(Q_{3/4-\delta}^+)} \leq C \left(\varepsilon^\theta + \|f\|_{L^\infty(Q_1^+)} \right).$$

□

Lemma 5.3 *Let $0 < \alpha < \min\{\frac{2}{2-\gamma}, 1\}$ with $2 + \alpha \notin \mathcal{D}$, and assume*

$$a^{ij}, b^i, c, f \in C_s^\alpha(\overline{Q_1^+}) \quad (i, j = 1, 2, \dots, n).$$

Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ is a solution of (1.1) satisfying $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. Then there exists an s -polynomial p of degree m corresponding to $\kappa := 2 + \alpha$ at O such that

$$|u(X) - p(X)| \leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{C_s^\alpha(\overline{Q_1^+})} \right) s[X, O]^{2+\alpha} \quad \text{for all } X \in \overline{Q_1^+},$$

where C is a positive constant depending only on $n, \lambda, \Lambda, \gamma, \alpha, \|a^{ij}\|_{C_s^\alpha(\overline{Q_1^+})}, \|b^i\|_{C_s^\alpha(\overline{Q_1^+})}$, and $\|c\|_{C_s^\alpha(\overline{Q_1^+})}$.

Proof By considering $u / (\|u\|_{L^\infty(Q_1^+)} + \varepsilon^{-1} \|f\|_{C_s^\alpha(\overline{Q_1^+})})$ for sufficiently small $\varepsilon > 0$, we may assume that $\|u\|_{L^\infty(Q_1^+)} \leq 1$ and $\|f\|_{C_s^\alpha(\overline{Q_1^+})} \leq \varepsilon$. By scaling we also assume that $[a^{ij}]_{C_s^\alpha(\overline{Q_1^+})} \leq \varepsilon, [b^i]_{C_s^\alpha(\overline{Q_1^+})} \leq \varepsilon, [c]_{C_s^\alpha(\overline{Q_1^+})} \leq \varepsilon$, and by the Hölder continuity of coefficients, we have

$$\begin{cases} |a^{ij}(X) - A^{ij}| \leq [a^{ij}]_{C_s^\alpha(\overline{Q_1^+})} s[X, O]^\alpha \leq \varepsilon \\ |b^i(X) - B^i| \leq [b^i]_{C_s^\alpha(\overline{Q_1^+})} s[X, O]^\alpha \leq \varepsilon \\ |c(X) - C^-| \leq [c]_{C_s^\alpha(\overline{Q_1^+})} s[X, O]^\alpha \leq \varepsilon \end{cases} \quad (5.1)$$

for all $X \in \overline{Q_1^+}$. By considering

$$\tilde{u}(X) = u(X) + \begin{cases} \frac{f(O)}{(2-\gamma)(1-\gamma)a^{nn}(O)}x_n^{2-\gamma} & (\gamma < 1) \\ \frac{f(O)}{a^{nn}(O)}x_n \log x_n & (\gamma = 1), \end{cases}$$

we may assume $f(O) = 0$. By Lemma 5.2, there exists a bounded solutions $h^1 \in C^\infty(Q_{3/4}^+) \cap C_s^\alpha(\overline{Q_{3/4}^+})$ of

$$\begin{cases} h_t^1 = L_0 h^1 & \text{in } Q_{3/4}^+ \\ h^1 = u & \text{on } \partial_p Q_{3/4}^+ \end{cases}$$

such that

$$\|u - h^1\|_{L^\infty(Q_{1/2}^+)} \leq C \left(\varepsilon^\theta + \|f\|_{L^\infty(Q_1^+)} \right) \leq 2C\varepsilon^\theta.$$

By Theorem 4.1, $h^1 \in C_s^{2+\alpha}(\overline{Q_{1/2}^+})$ and

$$\|h^1\|_{C_s^{2+\alpha}(\overline{Q_{1/2}^+})} \leq C, \tag{5.2}$$

where C is a universal constant. Let $r \in (0, 1/2)$ be a fixed number and choose now ε small enough such that $2C\varepsilon^\theta < r^{2+\alpha}$. Then,

$$\|u - h^1\|_{L^\infty(Q_r^+)} \leq r^{2+\alpha}. \tag{5.3}$$

Let us now define

$$u^2(X) := \frac{(u - h^1)(rX)}{r^{2+\alpha}} \quad \text{and} \quad \tilde{f}^2(X) := \frac{(f + Lh^1 - L_0h^1)(rX)}{r^\alpha}.$$

Then, $\|u^2\|_{L^\infty(Q_1^+)} \leq 1$ and

$$u_t^2 = L^2 u^2 + \tilde{f}^2 \quad \text{in } Q_1^+,$$

where

$$\begin{aligned} L^2 &= a^{i'j'}(rX)D_{i'j'} + 2x_n^{\gamma/2}a^{i'n}(rX)D_{i'n} + x_n^\gamma a^{nn}(rX)D_{nn} \\ &\quad + rb^{i'}(rX)D_{i'} + rx_n^{\gamma/2}b^n(rX)D_n + r^2c(rX). \end{aligned}$$

From (5.1), (5.2), and (5.3), we can see that the same hypotheses as before are fulfilled. Replacing the decomposition of solutions with Lemma 5.2, as in the proof of

Lemma 4.9, we can show that there exists an s -polynomial p^∞ of degree m corresponding to $(2 + \alpha)$ at O such that

$$|u(X) - p^\infty(X)| \leq Cs[X, O]^{2+\alpha} \quad \text{for all } X \in \overline{Q_1^+}. \quad \square$$

Satisfying the assumptions of Lemma 5.3, we know that $u \in C_s^{\tilde{\alpha}}(\overline{Q_{1/2}^+})$ for any $0 < \tilde{\alpha} < 1$. Applying the approximation lemma again to $\tilde{\alpha}$, we have $u \in C_s^{2+\tilde{\alpha}}(\overline{Q_{1/2}^+})$ for any $0 < \tilde{\alpha} < 1$ with $2 + \tilde{\alpha} \notin \mathcal{D}$.

5.2 Higher regularity of solutions for equations with s -polynomial coefficients

In the previous section, $C_s^{2+\alpha}$ -regularity can be obtained by freezing coefficients method. However due to the degeneracy/singular order of (1.1), higher regularity cannot be obtained inductively like the classical bootstrap argument. We solve this problem by considering freezing coefficients by s -polynomial. In this section, we consider equations of the form

$$u_t = L_p u + f, \tag{5.4}$$

where the operator L_p is given as

$$L_p = P^{i'j'}(X)D_{i'j'} + 2x_n^{\gamma/2}P^{i'n}(X)D_{i'n} + x_n^\gamma P^{nn}(X)D_{nn} \\ + Q^{i'}(X)D_{i'} + x_n^{\gamma/2}Q^n(X)D_n + R(X)$$

and the coefficients P^{ij} are s -polynomials at O satisfying the following condition

$$\lambda|\xi|^2 \leq P^{ij}(X)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for any } X \in Q_1^+, \xi \in \mathbb{R}^n$$

and the coefficients Q^1, \dots, Q^n, R are s -polynomials at O satisfying

$$\sum_{i=1}^n \|Q^i\|_{L^\infty(Q_1^+)} + \|R\|_{L^\infty(Q_1^+)} \leq \Lambda \quad \text{and} \quad R(X) < 0 \quad \text{for any } X \in Q_1^+.$$

Lemma 5.4 *Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$ and assume that the coefficients P^{ij} , Q^i , and R are s -polynomials at O of degree μ corresponding to $(k + \alpha)$ and the function $f \in C_s^{k,\alpha}(\overline{Q_1^+})$ is of the form*

$$f(X) = \begin{cases} \sum_{\substack{\frac{2i}{2-\gamma} + j < k + \alpha \\ i, j \in \mathbb{N}_0}} \tilde{f}^{ij}(x', t)x_n^{i + \frac{2-\gamma}{2}j} & (\gamma < 1) \\ \sum_{\substack{i \leq j < k + \alpha \\ i, j \in \mathbb{N}_0}} \tilde{f}^{ij}(x', t)(\log x_n)^i x_n^{j/2} & (\gamma = 1), \end{cases}$$

where each \tilde{f}^{ij} is a smooth function for (x', t) . Then there is a function $h \in C_s^{k+2,\alpha}(\overline{Q_1^+})$ of the form

$$h(X) = \begin{cases} \sum_{\substack{\frac{2i}{2-\gamma} + j < k + \alpha \\ i, j \in \mathbb{N}_0}} \tilde{h}^{ij}(x', t) x_n^{i + \frac{2-\gamma}{2}(j+2)} & (\gamma < 1) \\ \sum_{\substack{i \leq j < k + \alpha \\ i, j \in \mathbb{N}_0}} \tilde{h}^{ij}(x', t) (\log x_n)^{i+1} x_n^{(j+2)/2} & (\gamma = 1) \end{cases} \tag{5.5}$$

such that

$$|(h_t - L_p h - f)(X)| \leq C \|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})} x_n^{\frac{2-\gamma}{2}(k+\alpha)} \text{ for all } X \in \overline{Q_1^+}$$

and

$$\|h\|_{C_s^{k+2,\alpha}(\overline{Q_1^+})} \leq C \|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})},$$

where each \tilde{h}^{ij} is a smooth function for (x', t) and C depends on $n, \gamma, k, \alpha, \|P^{ij}\|_{C_s^{k,\alpha}(\overline{Q_1^+})}, \|Q^i\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$, and $\|R\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$.

Proof If $f \equiv 0$, obviously the function we are looking for is $h \equiv 0$. Thus, it is sufficient to consider only the case $f \not\equiv 0$, and $\|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})} = 1$ can be assumed. The desired function h can be obtained by comparing $(h_t - L_p h)$ and f by terms for x_n of both sides. If the operator $(\partial_t - L_p)$ is applied to (5.5), then $(h_t - L_p h)$ is expressed as follows:

$$(h_t - L_p h)(X) = \begin{cases} \sum_{\substack{\frac{2i}{2-\gamma} + j < k + \alpha + \mu + 2 \\ i, j \in \mathbb{N}_0}} g^{ij}(x', t) x_n^{i + \frac{2-\gamma}{2}j} & (\gamma < 1) \\ \sum_{\substack{i \leq j < k + \alpha + \mu + 2 \\ i, j \in \mathbb{N}_0}} g^{ij}(x', t) (\log x_n)^i x_n^{j/2} & (\gamma = 1), \end{cases}$$

where each g^{ij} is an unknown smooth function

We will find \tilde{h}^{ij} inductively from the process of tracing g^{ij} . Let us introduce some symbols for computational convenience. We add an auxiliary spacial dimension like

$$x^* = (x_1, \dots, x_n, z) \in \mathbb{R}^{n+1} \text{ and } X^* = (x^*, t).$$

Also, we define a modified version of the function u and the operator L_p as follows: The modified function u^* of u is given by the expression

$$u^*(X^*) = \begin{cases} \sum_{i,j \in \mathbb{N}_0} \tilde{u}^{ij}(x', t) z^i x_n^{\frac{2-\gamma}{2}j} & (\gamma < 1) \\ \sum_{i,j \in \mathbb{N}_0} \tilde{u}^{ij}(x', t) z^i x_n^{j/2} & (\gamma = 1) \end{cases}$$

$$\text{for } u(X) = \begin{cases} \sum_{i,j \in \mathbb{N}_0} \tilde{u}^{ij}(x', t) x_n^{i + \frac{2-\gamma}{2}j} & (\gamma < 1) \\ \sum_{i,j \in \mathbb{N}_0} \tilde{u}^{ij}(x', t) (\log x_n)^i x_n^{j/2} & (\gamma = 1) \end{cases}$$

and the modified operator L_p^* of L_p is given by the expression

$$L_p^* = P^{i'j'^*} D_{i'j'} + 2P^{i'n^*} D_{i'} \bar{D}_n^* + P^{nn^*} \bar{D}_{nn}^* + Q^{i'^*} D_{i'} + Q^{n^*} \bar{D}_n^* + R^*,$$

where

$$\bar{D}_n^* = \begin{cases} x_n^{\gamma/2} D_n + z x_n^{-\frac{2-\gamma}{2}} \partial_z & (\gamma < 1) \\ x_n^{1/2} D_n + x_n^{-1/2} \partial_z & (\gamma = 1), \end{cases}$$

$$\bar{D}_{nn}^* = \begin{cases} x_n^\gamma D_n^2 + 2z x_n^{\gamma-1} D_n \partial_z + z^2 x_n^{\gamma-2} \partial_z^2 & (\gamma < 1) \\ x_n D_n^2 + 2D_n \partial_z + x_n^{-1} \partial_z^2 - x_n^{-1} \partial_z & (\gamma = 1). \end{cases}$$

Then, we can revert the modified version to the original version as follows:

$$(u_t - L_p u)(X) = \begin{cases} (u_t^* - L_p^* u^*)(X^*) \Big|_{z=x_n} & (\gamma < 1) \\ (u_t^* - L_p^* u^*)(X^*) \Big|_{z=\log x_n} & (\gamma = 1), \end{cases}$$

$$u(X) = \begin{cases} u^*(X^*) \Big|_{z=x_n} & (\gamma < 1) \\ u^*(X^*) \Big|_{z=\log x_n} & (\gamma = 1). \end{cases}$$

It is enough to find \tilde{h}^{ij} in the modified version. For arbitrary fixed integers $l, m \in \mathbb{N}_0$, the function g^{lm} in $(h_t^* - L_p^* h^*)$ is the sum of the coefficient functions of

$$z^l x_n^{\frac{2-\gamma}{2}m} \tag{5.6}$$

in the functions obtained by expanding the following operators

$$(h_i^* - L_p^* h^*)(X^*) = \begin{cases} \sum_{\substack{\frac{2i}{2-\gamma} + j < k + \alpha \\ i, j \in \mathbb{N}_0}} (\partial_t - L_p^*) \left(\tilde{h}^{ij}(x', t) z^i x_n^{\frac{2-\gamma}{2}(j+2)} \right) & (\gamma < 1) \\ \sum_{\substack{i \leq j < k + \alpha \\ i, j \in \mathbb{N}_0}} (\partial_t - L_p^*) \left(\tilde{h}^{ij}(x', t) z^{i+1} x_n^{(j+2)/2} \right) & (\gamma = 1). \end{cases} \tag{5.7}$$

We consider several cases according to the range of i, j . The first case is when $j > m$. Then, in the expansion of (5.7), each monomial for (z, x_n) has the order of x_n greater than m . Thus, the expansion of (5.7) cannot have the monomial for (z, x_n) of the form (5.6) and hence the coefficient function of (5.6) in (5.7) is identically zero.

The second case is when $j = m$. For $\gamma < 1$, the expansion of (5.7) cannot have the monomial for (z, x_n) of the form (5.6) if $i > l$. Also, \tilde{h}^{lm} will be obtained by inductively finding functions $\tilde{h}^{0m}, \tilde{h}^{1m}, \tilde{h}^{2m}, \dots, \tilde{h}^{(l-1)m}$, it is sufficient to consider only the case $i = l$ for now. In this case, in the expansion of (5.7), the coefficient function for the monomial of the form (5.6) can be obtained as follows:

$$-P^{nn^*}(x', 0, 0, t) \bar{D}_{nn}^* \left(\tilde{h}^{lm}(x', t) z^l x_n^{\frac{2-\gamma}{2}(m+2)} \right) = S^{lm}(x', t) \tilde{h}^{lm}(x', t) z^l x_n^{\frac{2-\gamma}{2}m},$$

where

$$S^{lm}(x', t) = -\left(l + \frac{2-\gamma}{2}(m+2) \right) \left(l + \frac{2-\gamma}{2}(m+2) - 1 \right) P^{nn}(x', 0, t).$$

For $\gamma = 1$, we will show that the functions $\tilde{h}^{0m}, \tilde{h}^{1m}, \tilde{h}^{2m}, \dots, \tilde{h}^{mm}$ are solutions of some system of equations. For $l = 0, 1, \dots, m$ and $m \neq 0$, in the expansion of (5.7), the coefficient function for the monomial of the form (5.6) can be obtained as follows:

$$\begin{cases} -P^{nn^*}(x', 0, 0, t) \left[(m+1)\tilde{h}^{0m} + 2\tilde{h}^{1m} \right] x_n^{m/2} & (l = 0) \\ -P^{nn^*}(x', 0, 0, t) \left[\frac{1}{4}m(m+2)\tilde{h}^{(l-1)m} \right. \\ \quad \left. + (l+1)(m+1)\tilde{h}^{lm} + (l+1)(l+2)\tilde{h}^{(l+1)m} \right] z^l x_n^{m/2} & (1 \leq l < m) \\ -P^{nn^*}(x', 0, 0, t) \left[\frac{1}{4}m(m+2)\tilde{h}^{(m-1)m} + (m+1)^2\tilde{h}^{mm} \right] z^m x_n^{m/2} & (l = m). \end{cases} \tag{5.8}$$

The last case is when $j < m$. Since we will find \tilde{h}^{lm} inductively, it may be assumed that all of the functions

$$\begin{cases} \tilde{h}^{im} & (\gamma < 1, i < l) \\ \tilde{h}^{ij} & (\gamma < 1, j < m) \\ \tilde{h}^{ij} & (\gamma = 1, j < m) \end{cases} \tag{5.9}$$

have been found.

Now, we consider a truncated function $h_{(lm)}^*$ of h^* as follows:

$$h_{(lm)}^*(X^*) = \begin{cases} \sum_{\substack{2i \\ 2-\gamma} + m < k + \alpha \\ i < l} \tilde{h}^{im}(x', t) z^i x_n^{\frac{2-\gamma}{2}(m+2)} + \sum_{\substack{2i \\ 2-\gamma} + j < k + \alpha \\ j < m} \tilde{h}^{ij}(x', t) z^i x_n^{\frac{2-\gamma}{2}(j+2)} & (\gamma < 1) \\ \sum_{\substack{i \leq j < k + \alpha \\ j < m}} \tilde{h}^{ij}(x', t) z^{i+1} x_n^{(j+2)/2} & (\gamma = 1). \end{cases}$$

As in finding Taylor polynomials, we can find the coefficient function of (5.6) in $(\partial_t - L_p^*)h_{(lm)}^*$ as

$$\psi^{lm}(x', t) := \frac{1}{\left(\frac{2-\gamma}{2}\right)^m l!m!} \partial_z^l \bar{D}_n^m (\partial_t - L_p^*)h_{(lm)}^*(X^*) \Big|_{(z, x_n)=(0,0)},$$

where $\bar{D}_n := x_n^{\gamma/2} D_n$. Since ψ^{lm} is expressed by the functions of (5.9), we already know it by the induction assumption.

For $\gamma < 1$, we can represent the coefficient function $g^{lm}(x', t)$ of (5.6) in $(h_t^* - L_p^*h^*)$ as

$$g^{lm}(x', t) = S^{lm}(x', t)\tilde{h}^{lm}(x', t) + \psi^{lm}(x', t).$$

Now, taking \tilde{h}^{lm} such that $g^{lm} \equiv \tilde{f}^{lm}$, we can determine \tilde{h}^{lm} inductively using the following formula:

$$\tilde{h}^{lm}(x', t) = \frac{\tilde{f}^{lm}(x', t) - \psi^{lm}(x', t)}{S^{lm}(x', t)}. \tag{5.10}$$

Since S^{lm} is nonvanishing smooth function, \tilde{h}^{lm} is also smooth function. Furthermore, we can compute

$$\tilde{h}^{00}(x', t) = \frac{\tilde{f}^{00}(x', t)}{(2-\gamma)(1-\gamma)P^{nm^*}(x', 0, 0, t)}$$

first and apply the recurrence relation (5.10) to index i , we have $\tilde{h}^{10}, \tilde{h}^{20}, \dots, \tilde{h}^{l0}, \dots$. Next, for each $i = 1, 2, \dots, l$, applying the recurrence relation (5.10) to index j again, we have all of the following functions sequentially:

$$\begin{matrix} \tilde{h}^{00}, & \tilde{h}^{10}, & \tilde{h}^{20}, & \dots & \tilde{h}^{l0}, & \dots \\ \tilde{h}^{01}, & \tilde{h}^{11}, & \tilde{h}^{21}, & \dots & \tilde{h}^{l1}, & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \tilde{h}^{0m}, & \tilde{h}^{1m}, & \tilde{h}^{2m}, & \dots & \tilde{h}^{lm}. & \end{matrix}$$

Since l, m were arbitrary, we found all \tilde{h}^{ij} .
 On the other hand, by using (5.8) for $\gamma = 1$, we can represent the coefficient function $g^{0m}(x', t), g^{1m}(x', t), \dots, g^{mm}(x', t)$ of (5.6) in $(h_i^* - L^*h^*)$ as

$$\begin{pmatrix} g^{0m} \\ g^{1m} \\ g^{2m} \\ \vdots \\ g^{mm} \end{pmatrix} = -P^{nn^*}(x', 0, 0, t)T^m \begin{pmatrix} \tilde{h}^{0m} \\ \tilde{h}^{1m} \\ \tilde{h}^{2m} \\ \vdots \\ \tilde{h}^{mm} \end{pmatrix} + \begin{pmatrix} \psi^{0m} \\ \psi^{1m} \\ \psi^{2m} \\ \vdots \\ \psi^{mm} \end{pmatrix},$$

where the tridiagonal matrix T^m is given by

$$\begin{pmatrix} m+1 & 1 \cdot 2 & & & & \\ \frac{1}{4}m(m+2) & 2(m+1) & 2 \cdot 3 & & & \\ & \frac{1}{4}m(m+2) & 3(m+1) & 3 \cdot 4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{4}m(m+2) & m(m+1) & m(m+1) \\ & & & & \frac{1}{4}m(m+2) & (m+1)^2 \end{pmatrix}.$$

Since the tridiagonal matrix T^m is invertible, we can determine $\tilde{h}^{0m}, \tilde{h}^{1m}, \dots, \tilde{h}^{mm}$ inductively using the following formula

$$\begin{pmatrix} \tilde{h}^{0m} \\ \tilde{h}^{1m} \\ \tilde{h}^{2m} \\ \vdots \\ \tilde{h}^{mm} \end{pmatrix} = -\frac{1}{P^{nn^*}(x', 0, 0, t)}(T^m)^{-1} \begin{pmatrix} \tilde{f}^{0m} - \psi^{0m} \\ \tilde{f}^{1m} - \psi^{1m} \\ \tilde{f}^{2m} - \psi^{2m} \\ \vdots \\ \tilde{f}^{mm} - \psi^{mm} \end{pmatrix}, \tag{5.11}$$

when we take $g^{im} \equiv \tilde{f}^{im}$ ($i = 0, 1, 2, \dots, m$). Since $P^{nn^*}(x', 0, 0, t)$ is nonvanishing standard polynomial, $\tilde{h}^{0m}, \tilde{h}^{1m}, \dots, \tilde{h}^{mm}$ are also smooth functions. Furthermore, we can compute

$$\tilde{h}^{00}(x', t) = \frac{\tilde{f}^{00}(x', t)}{P^{nn^*}(x', 0, 0, t)}$$

first and apply the recurrence relation (5.11), we have \tilde{h}^{01} and \tilde{h}^{11} . Next, if we repeat this process, we can find the functions of each row of the following array from all the functions obtained in the previous step.

$$\begin{matrix} \tilde{h}^{02}, & \tilde{h}^{12}, & \tilde{h}^{22}, \\ \tilde{h}^{03}, & \tilde{h}^{13}, & \tilde{h}^{23}, & \tilde{h}^{33}, \\ \tilde{h}^{04}, & \tilde{h}^{14}, & \tilde{h}^{24}, & \tilde{h}^{34}, & \tilde{h}^{44}, \\ \vdots & & & & \\ \tilde{h}^{0m}, & \tilde{h}^{1m}, & \tilde{h}^{2m}, & \tilde{h}^{3m}, & \tilde{h}^{4m}, & \dots, & \tilde{h}^{mm}. \end{matrix}$$

Since l, m were arbitrary, we found all \tilde{h}^{ij} .

Finally, In (5.10) and (5.11), the function h^* is constructed so that all terms of f^* are deleted in $(h_t^* - L_p^* h^* - f^*)$. Thus, we have

$$(h_t - L_p h - f)(X) = \begin{cases} \sum_{\substack{2i + j \geq k + \alpha \\ i, j \in \mathbb{N}_0}} g^{ij}(x', t) x_n^{i + \frac{2-\gamma}{2} j} & (\gamma < 1) \\ \sum_{\substack{i \leq j, j > k + \alpha \\ i, j \in \mathbb{N}_0}} g^{ij}(x', t) (\log x_n)^i x_n^{j/2} & (\gamma = 1). \end{cases}$$

This implies that

$$|(h_t - L_p h - f)(X)| \leq C x_n^{\frac{2-\gamma}{2}(k+\alpha)} \quad \text{for all } X \in \overline{Q_1^+}$$

and $\|h\|_{C_s^{k+2,\alpha}(\overline{Q_1^+})} \leq C$, where C depends on $n, \gamma, k, \alpha, \|P^{ij}\|_{C_s^{k,\alpha}(\overline{Q_1^+})}, \|Q^i\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$, and $\|R\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$. □

Lemma 5.5 *Let $u \in C^\infty(Q_1^+) \cap C(\overline{Q_1^+})$ be a solution of (5.4) satisfying $\|u\|_{L^\infty(Q_1^+)} \leq 1$ and $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. If for each $i, j, k = 1, 2, \dots, n - 1$,*

$$\begin{aligned} \|f\|_{L^\infty(Q_1^+)} &\leq 1, & \|f_i\|_{L^\infty(Q_1^+)} &\leq 1, & \|f_{ij}\|_{L^\infty(Q_1^+)} &\leq 1, \\ \|f_{ijk}\|_{L^\infty(Q_1^+)} &\leq 1, & \|f_t\|_{L^\infty(Q_1^+)} &\leq 1, & \|f_{kt}\|_{L^\infty(Q_1^+)} &\leq 1, \end{aligned}$$

and

$$\begin{cases} |f(x, t)| \leq M x_n^\theta & \text{for all } (x, t) \in \overline{Q_1^+} \\ |f_k(x, t)| \leq M x_n^\theta & \text{for all } (x, t) \in \overline{Q_1^+} \end{cases}$$

for some $\theta \geq \gamma/2$ and $M > 0$, then u_n is well-defined on $\overline{Q_{1/2}^+}$ and

$$|u_n(x', x_n, t) - u_n(x', y_n, t)| \leq \begin{cases} C|x_n - y_n|^{1-\gamma/2} & (0 < \gamma \leq 1) \\ C|x_n - y_n| & (\gamma \leq 0) \end{cases}$$

for all $(x', x_n, t), (x', y_n, t) \in \overline{Q_{1/2}^+}$, where C is a positive constant depending only on $n, \lambda, \Lambda, \gamma, k, \alpha, \|D_{x'}^\beta \partial_t^k P^{ij}\|_{L^\infty(Q_1^+)}, \|D_{x'}^\beta \partial_t^k Q^i\|_{L^\infty(Q_1^+)}$, and $\|D_{x'}^\beta \partial_t^k R\|_{L^\infty(Q_1^+)}$ ($\beta \in \mathbb{N}_0^{n-1}, k \in \mathbb{N}_0, |\beta| + 2k \leq 3$).

Proof Differentiate both sides of (5.4) with respect to x_k ($k \neq n$), then

$$\begin{cases} v_t = L_p v + F & \text{in } Q_{3/4}^+ \\ v = 0 & \text{on } \{X \in \partial_p Q_1^+ : x_n = 0\}, \end{cases} \tag{5.12}$$

where $v = u_k$ and

$$F = P_k^{i'j'} u_{i'j'} + 2x_n^{\gamma/2} P_k^{i'n} u_{i'n} + x_n^\gamma P_k^{nn} u_{nn} + Q_k^{i'} u_{i'} + x_n^{\gamma/2} Q_k^n u_n + R_k u + f_k.$$

By Theorem 5.1, v is bounded solution of (5.12) with bounded forcing term F , we can apply Lemma 3.1 to v , we have

$$|u_k(x, t)| = |v(x, t)| \leq \begin{cases} Cx_n & (\gamma < 1) \\ -Cx_n \log x_n & (\gamma = 1) \end{cases}$$

for all $(x, t) \in \overline{Q_{1/2}^+}$. We can obtain the same inequality not only for u_k but also for u_t and u_{ij} ($i, j = 1, 2, \dots, n - 1$). An integration by parts yields

$$\begin{aligned} & \left| \int_0^z x_n^{-\gamma/2} P^{i'n}(x, t) u_{i'n}(x, t) dx_n \right| \\ & \leq \left| z^{-\gamma/2} P^{i'n}(x', z, t) u_{i'n}(x', z, t) \right| + C \int_0^z x_n^{-\gamma/2-1} |u_{i'n}(x, t)| dx_n \\ & \leq \begin{cases} Cz^{1-\gamma/2} & (\gamma < 1) \\ -C\sqrt{z} \log z & (\gamma = 1) \end{cases} \end{aligned} \tag{5.13}$$

for all $(x', z, t) \in \overline{Q_{1/2}^+}$. Similarly, we also have

$$\left| \int_0^z x_n^{-\gamma/2} Q^n(x, t) u_n(x, t) dx_n \right| \leq \begin{cases} Cz^{1-\gamma/2} & (\gamma < 1) \\ -C\sqrt{z} \log z & (\gamma = 1) \end{cases} \tag{5.14}$$

for all $(x', z, t) \in \overline{Q_{1/2}^+}$. Combining (5.4), (5.13), and (5.14) gives

$$\begin{aligned} \left| \int_0^z u_{nn} dx_n \right| &\leq \frac{1}{\lambda} \left(\left| \int_0^z x_n^{-\gamma} (u_t - P^{i'j'} u_{i'j'} - Q^{i'} u_{i'} - Ru) dx_n \right| \right. \\ &\quad \left. + \left| \int_0^z 2x_n^{-\gamma/2} P^{i'n} u_{i'n} dx_n \right| + \left| \int_0^z x_n^{-\gamma/2} Q^n u_n dx_n \right| + \left| \int_0^z x_n^{-\gamma} f dx_n \right| \right) \\ &\leq \begin{cases} Cz^{1-\gamma/2} & (\gamma < 1) \\ -C\sqrt{z} \log z & (\gamma = 1) \end{cases} \end{aligned} \tag{5.15}$$

for all $(x', z, t) \in \overline{Q_{1/2}^+}$ and hence for $\varepsilon < 2^{-\frac{2}{2-\gamma}}$,

$$\begin{aligned} |u_n(x', \varepsilon, t)| &= \left| u_n(x', 2^{-\frac{2}{2-\gamma}}, t) - \int_0^{2^{-\frac{2}{2-\gamma}}} u_{nn}(x, t) dx_n + \int_0^\varepsilon u_{nn}(x, t) dx_n \right| \\ &\leq \begin{cases} |u_n(x', 2^{-\frac{2}{2-\gamma}}, t)| + C(2^{-1} + \varepsilon^{1-\gamma/2}) & (\gamma < 1) \\ |u_n(x', 1/4, t)| + C(\log 2 - \sqrt{\varepsilon} \log \varepsilon) & (\gamma = 1). \end{cases} \end{aligned}$$

Therefore, we have $|u_n| \leq C$ in $Q_{1/2}^+$. This implies that even though $\gamma = 1$, we have the following Lipschitz estimate

$$|u(x, t)| \leq Cx_n \quad \text{for all } (x, t) \in \overline{Q_{1/2}^+}.$$

Furthermore $v = u_k$ ($k \neq n$) satisfies assumptions of Lemma 5.5 again, we have $|u_{kn}| \leq C$ in $Q_{1/2}^+$. Thus, (5.15) gives

$$\begin{aligned} &|u_n(x', x_n, t) - u_n(x', y_n, t)| \\ &\leq C(|x_n^{2-\gamma} - y_n^{2-\gamma}| + |x_n^{1-\gamma/2} - y_n^{1-\gamma/2}| + |x_n^{\theta-\gamma+1} - y_n^{\theta-\gamma+1}|) \\ &\leq \begin{cases} C|x_n - y_n|^{1-\gamma/2} & (0 < \gamma \leq 1) \\ C|x_n - y_n| & (\gamma \leq 0) \end{cases} \end{aligned} \tag{5.16}$$

for all $x_n, y_n \in (0, 2^{-\frac{2}{2-\gamma}})$.

As in the proof of Lemma 4.5, we can extend (5.16) on $\overline{Q_{1/2}^+}$. □

Lemma 5.6 *Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$, and $u \in C^\infty(Q_1^+) \cap C(\overline{Q_1^+})$ be a solution of (5.4) with $f \equiv 0$. If the coefficients P^{ij} , Q^i , and R are s -polynomials at O of degree μ corresponding to $\kappa = k + \alpha$, $\|u\|_{L^\infty(Q_1^+)} \leq 1$, and $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$,*

then for any $M, N \in \mathbb{N}$, there exists a function v of the form

$$v(X) = \begin{cases} u_n(x', 0, t)x_n + \sum_{\substack{2i \\ 2-\gamma} + j < \frac{2M}{2-\gamma} + N \\ i, j \in \mathbb{N}} \tilde{v}^{ij}(x', t)x_n^{i + \frac{2-\gamma}{2}j} & \text{if } (M, N) \neq (1, 1) \\ u_n(x', 0, t)x_n & \text{if } (M, N) = (1, 1) \end{cases} \tag{5.17}$$

such that

$$|(v_t - L_p v)(X)| \leq Cx_n^{M + \frac{2-\gamma}{2}(N-2)} \text{ for all } X \in \overline{Q_{1/2}^+}$$

and

$$|u(X) - v(X)| \leq Cx_n^{M + \frac{2-\gamma}{2}N} \text{ for all } X \in \overline{Q_{1/2}^+},$$

where each \tilde{v}^{ij} is a smooth function for (x', t) and C depends on $n, \gamma, \alpha, M, N, \|P^{ij}\|_{C_s^{k,\alpha}(\overline{Q_1^+})}, \|Q^i\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$, and $\|R\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$.

Proof For the modified function v^* of v and the modified operator L_p^* of L_p , the function $(v_t^* - L_p^* v^*)$ is expressed as follows:

$$(v_t^* - L_p^* v^*)(X^*) = \sum_{\substack{2i \\ 2-\gamma} + j < \frac{2M}{2-\gamma} + N + \mu + 2 \\ i, j \in \mathbb{N}} g^{ij}(x', t)z^i x_n^{\frac{2-\gamma}{2}(j-2)}, \tag{5.18}$$

where each g^{ij} is an unknown smooth function.

For arbitrary fixed integers $l, m \in \mathbb{N}$, the function g^{lm} in $(\partial_t - L_p^*)v^*$ is the sum of the coefficient functions of

$$z^l x_n^{\frac{2-\gamma}{2}(m-2)} \tag{5.19}$$

in the functions obtained by expanding the following operators

$$(v_t^* - L_p^* v^*)(X^*) = (\partial_t - L_p^*)\left(u_n(x', 0, t)z\right) + \sum_{\substack{2i \\ 2-\gamma} + j < \frac{2M}{2-\gamma} + N \\ i, j \in \mathbb{N}} (\partial_t - L_p^*)\left(\tilde{v}^{ij}(x', t)z^i x_n^{\frac{2-\gamma}{2}j}\right). \tag{5.20}$$

Since we will find \tilde{v}^{ij} in the same way as Lemma 5.4, it is sufficient to consider only the case $(i, j) = (l, m)$ for now. In this case, in the expansion of (5.20), the coefficient

function for the monomial of the form (5.19) can be obtained as follows:

$$-P^{nn^*}(x', 0, 0, t)\bar{D}_{nn}^* \left(\tilde{v}^{lm}(x', t)z^l x_n^{\frac{2-\gamma}{2}m} \right) = S^{lm}(x', t)\tilde{v}^{lm}(x', t)z^l x_n^{\frac{2-\gamma}{2}(m-2)},$$

where

$$S^{lm}(x', t) = -\left(l + \frac{2-\gamma}{2}m\right)\left(l + \frac{2-\gamma}{2}m - 1\right)P^{nn^*}(x', 0, 0, t).$$

Now, we consider a truncated function $v_{(lm)}^*$ of v^* as follows:

$$\begin{aligned} v_{(lm)}^*(X^*) &= u_n(x', 0, t)z + \sum_{\substack{\frac{2i}{2-\gamma} + m < \frac{2M}{2-\gamma} + N \\ 1 \leq i < l}} \tilde{v}^{im}(x', t)z^i x_n^{\frac{2-\gamma}{2}m} \\ &+ \sum_{\substack{\frac{2i}{2-\gamma} + j < \frac{2M}{2-\gamma} + N \\ i \in \mathbb{N}, 1 \leq j < m}} \tilde{v}^{ij}(x', t)z^i x_n^{\frac{2-\gamma}{2}j}. \end{aligned}$$

If $j = 1$, the order of x_n is negative in (5.18), so considering $x_n^{\frac{2-\gamma}{2}}(\partial_t - L_p^*)v_{(lm)}^*$, we can find the coefficient function of (5.19) in $x_n^{\frac{2-\gamma}{2}}(\partial_t - L_p^*)v_{(lm)}^*$ as

$$\psi^{lm}(x', t) := \frac{1}{\left(\frac{2-\gamma}{2}\right)^{m-1} l!(m-1)!} \partial_z^l \bar{D}_n^{m-1} \left(x_n^{\frac{2-\gamma}{2}}(\partial_t - L_p^*)v_{(lm)}^*(X^*) \right) \Big|_{(z, x_n)=(0,0)},$$

where $\bar{D}_n := x_n^{\gamma/2}D_n$. Thus, we can represent the function $g^{lm}(x', t)$ in $(\partial_t - L_p^*)v^*$ as

$$g^{lm}(x', t) = S^{lm}(x', t)\tilde{v}^{lm}(x', t) + \psi^{lm}(x', t).$$

Now, taking \tilde{v}^{lm} such that $g^{lm} \equiv 0$, we can determine \tilde{v}^{lm} inductively using the following formula:

$$\tilde{v}^{lm}(x', t) = -\frac{\psi^{lm}(x', t)}{S^{lm}(x', t)}. \tag{5.21}$$

Since l, m were arbitrary, we found all \tilde{v}^{ij} .

Using the recurrence relation (5.21), for any $M, N \in \mathbb{N}$, we can construct the function v^* such that all monomials of the form

$$g^{ij}(x', t)z^i x_n^{\frac{2-\gamma}{2}(j-2)} \left(\frac{2i}{2-\gamma} + j < \frac{2M}{2-\gamma} + N \right)$$

are deleted in $(v_t^* - L_p^* v^*)$. Thus, we have

$$(v_t - L_p v)(X) = \sum_{\substack{\frac{2i}{2-\gamma} + j \geq \frac{2M}{2-\gamma} + N \\ i, j \in \mathbb{N}}} g^{ij}(x', t) x_n^{i + \frac{2-\gamma}{2}(j-2)}.$$

This implies that

$$|D_{(x',t)}^\beta (v_t - L_p v)(X)| \leq C x_n^{M + \frac{2-\gamma}{2}(N-2)} \tag{5.22}$$

for all $X \in \overline{Q_{3/4}^+}$ and $\beta \in \mathbb{N}_0^n$, where C depends on $n, \gamma, \alpha, \beta, M, N, \|P^{ij}\|_{C_s^{k,\alpha}(\overline{Q_1^+})}, \|Q^i\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$, and $\|R\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$.

We now claim that for any $i_1, i_2 \in \mathbb{N}$ with $i_1 + \frac{2-\gamma}{2}i_2 \leq M + \frac{2-\gamma}{2}N$, the following inequalities hold:

$$|D_{(x',t)}^\beta (u - v)(X)| \leq C x_n^{i_1 + \frac{2-\gamma}{2}i_2} \quad \text{and} \quad |D_{(x',t)}^\beta (u_n - v_n)(X)| \leq C x_n^{i_1 - 1 + \frac{2-\gamma}{2}i_2} \tag{5.23}$$

for all $X \in \overline{Q_{1/2}^+}$ and $\beta \in \mathbb{N}_0^n$. The proof is by induction on M and N . Suppose first $(i_1, i_2) = (1, 1)$. From Theorem 5.1 and Lemma 5.5, Remark 4.6 also holds for equations with s -polynomial coefficients. Thus, by Remark 4.6 and (5.17), we know that

$$|D_{(x',t)}^\beta (u - v)(X)| \leq C x_n \quad \text{and} \quad |D_{(x',t)}^\beta (u_n - v_n)(X)| \leq C$$

for all $X \in \overline{Q_{1/2}^+}$ and $\beta \in \mathbb{N}_0^n$. The operator $x_n^\gamma P^{nn} D_{nn}$ can be represented as

$$\begin{aligned} x_n^\gamma P^{nn} D_{nn} &= \partial_t - P^{i'j'} D_{i'j'} - 2x_n^{\gamma/2} P^{i'n} D_{i'n} \\ &\quad - Q^{i'} D_{i'} - x_n^{\gamma/2} Q^n D_n - R - (\partial_t - L_p) \end{aligned}$$

and hence we know that

$$\begin{aligned} x_n^\gamma D_{nn} D_{(x',t)}^\beta (u - v) &= D_{(x',t)}^\beta (x_n^\gamma D_{nn} (u - v)) \\ &= D_{(x',t)}^\beta \left(\frac{1}{P^{nn}} \partial_t - \frac{P^{i'j'}}{P^{nn}} D_{i'j'} - \frac{Q^{i'}}{P^{nn}} D_{i'} - \frac{R}{P^{nn}} \right) (u - v) \\ &\quad - x_n^{\gamma/2} D_{(x',t)}^\beta \left(2 \frac{P^{i'n}}{P^{nn}} D_{i'} + \frac{Q^n}{P^{nn}} \right) (u_n - v_n) \\ &\quad + D_{(x',t)}^\beta \left(\frac{1}{P^{nn}} (v_t - L_p v) \right). \end{aligned} \tag{5.24}$$

Since P^{nn} is nonvanishing smooth function for x' and t , by Remark 2.1, (5.22), and (5.24), we have

$$|x_n^\gamma D_{nn} D_{(x',t)}^\beta (u - v)(X)| \leq C(x_n + x_n^{\gamma/2} + x_n^{\gamma/2}) \leq Cx_n^{\gamma/2} \tag{5.25}$$

for all $X \in \overline{Q_{1/2}^+}$ and $\beta \in \mathbb{N}_0^n$. Since $u = v$ on $\{x_n = 0\}$ and $u_n = v_n$ on $\{x_n = 0\}$, if we divide (5.25) by both sides x_n^γ and integrate on the x_n -variable from 0 to x_n , we have

$$|D_{(x',t)}^\beta (u - v)(X)| \leq Cx_n^{1+\frac{2-\gamma}{2}} \quad \text{and} \quad |D_{(x',t)}^\beta (u_n - v_n)(X)| \leq Cx_n^{\frac{2-\gamma}{2}}$$

for all $X \in \overline{Q_{1/2}^+}$ and $\beta \in \mathbb{N}_0^n$.

Now, let us prove (5.23) for fixed $i_1 \leq M$ by using induction on i_2 . We assume (5.23) is valid for some $i_2 \in \mathbb{N}$ with $i_2 < N + \frac{2}{2-\gamma}(M - i_1)$. From Remark 2.1, (5.22), (5.24) and induction assumption, we have

$$\begin{aligned} |x_n^\gamma D_{nn} D_{(x',t)}^\beta (u - v)(X)| &\leq C \left(x_n^{i_1 + \frac{2-\gamma}{2}i_2} + x_n^{i_1 + \frac{2-\gamma}{2}(i_2-1)} + x_n^{M + \frac{2-\gamma}{2}(N-2)} \right) \\ &\leq Cx_n^{i_1 + \frac{2-\gamma}{2}(i_2-1)} \end{aligned}$$

for all $X \in \overline{Q_{1/2}^+}$ and $\beta \in \mathbb{N}_0^n$. Then, we know that

$$|D_{(x',t)}^\beta (u - v)(X)| \leq Cx_n^{i_1 + \frac{2-\gamma}{2}(i_2+1)} \quad \text{and} \quad |D_{(x',t)}^\beta (u_n - v_n)(X)| \leq Cx_n^{i_1-1 + \frac{2-\gamma}{2}(i_2+1)}$$

for all $X \in \overline{Q_{1/2}^+}$ and $\beta \in \mathbb{N}_0^n$ in the same way as for $(i_1, i_2) = (1, 1)$.

Next, let us prove (5.23) completely again using induction on i_1 . We assume that (5.23) is valid for some $i_1, i_2 \in \mathbb{N}$ with $i_1 < M$. Then, by the induction process on i_2 , we can see that (5.23) holds for (i_1, j_2) , where

$$j_2 \geq N + \frac{2}{2-\gamma}(M - i_1) \geq 1 + \frac{2}{2-\gamma}.$$

It implies that

$$|D_{(x',t)}^\beta (u - v)(X)| \leq Cx_n^{i_1 + \frac{2-\gamma}{2}j_2} \leq Cx_n^{i_1+1 + \frac{2-\gamma}{2}}$$

and

$$|D_{(x',t)}^\beta (u_n - v_n)(X)| \leq Cx_n^{i_1-1 + \frac{2-\gamma}{2}j_2} \leq Cx_n^{i_1 + \frac{2-\gamma}{2}}.$$

Thus, we obtain (5.23) for $(i_1 + 1, 1)$. Therefore, we conclude that (5.23) holds for all $i_1, i_2 \in \mathbb{N}$ with $i_1 + \frac{2-\gamma}{2}i_2 \leq M + \frac{2-\gamma}{2}N$. □

Lemma 5.7 Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$, and $u \in C^\infty(Q_1^+) \cap C(\overline{Q_1^+})$ be a solution of (5.4) with $f = 0$. If the coefficients P^{ij} , Q^i , and R are s -polynomials at O of degree μ corresponding to $(k + \alpha)$, $\|u\|_{L^\infty(Q_1^+)} \leq 1$, $\|u\|_{L^\infty(\overline{Q_1^+})} \leq 1$, and $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$, then for each $M, N \in \mathbb{N}$, there exists an s -polynomial p of degree m corresponding to $\kappa := N + \frac{2M}{2-\gamma}$ at O such that

$$|(p_t - L_p p)(X)| \leq Cx_n^{M + \frac{2-\gamma}{2}(N-2)} \quad \text{for all } X \in \overline{Q_{1/2}^+}$$

and

$$|u(X) - p(X)| \leq Cs[X, O]^{N + \frac{2M}{2-\gamma}} \quad \text{for all } X \in \overline{Q_1^+},$$

where C is a positive constant depending only on $n, \lambda, \Lambda, \gamma, M, N, \|P^{ij}\|_{C_s^{k,\alpha}(\overline{Q_1^+})}, \|Q^i\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$, and $\|R\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$.

Proof By Lemma 5.6, for any $M, N \in \mathbb{N}$, there exists a function v of the form (5.17) such that

$$|u(X) - v(X)| \leq Cx_n^{M + \frac{2-\gamma}{2}N}$$

for all $X \in \overline{Q_{1/2}^+}$ and $\beta \in \mathbb{N}_0^n$. Since each $\tilde{v}^{ij}(x', t)$ is a smooth function, by Taylor theorem, there exists Taylor polynomial

$$T^{ij}(x', t) = \sum_{|\beta|+2k \leq 2M+N-1} \frac{D_{x'}^\beta \partial_t^k \tilde{v}^{ij}(O)}{\beta!k!} x'^\beta t^k$$

of $\tilde{v}^{ij}(x', t)$ at O such that

$$|\tilde{v}^{ij}(x', t) - T^{ij}(x', t)| \leq C \sum_{|\beta|+2k=2M+N} |x_1|^{\beta_1} \dots |x_{n-1}|^{\beta_{n-1}} |t|^k$$

for all $|x_i| < 1/2$ ($i = 1, 2, \dots, n - 1$) and $-1/4 < t \leq 0$. This implies that

$$\begin{aligned} & \left| v(X) - \sum_{\substack{\frac{2i}{2-\gamma} + j < \frac{2M}{2-\gamma} + N \\ i, j \in \mathbb{N} \text{ or } (i, j) = (1, 0)}} T^{ij}(x', t)x_n^{i + \frac{2-\gamma}{2}j} \right| \\ & \leq C \sum_{\substack{\frac{2i}{2-\gamma} + j < \frac{2M}{2-\gamma} + N \\ i, j \in \mathbb{N} \text{ or } (i, j) = (1, 0)}} \sum_{|\beta|+2k=2M+N} |x_1|^{\beta_1} \dots |x_{n-1}|^{\beta_{n-1}} |t|^k x_n^{i + \frac{2-\gamma}{2}j} \\ & \leq Cs[X, O]^{2M+N + \frac{2}{2-\gamma}} \end{aligned} \tag{5.26}$$

for all $X \in \overline{Q_{1/2}^+}$. Now put

$$p(X) = \sum_{|\beta|+2k+\frac{2i}{2-\gamma}+j < N+\frac{2M}{2-\gamma}} \frac{D_{x'}^\beta \partial_t^k \tilde{v}^{ij}(O)}{\beta!k!} x'^\beta x_n^{i+\frac{2-\gamma}{2}j} t^k \tag{5.27}$$

which is an s -polynomial of degree m corresponding to $\kappa = N + \frac{2M}{2-\gamma}$. Then, from (5.22), we know that

$$|(p_t - L_p p)(X)| \leq Cx_n^{M+\frac{2-\gamma}{2}(N-2)} \text{ for all } X \in \overline{Q_{1/2}^+}.$$

Combining (5.23) and (5.26) gives

$$\begin{aligned} |u(X) - p(X)| &\leq |u(X) - v(X)| + \left| v(X) - \sum_{\substack{\frac{2i}{2-\gamma}+j < \frac{2M}{2-\gamma}+N \\ i,j \in \mathbb{N} \text{ or } (i,j)=(1,0)}} T^{ij}(x', t)x_n^{i+\frac{2-\gamma}{2}j} \right| \\ &+ \left| \sum_{\substack{\frac{2i}{2-\gamma}+j < \frac{2M}{2-\gamma}+N \\ i,j \in \mathbb{N} \text{ or } (i,j)=(1,0)}} T^{ij}(x', t)x_n^{i+\frac{2-\gamma}{2}j} - p(x, t) \right| \\ &\leq Cx_n^{M+\frac{2-\gamma}{2}N} + Cs[X, O]^{2M+N+\frac{2}{2-\gamma}} \\ &+ \sum_{|\beta|+2k+\frac{2i}{2-\gamma}+j \geq N+\frac{2M}{2-\gamma}} \frac{|D_{x'}^\beta \partial_t^k \tilde{v}^{ij}(O)|}{\beta!k!} |x_1|^{\beta_1} \dots |x_{n-1}|^{\beta_{n-1}} x_n^{i+\frac{2-\gamma}{2}j} |t|^k \\ &\leq Cs[X, O]^{N+\frac{2M}{2-\gamma}} \end{aligned}$$

for all $X \in \overline{Q_{1/2}^+}$ which is extensible throughout Q_1^+ . □

5.3 Generalized coefficient freezing method

In this section, we prove the main theorem. By combining $C_s^{2+\alpha}$ -regularity of solutions for (1.1) and boundary $C_s^{k,2+\alpha}$ -regularity of solutions for (5.4), boundary $C_s^{k,2+\alpha}$ -regularity of solutions for (1.1) can be obtained, and finally we have Theorem 2.2 by combining Lemma 2.10, Theorem 2.11 and Lemma 5.8.

Lemma 5.8 *Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$ with $k + 2 + \alpha \notin \mathcal{D}$, and assume*

$$a^{ij}, b^i, c, f \in C_s^{k,\alpha}(\overline{Q_1^+}) \quad (i, j = 1, 2, \dots, n).$$

Suppose $u \in C^2(Q_1^+) \cap C(\overline{Q_1^+})$ is a solution of (1.1) satisfying $u = 0$ on $\{X \in \partial_p Q_1^+ : x_n = 0\}$. Then there exists an s -polynomial p of degree m corresponding to

$\kappa := k + 2 + \alpha$ at O such that

$$|u(X) - p(X)| \leq C \left(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})} \right) s[X, O]^{k+2+\alpha} \text{ for all } X \in \overline{Q_1^+},$$

where C is a positive constant depending only on $n, \lambda, \Lambda, \gamma, k, \alpha, \|a^{ij}\|_{C_s^{k,\alpha}(\overline{Q_1^+})}, \|b^i\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$, and $\|c\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$.

Proof By considering $u/(\|u\|_{L^\infty(Q_1^+)} + \varepsilon^{-1}\|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})})$ for $\varepsilon > 0$, we may assume without loss of generality that $\|u\|_{L^\infty(Q_1^+)} \leq 1$ and $\|f\|_{C_s^{k,\alpha}(\overline{Q_1^+})} \leq \varepsilon$. By scaling and the $C_s^{k,\alpha}$ -Hölder continuity for a^{ij}, b^i, c , and f , we also assume that

$$|a^{ij}(X) - P^{ij}(X)| \leq \varepsilon s[X, O]^{k+\alpha}, \tag{5.28}$$

$$|b^i(X) - Q^i(X)| \leq \varepsilon s[X, O]^{k+\alpha}, \tag{5.29}$$

$$|c(X) - R(X)| \leq \varepsilon s[X, O]^{k+\alpha}, \tag{5.30}$$

$$|f(X) - F(X)| \leq \varepsilon s[X, O]^{k+\alpha} \tag{5.31}$$

for all $X \in \overline{Q_1^+}$, where P^{ij}, Q^i, R , and F are s -polynomials with degree \tilde{m} corresponding to $(k + \alpha)$ at O .

Let L_p be the operator given by

$$\begin{aligned} L_p &= P^{i'j'}(X)D_{i'j'} + 2x_n^{\gamma/2}P^{i'n}(X)D_{i'n} + x_n^\gamma P^{nn}(X)D_{nn} \\ &\quad + Q^{i'}(X)D_{i'} + x_n^{\gamma/2}Q^n(X)D_n + R(X). \end{aligned}$$

From (2.1) and (5.28), we know that

$$(\lambda - \varepsilon n^2)|\xi|^2 \leq P^{ij}(X)\xi_i\xi_j \leq (\Lambda + \varepsilon n^2)|\xi|^2 \text{ for all } X \in \overline{Q_1^+}, \xi \in \mathbb{R}^n$$

and hence for sufficiently small $\varepsilon > 0$, we have

$$\frac{\lambda}{2}|\xi|^2 \leq P^{ij}(X)\xi_i\xi_j \leq 2\Lambda|\xi|^2 \text{ for all } X \in \overline{Q_1^+}, \xi \in \mathbb{R}^n.$$

By Lemma 5.4, there exists a function $h \in C_s^{k+2,\alpha}(\overline{Q_1^+})$ of the form (5.5) such that

$$|(F + L_p h - h_t)(X)| \leq C\varepsilon s[X, O]^{k+\alpha} \text{ for all } X \in \overline{Q_1^+} \tag{5.32}$$

and $\|h\|_{C_s^{k+2,\alpha}(\overline{Q_1^+})} \leq C\varepsilon$, where C depends on $n, \gamma, k, \alpha, \|a^{ij}\|_{C_s^{k,\alpha}(\overline{Q_1^+})}, \|b^i\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$, and $\|c\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$.

By Theorem 5.1, we know that $u \in C_s^{2+\alpha}(\overline{Q_{1/2}^+})$. Now, we decompose $(u - h)$ into the sum of v^1 and w^1 such that

$$\begin{cases} v_t^1 = L_p v^1 & \text{in } Q_{1/2}^+ \\ v^1 = u - h & \text{on } \partial_p Q_{1/2}^+ \end{cases} \quad \text{and} \quad \begin{cases} w_t^1 = L_p w^1 + \tilde{f}^1 & \text{in } Q_{1/2}^+ \\ w^1 = 0 & \text{on } \partial_p Q_{1/2}^+, \end{cases}$$

where $\tilde{f}^1 = f + L_p h - h_t + Lu - L_p u$. Then, combining (5.28)-(5.32) leads us to the estimate

$$\begin{aligned} |\tilde{f}^1(X)| &\leq |f(X) - F(X)| + |(F + L_p h - h_t)(X)| + |(Lu - L_p u)(X)| \\ &\leq \varepsilon s[X, O]^{k+\alpha} + C\varepsilon s[X, O]^{k+\alpha} + C\varepsilon \|u\|_{C_s^{2,\alpha}(\overline{Q_{1/2}^+})} s[X, O]^{k+\alpha} \\ &\leq C\varepsilon s[X, O]^{k+\alpha} \end{aligned} \tag{5.33}$$

for all $X \in \overline{Q_{1/2}^+}$. By Lemma 2.5, for fixed $r \in (0, 1/2)$, (5.33) gives

$$|w^1(X)| \leq C \|\tilde{f}^1\|_{L^\infty(Q_{1/2}^+)} \leq C\varepsilon \quad \text{for all } X \in \overline{Q_r^+}$$

and hence, we have

$$|u(X) - v^1(X) - h(X)| = |w^1(X)| \leq C\varepsilon \quad \text{for all } X \in \overline{Q_r^+}.$$

Choose now ε small enough such that $C\varepsilon < r^{k+2+\alpha}$. Then,

$$|u(X) - v^1(X) - h(X)| \leq r^{k+2+\alpha} \quad \text{for all } X \in \overline{Q_r^+}.$$

Let us now define

$$u^2(X) := \frac{(u - v^1 - h)(rX)}{r^{k+2+\alpha}} \quad \text{and} \quad \tilde{f}^2(X) := \frac{\tilde{f}^1(rX)}{r^{k+\alpha}}.$$

Then, $\|u^2\|_{L^\infty(Q_1^+)} \leq 1$ and

$$u_t^2 = L_p^2 u^2 + \tilde{f}^2 \quad \text{in } Q_1^+,$$

where

$$\begin{aligned} L_p^2 &= P_{(2)}^{i'j'}(X)D_{i'j'} + 2x_n^{\gamma/2} P_{(2)}^{i'n}(X)D_{i'n} + x_n^\gamma P_{(2)}^{nn}(X)D_{nn} \\ &\quad + Q_{(2)}^{i'}(X)D_{i'} + x_n^{\gamma/2} Q_{(2)}^n(X)D_n + R_{(2)}(X) \end{aligned}$$

with

$$P_{(2)}^{ij}(X) = P^{ij}(rX), \quad Q_{(2)}^i(X) = rQ^i(rX), \quad \text{and} \quad R_{(2)}(X) = r^2R(rX).$$

Furthermore, (5.33) leads us to the estimate

$$|\tilde{f}^2(X)| = r^{-(k+\alpha)} |\tilde{f}^1(rX)| \leq C\varepsilon s[X, O]^{k+\alpha} \quad \text{for all } X \in \overline{Q_{1/2}^+}.$$

That is, the same hypotheses as before are fulfilled. Repeating the same procedure, we decompose u^2 into the sum of v^2 and w^2 such that

$$\begin{cases} v_i^2 = L_p^2 v^2 & \text{in } Q_{1/2}^+ \\ v^2 = u^2 & \text{on } \partial_p Q_{1/2}^+, \end{cases} \quad \begin{cases} w_i^2 = L_p^2 w^2 + \tilde{f}^2 & \text{in } Q_{1/2}^+ \\ w^2 = 0 & \text{on } \partial_p Q_{1/2}^+, \end{cases}$$

and

$$|u^2(X) - v^2(X)| \leq r^{k+2+\alpha} \quad \text{for all } X \in \overline{Q_r^+}.$$

By substituting back, we have

$$|u(X) - h(X) - v^1(X) - r^{k+2+\alpha} v^2(r^{-1}X)| \leq r^{2(k+2+\alpha)} \quad \text{for all } X \in \overline{Q_{r^2}^+}.$$

Continuing iteratively, for each integer $l \geq 3$, let us define the sequence of functions $\{u^l\}$ inductively as follows:

$$u^l(X) := \frac{(u^{l-1} - v^{l-1})(rX)}{r^{k+2+\alpha}} \quad \text{and} \quad \tilde{f}^l(X) := \frac{\tilde{f}^{l-1}(rX)}{r^{k+\alpha}}.$$

Then, $\|u^l\|_{L^\infty(Q_1^+)} \leq 1$ and

$$u_i^l = L_p^l u^l + \tilde{f}^l \quad \text{in } Q_1^+,$$

where

$$\begin{aligned} L_p^l &= P_{(l)}^{i'j'}(X)D_{i'j'} + 2x_n^{\gamma/2} P_{(l)}^{i'n}(X)D_{i'n} + x_n^\gamma P_{(l)}^{nn}(X)D_{nn} \\ &\quad + Q_{(l)}^{i'}(X)D_{i'} + x_n^{\gamma/2} Q_{(l)}^n(X)D_n + R_{(l)}(X) \end{aligned}$$

with

$$P_{(l)}^{ij}(X) = P_{(l-1)}^{ij}(rX), \quad Q_{(l)}^i(X) = rQ_{(l-1)}^i(rX), \quad \text{and} \quad R_{(l)}(X) = r^2R_{(l-1)}(rX).$$

Furthermore,

$$|\tilde{f}^l(X)| \leq C\varepsilon s[X, O]^{k+\alpha} \quad \text{for all } X \in \overline{Q_{1/2}^+}$$

and hence we decompose u^l into the sum of v^l and w^l such that

$$\begin{cases} v_t^l = L_p^l v^l & \text{in } Q_{1/2}^+ \\ v^l = u^l & \text{on } \partial_p Q_{1/2}^+, \end{cases} \quad \begin{cases} w_t^l = L_p^l w^l + \tilde{f}^l & \text{in } Q_{1/2}^+ \\ w^l = 0 & \text{on } \partial_p Q_{1/2}^+, \end{cases}$$

and

$$|u^l(X) - v^l(X)| \leq r^{k+2+\alpha} \quad \text{for all } X \in \overline{Q_r^+}.$$

By substituting back, we have

$$\left| u(X) - h(X) - \sum_{i=1}^l r^{(i-1)(k+2+\alpha)} v^i(r^{-i+1}X) \right| \leq r^{l(k+2+\alpha)} \quad \text{for all } X \in \overline{Q_r^+}. \tag{5.34}$$

We choose $M, N \in \mathbb{N}$ such that

$$k + 2 + \alpha < N + \frac{2M}{2 - \gamma} =: \min \left\{ i + \frac{2j}{2 - \gamma} \in (k + 2 + \alpha, \infty) : i, j \in \mathbb{N} \right\}.$$

By Lemma 5.7 and (5.27), for each $l \in \mathbb{N}$, there exists an s -polynomial $p^l(X)$ of the form

$$p^l(X) = \sum_{|\beta| + \frac{2i_1}{2-\gamma} + i_2 + 2j < N + \frac{2M}{2-\gamma}} A_l^{\beta i j k} x_n^{\beta} x_n^{i_1 + \frac{2-\gamma}{2} i_2} t^j$$

such that

$$|v^l(X) - p^l(X)| \leq Cs[X, O]^{N + \frac{2M}{2-\gamma}} \quad \text{for all } X \in \overline{Q_1^+},$$

where C is a positive constant depending only on $n, \lambda, \Lambda, \gamma, M, N, \|a^{ij}\|_{C_s^{k,\alpha}(\overline{Q_1^+})}, \|b^i\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$, and $\|c\|_{C_s^{k,\alpha}(\overline{Q_1^+})}$. As in the proof of Lemma 4.9, we can see

$$\left| u(X) - h(X) - \sum_{i=1}^l r^{(i-1)(k+2+\alpha)} p^i(r^{-i+1}X) \right| \leq Cr^{l(k+2+\alpha)}$$

for all $X \in \overline{Q_r^+}$. Now put

$$P^l(X) = \sum_{i=1}^l r^{(i-1)(k+2+\alpha)} p^i(r^{-i+1}X).$$

Then, (5.34) gives

$$|P^l(X) - P^{l-1}(X)| \leq Cr^{l(k+2+\alpha)} \quad \text{for all } X \in \overline{Q_r^+}.$$

Since $k + 2 + \alpha \notin \mathcal{D}$, each $p^l(X)$ has degree m corresponding to $\kappa = k + 2 + \alpha$, so we can consider the rescaling s -polynomial

$$\begin{aligned} \tilde{P}(X) &:= P^l(r^l X) - P^{l-1}(r^l X) \\ &= r^{(l-1)(k+2+\alpha)} \sum_{|\beta| + \frac{2i_1}{2-\gamma} + i_2 + 2j < k+2+\alpha} r^{|\beta| + \frac{2i_1}{2-\gamma} + i_2 + 2j} A_l^{\beta i_1 i_2 j} x^{\beta} x_n^{i_1 + \frac{2-\gamma}{2} i_2} t^j. \end{aligned}$$

Finally, as in the proof of Lemma 5.7, if h is approximated with an s -polynomial, we can show that there exists an s -polynomial p of degree m corresponding to $\kappa = k + 2 + \alpha$ at O such that

$$|u(X) - p(X)| \leq Cs[X, O]^{k+2+\alpha} \quad \text{for all } X \in \overline{Q_1^+}. \quad \square$$

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

References

- Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. *Commun. Partial Differ. Equ.* **32**(7–9), 1245–1260 (2007)
- Cox, J.: Notes on option pricing i: Constant elasticity of variance diffusions. Unpublished note, Stanford University, Graduate School of Business (1975)
- Cox, J.C., Ross, S.A.: The valuation of options for alternative stochastic processes. *J. Financ. Econ.* **3**(1–2), 145–166 (1976)
- Daskalopoulos, P., Hamilton, R.: The free boundary in the Gauss curvature flow with flat sides. *J. Reine Angew. Math.* **510**, 187–227 (1999)
- Daskalopoulos, P., Savin, O.: On Monge–Ampère equations with homogeneous right-hand sides. *Commun. Pure Appl. Math.* **62**(5), 639–676 (2009)
- Kim, T., Lee, K.-A., Yun, H.: Higher regularity up to boundary for degenerate parabolic equations. *J. Differ. Equ.* **348**, 223–277 (2023)
- Le, N.Q., Savin, O.: Schauder estimates for degenerate Monge–Ampère equations and smoothness of the eigenfunctions. *Invent. Math.* **207**(1), 389–423 (2017)
- Lieberman, G.M.: *Second Order Parabolic Differential Equations*. World Scientific Publishing Co., Inc, River Edge (1996)
- McShane, E.J.: Extension of range of functions. *Bull. Am. Math. Soc.* **40**(12), 837–842 (1934)
- Wang, L.: On the regularity theory of fully nonlinear parabolic equations. II. *Commun. Pure Appl. Math.* **45**(2), 141–178 (1992)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.