Mathematische Annalen
https://doi.org/10.1007/s00208-024-02824-w

Mathematische Annalen

Linear stability of compact shrinking Ricci solitons

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Received: 23 March 2023 / Revised: 23 March 2023 / Accepted: 29 January 2024

Abstract
In this paper, we continue investigating the second variation of Perelman's ν-entropy for compact shrinking Ricci solitons. In particular, we improve some of our previous work in Cao and Zhu (Math Ann 353(3):747–763, 2012), as well as the more recent work in Mehrmohamadi and Razavi (arXiv:2104.08343, 2021), and obtain a necessary and sufficient condition for a compact shrinking Ricci soliton to be linearly stable. Our work also extends similar results of Hamilton, Ilmanen and the first author in Cao et al. (arXiv:math.DG/0404165, 2004) (see also Cao and He in J Reine Angew Math, 2015:229–246, 2015) for positive Einstein manifolds to the compact shrinking Ricci soliton case.

1 Introduction

This is a sequel to our previous paper [11], in which we derived the second variation formula of Perelman's ν-entropy for compact shrinking Ricci solitons and obtained certain necessary condition for the linear stability of compact Ricci shrinkers.

Recall that a complete Riemannian manifold (M^n, g) is called a *shrinking Ricci soliton* if there exists a smooth vector field V on M^n such that the Ricci tensor Rc of the metric g satisfies the equation

The first author was partially supported by a Simons Foundation Collaboration Grant. The second author was partially supported by NSFC Grant No. 11971168, Shanghai Science and Technology Innovation Program Basic Research Project STCSM 20JC1412900, and the Science and Technology Commission of Shanghai Municipality No. 22DZ2229014.

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Published online: 10 March 2024

$$Rc + \frac{1}{2}\mathscr{L}_V g = \frac{1}{2\tau} g,$$

where $\tau > 0$ is a constant and $\mathscr{L}_V g$ denotes the Lie derivative of g in the direction of V. If V is the gradient vector field ∇f of a smooth function f, then we have a *gradient shrinking Ricci soliton* given by

$$Rc + \nabla^2 f = \frac{1}{2\tau} g, \tag{1.1}$$

for some constant $\tau > 0$. Here, $\nabla^2 f$ denotes the Hessian of f, and f is called a *potential function* of the Ricci soliton. Clearly, when f is a constant we have an Einstein metric of positive scalar curvature. Thus, gradient shrinking Ricci solitons include positive Einstein manifolds as a special case. In the following, we use (M^n, g, f) to denote a gradient shrinking Ricci soliton.

Gradient shrinking Ricci solitons are self-similar solutions to Hamilton's Ricci flow, and often arise as Type I singularity models in the Ricci flow as shown by Naber [35], Enders–Müller–Topping [20] and Cao–Zhang [12]; see also Zhang [42]. As such, they play a significant role in the study of the formation of singularities in the Ricci flow and its applications. Therefore, it is very important to either classify, if possible, or understand the geometry of gradient shrinking Ricci solitons.

Hamilton [26] showed that any 2-dimensional complete gradient shrinking Ricci soliton is isometric to either $\mathbb{S}^2$, or $\mathbb{RP}^2$, or the Gaussian shrinking soliton on $\mathbb{R}^2$. In dimension $n = 3$, by using the Hamilton-Ivey curvature pinching, Ivey [27] proved that a compact shrinking soliton must be a spherical space form $\mathbb{S}^3/\Gamma$. Furthermore, for $n = 3$, a complete classification follows from the works of Perelman [39], Naber [35], Ni–Wallach [36], and Cao–Chen–Zhu [7] that any three-dimensional complete gradient shrinking Ricci soliton is either isometric to the Gaussian soliton $\mathbb{R}^3$ or a finite quotient of either $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$.

However, in dimension $n \geq 4$, there do exist non-Einstein and non-product gradient shrinking Ricci solitons. Specifically, in dimension $n = 4$, Koiso [29] and the first author [4] independently constructed a gradient Kähler–Ricci shrinking soliton on $\mathbb{CP}^2\#(-\mathbb{CP}^2)$, and Wang–Zhu [40] found another one on $\mathbb{CP}^2\#(-2\mathbb{CP}^2)$. In the noncompact case, Feldman–Ilmanen–Knopf [21] constructed a $U(2)$-invariant gradient shrinking Kähler–Ricci soliton on the tautological line bundle $\mathcal{O}(-1)$ of $\mathbb{CP}^1$, i.e., the blow-up of $\mathbb{C}^2$ at the origin. Very recently, a noncompact toric gradient shrinking Kähler–Ricci soliton on the blowup of $\mathbb{CP}^1 \times \mathbb{C}$ at one point was found by Bamler–Cifarelli–Conlon–Deruelle [2]. These are the only known examples of nontrivial (i.e., non-Einstein) and non-product complete shrinking Ricci solitons in dimension 4 so far. We remark that the constructions in [4, 21, 29, 40] all extend to higher dimensions. For additional examples in higher dimensions, see, e.g., Angenent–Knopf [1], Dancer–Wang [19], Futaki–Wang [22], and Yang [41].

Ricci solitons can be viewed as fixed points of the Ricci flow, as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. In [38], Perelman introduced the $\mathcal{W}$-functional

$$\mathcal{W}(\hat{g}, \hat{f}, \hat{\tau}) = \int_M [\hat{\tau}(\hat{R} + |\nabla \hat{f}|^2) + \hat{f} - n](4\pi\hat{\tau})^{-\frac{n}{2}} e^{-\hat{f}} d\hat{V},$$

on any compact manifold M^n, where $\hat{g}$ is a Riemannian metric on M, $\hat{R}$ is its scalar curvature, $\hat{f}$ is any smooth function on M^n, and $\hat{\tau} > 0$ is a positive parameter. The associated ν-entropy is defined by

$$\nu(\hat{g}) = \inf \left\{ \mathcal{W}(\hat{g}, \hat{f}, \hat{\tau}) : \hat{f} \in C^\infty(M), \hat{\tau} > 0, (4\pi\hat{\tau})^{-\frac{n}{2}} \int_M e^{-\hat{f}} d\hat{V} = 1 \right\},$$

which is always attained by some $\hat{f}$ and $\hat{\tau}$. Furthermore, Perelman showed that the ν-entropy is monotone increasing under the Ricci flow, and its critical points are precisely given by gradient shrinking Ricci solitons (M^n, g, f) satisfying (1.1). In particular, it follows that all compact shrinking Ricci solitons are necessarily gradient ones.

By definition, a compact shrinking Ricci soliton (M^n, g, f) is *linearly stable* (or ν-stable) if the second variation of the ν-entropy is nonpositive at g. In [8], Hamilton, Ilmanen and the first author initiated the study of linear stability of compact shrinking Ricci solitons. They obtained the second variation formula of Perelman's ν-entropy for positive Einstein manifolds and investigated their linear stability. Among other results, they showed that, while the round sphere $\mathbb{S}^n$ is linearly stable and the complex projective space $\mathbb{CP}^n$ is neutrally linearly stable[1], many known positive Einstein manifolds are unstable. In particular, all product Einstein manifolds and Fano Kähler–Einstein manifolds with Hodge number $h^{1,1} > 1$ are unstable. More recently, a complete description of the linear stability (or instability) of irreducible symmetric spaces of compact type was provided by He and the first author [9]. Meanwhile, in [11], we derived the second variation formula of Perelman's ν-entropy for compact shrinking Ricci solitons which we now recall.

Let (M^n, g, f) be a compact shrinking Ricci soliton satisfying (1.1) and $\mathrm{Sym}^2(T^*M)$ denote the space of symmetric (covariant) 2-tensors on M. For any $h = h_{ij} \in \mathrm{Sym}^2(T^*M)$, consider the variation $g(s) = g + sh$ and let

$$\mathrm{div}_f h = e^f \mathrm{div}(e^{-f} h) = \mathrm{div}\, h - h(\nabla f, \cdot), \tag{1.2}$$

$\mathrm{div}_f^\dagger$ be the adjoint of div_f with respect to the weighted L^2-inner product

$$(\cdot, \cdot)_f = \int_M < \cdot, \cdot > e^{-f} dV, \tag{1.3}$$

$$\Delta_f h := \Delta h - \nabla f \cdot \nabla h, \tag{1.4}$$

and

$$\mathcal{L}_f h = \frac{1}{2}\Delta_f h + Rm(h, \cdot) = \frac{1}{2}\Delta_f h_{ik} + R_{ijkl} h_{jl}. \tag{1.5}$$

[1] Recently, Knopf and Sesum [28] showed that $\mathbb{CP}^n$ is not a local maximum of the ν-entropy, hence is dynamically unstable as first shown by Kröncke [31].

Then the second variation $\delta_g^2\nu(h,h)$ of the ν-entropy is given in [11] by

$$\delta_g^2\nu(h,h) = \frac{d^2}{ds^2}\bigg|_{s=0} \nu(g(s)) = \frac{1}{(4\pi\tau)^{n/2}}\int_M < N_f h, h > e^{-f} dV,$$

where the *Jacobi operator* (also known as the *stability operator*) N_f is defined by

$$N_f h := \mathcal{L}_f h + \mathrm{div}_f^\dagger \,\mathrm{div}_f\, h + \frac{1}{2}\nabla^2 \hat{v}_h - Rc\,\frac{\int_M < Rc, h > e^{-f}\, dV}{\int_M Re^{-f}\, dV}, \tag{1.6}$$

and $\hat{v}_h$ is the unique solution of

$$\Delta_f \hat{v}_h + \frac{\hat{v}_h}{2\tau} = \mathrm{div}_f\,\mathrm{div}_f\, h, \qquad \int_M \hat{v}_h e^{-f}\, dV = 0.$$

For more details, we refer the reader to our previous paper [11] or Sect. 2 below. Note that $Sym^2(T^*M)$ admits the following standard direct sum decomposition:

$$\mathrm{Sym}^2(T^*M) = \mathrm{Im}(\mathrm{div}_f^\dagger) \oplus \mathrm{Ker}(\mathrm{div}_f). \tag{1.7}$$

The first factor

$$\begin{aligned}\mathrm{Im}(\mathrm{div}_f^\dagger) &= \{\mathrm{div}_f^\dagger(\omega) \mid \omega \in \Omega^1(M)\}\\ &= \{\mathscr{L}_X g \mid X = \omega^\sharp \in \mathscr{X}(M)\}\end{aligned}$$

represents deformations $g(s)$ of g by diffeomorphisms. Since the ν-entropy is invariant under diffeomorphisms, the second variation vanishes on this factor.

In [11], we observed that $\mathrm{div}_f(Rc) = 0$ and showed that Rc is an eigen-tensor of $\mathcal{L}_f$ with eigenvalue[2] $1/2\tau$, i.e., $\mathcal{L}_f Rc = \frac{1}{2\tau}Rc$. Moreover, for any linearly stable compact shrinking Ricci soliton, we proved that $1/2\tau$ is the only positive eigenvalue of $\mathcal{L}_f$ on $\mathrm{Ker}(\mathrm{div}_f)$ with multiplicity one. Very recently, Mehrmohamadi and Razavi [32] made some new progress. In particular, they showed that N_f vanishes on $\mathrm{Im}(\mathrm{div}_f^\dagger)$, extending a similar result in [8, 9] for positive Einstein manifolds to the compact shrinking Ricci soliton case. In addition, in terms of the operator $\mathcal{L}_f$, they showed that (i) if a compact shrinking Ricci soliton (M^n, g, f) is linearly stable, then the eigenvalues of $\mathcal{L}_f$ on $\mathrm{Sym}^2(T^*M)$, other than $\frac{1}{2\tau}$ with multiplicity one, must be less than or equal to $\frac{1}{4\tau}$; (ii) if a compact shrinking soliton (M^n, g, f) has $\mathcal{L}_f \le 0$ on $\mathrm{Sym}^2(T^*M)$, except on scalar multiples of Rc, then (M^n, g, f) is linearly stable (see Theorems 1.3 and 1.4 in [32], respectively).

Clearly, the nonpositivity of the second variation of ν, i.e., $\delta_g^2\nu(h,h) \le 0$, is implied by the nonpositivity of the stability operator N_f on the space $\mathrm{Sym}^2(T^*M)$ of symmetric 2-tensors. Thus, studying linear stability of compact shrinking Ricci solitons

[2] Note the different sign convention we used in [11] for eigenvalues of $\mathcal{L}_f$: In [11], λ is an eigenvalue of $\mathcal{L}_f$ if $-\mathcal{L}_f h = \lambda h$ for some symmetric 2-tensor $h \ne 0$.

requires a closer look into the eigenvalues and eigenspaces of N_f, especially its leading term $\mathcal{L}_f$ defined by (1.5), acting on $\mathrm{Sym}^2(T^*M)$. Since $\mathrm{div}_f(Rc) = 0$, we can further decompose $\mathrm{Ker}(\mathrm{div}_f)$ as

$$\mathrm{Ker}(\mathrm{div}_f) = \mathbb{R} \cdot \mathrm{Rc} \oplus \mathrm{Ker}(\mathrm{div}_f)_0,$$

where $\mathbb{R} \cdot \mathrm{Rc} = \{\rho Rc \mid \rho \in \mathbb{R}\}$ is the one dimensional subspace generated by the Ricci tensor Rc, and

$$\mathrm{Ker}(\mathrm{div}_f)_0 = \{h \in \mathrm{Ker}(\mathrm{div}_f) \,\Big|\, \int_M < h, Rc > e^{-f}\, dV = 0\} \tag{1.8}$$

denotes the orthogonal complement of $\mathbb{R}\cdot\mathrm{Rc}$ in $\mathrm{Ker}(\mathrm{div}_f)$ with respect to the weighted inner product (1.3). Accordingly, we can refine the decomposition of $\mathrm{Sym}^2(T^*M)$ in (1.7) by

$$\mathrm{Sym}^2(T^*M) = \mathrm{Im}(\mathrm{div}_f^{\dagger}) \oplus \mathbb{R} \cdot \mathrm{Rc} \oplus \mathrm{Ker}(\mathrm{div}_f)_0. \tag{1.9}$$

In this paper, by exploring decomposition (1.9), we are able to further improve our previous work in [11] and the work of Mehrmohamadi and Razavi [32]. Our main results are as follows.

Theorem 1.1 *Let (M^n, g, f) be a compact shrinking Ricci soliton satisfying Eq.* (1.1). *Then,*

(i) *the decomposition of* $\mathrm{Sym}^2(T^*M)$ *in* (1.9) *is both invariant under $\mathcal{L}_f$ and orthogonal with respect to the second variation $\delta_g^2 \nu$ of the ν-entropy.*

(ii) *the eigenvalues of $\mathcal{L}_f$ on* $\mathrm{Im}(\mathrm{div}_f^{\dagger})$ *are strictly less than* $\frac{1}{4\tau}$.

Theorem 1.2 *A compact shrinking Ricci soliton (M^n, g, f) is linearly stable if and only if $\mathcal{L}_f \leq 0$ on* $\mathrm{Ker}(\mathrm{div}_f)_0$.

Remark 1.1 Theorems 1.1 and 1.2 above are extensions of similar results by Hamilton, Ilmanen and the first author in [8] (see also Theorem 1.1 in [9]) for positive Einstein manifolds.

While there have been a lot of progress in recent years in understanding geometry of general higher dimensional ($n \geq 4$) complete noncompact gradient shrinking Ricci solitons, especially in dimension four, e.g., [10, 13, 14, 16, 30, 33, 34] and [2, 18], very little is known about the geometry of general compact shrinking Ricci solitons in dimension $n = 4$ or higher. On the other hand, for possible applications of the Ricci flow to topology, one is mostly interested in the classification of stable shrinking solitons, since unstable ones could be perturbed away hence may not represent generic singularities of the Ricci flow. Thus, exploring the variational structure of compact Ricci shrinkers becomes rather significant.

We point out that Hall and Murphy [24] have proven that compact shrinking Kähler–Ricci solitons with Hodge number $h^{1,1} > 1$ are unstable, thus extending the result of Cao–Hamilton–Ilmanen [8] for Fano Kähler–Einstein manifolds to the shrinking Kähler–Ricci soliton case. In particular, the Cao–Koiso soliton on $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ and Wang–Zhu soliton on $\mathbb{CP}^2\#(-2\mathbb{CP}^2)$ are unstable. In addition, Hall–Haslhofer–Siepmann [23] and Hall–Murphy [25] have shown that the Page metric [37] on $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ is unstable. Most recently, Biquard and Ozuch [3] proved that the Chen–LeBrun–Weber metric [15] on $\mathbb{CP}^2\#(-2\mathbb{CP}^2)$ is also unstable. We hope our new results in this paper will play a significant role in future study of linear stability of shrinking Ricci solitons, especially in classifying compact 4-dimensional linearly stable shrinking Ricci solitons.

2 Preliminaries

In this section, we fix our notation and recall some useful facts that will be used in the proof of Theorem 1.1. First of all, by scaling the metric g, we may assume that $\tau = 1$ in Eq. (1.1) so that

$$Rc + \nabla^2 f = \frac{1}{2} g. \tag{2.1}$$

We also normalize f so that

$$(4\pi)^{-\frac{n}{2}} \int_M e^{-f} \, dV = 1.$$

From now on, we shall assume that (M^n, g, f) is a compact shrinking Ricci soliton satisfying (2.1).

As in [11], for any symmetric 2-tensor $h = h_{ij}$ and 1-form $\omega = \omega_i$, we denote

$$\operatorname{div} \omega := \nabla_i \omega_i, \qquad (\operatorname{div} h)_i := \nabla_j h_{ji}.$$

Moreover, as done in [6, 11], we define $\operatorname{div}_f(\cdot) := e^f \operatorname{div}(e^{-f}(\cdot))$, or more specifically,

$$\operatorname{div}_f \omega = \operatorname{div} \omega - \omega(\nabla f) = \nabla_i \omega_i - \omega_i \nabla_i f, \tag{2.2}$$

and

$$\operatorname{div}_f h = \operatorname{div} h - h(\nabla f, \cdot) = \nabla_j h_{ij} - h_{ij} \nabla_j f. \tag{2.3}$$

We also define the operator $\operatorname{div}_f^\dagger$ on functions by

$$\operatorname{div}_f^\dagger u = -\nabla u, \qquad u \in C^\infty(M) \tag{2.4}$$

and on 1-forms by

$$(\operatorname{div}_f^\dagger \omega)_{ij} = -\frac{1}{2}(\nabla_i \omega_j + \nabla_j \omega_i) = -\frac{1}{2}\mathscr{L}_{\omega^\sharp} g_{ij}, \tag{2.5}$$

where $\omega^\sharp$ is the vector field dual to ω and $\mathscr{L}$ denotes the Lie derivative, so that

$$\int_M e^{-f} \left\langle \operatorname{div}_f^\dagger \omega, h \right\rangle dV = \int_M e^{-f} \left\langle \omega, \operatorname{div}_f h \right\rangle dV, \tag{2.6}$$

for any symmetric 2-tensor h.

Clearly, $\operatorname{div}_f^\dagger$ is just the adjoint of div_f with respect to the weighted L^2-inner product

$$(\cdot,\cdot)_f = \int_M < \cdot,\cdot > e^{-f} dV. \tag{2.7}$$

Remark 2.1 If we denote by div* the adjoint of div with respect to the usual L^2-inner product

$$(\cdot,\cdot) = \int_M < \cdot,\cdot > dV, \tag{2.8}$$

then, as pointed out in [6], one can easily verify that

$$\operatorname{div}_f^\dagger = \operatorname{div}^* . \tag{2.9}$$

Finally, we denote

$$\Delta_f := e^f \operatorname{div}(e^{-f} \nabla) = \Delta - \nabla f \cdot \nabla, \tag{2.10}$$

which is self-adjoint with respect to the weighted L^2-inner product (2.7),

$$Rm(h,\cdot)_{ik} := R_{ijkl} h_{jl},$$

and define the operator

$$\mathcal{L}_f h = \frac{1}{2}\Delta_f h + Rm(h,\cdot) \tag{2.11}$$

on the space of symmetric 2-tensors. It is easy to see that, like Δ_f, $\mathcal{L}_f$ is a self-adjoint operator with respect to the weighted L^2-inner product (2.7).

Now we restate the second variation of the ν-entropy derived in [11] with $\tau = 1$.

Theorem 2.1 [11] *Let* (M^n, g, f) *be a compact shrinking Ricci soliton satisfying* (2.1). *For any symmetric 2-tensor* $h = h_{ij}$, *consider the variation* $g(s) = g_{ij} + sh_{ij}$. *Then the second variation* $\delta_g^2 \nu(h, h)$ *is given by*

$$\delta_g^2 \nu(h, h) = \frac{d^2}{ds^2}\bigg|_{s=0} \nu(g(s)) = \frac{1}{(4\pi)^{n/2}} \int_M \langle N_f h, h \rangle e^{-f} dV, \tag{2.12}$$

where the stability operator N_f *is given by*

$$N_f h := \mathcal{L}_f h + \mathrm{div}_f^\dagger \, \mathrm{div}_f \, h + \frac{1}{2}\nabla^2 \hat{v}_h - Rc \, \frac{\int_M < Rc, h > e^{-f} \, dV}{\int_M Re^{-f} \, dV}, \tag{2.13}$$

and the function $\hat{v}_h$ *is the unique solution of*

$$\Delta_f \hat{v}_h + \frac{\hat{v}_h}{2} = \mathrm{div}_f \, \mathrm{div}_f \, h, \qquad \int_M \hat{v}_h e^{-f} \, dV = 0. \tag{2.14}$$

Next, we recall the following facts (see, e.g., Lemmas 3.1 and 3.2 in [11]).

Lemma 2.1 [11] *Let* (M^n, g, f) *be a compact shrinking Ricci soliton satisfying* (2.1). *Then,*

(i) $Rc \in \mathrm{Ker}(\mathrm{div}_f)$;
(ii) $\mathcal{L}_f(Rc) = \frac{1}{2} Rc$.

We shall also need the following useful identities found by Mehrmohamadi-Razavi [32]; see also Colding and Minicozzi [17], in which they derived more general versions of identities (2.15)–(2.20) that are valid for smooth metric measure spaces.

Lemma 2.2 [17, 32] *Let* (M^n, g, f) *be a compact shrinking Ricci soliton satisfying* (2.1). *Then, for any function* u, *1-form* ω *and symmetric 2-tensor* h, *the following identities hold*

$$\nabla \Delta_f u = \Delta_f \nabla u - \frac{1}{2}\nabla u, \tag{2.15}$$

$$\mathrm{div}_f \, \Delta_f \omega = \Delta_f \, \mathrm{div}_f \, \omega + \frac{1}{2} \mathrm{div}_f \, \omega, \tag{2.16}$$

$$\mathrm{div}_f^\dagger \, \Delta_f \omega = 2\mathcal{L}_f \, \mathrm{div}_f^\dagger \, \omega - \frac{1}{2} \mathrm{div}_f^\dagger \, \omega, \tag{2.17}$$

$$2\mathcal{L}_f(\mathscr{L}_{\omega^\sharp} g) = \mathscr{L}_{(\Delta_f \omega)^\sharp} g + \frac{1}{2}\mathscr{L}_{\omega^\sharp} g, \tag{2.18}$$

$$2\, \mathrm{div}_f \, \mathcal{L}_f h = \Delta_f \, \mathrm{div}_f \, h + \frac{1}{2} \mathrm{div}_f \, h, \tag{2.19}$$

$$\mathrm{div}_f(\mathscr{L}_{\omega^\sharp} g) = -2\, \mathrm{div}_f \, \mathrm{div}_f^\dagger \, \omega = \Delta_f \omega + \nabla(\mathrm{div}_f \, \omega) + \frac{1}{2}\omega. \tag{2.20}$$

For the readers' convenience and the sake of completeness, we provide a quick proof here.

Proof The above identities follow from direct computations given below.

• For (2.15):

$$\begin{aligned}\nabla_i \Delta_f u &= \nabla_i \nabla_j \nabla_j u - \nabla_i \nabla_j f \nabla_j u - \nabla_j f \nabla_i \nabla_j u \\ &= \Delta \nabla_i u + R_{ijjk} \nabla_k u - \frac{1}{2} \nabla_i u + R_{ij} \nabla_j u - \nabla_j f \nabla_j \nabla_i u \\ &= \Delta_f \nabla_i u - \frac{1}{2} \nabla_i u.\end{aligned}$$

• For (2.16): It follows from (2.15) that

$$\begin{aligned}\int_M u \,\mathrm{div}_f(\Delta_f \omega)\, e^{-f} dV &= \int_M - < \Delta_f \nabla u, \omega > e^{-f}\, dV \\ &= \int_M - \left\langle \nabla(\Delta_f u) + \frac{1}{2} \nabla u, \omega \right\rangle e^{-f} dV \\ &= \int_M u \left(\Delta_f \,\mathrm{div}_f\, \omega + \frac{1}{2}\, \mathrm{div}_f\, \omega \right) e^{-f} dV.\end{aligned}$$

• For (2.17):

$$2\mathcal{L}_f \,\mathrm{div}_f^{\dagger}\, \omega = -\frac{1}{2} \Delta_f (\nabla_i \omega_j + \nabla_j \omega_i) - R_{ikjl}(\nabla_k \omega_l + \nabla_l \omega_k).$$

Notice that

$$\begin{aligned}\Delta_f \nabla_i \omega_j &= \nabla_k \nabla_k \nabla_i \omega_j - \nabla_k f \nabla_k \nabla_i \omega_j \\ &= \nabla_k (\nabla_i \nabla_k \omega_j + R_{kijl} \omega_l) - \nabla_k f (\nabla_i \nabla_k \omega_j + R_{kijl} \omega_l) \\ &= \nabla_i \Delta \omega_j + R_{il} \nabla_l \omega_j + R_{kijl} \nabla_k \omega_l + \nabla_k R_{kijl} \omega_l + R_{kijl} \nabla_k \omega_l \\ &\quad - \nabla_i (\nabla_k f \nabla_k \omega_j) + \nabla_i \nabla_k f \nabla_k \omega_j + R_{kijl} \nabla_k f \omega_l \\ &= \nabla_i \Delta_f \omega_j - 2 R_{ikjl} \nabla_k \omega_l + \frac{1}{2} \nabla_i \omega_j.\end{aligned}$$

• For (2.18): According to (2.5), (2.18) is equivalent to (2.17).

• For (2.19): Similar to the proof of (2.16), (2.19) is the adjoint of (2.17) with respect to the inner product (2.7).

• For (2.20):

$$\begin{aligned}\mathrm{div}_f (\mathscr{L}_{\omega^\sharp} g)_j &= \nabla_i (\nabla_i \omega_j + \nabla_j \omega_i) - \nabla_i f (\nabla_i \omega_j + \nabla_j \omega_i) \\ &= \Delta_f \omega_j + \nabla_j \nabla_i \omega_i + R_{jk} \omega_k - \nabla_j (\nabla_i f \omega_i) + \nabla_j \nabla_i f \omega_i \\ &= \Delta_f \omega_j + \nabla_j \,\mathrm{div}_f\, \omega + \frac{1}{2} \omega_j.\end{aligned}$$

□

Remark 2.2 Some of the identities in Lemma 2.2 were first obtained in [9] for positive Einstein manifolds.

For positive Einstein manifolds, He and the first author also showed in [9] that the restriction of N_f to the subspace $\mathrm{Im}(\mathrm{div}_f^\dagger)$ is zero, i.e., $N_f|_{\mathrm{Im}(\mathrm{div}_f^\dagger)} = 0$, a fact first noted in Cao–Hamilton–Ilmanen [8]. By using identities (2.16), (2.18) and (2.20) in Lemma 2.2, Mehrmohamadi and Razavi [32] were able to generalize this to the case of compact shrinking Ricci solitons.

Lemma 2.3 [32] *Let (M^n, g, f) be a compact shrinking Ricci soliton satisfying* (2.1). *Then, we have*

$$N_f|_{\mathrm{Im}(\mathrm{div}_f^\dagger)} = 0.$$

Proof Notice that, according to (2.20) and (2.16),

$$\begin{aligned}\mathrm{div}_f\,\mathrm{div}_f(\mathscr{L}_{\omega^\sharp} g) &= \mathrm{div}_f\left(\Delta_f\omega + \nabla\,\mathrm{div}_f\,\omega + \frac{1}{2}\omega\right)\\ &= 2\Delta_f(\mathrm{div}_f\,\omega) + \mathrm{div}_f\,\omega.\end{aligned} \tag{2.21}$$

Thus, if we denote by $\xi = \mathscr{L}_{\omega^\sharp} g$, then according to (2.14)

$$\hat{v}_\xi = 2\,\mathrm{div}_f\,\omega. \tag{2.22}$$

Now, by (2.5), (2.18), (2.20) and (2.22), we obtain

$$\begin{aligned}-2N_f(\mathrm{div}_f^\dagger\,\omega) &= N_f(\mathscr{L}_{\omega^\sharp} g)\\ &= \mathcal{L}_f(\mathscr{L}_{\omega^\sharp} g) + \mathrm{div}_f^\dagger\,\mathrm{div}_f(\mathscr{L}_{\omega^\sharp} g) + \nabla^2(\mathrm{div}_f\,\omega)\\ &= \frac{1}{2}\mathscr{L}_{(\Delta_f\omega)^\sharp} g + \frac{1}{4}\mathscr{L}_{\omega^\sharp} g + \mathrm{div}_f^\dagger(\Delta_f\omega) + \mathrm{div}_f^\dagger\big(\nabla(\mathrm{div}_f\,\omega)\big)\\ &\quad + \frac{1}{2}\,\mathrm{div}_f^\dagger\,\omega + \nabla^2(\mathrm{div}_f\,\omega)\\ &= 0.\end{aligned} \tag{2.23}$$

□

3 Proof of the main theorems

In this section, we prove Theorems 1.1 and 1.2 stated in the introduction. Once again, by scaling the metric g, we normalize $\tau = 1$ and assume that (M^n, g, f) is a compact shrinking Ricci soliton satisfying

$$Rc + \nabla^2 f = \frac{1}{2} g. \tag{3.1}$$

First of all, recall that we have the following direct sum decomposition

$$\mathrm{Sym}^2(T^*M) = \mathrm{Im}(\mathrm{div}_f^\dagger) \oplus \mathbb{R}\cdot\mathrm{Rc} \oplus \mathrm{Ker}(\mathrm{div}_f)_0, \tag{3.2}$$

where $\mathbb{R}\cdot \mathrm{Rc}$ is the one dimensional subspace generated by the Ricci tensor Rc and $\mathrm{Ker}(\mathrm{div}_f)_0$, as defined in (1.8), denotes the orthogonal complement of $\mathbb{R}\cdot \mathrm{Rc}$ in $\mathrm{Ker}(\mathrm{div}_f)$ with respect to the weighted inner product $\int_M <\cdot,\cdot> e^{-f}\, dV$.

We divide the proof of Theorem 1.1 into two propositions.

Proposition 3.1 *The subspaces* $\mathrm{Im}(\mathrm{div}_f^\dagger)$, $\mathbb{R}\cdot\mathrm{Rc}$, $\mathrm{Ker}(\mathrm{div}_f)_0$ *are invariant subspaces of the linear operator* $\mathcal{L}_f$. *Moreover, (3.2) is an orthogonal decomposition with respect to the quadratic form* $\delta_g^2\nu(h,h)$ *of the second variation in Theorem 2.1.*

Proof Firstly, by (2.17),

$$\mathcal{L}_f(\mathrm{div}_f^\dagger\,\omega) = \frac{1}{2}\,\mathrm{div}_f^\dagger\left(\Delta_f\omega + \frac{1}{2}\omega\right) \in \mathrm{Im}(\mathrm{div}_f^\dagger).$$

This shows that $\mathrm{Im}(\mathrm{div}_f^\dagger)$ is invariant under $\mathcal{L}_f$.

Next, from Lemma 2.1(ii), we have

$$\mathcal{L}_f Rc = \frac{1}{2}Rc.$$

Hence, $\mathbb{R}\cdot Rc$ is an invariant subspace of $\mathcal{L}_f$.

Finally, for any $h\in \mathrm{Ker}(\mathrm{div}_f)_0$, it follows from (2.19) that

$$\mathrm{div}_f(\mathcal{L}_f h) = \frac{1}{2}\left(\Delta_f\,\mathrm{div}_f\,h + \frac{1}{2}\,\mathrm{div}_f\,h\right) = 0.$$

Moreover, since $\mathcal{L}_f Rc = \frac{1}{2}Rc$, it follows that

$$\begin{aligned}\int_M < \mathcal{L}_f h, Rc > \; e^{-f} dV &= \int_M < h, \mathcal{L}_f Rc > \; e^{-f} dV\\ &= \frac{1}{2}\int_M < h, Rc > \; e^{-f} dV = 0,\end{aligned}$$

i.e., $\mathcal{L}_f h\in \mathrm{Ker}(\mathrm{div}_f)_0$. Therefore, $\mathrm{Ker}(\mathrm{div}_f)_0$ is also invariant under $\mathcal{L}_f$.

Furthermore, the invariant subspace property just demonstrated together with the fact that $\mathrm{Im}(\mathrm{div}_f^\dagger)$, $\mathbb{R}\cdot\mathrm{Rc}$, and $\mathrm{Ker}(\mathrm{div}_f)_0$ are mutually orthogonal to each other (with respect to the weighted inner product) immediately imply that the decomposition (1.9) of $\mathrm{Sym}^2(T^*M)$ is also orthogonal with respect to the second variation $\delta_g^2\nu(h,h)$ of the ν-entropy. □

Proposition 3.2 *Let* (M^n, g, f) *be a compact shrinking Ricci soliton satisfying (3.1). Then, the eigenvalues of* $\mathcal{L}_f$ *on* $\mathrm{Im}(\mathrm{div}_f^\dagger)$ *are strictly less than* $\frac{1}{4}$.

Proof Suppose that λ is an eigenvalue of $\mathcal{L}_f$ on $\mathrm{Im}(\mathrm{div}_f^\dagger)$, and

$$\mathcal{L}_f(\mathscr{L}_{\omega^\sharp}g) = \lambda\mathscr{L}_{\omega^\sharp}g$$

for some $\mathscr{L}_{\omega^\sharp} g \equiv -2\,\mathrm{div}_f^\dagger\,\omega \in \mathrm{Im}(\mathrm{div}_f^\dagger)$ with $\mathscr{L}_{\omega^\sharp} g \neq 0$. We need to show $\lambda < \frac{1}{4}$.

Since $N_f = 0$ on $\mathrm{Im}(\mathrm{div}_f^\dagger)$ by Lemma 2.3, from (2.22) and (2.23), we have

$$
\begin{aligned}
0 &= N_f(\mathscr{L}_{\omega^\sharp} g) \\
&= \mathcal{L}_f(\mathscr{L}_{\omega^\sharp} g) + \mathrm{div}_f^\dagger\,\mathrm{div}_f\,\mathscr{L}_{\omega^\sharp} g + \nabla^2\,\mathrm{div}_f\, w \\
&= \lambda \mathscr{L}_{\omega^\sharp} g + \mathrm{div}_f^\dagger\,\mathrm{div}_f\,\mathscr{L}_{\omega^\sharp} g + \nabla^2\,\mathrm{div}_f\,\omega \\
&= -2\lambda\,\mathrm{div}_f^\dagger\,\omega - 2\,\mathrm{div}_f^\dagger\,\mathrm{div}_f\,\mathrm{div}_f^\dagger\,\omega - \mathrm{div}_f^\dagger\,\nabla\,\mathrm{div}_f\,\omega \\
&= -\,\mathrm{div}_f^\dagger(2\lambda\omega + 2\,\mathrm{div}_f\,\mathrm{div}_f^\dagger\,\omega + \nabla\,\mathrm{div}_f\,\omega).
\end{aligned}
\tag{3.3}
$$

Claim. The following identity holds,

$$
\Delta_f\,\mathrm{div}_f\,\omega = (2\lambda - 1)\,\mathrm{div}_f\,\omega. \tag{3.4}
$$

Indeed, it follows from (2.18) that

$$
2\mathcal{L}_f(\mathscr{L}_{\omega^\sharp} g) = \mathscr{L}_{(\Delta_f\omega + \frac{1}{2}\omega)^\sharp} g.
$$

From (2.21), we know that

$$
\hat{v}_{\mathscr{L}_{\omega^\sharp} g} = 2\,\mathrm{div}_f\,\omega.
$$

Here, for any symmetric 2-tensor h, $\hat{v}_h$ is given by (2.14). Hence,

$$
\begin{aligned}
2\hat{v}_{\mathcal{L}_f(\mathscr{L}_{\omega^\sharp} g)} &= \hat{v}_{\mathscr{L}_{(\Delta_f\omega + \frac{1}{2}w)^\sharp} g} \\
&= 2\,\mathrm{div}_f(\Delta_f\omega + \frac{1}{2}\omega) \\
&= 2(\Delta_f\,\mathrm{div}_f\,\omega + \mathrm{div}_f\,\omega),
\end{aligned}
$$

where, in the last step above, we have used (2.16).

Since $\mathcal{L}_f(\mathscr{L}_{\omega^\sharp} g) = \lambda \mathscr{L}_{\omega^\sharp} g$, we get

$$
\begin{aligned}
\Delta_f\,\mathrm{div}_f\,\omega + \mathrm{div}_f\,\omega &= \hat{v}_{\mathcal{L}_f(\mathscr{L}_{\omega^\sharp} g)} \\
&= \lambda \hat{v}_{\mathscr{L}_{\omega^\sharp} g} \\
&= 2\lambda\,\mathrm{div}_f\,\omega,
\end{aligned}
$$

i.e.,

$$
\Delta_f\,\mathrm{div}_f\,\omega = (2\lambda - 1)\,\mathrm{div}_f\,\omega.
$$

This proves the Claim.

Now, we divide the rest of our argument into two cases.

Case 1: $\operatorname{div}_f \omega$ is not a constant.

In this case, by the Claim, $\operatorname{div}_f \omega$ is an eigenfunction of Δ_f with eigenvalue $1-2\lambda$. On the other hand, from [11], we know that the first eigenvalue of Δ_f is greater than $1/2$. Thus, $1-2\lambda > \frac{1}{2}$; hence $\lambda < \frac{1}{4}$.

Case 2: $\operatorname{div}_f \omega$ is a constant.

In this case, we have

$$\begin{aligned}\int_M |\operatorname{div}_f \omega|^2 e^{-f} dV &= \int_M < \omega, \operatorname{div}_f^\dagger \operatorname{div}_f \omega > e^{-f} dV \\ &= -\int_M < \omega, \nabla(\operatorname{div}_f \omega) > e^{-f} dV \\ &= 0.\end{aligned}$$

It follows that $\operatorname{div}_f \omega = 0$. So (3.3) becomes

$$\operatorname{div}_f^\dagger(\lambda\omega + \operatorname{div}_f \operatorname{div}_f^\dagger \omega) = 0.$$

Multiplying both sides of the above identity by $\operatorname{div}_f^\dagger \omega$ and integrating yields

$$\int_M \left(\lambda |\operatorname{div}_f^\dagger \omega|^2 + |\operatorname{div}_f \operatorname{div}_f^\dagger \omega|^2\right) e^{-f} dV = 0.$$

Since $\operatorname{div}_f^\dagger \omega = -\frac{1}{2}\mathscr{L}_{\omega^\sharp} g \neq 0$ by assumption, we have $\lambda \le 0 < 1/4$.

Therefore, we have shown that $\lambda < \frac{1}{4}$. This concludes the proof of Proposition 3.2 and Theorem 1.1. □

Finally, we are ready to prove Theorem 1.2.

Proof By Theorem 2.1, a compact shrinking Ricci soliton (M^n, g, f) is linearly stable if and only if

$$\delta_g^2 \nu(h,h) := \frac{1}{(4\pi)^{n/2}} \int_M < N_f h, h > e^{-f} dV \le 0$$

for every $h \in \operatorname{Sym}^2(T^*M) = \operatorname{Im}(\operatorname{div}_f^\dagger) \oplus \mathbb{R}\cdot \operatorname{Rc} \oplus \operatorname{Ker}(\operatorname{div}_f)_0$.

However, by Theorem 1.1(i) (i.e., Proposition 3.1), we have

$$\begin{aligned}\int_M < N_f h, h > e^{-f} dV &= \int_M < N_f h_1, h_1 > e^{-f} dV + \int_M < N_f h_2, h_2 > e^{-f} dV \\ &\quad + \int_M < N_f h_0, h_0 > e^{-f} dV \\ &= \int_M < N_f h_2, h_2 > e^{-f} dV + \int_M < N_f h_0, h_0 > e^{-f} dV,\end{aligned}$$

where

$$h = h_1 + h_2 + h_0, \quad \text{with } h_1 \in \mathrm{Im}(\mathrm{div}_f^\dagger),\ h_2 \in \mathbb{R}\cdot \mathrm{Rc},\ h_0 \in \mathrm{Ker}(\mathrm{div}_f)_0,$$

and, in the last equality, we have used the fact that $\delta_g^2 \nu(h_1, h_1) = 0$ for $h_1 \in \mathrm{Im}(\mathrm{div}_f^\dagger)$ due to the diffeomorphism invariance of the ν-entropy.

On the other hand, since $\mathrm{div}_f\, Rc = 0$ and $\mathcal{L}_f Rc = \frac{1}{2} Rc$ by Lemma 2.1, we obtain

$$\begin{aligned} N_f(Rc) &= \mathcal{L}_f Rc - \frac{\int_M |Rc|^2 e^{-f} dV}{\int_M R\, e^{-f} dV} Rc \\ &= \mathcal{L}_f Rc - \frac{1}{2} Rc = 0, \end{aligned}$$

where we have used the fact that

$$\int_M |Rc|^2 e^{-f} dV = \frac{1}{2}\int_M R\, e^{-f} dV,$$

because the scalar curvature R satisfies the well-known equation $\Delta_f R = R - 2|Rc|^2$. Hence, $N_f = 0$ on $\mathbb{R}\cdot \mathrm{Rc}$, and it follows that

$$\int_M < N_f h_2, h_2 > e^{-f} dV = 0.$$

Also, as $N_f = \mathcal{L}_f$ on $\mathrm{Ker}(\mathrm{div}_f)_0$, we immediately conclude that

$$\begin{aligned} \int_M < N_f h, h > e^{-f} dV &= \int_M < N_f h_0, h_0 > e^{-f} dV \\ &= \int_M < \mathcal{L}_f h_0, h_0 > e^{-f} dV. \end{aligned}$$

Therefore, $\delta_g^2 \nu(h, h) \leq 0$ if and only if

$$\int_M < \mathcal{L}_f h_0, h_0 > e^{-f} dV \leq 0.$$

This finishes the proof of Theorem 1.2. □

Remark 3.1 In the proof of Theorem 1.2, if we use Lemma 2.3 instead of Theorem 1.1 (i) then we would get the following more explicit information about the Jacobi operator N_f.

Proposition 3.3

$$N_f = \begin{cases} 0, & on\ \mathrm{Im}(\mathrm{div}_f^\dagger); \\ 0, & on\ \mathbb{R}\cdot \mathrm{Rc}; \\ \mathcal{L}_f & on\ \mathrm{Ker}(\mathrm{div}_f)_0. \end{cases} \tag{3.5}$$

In particular, $N_f \leq 0$ *on* $\mathrm{Sym}^2(T^*M)$ *if and only if* $\mathcal{L}_f \leq 0$ *on* $\mathrm{Ker}(\mathrm{div}_f)_0$.

Remark 3.2 Suppose $\xi = \mathscr{L}_{\omega^\sharp} g$ is an eigen-tensor of $\mathcal{L}_f$ for some 1-form ω, with

$$\mathcal{L}_f \xi = \lambda \xi.$$

Then one can show that $\mathrm{div}_f^\dagger \mathrm{div}_f \xi$ and $\nabla^2 \mathrm{div}_f \omega$ are also eigen-tensors of $\mathcal{L}_f$ with the same eigenvalue, i.e.,

$$\mathcal{L}_f(\mathrm{div}_f^\dagger \mathrm{div}_f \xi) = \lambda(\mathrm{div}_f^\dagger \mathrm{div}_f \xi),$$

and

$$\mathcal{L}_f(\nabla^2 \mathrm{div}_f \omega) = \lambda(\nabla^2 \mathrm{div}_f \omega).$$

Indeed, if $\mathcal{L}_f(\xi) = \lambda\xi$ then, by using the identity

$$\mathrm{div}_f^\dagger \mathrm{div}_f(\mathcal{L}_f h) = \mathcal{L}_f(\mathrm{div}_f^\dagger \mathrm{div}_f h) \tag{3.6}$$

shown in [32], we have

$$\begin{aligned}\mathcal{L}_f(\mathrm{div}_f^\dagger \mathrm{div}_f \xi) &= \mathrm{div}_f^\dagger \mathrm{div}_f(\mathcal{L}_f \xi) \\ &= \lambda(\mathrm{div}_f^\dagger \mathrm{div}_f \xi).\end{aligned}$$

On the other hand, by setting $u = \mathrm{div}_f \omega$ and combining (3.4) with (2.18) and (2.15), we get

$$\begin{aligned}2\mathcal{L}_f(\nabla^2 u) &= \mathcal{L}_f(\mathscr{L}_{\nabla u} g) \\ &= \frac{1}{2}\mathscr{L}_{(\Delta_f(du))^\sharp} g + \frac{1}{2}\mathscr{L}_{\frac{1}{2}\nabla u} g \\ &= \frac{1}{2}\mathscr{L}_{\nabla(\Delta_f u + u)} g \\ &= \frac{1}{2}\mathscr{L}_{2\lambda \nabla u} g \\ &= 2\lambda \nabla^2 u.\end{aligned}$$

To conclude our paper, we mention two open problems.

Conjecture 1 (Hamilton; 2004 [5, 6]) $\mathbb{S}^4$ and $\mathbb{CP}^2$ are the only ν-stable four-dimensional positive Einstein manifolds.

Conjecture 2 (Cao; 2006 [5, 6]) A ν-stable compact shrinking Ricci soliton is necessarily Einstein, at least in dimension four.

Remark 3.3 Besides $\mathbb{S}^4$ and $\mathbb{CP}^2$, the other known positive Einstein 4-manifolds are the Kähler–Einstein manifolds $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2\#(-k\mathbb{CP}^2)$ $(3 \leq k \leq 8)$, and the (non-Kähler Einstein but conformally Kähler) Page metric [37] on $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ and Chen–LeBrun–Weber metric [15] on $\mathbb{CP}^2\#(-2\mathbb{CP}^2)$. Note that, for $n > 4$, He and the first author [9] have found a strictly stable positive Einstein manifold, other than the round sphere $\mathbb{S}^n$, in dimension 8.

Acknowledgements We thank Professor Detang Zhou for his interest in our work and for his helpful question that led to a simpler proof of Proposition 3.2.

Data Availability No data associate for the submission.

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