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L^p bounds for Stein's spherical maximal operators

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Abstract

Let $\mathfrak{M}^\alpha$ be the spherical maximal operators of complex order α on $\mathbb{R}^n$. In this article we show that when $n \geq 2$, suppose

$$\|\mathfrak{M}^\alpha f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

holds for some α and $p \geq 2$, then we must have that $\mathrm{Re}\,\alpha \geq \max\{1/p - (n-1)/2,\ -(n-1)/p\}$. In particular, when $n = 2$, we prove that $\|\mathfrak{M}^\alpha f\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$ if $\mathrm{Re}\,\alpha > \max\{1/p - 1/2,\ -1/p\}$, and consequently the range of α is sharp in the sense that the estimate fails for $\mathrm{Re}\,\alpha < \max\{1/p - 1/2, -1/p\}$.

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1 Introduction

In 1976 Stein [19] introduced the spherical maximal means $\mathfrak{M}^\alpha f(x) = \sup_{t>0} |\mathfrak{M}_t^\alpha f(x)|$ of (complex) order α, where

$$\mathfrak{M}_t^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{|y|\leq 1} \left(1 - |y|^2\right)^{\alpha-1} f(x - ty)\,\mathrm{d}y. \tag{1.1}$$

These means are defined a priori only for $\mathrm{Re}\,\alpha > 0$, but the definition can be extended to all complex α by analytic continuation. In the case $\alpha = 1$, $\mathfrak{M}^\alpha$ corresponds to the

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Hardy–Littlewood maximal operator and in the case $\alpha = 0$, one recovers the spherical maximal means $\mathfrak{M} f(x) = \sup_{t>0} |\mathfrak{M}_t f(x)|$ in which

$$\mathfrak{M}_t f(x) = c_n \int_{\mathbb{S}^{n-1}} f(x - ty)\, \mathrm{d}\sigma(y), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{1.2}$$

where c_n is a constant depending only on n, $\mathbb{S}^{n-1}$ denotes the standard unit sphere in $\mathbb{R}^n$ and $\mathrm{d}\sigma$ is the induced Lebesgue measure on the unit sphere $\mathbb{S}^{n-1}$. In [19, Theorem 2], Stein showed that

$$\|\mathfrak{M}^\alpha f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \tag{1.3}$$

in the following circumstances:

$$\mathrm{Re}\, \alpha > 1 - n + \frac{n}{p} \quad \text{when } 1 < p \leq 2; \tag{1.4}$$

or

$$\mathrm{Re}\, \alpha > \frac{2-n}{p} \qquad \text{when } 2 \leq p \leq \infty. \tag{1.5}$$

The above maximal theorem tells us that when $\alpha = 0$ and $n \geq 3$, the maximal operator $\mathfrak{M}$ is bounded on $L^p(\mathbb{R}^n)$ for the range of $p > n/(n-1)$. This range of p is sharp, as has been pointed out in [19, 21], no such result can hold for $p \leq n/(n-1)$ if $n \geq 2$.

Some 10 years passed before Bourgain [1] finally proved that the maximal operator $\mathfrak{M}$ is bounded on $L^p(\mathbb{R}^2)$ for $p > 2$. Bourgain's theorem says that there exists $\epsilon(p) > 0$ such that

$$\|\mathfrak{M}^\alpha f\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}, \quad \mathrm{Re}\, \alpha > -\epsilon(p), \quad 2 < p < \infty. \tag{1.6}$$

This result cannot hold even for $\alpha = 0$ when $p = 2$, see [19]. An alternative proof of Bourgain's result was subsequently found by Mockenhaupt, Seeger and Sogge [11], who used a local smoothing estimate for the solutions of the wave operator. In 2017, Miao, Yang and Zheng [10] improved certain range of α for L^p-bounds for the operator $\mathfrak{M}^\alpha$ by using the Bourgain–Demeter decoupling theorem [2]. All these refinements can be stated altogether as follows: For $n \geq 2$ and $p \geq 2$, (1.3) holds whenever

$$\mathrm{Re}\, \alpha > \max \left\{ \frac{1-n}{4} + \frac{3-n}{2p}, \frac{1-n}{p} \right\}. \tag{1.7}$$

The above range α in (1.7) for $p > 2$ is strictly wider than the range of α in (1.5). However, the range α in (1.7) is not optimal.

As mentioned above, the proof of the range of α in (1.7) relies on the progress concerning Sogge's local smoothing conjecture, as originally formulated by Sogge [18]: For $n \geq 2$ and $p \geq 2n/(n-1)$, one has

$$\|u\|_{L^p(\mathbb{R}^n \times [1,2])} \leq C \left(\|f\|_{W^{\gamma,p}(\mathbb{R}^n)} + \|g\|_{W^{\gamma-1,p}(\mathbb{R}^n)} \right), \qquad \text{if } \gamma > \frac{n-1}{2} - \frac{n}{p}, \tag{1.8}$$

where

$$u(x,t)=\cos(t\sqrt{-\Delta})f(x)+\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g(x)$$

is the solution to the Cauchy problem for the wave equation in $\mathbb{R}^n\times\mathbb{R}$:

$$\begin{cases}\big((\partial/\partial t)^2-\Delta\big)\,u(x,t)=0,\\ u|_{t=0}=f,\\ (\partial/\partial t)u|_{t=0}=g.\end{cases}\tag{1.9}$$

The local smoothing conjecture has been studied in numerous papers, see for instance [2, 4, 7, 8, 10, 11, 17, 24] and the references therein. When $n=2$, sharp results follow by the work of Guth, Wang and Zhang [7]. When $n\geq 3$, the conjecture holds for all $p\geq 2(n+1)/(n-1)$ by the Bourgain–Demeter decoupling theorem [2] and the method of [24].

The aim of this article is to prove the following result.

Theorem 1.1 *Let* $p\geq 2$.

(i) *Let* $n\geq 2$. *Suppose* (1.3) *holds for some* $\alpha\in\mathbb{C}$. *Then we must have*

$$\operatorname{Re}\alpha\geq\max\left\{\frac{1}{p}-\frac{n-1}{2},\ -\frac{n-1}{p}\right\}.$$

(ii) *Let* $n=2$. *Then the estimate* (1.3) *holds if*

$$\operatorname{Re}\alpha>\max\left\{\frac{1}{p}-\frac{1}{2},\ -\frac{1}{p}\right\},$$

and consequently the range of α *is sharp in the sense that the estimate fails for* $\operatorname{Re}\alpha<\max\{1/p-1/2,-1/p\}$.

Let $p\geq 2$ and $\alpha=(3-n)/2$. For an appropriate constant c_n, we have that $c_n t(\mathfrak{M}_t^{\alpha}g)(x)=u(x,t)$, where u is the solution to the wave equation (1.9) with $f=0$, see [20, 4.10, p.519]. As a consequence of (i) of Theorem 1.1, we have the following corollary.

Corollary 1.2 *Let* $n\geq 4$. *Then*

$$\left\|\sup_{t>0}\left|\frac{u(x,t)}{t}\right|\right\|_{L^p(\mathbb{R}^n)}\leq C_p\|g\|_{L^p(\mathbb{R}^n)}$$

can not hold whenever $p>2(n-1)/(n-3)$.

We would like to mention that for the range α in (1.5), it is commented in [20, 4.10, p.519] that the optimal results for $p>2$ and $n\geq 2$ "are still a mystery". Our

Theorem 1.1 gives an affirmative answer in dimension $n = 2$ to show sharpness of $\mathrm{Re}\,\alpha > \max\{1/p - 1/2,\ -1/p\}$ in the estimate (1.3) except the borderline.

The proof of (ii) of Theorem 1.1 can be shown by applying the work of Guth-Wang-Zhang [7] on local smoothing estimates along with the techniques previously used in [11] and [10]. The main contribution of this article is to show (i) of Theorem 1.1. From the asymptotic expansion of Fourier multiplier of the operator $\mathfrak{M}_t^\alpha$, it is seen that $\mathfrak{M}_t^\alpha$ are essentially the sum of half-wave operators $e^{it\sqrt{-\Delta}}$ and $e^{-it\sqrt{-\Delta}}$, and hence the complexity of the operator $\mathfrak{M}_t^\alpha$ comes from the interference between the operators $e^{it\sqrt{-\Delta}}$ and $e^{-it\sqrt{-\Delta}}$. To show the necessity of L^p-boundedness of $\mathfrak{M}_t^\alpha$, we make the following observations. For the case $p > 2n/(n-1)$ we note that by the stationary phase argument, two waves $e^{it\sqrt{-\Delta}}f$ and $e^{-it\sqrt{-\Delta}}f$ concentrate on the opposite parts of sphere $\{x \in \mathbb{R}^n : |x| = t\}$, respectively, when $\hat{f}$ is supported on a small cone. For the case $2 \le p \le 2n/(n-1)$, we let f be a wave packet of direction $\nu \in S^{n-1}$, then one can regard $e^{\pm it\sqrt{-\Delta}}f(x)$ as the translations $f(x \pm t\nu)$ of $f(x)$, which concentrate on the opposite parts of sphere $\{x \in \mathbb{R}^n :\ |x| = t\}$. In Sect. 3, we construct two examples such that there is no interference between $e^{it\sqrt{-\Delta}}f$ and $e^{-it\sqrt{-\Delta}}f$ to obtain the desired range of α in (i) of Theorem 1.1.

The paper is organized as follows. In Sect. 2, we give some preliminary results including the properties of the Fourier multiplier associated to the spherical operators $\mathfrak{M}_t^\alpha$ by using asymptotic expansions of Bessel functions. The proof of (i) of Theorem 1.1 will be given in Sect. 3 by constructing two examples to obtain the necessarity of L^p-bounds for the maximal operator $\mathfrak{M}^\alpha$. In Sect. 4 we will give the proof of (ii) of Theorem 1.1.

2 Preliminary results

We begin with recalling the spherical function $\mathfrak{M}_t^\alpha f(x) = f * m_{\alpha,t}(x)$ where $m_{\alpha,t}(x) = t^{-n}m_\alpha(t^{-1}x)$ and

$$m_\alpha(x) = \Gamma(\alpha)^{-1}\big(1 - |x|^2\big)_+^{\alpha-1},$$

where $\Gamma(\alpha)$ is the Gamma function and $(r)_+ = \max\{0, r\}$ for $r \in \mathbb{R}$. Define the Fourier transform of f by $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\,\mathrm{d}x$. It follows by [22, p.171] that the Fourier transform of m_α is given by

$$\widehat{m_\alpha}(\xi) = \pi^{-\alpha+1}|\xi|^{-n/2-\alpha+1}J_{n/2+\alpha-1}\big(2\pi|\xi|\big). \tag{2.1}$$

Here J_β denotes the Bessel function of order β. For any complex number β, we can obtain the complete asymptotic expansion

$$J_\beta(r) \sim r^{-1/2}e^{ir}\sum_{j=0}^{\infty} b_j r^{-j} + r^{-1/2}e^{-ir}\sum_{j=0}^{\infty} d_j r^{-j}, \qquad r \ge 1 \tag{2.2}$$

for suitable coefficients b_j and d_j with $b_0, d_0 \neq 0$. Note that when β is a positive integer, (2.2) is given in [20, (15), p.338]. For general β, we refer it to [23, (1). 7.21, p.199].

Then there exists an error terms $E_{N,1}(r)$, $E_{N,2}(r)$ and $E(r)$ such that for any given $N \geq 1$ and $r \geq 1$,

$$J_\beta(r) = r^{-1/2}e^{ir}\left(\sum_{j=0}^{N-1} b_j r^{-j} + E_{N,1}(r)\right) + r^{-1/2}e^{-ir}\left(\sum_{j=0}^{N-1} d_j r^{-j} + E_{N,2}(r)\right) + E(r), \quad (2.3)$$

where

$$\left|\left(\frac{d}{dr}\right)^k E_{N,1}(r)\right| + \left|\left(\frac{d}{dr}\right)^k E_{N,2}(r)\right| + \left|\left(\frac{d}{dr}\right)^k E(r)\right| \leq C_k r^{-N-k} \quad (2.4)$$

for all $k \in \mathbb{Z}_+$. We rewrite (2.1) as

$$\begin{aligned}
\widehat{m_\alpha}(\xi) &= \varphi(|\xi|)\widehat{m_\alpha}(\xi) + (1-\varphi(|\xi|))\widehat{m_\alpha}(\xi) \\
&= [\varphi(|\xi|)\widehat{m_\alpha}(\xi) + \mathcal{E}(|\xi|)] \\
&\quad + \left[e^{2\pi i|\xi|}\mathcal{E}_{N,1}(|\xi|) + e^{-2\pi i|\xi|}\mathcal{E}_{N,2}(|\xi|)\right] \\
&\quad + |\xi|^{-(n-1)/2-\alpha}\left[e^{2\pi i|\xi|}a_1(|\xi|) + e^{-2\pi i|\xi|}a_2(|\xi|)\right], \quad (2.5)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{E}(r) &= (2\pi)^{1/2}c(\pi,\alpha)(1-\varphi(r))r^{-(n-2)/2-\alpha}E(2\pi r), \\
\mathcal{E}_{N,\ell}(r) &= c(\pi,\alpha)E_{N,\ell}(2\pi r)(1-\varphi(r))r^{-(n-1)/2-\alpha}, \quad \ell = 1, 2, \\
a_1(r) &= c(\pi,\alpha)\sum_{j=0}^{N-1} b_j(2\pi r)^{-j}(1-\varphi(r)), \\
a_2(r) &= c(\pi,\alpha)\sum_{j=0}^{N-1} d_j(2\pi r)^{-j}(1-\varphi(r)) \quad (2.6)
\end{aligned}$$

with $c(\pi,\alpha) = 2^{-1/2}\pi^{-\alpha+1/2}$. Here $\varphi \in C_0^\infty(\mathbb{R})$ is an even function, identically equals 1 on $B(0, M)$ and supported on $B(0, 2M)$, where $M = M(N)$ is large enough such that $|a_2(r)| \geq c_{low} > 0$ for $|r| \geq M$. Then we can split the Fourier multiplier of the operator $\mathfrak{M}_1^\alpha$ into three parts as in (2.5) above. Firstly, we note that $\varphi(|\xi|)\widehat{m_\alpha}(\xi)$ is smooth and compactly supported and $\mathcal{E}(|\xi|) \in \mathscr{S}(\mathbb{R}^n)$. It is seen that $\sup_{t>0}|\widehat{m_\alpha}(tD)\varphi(t|D|)f|$ and $\sup_{t>0}|\mathcal{E}(t|D|)f|$ are bounded by the Hardy–Littlewood maximal function. Then for $p > 1$,

$$\left\|\sup_{t>0}|\widehat{m_\alpha}(tD)\varphi(t|D|))f|\right\|_{L^p(\mathbb{R}^n)}+\left\|\sup_{t>0}|\mathcal{E}(t|D|)f|\right\|_{L^p(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.7}$$

Secondly, we define

$$\mathscr{E}_N f(x,t)=\int_{\mathbb{R}^n}e^{2\pi i(x\cdot\xi+t|\xi|)}\mathcal{E}_{N,1}\big(t|\xi|\big)\hat{f}(\xi)\,\mathrm{d}\xi+\int_{\mathbb{R}^n}e^{2\pi i(x\cdot\xi-t|\xi|)}\mathcal{E}_{N,2}\big(t|\xi|\big)\hat{f}(\xi)\,\mathrm{d}\xi.$$

Then we have the following lemma.

Lemma 2.1 *Let $p\ge 2$. There exists a constant $C>0$ such that*

$$\left\|\sup_{t\in[1,2]}|\mathscr{E}_N f(\cdot,t)|\right\|_{L^p(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)}, \tag{2.8}$$

when

$$N>-\frac{n-2}{p}-\operatorname{Re}\alpha.$$

The proof of Lemma 2.1 is based on the following elementary result (see [17, Lemma 2.4.2]).

Lemma 2.2 *Let F be a smooth function defined on $\mathbb{R}^n\times[1,2]$. Then for $p>1$ and $1/p+1/p'=1$,*

$$\left\|\sup_{1\le t\le 2}|F(\cdot,t)|\right\|_{L^p(\mathbb{R}^n)}\le C_p\left(\|F(\cdot,1)\|_{L^p(\mathbb{R}^n)}+\|F\|^{1-1/p}_{L^p(\mathbb{R}^n\times[1,2])}\|\partial_t F\|^{1/p}_{L^p(\mathbb{R}^n\times[1,2])}\right).$$

Proof of Lemma 2.1 We fix a function φ as in (2.5). Let $\psi(r):=\varphi(r)-\varphi(2r)$ and $\psi_j(r):=\psi(2^{-j}r)$, for $j\ge 1$. So we have

$$1\equiv\varphi(r)+\sum_{j\ge 1}\psi_j(r),\quad r\ge 0. \tag{2.9}$$

For $j\ge 1$, define

$$\mathscr{E}_{N,j}f(x,t)=\int_{\mathbb{R}^n}\left(e^{2\pi i(x\cdot\xi+t|\xi|)}\mathcal{E}_{N,1}\big(t|\xi|\big)+e^{2\pi i(x\cdot\xi-t|\xi|)}\mathcal{E}_{N,2}\big(t|\xi|\big)\right)\psi_j(t|\xi|)\hat{f}(\xi)d\xi.$$

To prove (2.8), it suffices to show that there exists a constant $\delta > 0$ such that for all $j \ge 1$,

$$\left\| \sup_{1\le t\le 2} |\mathscr{E}_{N,j} f(\cdot, t)| \right\|_{L^p(\mathbb{R}^n)} \le C2^{-\delta j} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.10}$$

Let us prove (2.10) by using Lemma 2.2. First, for each fixed $t \in [1, 2]$, $\mathscr{E}_{N,j} f$ are the sum of two Fourier integral operators of order $-(n-1)/2 - \mathrm{Re}\,\alpha - N$ with phase $x \cdot \xi \pm t|\xi|$. By [20, Theorem 2, Chapter IX] and the fact that $e^{it\sqrt{-\Delta}}$ is local at scale t, we have

$$\sup_{1\le t\le 2} \left\| \mathscr{E}_{N,j} f(\cdot, t) \right\|_{L^p(\mathbb{R}^n)} \le C2^{-((n-1)/2+\mathrm{Re}\,\alpha+N)j} 2^{(n-1)(1/2-1/p)j} \|f\|_{L^p(\mathbb{R}^n)}, \tag{2.11}$$

see also [16, Corollary 2.4]. Next, we write $\partial_t \mathscr{E}_{N,j} f(x, t)$ as the sum of following terms,

$$\begin{aligned}
&\pm 2\pi i t^{-1} \int e^{2\pi i(x\cdot\xi \pm t|\xi|)} t|\xi| \mathcal{E}_{N,1}\big(t|\xi|\big) \psi_j(t|\xi|) \hat{f}(\xi) d\xi;\\
&\pm 2\pi i t^{-1} \int e^{2\pi i(x\cdot\xi \pm t|\xi|)} t|\xi| \mathcal{E}_{N,2}\big(t|\xi|\big) \psi_j(t|\xi|) \hat{f}(\xi) d\xi;\\
&t^{-1} \int e^{2\pi i(x\cdot\xi \pm t|\xi|)} t|\xi| (\mathcal{E}_{N,1}\psi_j)'\big(t|\xi|\big) \hat{f}(\xi) d\xi;\\
&t^{-1} \int e^{2\pi i(x\cdot\xi \pm t|\xi|)} t|\xi| (\mathcal{E}_{N,2}\psi_j)'\big(t|\xi|\big) \hat{f}(\xi) d\xi.
\end{aligned}$$

By (2.4), we see that for each fixed $t \in [1, 2]$, they are Fourier integral operators of order no more than $-(n-1)/2 - \mathrm{Re}\,\alpha - N + 1$. By [20, Theorem 2, Chapter IX] again,

$$\sup_{1\le t\le 2} \left\| \partial_t \mathscr{E}_{N,j} f(\cdot, t) \right\|_{L^p(\mathbb{R}^n)} \le C2^{-((n-1)/2+\mathrm{Re}\,\alpha+N-1)j} 2^{(n-1)(1/2-1/p)j} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.12}$$

Lemma 2.2, together with (2.11) and (2.12), gives

$$\left\| \sup_{1\le t\le 2} |\mathscr{E}_{N,j} f(\cdot, t)| \right\|_{L^p(\mathbb{R}^n)} \le C2^{-((n-1)/2+\mathrm{Re}\,\alpha+N-1/p)j} 2^{(n-1)(1/2-1/p)j} \|f\|_{L^p(\mathbb{R}^n)}.$$

Choosing $N > -(n-2)/p - \mathrm{Re}\,\alpha$ and letting $\delta = N + (n-2)/p + \mathrm{Re}\,\alpha$, we obtain estimate (2.10). The proof of Lemma 2.1 is complete. □

Finally, we define

$$\mathscr{A}_t f(x) = \int_{\mathbb{R}^n} \left(e^{2\pi i(x\cdot\xi + t|\xi|)} a_1(t|\xi|) + e^{2\pi i(x\cdot\xi - t|\xi|)} a_2(t|\xi|) \right) \hat{f}(\xi)\, \mathrm{d}\xi. \tag{2.13}$$

From (2.5), (2.7) and Lemma 2.2, we see that the L^p-boundness of the operator $\mathfrak{M}_t^\alpha$ reduces to boundedness of the operator $\mathscr{A}_t$ on Sobolev spaces, which will be investigated in Sect. 3 below.

3 Proof of (i) of Theorem 1.1

To prove (i) of Theorem 1.1, we need to show the following proposition.

Proposition 3.1 *Let $n \geq 2$ and $p \geq 2$. Suppose*

$$\left\|\mathfrak{M}_1^\alpha f\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{3.1}$$

holds for some $\alpha \in \mathbb{C}$. Then, we have

$$\operatorname{Re}\alpha \geq -\frac{n-1}{p}.$$

Let us prove Proposition 3.1. Fix $N > -(n-2)/p - \operatorname{Re}\alpha$ as in Lemma 2.1. By (2.5), (2.7) and Lemma 2.1, we see that the proof of Proposition 3.1 reduces to the following lemma.

Lemma 3.2 *Let $n \geq 2$ and $1 < p < \infty$. Let $\mathscr{A}_1$ be an operator given in (2.13). Suppose*

$$\|\mathscr{A}_1 f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{W^{s,p}(\mathbb{R}^n)} \tag{3.2}$$

holds for some $s \in \mathbb{R}$. Then, we have

$$s \geq (n-1)\left|\frac{1}{2}-\frac{1}{p}\right|.$$

Proof Let $\widehat{\gamma_\beta}(\xi) := (1+|\xi|^2)^{-\beta/2}$ with $\beta > (n-1)/2$. Recall that φ is a function in (2.5). Let w belong to S^0 (a symbol of order zero) satisfying $|w(r)| \geq c > 0$ on $\mathbb{R}$ for some constant c. Moreover, w equals $\left(\sum_{j\geq 0}^{N-1} d_j r^{-j}\right)^{-1}$ on supp $(1-\varphi)$, and equals constant near zero. Assume that $\chi(\xi) \in C^\infty(\mathbb{R}^n\backslash\{0\})$ is homogeneous of order 0 and vanishes if $|\frac{\xi}{|\xi|} - v_1| \geq 10^{-2}$, where $v_1 := (1, 0, \ldots, 0)$. Define

$$\hat{f}_{\beta,R}(\xi) = w(|\xi|)\varphi_R(|\xi|)\chi(\xi)\widehat{\gamma_\beta}(\xi),$$

where $\varphi_R(\cdot) := \varphi(\cdot/R)$, and R is a large positive number. Since $w(|\xi|) \in S^0$ and χ is a Hörmander multiplier, $w(|D|)$ and $\chi(D)$ are bounded on $L^p(\mathbb{R}^n)$. And $\varphi_R(|D|)$ is bounded on $L^p(\mathbb{R}^n)$ uniformly in R. So we have

$$\|f_{\beta,R}\|_{W^{s,p}(\mathbb{R}^n)} = \|w(|D|)\varphi_R(|D|)\chi(D)\gamma_{\beta-s}\|_{L^p(\mathbb{R}^n)} \leq C\|\gamma_{\beta-s}\|_{L^p(\mathbb{R}^n)}, \tag{3.3}$$

where $C > 0$ is a constant independent of R. On the other hand, it follows by [6, Proposition 1.2.5] that

$$|\gamma_{\beta-s}(x)| \le \begin{cases} C|x|^{-n+\beta-s} & \text{if } |x| \le 2, \\ Ce^{-|x|/2} & \text{if } |x| \ge 2 \end{cases}$$

when $0 < \beta - s < n$. From this, we see that $\|f_{\beta,R}\|_{W^{s,p}(\mathbb{R}^n)} < \infty$ whenever $0 < \beta - s < n$ and $(-n+\beta-s)p > -n$.

Now we turn to estimate $\|\mathscr{A}_1 f_{\beta,R}\|_{L^p(\mathbb{R}^n)}$. By using polar coordinate,

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi+|\xi|)} a_1(\xi)\hat{f}_{\beta,R}(\xi)\,\mathrm{d}\xi \\ &\quad= \int_0^\infty \int_{S^{n-1}} e^{2\pi i(x\cdot r\theta+r)} a_1(r)w(r)\chi(\theta)(1+r^2)^{-\beta/2} r^{n-1}\varphi_R(r)\,\mathrm{d}\sigma(\theta)\mathrm{d}r \\ &\quad= \int_0^\infty e^{2\pi i r}\widehat{\chi\,\mathrm{d}\sigma}(-rx)a_1(r)w(r)(1+r^2)^{-\beta/2}r^{n-1}\varphi_R(r)\,\mathrm{d}r. \end{aligned} \tag{3.4}$$

Note that $\chi(\xi)$ vanishes if $|\frac{\xi}{|\xi|} - v_1| \ge 10^{-2}$. By the expansion in [20, p. 360], we can write that for $|x| \ge 1$ and $|\frac{x}{|x|} - v_1| \le 10^{-2}$,

$$\widehat{\chi\,\mathrm{d}\sigma}(-x) = e^{2\pi i|x|}h(-x) + e(-x), \tag{3.5}$$

where e belongs to $S^{-\infty}$ and $h \in S^{-(n-1)/2}$ can be splitted into two terms:

$$h(x) = c_0|x|^{-(n-1)/2}\chi(-x/|x|) + \tilde{e}(x), \quad \tilde{e} \in S^{-(n+1)/2} \tag{3.6}$$

for all $|x| \ge 1$. Hence, if $|\frac{x}{|x|} - v_1| \le 10^{-2}$, we then have

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi+|\xi|)} a_1(\xi)\hat{f}_{\beta,R}(\xi)\,\mathrm{d}\xi \\ &\quad= \int_0^\infty e^{2\pi i r(|x|+1)}h(-rx)a_1(r)w(r)(1+r^2)^{-\beta/2}r^{n-1}\varphi_R(r)\,\mathrm{d}r \\ &\qquad+ \int_0^\infty e^{2\pi i r}e(-rx)a_1(r)w(r)(1+r^2)^{-\beta/2}r^{n-1}\varphi_R(r)\,\mathrm{d}r. \end{aligned} \tag{3.7}$$

From (2.6), we have that $a_1 = 0$ near the origin. Since $\beta > (n-1)/2$, we see that if $|\frac{x}{|x|} - v_1| \le 10^{-2}$ and $1/2 \le |x| \le 2$,

$$\left|\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi+|\xi|)} a_1(\xi)\hat{f}_{\beta,R}(\xi)\,\mathrm{d}\xi\right| \le C \tag{3.8}$$

for some constant $C > 0$ independent of R.

Next we calculate

$$\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi-|\xi|)} a_2(\xi)\hat{f}_{\beta,R}(\xi)\,\mathrm{d}\xi$$

when $|\frac{x}{|x|}-v_1|\le 10^{-2}$ and $1<|x|\le 1+\varepsilon$ ($\varepsilon>0$ is a small constant that will be chosen later). As (3.4) and (3.7), we write

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi-|\xi|)} a_2(\xi)\hat{f}_{\beta,R}(\xi)\,\mathrm{d}\xi\\
&\quad = C\int_0^\infty\int_{S^{n-1}} e^{2\pi i(x\cdot r\theta-r)}(1-\varphi(r))\chi(\theta)(1+r^2)^{-\beta/2}r^{n-1}\varphi_R(r)\,\mathrm{d}\sigma(\theta)\mathrm{d}r\\
&\quad = C\int_0^\infty e^{-2\pi ir}\widehat{\chi\,\mathrm{d}\sigma}(-rx)(1-\varphi(r))(1+r^2)^{-\beta/2}r^{n-1}\varphi_R(r)\,\mathrm{d}r\\
&\quad = C\int_0^\infty e^{2\pi ir(|x|-1)}h(-rx)(1-\varphi(r))(1+r^2)^{-\beta/2}r^{n-1}\varphi_R(r)\,\mathrm{d}r\\
&\qquad + C\int_0^\infty e^{-2\pi ir}e(-rx)(1-\varphi(r))(1+r^2)^{-\beta/2}r^{n-1}\varphi_R(r)\,\mathrm{d}r.
\end{aligned}$$

The second term is bounded since $e\in S^{-\infty}$. Now we use (3.6) to write

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi-|\xi|)} a_2(\xi)\hat{f}_{\beta,R}(\xi)\,\mathrm{d}\xi\\
&\quad = C\int_0^\infty e^{2\pi ir(|x|-1)}\left[c_0(r|x|)^{-\frac{n-1}{2}}\chi(x/|x|)+\tilde{e}(-rx)\right]\\
&\qquad (1-\varphi(r))(1+r^2)^{-\beta/2}r^{n-1}\varphi_R(r)\,\mathrm{d}r+O(1).
\end{aligned}$$

To continue, we need the following result.

Lemma 3.3 *Let g be a function satisfying $|g^{(k)}(r)|\le Cr^{m-k}$, $r\ge 1$ for some $m\in\mathbb{R}$ and for all $k\in\mathbb{Z}_+$. Then for all $\tau\neq 0$, we have*

$$\left|\int_0^\infty e^{2\pi ir\tau}g(r)(1-\varphi(r))\varphi_R(r)\,dr\right|\le C|\tau|^{-m-1} \tag{3.9}$$

for some constant $C>0$ independent of R and τ.

Proof By (2.9), we write

$$\int_0^\infty e^{2\pi ir\tau}g(r)(1-\varphi(r))\varphi_R(r)\,\mathrm{d}r=\sum_{j\ge 1}\int_0^\infty e^{2\pi ir\tau}g(r)\psi_j(r)\varphi_R(r)\,\mathrm{d}r.$$

For each j and N, integration by parts shows

$$\begin{aligned}
&\left|\int_0^\infty e^{2\pi i r\tau} g(r)\psi_j(r)\varphi_R(r)\,\mathrm{d}r\right| \\
&\quad = (2\pi)^N|\tau|^{-N}\left|\int_0^\infty e^{2\pi i r\tau}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^N \big(g(r)\psi_j(r)\varphi_R(r)\big)\,\mathrm{d}r\right| \\
&\quad \le C|\tau|^{-N}\int_{2^{j-1}\le r\le 2^{j+1}} r^{m-N}\mathrm{d}r \\
&\quad \le C|\tau|^{-N}2^{j(m-N+1)}, \qquad (3.10)
\end{aligned}$$

where we applied the condition on g and for all $k\in\mathbb{Z}_+$

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}r^k}\big(\varphi_R(r)\big)\right| \le C_k r^{-k}$$

for some constant $C_k>0$ independent of R and r.

Set $N=0$ for $2^j\le|\tau|^{-1}$, and $N>m+1$ otherwise. From this, it follows that

$$\begin{aligned}
&\left|\int_0^\infty e^{2\pi i r\tau} g(r)(1-\varphi(r))\varphi_R(r)\,\mathrm{d}r\right| \\
&\quad \le C\sum_{2^j\le|\tau|^{-1}} 2^{j(m+1)} + C\sum_{2^j\ge|\tau|^{-1}} |\tau|^{-N}2^{j(m-N+1)} \\
&\quad \le C|\tau|^{-m-1}.
\end{aligned}$$

This proves Lemma 3.3. □

Back to the proof of Lemma 3.2. By Lemma 3.3,

$$\int_0^\infty e^{2\pi i r(|x|-1)}\tilde{e}(-rx)(1-\varphi(r))(1+r^2)^{-\beta/2}r^{n-1}\varphi_R(r)\,\mathrm{d}r = O\left(\big||x|-1\big|^{\beta-(n-1)/2}\right).$$

Finally, for $|\frac{x}{|x|}-v_1|\le 10^{-2}$ and $1<|x|\le 1+\varepsilon$, let us estimate

$$\int_0^\infty e^{2\pi i r(|x|-1)}(r|x|)^{-\frac{n-1}{2}}(1-\varphi(r))(1+r^2)^{-\beta/2}r^{n-1}\varphi_R(r)\,\mathrm{d}r.$$

Note that by Lemma 3.3 again,

$$\begin{aligned}
&|x|^{-\frac{n-1}{2}}\int_0^\infty e^{2\pi i r(|x|-1)}(1-\varphi(r))r^{\frac{n-1}{2}}((1+r^2)^{-\beta/2}-r^{-\beta})\varphi_R(r)\,\mathrm{d}r \\
&\quad = O\left(\big||x|-1\big|^{\beta-(n-1)/2+1}\right).
\end{aligned}$$

For the term $|x|^{-\frac{n-1}{2}}\int_0^\infty e^{2\pi ir(|x|-1)}(1-\varphi(r))r^{-\beta+\frac{n-1}{2}}\varphi_R(r)\,\mathrm{d}r$, we use scaling to obtain that if $-\beta+\frac{n-1}{2}>-1$,

$$
\begin{aligned}
&|x|^{-\frac{n-1}{2}}\int_0^\infty e^{2\pi ir(|x|-1)}(1-\varphi(r))r^{-\beta+\frac{n-1}{2}}\varphi_R(r)\,\mathrm{d}r\\
&\quad=|x|^{-\frac{n-1}{2}}\int_0^\infty e^{2\pi ir(|x|-1)}r^{-\beta+\frac{n-1}{2}}\varphi_R(r)\,\mathrm{d}r+O(1)\\
&\quad=|x|^{-\frac{n-1}{2}}(|x|-1)^{\beta-\frac{n+1}{2}}\int_0^\infty e^{2\pi ir}r^{-\beta+\frac{n-1}{2}}\varphi\left(\frac{r}{(|x|-1)R}\right)\mathrm{d}r+O(1).
\end{aligned}
$$

Note that $1<|x|\le 1+\varepsilon$. When $\beta>\frac{n-1}{2}$ and $-\beta+\frac{n-1}{2}>-1$,

$$
\lim_{R\to\infty}\int_0^\infty e^{2\pi ir}r^{-\beta+\frac{n-1}{2}}\varphi\left(\frac{r}{(|x|-1)R}\right)\mathrm{d}r=C_0,
$$

where C_0 is a non-zero constant. Hence, there exist $C>0$ and $\varepsilon_1\in(0,1/2)$ such that if $1<|x|\le 1+\varepsilon_1$,

$$
\liminf_{R\to\infty}|x|^{-\frac{n-1}{2}}\left|\int_0^\infty e^{2\pi ir(|x|-1)}(1-\varphi(r))r^{-\beta+\frac{n-1}{2}}\varphi_R(r)\,\mathrm{d}r\right|\ge C\big||x|-1\big|^{\beta-\frac{n+1}{2}}.
$$

Furthermore, we can find $0<\varepsilon\le\varepsilon_1$ such that for $1<|x|\le 1+\varepsilon$,

$$
\begin{aligned}
&\liminf_{R\to\infty}\left|\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi-|\xi|)}a_2(\xi)\hat{f}_{\beta,R}(\xi)\,\mathrm{d}\xi\right|\\
&\qquad\ge C\big||x|-1\big|^{\beta-\frac{n+1}{2}}-O\left(\big||x|-1\big|^{\beta-(n-1)/2+1}\right)\\
&\qquad\ge\frac{C}{2}\big||x|-1\big|^{\beta-\frac{n+1}{2}}.
\end{aligned}
$$

This, together with (3.8), tells us

$$
\liminf_{R\to\infty}\|\mathscr{A}_1 f_{\beta,R}\|_{L^p(\Omega_\varepsilon)}\ge\|\liminf_{R\to\infty}|\mathscr{A}_1 f_{\beta,R}|\,\|_{L^p(\Omega_\varepsilon)}=\infty,\tag{3.11}
$$

if $\beta>\frac{n-1}{2}$, $-\beta+\frac{n-1}{2}>-1$, and $\left(\beta-\frac{n+1}{2}\right)p\le -1$. Here we applied Fatou's lemma and $\Omega_\varepsilon:=\{x\in\mathbb{R}^n:|\frac{x}{|x|}-v_1|\le 10^{-2}, 1<|x|\le 1+\varepsilon\}$.

Therefore, we have $\sup_{R>0} \|f_{\beta,R}\|_{W^{s,p}(\mathbb{R}^n)} < \infty$ and $\liminf_{R\to\infty} \|\mathscr{A}_1 f_{\beta,R}\|_{L^p(\mathbb{R}^n)} = \infty$ provided that

$$\begin{cases} 0 < \beta - s < n, \\ (-n+\beta-s)p > -n, \\ \beta > \frac{n-1}{2}, \\ -\beta + \frac{n-1}{2} > -1, \\ \left(\beta - \frac{n+1}{2}\right)p \le -1, \end{cases} \tag{3.12}$$

which is solvable when

$$-(n+1)/2 < s < (n-1)(1/p - 1/2). \tag{3.13}$$

Hence, if (3.2) holds, then we must have $s \ge (n-1)(1/p-1/2)$ or $s \le -(n+1)/2$. However, once (3.2) holds for some $s_0 \le -(n+1)/2$, it holds for all $s \ge s_0$, which is in contradiction with (3.13). So the only possible range of s where (3.2) holds is $s \ge (n-1)(1/p-1/2)$. By duality,

$$\|(\mathscr{A}_1)^* f\|_{L^{p'}(\mathbb{R}^n)} \le C\|f\|_{W^{s,p'}(\mathbb{R}^n)}.$$

Because $(\mathscr{A}_1)^*$ is essentially the same as $\mathscr{A}_1$, we must have $s \ge (n-1)(1/p'-1/2) = (n-1)(1/2-1/p)$ by the previous counterexample. This proves Lemma 3.2, and then the proof of Proposition 3.1 is complete. □

Next, let us prove the following result.

Proposition 3.4 *Let $n \ge 2$ and $p \ge 2$. Suppose*

$$\left\| \sup_{1\le t\le 2} |\mathfrak{M}_t^\alpha f| \right\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{3.14}$$

holds for some $\alpha \in \mathbb{C}$. Then, we have

$$\operatorname{Re}\alpha \ge \frac{1}{p} - \frac{n-1}{2}.$$

Let us prove Proposition 3.4. Fix $N > -(n-2)/p - \operatorname{Re}\alpha$ as in Lemma 2.1. By (2.7) and Lemma 2.1, the proof of Proposition 3.4 reduces to show the following lemma.

Lemma 3.5 *Let $n \ge 2$ and $p > 1$. Suppose*

$$\left\| \sup_{1\le t\le 2} |\mathscr{A}_t f| \right\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{W^{s,p}(\mathbb{R}^n)} \tag{3.15}$$

holds for some $s \in \mathbb{R}$. Then, we have $s \ge 1/p$.

Proof Let $\delta > 0$ be a small number to be chosen later, and denote $\xi = (\xi_1, \xi') \in \mathbb{R}^n$. For a given large $j \in \mathbb{N}$, we let $\hat{f} \geq 0$ be a smooth cut-off of the set

$$\left\{(\xi_1, \xi') \in \mathbb{R}^n : |\xi_1 - 2^j| \leq \delta 2^{j-1}, |\xi'| \leq \delta 2^{j/2}\right\} \tag{3.16}$$

such that $\left|\partial_\xi^\beta \hat{f}(\xi)\right| \leq C_{\delta,\beta} 2^{-j|\beta'|/2} 2^{-j|\beta_1|}$ for any $\beta = (\beta_1, \beta') \in \mathbb{Z}_+^n$. By a simple calculation, we see that

$$|\xi| - \xi_1 \leq C\delta^2 \tag{3.17}$$

in the support of $\hat{f}$. Let j be large enough such that $(1 - \varphi(t|\xi|))\hat{f}(\xi) = \hat{f}(\xi)$ for all $t \in [1, 2]$, $\xi \in \mathbb{R}^n$ and

$$\inf_{\xi \in \operatorname{supp} \hat{f}} |a_2(\xi)| \geq c_{low} > 0. \tag{3.18}$$

Note by [20, Chapter IX, Section 4] we have

$$\sup_{1 \leq t \leq 2} \left|\partial_\xi^\beta \left(e^{2\pi i t(|\xi| - \xi_1)} a_1(t|\xi|)\hat{f}(\xi)\right)\right| \leq C_{\delta,\beta} 2^{-j|\beta'|/2} 2^{-j|\beta_1|}.$$

Then for $1 \leq t \leq 2$ and $x_1 > 0$, we use integration by parts to bound that

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi + t|\xi|)} a_1(t|\xi|)\hat{f}(\xi)\, \mathrm{d}\xi\right| \\ &\quad = \left|\int_{\mathbb{R}^n} e^{2\pi i(x + tv_1)\cdot\xi} \left(e^{2\pi i t(|\xi| - \xi_1)} a_1(t|\xi|)\hat{f}(\xi)\right) \mathrm{d}\xi\right| \\ &\quad \leq C_\delta 2^{-jN} 2^{j\frac{n+1}{2}} (x_1 + t)^{-N} \leq C_\delta 2^{-jN} 2^{j\frac{n+1}{2}}, \end{aligned} \tag{3.19}$$

where $v_1 = (1, 0, \ldots, 0)$, $N \geq 1$ and the constant C_δ is independent of j and t.

As for $\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi - t|\xi|)} a_2(t|\xi|)\hat{f}(\xi)\, \mathrm{d}\xi$ with $1 \leq t \leq 2$, we split it into three terms

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi - t|\xi|)} a_2(t|\xi|)\hat{f}(\xi)\, \mathrm{d}\xi \\ &\quad = \int_{\mathbb{R}^n} e^{2\pi i(x - tv_1)\cdot\xi} \left(e^{2\pi i t(-|\xi| + \xi_1)} - 1\right) a_2(t|\xi|)\hat{f}(\xi)\, \mathrm{d}\xi \\ &\qquad + \int_{\mathbb{R}^n} (e^{2\pi i(x - tv_1)\cdot\xi} - 1) a_2(t|\xi|)\hat{f}(\xi)\, \mathrm{d}\xi \\ &\qquad + \int_{\mathbb{R}^n} a_2(t|\xi|)\hat{f}(\xi)\, \mathrm{d}\xi. \end{aligned} \tag{3.20}$$

By (3.17), the first term of (3.20) is bounded by

$$C\int_{\mathbb{R}^n} \left|t(-|\xi| + \xi_1)\right| \hat{f}(\xi)\, \mathrm{d}\xi \leq C\delta^2 \int_{\mathbb{R}^n} \hat{f}(\xi)\, \mathrm{d}\xi \leq C\delta^{n+2} 2^{j(n+1)/2}.$$

If $|x_1 - t| \le \delta 2^{-j}$ and $|x'| \le 2^{-j/2}$, by the support condition (3.16) of $\hat{f}$, we have

$$\big|(x - tv_1) \cdot \xi\big| \le C\delta, \text{ for all } \xi \in \operatorname{supp} \hat{f},$$

which implies the second term of (3.20) is bounded by

$$C \int_{\mathbb{R}^n} \big|(x - tv_1) \cdot \xi\big| \hat{f}(\xi)\, \mathrm{d}\xi \le C\delta \int_{\mathbb{R}^n} \hat{f}(\xi)\, \mathrm{d}\xi \le C\delta^{n+1} 2^{j(n+1)/2}.$$

By (3.18), we have

$$\left|\int_{\mathbb{R}^n} a_2(t|\xi|) \hat{f}(\xi)\, \mathrm{d}\xi\right| \ge \frac{c_{low}}{2} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \mathrm{d}\xi \ge C_L \delta^n 2^{j\frac{n+1}{2}}.$$

Then by (3.20) and the above estimates, if $\delta \le \min\{\frac{C_L}{2C_U}, 1\}$, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi - t|\xi|)} a_2(t|\xi|) \hat{f}(\xi)\, \mathrm{d}\xi\right| \\ &\quad \ge \left|\int_{\mathbb{R}^n} a_2(t|\xi|) \hat{f}(\xi)\, \mathrm{d}\xi\right| - C_U \delta^{n+1} 2^{j\frac{n+1}{2}} \ge \frac{C_L}{2} \delta^n 2^{j\frac{n+1}{2}} \end{aligned} \tag{3.21}$$

if $|x_1 - t| \le \delta 2^{-j}$ and $|x'| \le 2^{-j/2}$. It then follows from (3.19) and (3.21) that

$$\sup_{1\le t\le 2} |\mathscr{A}_t f| \ge \frac{C_L}{2} \delta^n 2^{j\frac{n+1}{2}} - C_\delta 2^{-jN} 2^{j\frac{n+1}{2}} \ge \frac{C_L}{4} \delta^n 2^{j\frac{n+1}{2}}, \tag{3.22}$$

when $1 \le x_1 \le 2$, $|x'| \le 2^{-j/2}$ and $j \ge \frac{1}{N} \log_2(\frac{4C_\delta}{\delta^n C_L} + 1)$.

Assume (3.15) is true. Then from the definition of f and (3.22), we have

$$\begin{aligned} \frac{C_L}{4} \delta^n 2^{(n+1)j/2 - (n-1)j/(2p)} &\le \left\| \sup_{1\le t\le 2} |\mathscr{A}_t f| \right\|_{L^p(\mathbb{R}^n)} \\ &\le C \|f\|_{W^{s,p}(\mathbb{R}^n)} \le C_\delta 2^{sj} 2^{(n+1)j/2 - (n+1)j/(2p)}. \end{aligned} \tag{3.23}$$

Let $j \to \infty$, then we obtain $s \ge 1/p$. This proves Lemma 3.5, and then the proof of Proposition 3.4 is complete. □

We finally present the endgame in the

Proof of (i) of Theorem 1.1 This is a consequence of Proposition 3.1 and Proposition 3.4. □

4 Proof of (ii) of Theorem 1.1

In this section, we give a criterion that allows us to derive L^p-boundedness for the maximal operator $\mathfrak{M}^\alpha$ on $\mathbb{R}^n$, $n \geq 2$. As a consequence, (ii) of Theorem 1.1 follows readily by applying the result of Guth, Wang and Zhang [7] on local smoothing estimate on $\mathbb{R}^2$. More precisely, we have the following result.

Proposition 4.1 *Let $n \geq 2$ and $p > 2$. If the local smoothing estimate*

$$\left\| e^{it\sqrt{-\Delta}} f \right\|_{L^p(\mathbb{R}^n \times [1,2])} \leq C_{n,p} \|f\|_{W^{s,p}(\mathbb{R}^n)} \tag{4.1}$$

holds for some $s \in \mathbb{R}$, then we have

$$\left\| \sup_{t>0} |\mathfrak{M}_t^\alpha f| \right\|_{L^p(\mathbb{R}^n)} \leq C_{n,p,\alpha} \|f\|_{L^p(\mathbb{R}^n)} \tag{4.2}$$

whenever $\mathrm{Re}\,\alpha > \max\left\{ -(n-1)/p,\ s-(n-1)/2+1/p \right\}$.

The proof of Proposition 4.1 is inspired by [10]. Let φ and $\{\psi_j\}_j$ be functions in (2.9). We write

$$\begin{aligned} \widehat{\mathfrak{M}_t^\alpha f}(\xi) &= \varphi(t|\xi|)\widehat{m_\alpha}(t\xi)\hat{f}(\xi) + \sum_{j\geq 1} \psi_j(t|\xi|)\widehat{m_\alpha}(t\xi)\hat{f}(\xi) \\ &=: \widehat{\mathfrak{M}_{0,t}^\alpha f}(\xi) + \sum_{j\geq 1} \widehat{\mathfrak{M}_{j,t}^\alpha f}(\xi). \end{aligned} \tag{4.3}$$

To prove Proposition 4.1, the first strategy is to show that if one modifies the definition so that for each operator $\mathfrak{M}_{j,t}^\alpha$, the supremum is taken over $1 \leq t \leq 2$, then the resulting maximal function is bounded on $L^p(\mathbb{R}^n)$.

Lemma 4.2 *Let $n \geq 2$ and $p > 2$. Under the assumption (4.1) of Proposition 4.1, there exist $\delta > 0$ and $C > 0$, such that for all $j \geq 1$,*

$$\left\| \sup_{t\in[1,2]} |\mathfrak{M}_{j,t}^\alpha f| \right\|_{L^p(\mathbb{R}^n)} \leq C 2^{-\delta j} \|f\|_{L^p(\mathbb{R}^n)}, \tag{4.4}$$

if $\mathrm{Re}\,\alpha > \max\left\{ -(n-1)/p, s-(n-1)/2+1/p \right\}$.

Proof By (2.5), (2.7) and (2.10), it suffices to show

$$\left\| \sup_{t\in[1,2]} |\mathscr{A}_{j,t} f| \right\|_{L^p(\mathbb{R}^n)} \leq C 2^{[\max\{(n-1)(1/2-1/p),\, s+1/p\}]j} \|f\|_{L^p(\mathbb{R}^n)}, \tag{4.5}$$

where $\widehat{\mathscr{A}_{j,t}f}(\xi) = \psi_j(t|\xi|)\widehat{\mathscr{A}_t f}(\xi)$ and $\mathscr{A}_t f$ is defined in (2.13). By (2.6), we can write

$$\mathscr{A}_{j,t}f(x) = C\sum_{\ell=0}^{N-1}\int_{\mathbb{R}^n}\Big(b_\ell e^{2\pi i(x\cdot\xi+t|\xi|)} + d_\ell e^{2\pi i(x\cdot\xi-t|\xi|)}\Big)|t\xi|^{-\ell}\psi_j(t|\xi|)\hat{f}(\xi)\,\mathrm{d}\xi,$$

which is a linear combination of

$$T_{\ell,j}f(x,t) := \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi\pm t|\xi|)}|t\xi|^{-\ell}\psi_j(t|\xi|)\hat{f}(\xi)\,\mathrm{d}\xi,\quad \ell = 0, 1, \ldots, N-1.$$

Hence, the proof of (4.5) reduces to showing that

$$\Big\|\sup_{t\in[1,2]}|T_{0,j}f(\cdot,t)|\Big\|_{L^p(\mathbb{R}^n)} \le C2^{[\max\{(n-1)(1/2-1/p),\, s+1/p\}]j}\|f\|_{L^p(\mathbb{R}^n)},\quad j \ge 1. \tag{4.6}$$

Now we apply Lemma 2.2 to deal with (4.6). First, it follows from [20, Theorem 2, Chapter IX] that

$$\|T_{0,j}f(\cdot,1)\|_{L^p(\mathbb{R}^n)} \le C2^{(n-1)(1/2-1/p)j}\|f\|_{L^p(\mathbb{R}^n)}. \tag{4.7}$$

Next, we observe that for any $1 \le t \le 2$ and $j \ge 1$, there holds

$$\big|\partial_\xi^\beta\big(\psi_j(t|\xi|)\big)\big| \le C(1+|\xi|)^{-|\beta|},$$

where β is any multi-index. So $\psi_j(t|\cdot|) \in S^0$ uniformly $1 \le t \le 2$ and $j \ge 1$, hence

$$\begin{aligned}\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi\pm t|\xi|)}\psi_j(t|\xi|)\hat{f}(\xi)\,\mathrm{d}\xi\Big|^p\,\mathrm{d}x\\ \le C\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi\pm t|\xi|)}\tilde{\psi}_j(\xi)\hat{f}(\xi)\,\mathrm{d}\xi\Big|^p\,\mathrm{d}x,\end{aligned} \tag{4.8}$$

where constant C is independent of t and j. Here $\tilde{\psi}_j$ equals to 1 if $|\xi| \in [2^{j-2}M, 2^{j+1}M]$ and vanishes if $|\xi| \notin [2^{j-3}M, 2^{j+2}M]$, so that $\tilde{\psi}_j$ equals to 1 on the support of $\psi_j(t|\cdot|)$ when $1 \le t \le 2$. Then we apply our assumption (4.1) on local smoothing estimate to (4.8) to obtain

$$\|T_{0,j}f\|_{L^p(\mathbb{R}^n\times[1,2])} \le C2^{sj}\|f\|_{L^p(\mathbb{R}^n)},$$

and by the same token, the operator

$$\partial_t T_{0,j}(x,t) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi\pm t|\xi|)}\big(\pm 2\pi i|\xi|\psi_j(t|\xi|) + |\xi|\psi_j'(t|\xi|)\big)\hat{f}(\xi)\,\mathrm{d}\xi.$$

satisfies

$$\|\partial_t T_{0,j} f\|_{L^p(\mathbb{R}^n\times[1,2])} \leq C2^{(s+1)j}\|f\|_{L^p(\mathbb{R}^n)}.$$

Thus, we use Lemma 2.2 to get

$$\left\|\sup_{t\in[1,2]} |T_{0,j} f(\cdot,t)|\right\|_{L^p(\mathbb{R}^n)} \leq C\left(2^{(n-1)(1/2-1/p)j} + 2^{(s+1/p)j}\right)\|f\|_{L^p(\mathbb{R}^n)},$$

which implies estimate (4.6). □

Finally, we can apply Lemma 4.2 to prove Proposition 4.1.

Proof of Proposition 4.1 By (4.3) and (2.7), (4.2) reduces to

$$\left\|\sup_{t>0} |\mathfrak{M}^\alpha_{j,t} f|\right\|_{L^p(\mathbb{R}^n)} \leq C2^{-\delta j}\|f\|_{L^p(\mathbb{R}^n)} \tag{4.9}$$

for some $\delta > 0$. Since $\ell^p \subseteq \ell^\infty$, we have

$$\left\|\sup_{t>0} |\mathfrak{M}^\alpha_{j,t} f|\right\|_{L^p(\mathbb{R}^n)} \leq \left(\sum_{k\in\mathbb{Z}} \left\|\sup_{t\in[2^k,2^{k+1}]} |\mathfrak{M}^\alpha_{j,t} f|\right\|^p_{L^p(\mathbb{R}^n)}\right)^{1/p}. \tag{4.10}$$

However, it follows from Lemma 4.2 and a rescaling $t \to 2^{-k}t$ that

$$\left\|\sup_{t\in[2^k,2^{k+1}]} |\mathfrak{M}^\alpha_{j,t} f|\right\|_{L^p(\mathbb{R}^n)} \leq C2^{-\delta j}\|f\|_{L^p(\mathbb{R}^n)}. \tag{4.11}$$

Then for $2^k \leq t \leq 2^{k+1}$, there must be $|\xi| \in [2^{j-k-2}M, 2^{j-k+1}M]$. This tells us that we can rewrite (4.11) as

$$\left\|\sup_{t\in[2^k,2^{k+1}]} |\mathfrak{M}^\alpha_{j,t} f|\right\|_{L^p(\mathbb{R}^n)} \leq C2^{-\delta j}\|P_{j-k} f\|_{L^p(\mathbb{R}^n)}.$$

This, together with (4.10), implies

$$\begin{aligned}\left\|\sup_{t>0} |\mathfrak{M}^\alpha_{j,t} f|\right\|_{L^p(\mathbb{R}^n)} &\leq C2^{-\delta j}\left(\sum_{k\in\mathbb{Z}} \|P_{j-k} f\|^p_{L^p(\mathbb{R}^n)}\right)^{1/p} \\ &= C2^{-\delta j}\left\|\left(\sum_{k\in\mathbb{Z}} |P_{j-k} f|^p\right)^{1/p}\right\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

$$\le C2^{-\delta j}\left\|\left(\sum_{k\in\mathbb{Z}}|P_{j-k}f|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)}$$

since $p > 2$. By the Littlewood–Paley inequality [5],

$$\left\|\left(\sum_{k\in\mathbb{Z}}|P_{j-k}f|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

This proves (4.9). Hence, the proof of Proposition 4.1 is complete. □

Remark 4.3 (i) In the dimension $n \ge 3$ Gao et al. [3] obtained improved local smoothing estimates for the wave equation, that is, (4.1) holds with $s = (n-1)(1/2-1/p)-\sigma$ for all $\sigma < 2/p - 1/2$ when

$$p > \begin{cases} \frac{2(3n+5)}{3n+1}, & \text{for } n \text{ odd;} \\ \frac{2(3n+6)}{3n+2}, & \text{for } n \text{ even.} \end{cases}$$

Applying Proposition 4.1, we get that (1.3) holds if $\mathrm{Re}\,\alpha > \alpha(p,n)$ where

$$\alpha(p,n) = \begin{cases} \max\left\{-\frac{n-1}{p}, -\frac{3}{8}(n-1) + \frac{5-n}{4p}, \frac{4(n-1)}{(3n+5)(n+3)} - \frac{n^2-5}{(n+3)p}\right\}, & \text{for } n \text{ odd;} \\ \max\left\{-\frac{n-1}{p}, -\frac{3n-2}{8} - \frac{n-6}{4p}, -\frac{n-1}{n+4} - \frac{n^2+n-6}{(n+4)p}\right\}, & \text{for } n \text{ even.} \end{cases} \tag{4.12}$$

The above range α in (4.12) for $p > 2$ is strictly wider than (1.7). However, the range p in (4.12) is not optimal. What happens when $n \ge 3$ (and $p > 2$) remains open.

(ii) Under the assumption (4.1) of Proposition 4.1, it follows by (4.4) that for $n \ge 2$ and $p > 2$,

$$\left\|\sup_{t\in[1,2]}|\mathfrak{M}_t^\alpha f|\right\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$$

provided that $\mathrm{Re}\,\alpha > \max\{-(n-1)/p, s-(n-1)/2+1/p\}$. It is interesting to describe the full range of (p,q) such that

$$\left\|\sup_{t\in[1,2]}|\mathfrak{M}_t^\alpha f|\right\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

For $\alpha = 0$, we refer it to [9, 13–15] and the references therein.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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