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Classification of minimal immersions of conformally flat 3-tori and 4-tori into spheres by the first eigenfunctions

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Abstract

The Clifford 2-torus in $\mathbb{S}^3$ and the equlaterial 2-torus in $\mathbb{S}^5$ are known as the only minimal immersions of 2-tori into spheres by the first eigenfunctions (called λ_1-minimal for short). For $n \geq 3$, the Clifford n-torus in $\mathbb{S}^{2n-1}$ might be the only known example of λ_1-minimal n-tori in the literature. By discussing the general construction of homogeneous minimal flat n-tori in spheres, we construct several new examples of λ_1-minimal flat 3-tori and 4-tori. In particular, the existence of 2-parameter family of non-congruent λ_1-minimal flat 4-tori is shown for the first time. We obtain the complete classification for λ_1-minimal immersions of conformally flat 3-tori and 4-tori in spheres, by some detailed investigations of shortest vectors in lattices, which could be of independent interests. Using them, we also solve the Berger's problem (finding the maximal value of the dilation-invariant functional $\lambda_1(g)V(g)^{\frac{2}{n}}$) among all flat 3-tori and 4-tori.

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1 Introduction

The study of minimal immersions of Riemannian manifolds into spheres is an interesting topic in differential geometry. It builds a deep link between the spectral theory and minimal submanifold theory. The famous theorem of Takahashi [31] states that the isometric immersion $x : (M^n, g) \to \mathbb{S}^m$ is minimal, if and only if the coordinate functions of x are eigenfunctions of the Laplacian with respect to the eigenvalue n.

Given a closed Riemannian manifold (M^n, g), we denote the volume and the first eigenvalues of (M^n, g) respectively by $V(g)$ and $\lambda_1(g)$. The dilation-invariant functional

$$\mathcal{L}_1(g) \triangleq \lambda_1(g) V(g)^{\frac{2}{n}}$$

on the set of all smooth Riemannian metrics is a basic functional considered in spectral theory. It was shown in [10] (see also [12]) that the critical metric of $\mathcal{L}_1(g)$ among all smooth Riemannian metrics on M^n (called λ_1-*critical metric*) admits an isometric minimal immersion of M^n in spheres by the first eigenfunctions (called λ_1-*minimal immersion*).

Moreover, it was proved by Hersch [15] that on the topological 2-sphere, among all smooth Riemannian metrics, 8π is the maximal value of $\mathcal{L}_1(g)$, which can only be attained by the standard metric. In 1973, Berger [1] obtained the maximal value of $\mathcal{L}_1(g)$ on the topological 2-torus among all flat metrics. This value is attained by the λ_1-minimal immersion of the equilateral 2-torus in $\mathbb{S}^5$. Since then, finding the uniform upper bound of $\mathcal{L}_1(g)$ among all smooth metrics is referred as *Berger's problem*. By introducing the conformal volume, Li and Yau [21] solved the case of $\mathbb{R}P^2$ in 1982, whose upper bound is attained by the λ_1-minimal immersion of $\mathbb{R}P^2$ in $\mathbb{S}^4$ (i.e., Veronese surface). Due to also the work of El Soufi and Ilias [9], on any closed manifold of dimension n, Li–Yau's conformal volume can be used to provide an upper bound for $\mathcal{L}_1(g)$ in the conformal class $[g]$. In the mean time, they also show that for the conformal manifold $(M^n, [g])$ admitting a λ_1-minimal immersion in spheres, the volume of such λ_1-minimal immersion is exactly equal to the conformal volume. Therefore it realizes the supremum for $\mathcal{L}_1(g)$ in $[g]$. Furthermore, for a given conformal manifold, Montiel–Ros [23] and El Soufi-Ilias [10] proved there exists at most one λ_1-minimal metric in the conformal class. Combining this with the existence of λ_1-maximal metric, Nadirashvili finally solved the Berger's problem completely for the topological 2-torus in [25]. We point out that for the case of Klein bottle, the Berger's problem has also been solved due to the work of Nadirashvili [25], Jakobson-Nadirashvili-Polterovich [17] and El Soufi-Giacomini-Jazar [13], where one λ_1-minimal immersion of the Klein bottle is presented (see [16, 17]). For surfaces of higher genus, we refer to [19, 22, 28] and references therein. In contrast to the dimension 2, Colbois and Dodziuk proved in [4] that there is no uniform bound for the functional $\mathcal{L}_1(g)$ on any closed manifold of dimension $n \geq 3$. This implies that one might consider the Berger's problem restricting to a given conformal class (see [11, 18, 27] and reference therein), for which the investigation of λ_1-minimal immersions of higher dimensional manifolds into spheres plays an important role.

In the literature, there have been several known classes of λ_1-minimal submanifolds in spheres. A famous conjecture of Yau states that any closed embedded minimal hypersurface in $\mathbb{S}^{n+1}$ is λ_1-minimal. Due to the work of Muto–Onita–Urakawa [24] and Tang–Yan [32] on this conjecture (see also [34]), we know all isoparametric hypersurfaces and some ones of their focal submanifolds form a class of λ_1-minimal submanifolds in spheres. Another class of examples is due to Takahashi [31], who proved that for any positive integer k, up to a dilation of the metric, any compact irreducible homogeneous Riemannian manifold can be immersed minimally into a certain sphere by the kth eigenfunctions (we call it λ_k*-minimal immersion* for short). Later, the case of sphere equipped with the constantly curved metric was investigated in detail by Do Carmo and Wallach [6]. It was proved that when $n \geq 3$, the linearly full λ_k-minimal immersion of n-sphere has rigidity if and only if $k \leq 3$. Moreover, they also proved that the immersion will span the full k-eigenspace when $k \leq 3$. In this paper, we will show that these two properties do not hold for minimal flat tori of dimension 4, even for the case of λ_1-minimal immersion. To be precise, we construct a λ_1-minimal flat 4-torus in $\mathbb{S}^{11}$, which has rigidity but does not span the whole eigenspace (see Example 4.9). Furthermore, a 2-parameter family of non-congruent λ_1-minimal flat 4-torus in $\mathbb{S}^{23}$ is also constructed, among which there is a 1-parameter family living in $\mathbb{S}^{15}$, neither rigid nor fully-spanning the eigenspace (see Example 1.1, Proposition 6.3 and Remark 6.4).

By Kenmotsu [20] and Bryant [2], all minimal flat 2-tori in spheres are homogeneous. Let Λ_n denote a lattice of rank n. In [2], Bryant proved that a flat torus $T^2 = \mathbb{R}/\Lambda_2$ admits minimal immersions in spheres, if and only if the Gram matrix of Λ_2 is rational (i.e., all entries are rational numbers) up to some dilation. This implies there are infinite non-congruent minimal flat 2-tori in spheres. But among them, there are only two λ_1-minimal ones: the Clifford 2-torus in $\mathbb{S}^3$, and the equilateral 2-torus in $\mathbb{S}^5$. This classification is due to the work of Montiel–Ros [23] and El Soufi-Ilias [10].

In contrast to the plentiful results on dimension 2 in the literature, minimal flat tori of higher dimension haven't been investigated so much, especially for those λ_1-minimal ones. As far as we know, the Clifford n-torus in $\mathbb{S}^{2n-1}$ is the only known λ_1-minimal example (see [26]). In this paper, we construct four non-congruent λ_1-minimal flat

Table 1 The classification of λ_1-minimal immersions of confromally flat 3-tori

$\mathbb{S}^n$	Total Numbers	Reducible	Irreducible	Examples	$\lambda_1(g)V(g)^{\frac{2}{n}}$
$\mathbb{S}^5$	1	1	0	E.g. 4.2	$4\pi^2$
$\mathbb{S}^7$	2	1	1	E.g. 4.2, 4.3	$\frac{4\sqrt[3]{4}}{\sqrt[3]{3}}\pi^2, 3\sqrt[3]{4}\pi^2$
$\mathbb{S}^9$	1	0	1	E.g. 4.4	$\frac{8\sqrt[3]{2}}{\sqrt[3]{9}}\pi^2$
$\mathbb{S}^{11}$	1	0	1	E.g. 4.5	$4\sqrt[3]{2}\pi^2$

3-tori, a 2-parameter family and another sixteen non-congruent λ_1-minimal flat 4-tori in spheres. Among them, the 2-parameter family described as follows is the most interesting.

Example 1.1 Denote by $\{e_i\}$ the standard basis of $\mathbb{R}^4$. The flat 4-torus

$$T^4 = \mathbb{R}^4/\mathrm{Span}_{\mathbb{Z}}\{e_1 - e_4, e_2 - e_4, e_3 - e_4, 2e_4\}$$

admits a 2-parameter family of non-congruent λ_1-minimal immersions in $\mathbb{S}^{23}$ given as follows:

$$\Big(a_1e^{i\pi(u_1+u_2+u_3+u_4)}, a_1e^{i\pi(u_1+u_2-u_3-u_4)}, a_1e^{i\pi(u_1-u_2+u_3-u_4)}, a_1e^{i\pi(-u_1+u_2+u_3-u_4)},$$
$$a_2e^{i\pi(u_1+u_2+u_3-u_4)}, a_2e^{i\pi(u_1+u_2-u_3+u_4)}, a_2e^{i\pi(u_1-u_2+u_3+u_4)}, a_2e^{i\pi(u_1-u_2-u_3-u_4)},$$
$$a_3e^{2i\pi u_1}, a_3e^{2i\pi u_2}, a_3e^{2i\pi u_3}, a_3e^{2i\pi u_4}\Big),$$

where $0 \le a_1 \le a_2 \le a_3$ and $a_1^2 + a_2^2 + a_3^2 = \frac{1}{4}$. See Sect. 6 for more details.

It turns out that the examples constructed in Sect. 4 exhaust all non-congruent λ_1-minimal immersions of conformally flat 3-tori and 4-tori into spheres.

Theorem 1 (1) *Up to congruence, there are five λ_1-minimal immersions of conformally flat* 3*-tori in spheres, see Table 1.*
(2) *Up to congruence, there are sixteen, as well as a* 2*-parameter family, λ_1-minimal immersions of conformally flat* 4*-tori in spheres, see Table 2.*

For the definition of irreducible and reducible appearing in Tables 1 and 2, see Sect. 4.

The construction of these examples is based on the variational characterizations (see Theorems 3.2 and 3.6) we obtained for homogeneous minimal flat tori in spheres. Roughly speaking, the construction of a homogeneous minimal flat n-torus in some sphere is equivalent to finding a 2-tuple $\{Y, Q\}$ (we call it *matrix data*) satisfying some constrains, where Q^{-1} is a Gram matrix of the lattice corresponding to this torus, and Y is a set of finite integer vectors in $\mathbb{Z}^n$ describing the linear relations between lattice vectors involved in the minimal immersion. Theoretically, all homogeneous minimal flat tori in spheres can be constructed by the approaches we provided (see Sect. 3.1).

Table 2 The classification of λ_1-minimal immersions of confromally flat 4-tori

$\mathbb{S}^n$	Total numbers	Reducible	Irreducible	Examples	$\lambda_1(g)V(g)^{\frac{2}{n}}$
$\mathbb{S}^7$	2	1	1	E.g. 4.6, 1.1	$4\pi^2, 4\sqrt{2}\pi^2$
$\mathbb{S}^9$	3	2	1	E.g. 4.6, 4.7	$\frac{4\sqrt{2}}{\sqrt[4]{3}}\pi^2, 2\sqrt[4]{27}\pi^2, \frac{16}{\sqrt[4]{125}}\pi^2$
$\mathbb{S}^{11}$	4	2	2	E.g. 4.6, 4.8, 4.9	$\frac{8}{\sqrt{3}}\pi^2, \frac{8}{\sqrt{3}}\pi^2, 2\sqrt{6}\pi^2, \frac{8\sqrt{2}}{\sqrt[4]{27}}\pi^2$
$\mathbb{S}^{13}$	3	1	2	E.g. 4.6, 4.10, 4.11	$4\sqrt[4]{2}\pi^2, \frac{8\sqrt{2}}{\sqrt[4]{27}}\pi^2, \frac{4\sqrt[4]{26\sqrt{13}-70}}{\sqrt{3}}\pi^2$
$\mathbb{S}^{15}$	1-family & 2	0	1-family & 2	E.g. 1.1, 4.12, 4.13	$4\sqrt{2}\pi^2, \frac{4\sqrt[4]{8}}{\sqrt[4]{3}}\pi^2, \frac{4\sqrt[4]{8\sqrt{3}+12}}{\sqrt{3}}\pi^2$
$\mathbb{S}^{17}$	2	0	2	E.g. 4.14, 4.15	$\frac{16}{3}\pi^2, 4\sqrt[4]{3}\pi^2$
$\mathbb{S}^{19}$	1	0	1	E.g. 4.16	$\frac{8}{\sqrt[4]{5}}\pi^2$
$\mathbb{S}^{23}$	2-family	0	2-family	E.g. 1.1	$4\sqrt{2}\pi^2$

To construct λ_1-minimal flat n-tori, we also need to deal with the problem of finding all shortest vectors in a lattice, which is important but difficult in the theory of lattice (or geometry of numbers). Fortunately, a result of Ryshkov [29] proved in Minkowski's reduction theory can be used in our construction to overcome this obstacle.

To classify all λ_1-minimal immersions of conformally flat 3-tori and 4-tori in spheres, it follows from the work of El Soufi and Ilias [10] that we only need to classify all λ_1-minimal immersions of flat 3-tori and 4-tori in spheres. It turns out that they are all homogeneous (in fact, a sufficient condition is given in Proposition 2.7 for general minimal flat tori in spheres to be homogeneous). Note that the moduli space of flat tori (modulo isometry) is $SL(n, \mathbb{Z}) \setminus GL(n, \mathbb{R}) \,/\, O(n)$ (see [36], or Sect. 2). The action of $SL(n, \mathbb{Z})$ makes the classification highly nontrivial. However, our variational characterization suggests that all we need to do is to find out all the possible integer sets Y, where Q is uniquely determined if it exists. To do this, a coarse classification to lattices of rank no more than 4 is given at first (see Theorem 5.8), from which some necessary constrains on Y can be obtained. Then after introducing some invariants to the set of shortest lattice vectors, we can determine all the possibilities of Y up to the action of $SL(n, \mathbb{Z})$.

The volumes for these λ_1-minimal flat tori we construct are also calculated (see Sect. 4). It follows from the theory of conformal volume that these λ_1-minimal metrics maximize the functional $\lambda_1(g)V(g)^{\frac{2}{n}}$ in their respective conformal classes. Among all λ_1-minimal flat 3-tori, the maximal value of $\lambda_1(g)V(g)^{\frac{2}{n}}$ is $4\sqrt[3]{2}\pi^2$, and it is $4\sqrt[2]{2}\pi^2$ among all λ_1-minimal flat 4-tori (see Tables 1 and 2). In fact, using the investigation about lattices given in Sects. 5 and 6, we can prove the following theorem, which can be seen as a generalization of Berger's result from flat 2-tori to flat 3 and 4-tori.

Theorem 2 *Consider the dilation-invariant functional $\lambda_1(g)V(g)^{\frac{2}{n}}$ on the topological n-torus.*

(1) *When $n = 3$, among all flat metrics,*

$$\lambda_1(g)V(g)^{\frac{2}{n}} \leq 4\sqrt[3]{2}\pi^2,$$

and the equality is attained by the λ_1-minimal flat 3-torus given in Example 4.5.

(2) *When $n = 4$, among all flat metrics,*

$$\lambda_1(g)V(g)^{\frac{2}{n}} \leq 4\sqrt{2}\pi^2,$$

and the equality is attained by those λ_1-minimal flat 4-tori given in Example 1.1.

Inspired by this theorem, it is natural to consider the Berger's problem on conformally flat 3-tori and 4-tori: whether $4\sqrt[3]{2}\pi^2$ and $4\sqrt{2}\pi^2$ are respectively the upper bounds of $\lambda_1(g)V(g)^{\frac{2}{n}}$ on the topological 3-torus and 4-torus among all smooth conformally flat metrics.

In [11], El Soufi and Ilias exhibited a class of flat n-tori for which the endowed flat metric maximizes $\lambda_1(g)V(g)^{\frac{2}{n}}$ on its conformal class (see Corollary 3.1 in their paper). Combining their result with our work (Theorems 1 and 5.8), we can partially solve the above problem.

Theorem 3 *Suppose g is a smooth Riemannian metric on the the topological n-torus. If g is conformal equivalent to a flat metric whose first eigenspace is of dimension no less than 2n, then* $\lambda_1(g)V(g)^{\frac{2}{n}} \leq 4\sqrt[3]{2}\pi^2$ *when* $n = 3$*, and* $\lambda_1(g)V(g)^{\frac{2}{n}} \leq 4\sqrt{\pi^2}$ *when* $n = 4$.

The paper is organized as follows. In Sect. 2, we firstly recall the basic spectral theory of flat tori, and then discuss the homogeneity of minimal flat tori in spheres. Section 3 is devoted to presenting our basic setup on homogeneous minimal flat tori, as well as the variational characterizations obtained for them. New examples of λ_1-minimal flat 3-tori and 4-tori are constructed in Sect. 4. We devote Sect. 5 to investigate the shortest vectors of lattices, where a coarse classification is given to lattices of rank no more than 4. The classification of λ_1-minimal immersions of conformally flat 3-tori and 4-tori are obtained in Sect. 6. A class of λ_1-minimal flat n-tori is presented in Sect. 7 as an application of our construction method in higher dimensions. Finally, Sect. 8 is devoted to discussing Berger's problem on conformally flat 3-tori and 4-tori, where Theorems 2 and 3 are proved.

2 On isometric minimal immersions of flat tori

In this section, we will firstly recall the basic theory of flat tori. Then a sufficient condition for minimal flat tori in spheres to be homogeneous will be given.

2.1 Flat tori and lattices

It is well known that a flat torus T^n of dimension n can be described as

$$T^n = \mathbb{R}^n/\Lambda_n,$$

where Λ_n is a lattice of rank n on $\mathbb{R}^n$. Set L_n to be the generator matrix of Λ_n, which means Λ_n can be generated by row vectors of L_n. Two tori $T^n = \mathbb{R}^n/\Lambda_n$ and $\widetilde{T}^n = \mathbb{R}^n/\widetilde{\Lambda}_n$ are isometric if and only if Λ_n and $\widetilde{\Lambda}_n$ are isometric, i.e., there exists an orthogonal matrix O and an unimodular matrix $U \in SL(n, \mathbb{Z})$, such that $L_n = U\,\widetilde{L}_n\,O$, where L_n (w.r.t. $\widetilde{L}_n$) is a generator matrix of lattice Λ_n (w.r.t. $\widetilde{\Lambda}_n$). It follows that the moduli space of flat n-tori is

$$SL(n, \mathbb{Z}) \setminus GL(n, \mathbb{R}) \,/\, O(n).$$

The dual lattice of Λ_n is defined to be a lattice Λ_n^*, whose generator matrix L_n^* satisfies $L_n(L_n^*)^t = I_n$. The spectrum of $T^n = \mathbb{R}^n/\Lambda_n$ is

$$\mathrm{Spec}(T^n) = \left\{4\pi^2|\xi|^2 \,\middle|\, \xi \in \Lambda_n^*\right\},$$

and $e^{2\pi\langle\xi,u\rangle i}$ is an eigenfunction corresponding to the eigenvalue $4\pi^2|\xi|^2$, where $u = (u_1, u_2, \cdots, u_n)$ is the coordinates of $\mathbb{R}^n$, such that the flat metric on $\mathbb{R}^n$ (T^n) can be written as $du_1^2 + du_2^2 + \cdots + du_n^2$.

2.2 Minimal homogeneous flat tori in spheres

Let $T^n = \mathbb{R}^n/\Lambda_n$ be a flat torus. Assume n is an eigenvalue of this torus, whose eigenspace is of dimension $2N$. It follows that there are exactly N distinct lattice vectors (up to ± 1) having the length $\frac{\sqrt{n}}{2\pi}$ in the dual lattice Λ_n^*, which are denoted by

$$\xi_1, \xi_2, \ldots, \xi_N.$$

By the theorem of Takahashi, any minimal isometric immersion of T^n in spheres can be expressed as follows:

$$x = \begin{pmatrix}\Theta_1 & \Theta_2 & \cdots & \Theta_N\end{pmatrix} A : T^n = \mathbb{R}^n/\Lambda_n \longrightarrow \mathbb{S}^{2N-1}, \tag{1}$$

where $\Theta_r = \begin{pmatrix}\cos\theta_r & \sin\theta_r\end{pmatrix}$, $\theta_r = 2\pi\langle\xi_r, u\rangle$ for $1 \le r \le N$, and A is a $2N \times 2N$ matrix. Write

$$AA^t = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix}, \quad A_{rs} = \begin{pmatrix} A_{rs}^{11} & A_{rs}^{12} \\ A_{rs}^{21} & A_{rs}^{22} \end{pmatrix},$$

we have the following conclusions.

Lemma 2.1 $\left\{\theta_r \pm \theta_s \,\middle|\, 1 \le r \ne s \le N\right\} \cap \left\{0, \pm 2\theta_j \,\middle|\, 1 \le j \le N\right\} = \varnothing.$

Proof This follows from the fact that $0 < |\xi_r \pm \xi_s| < |2\xi_j|$ for $r \ne s$. □

Lemma 2.2 *For every r, $1 \le r \le N$, we have*

$$A_{rr} = \begin{pmatrix} a_r & \\ & a_r \end{pmatrix}.$$

Proof By definition, we have $A_{rr}^{12} = A_{rr}^{21}$. From $|x| = 1$, we can obtain that

$$\begin{aligned} 2 = &\sum_{1\le r\le N} [(A_{rr}^{11} - A_{rr}^{22})\cos(2\theta_r) + (A_{rr}^{11} + A_{rr}^{22}) + (A_{rr}^{12} + A_{rr}^{21})\sin(2\theta_r)) \\ &+ \sum_{1\le r\ne s\le N} [(A_{rs}^{11} - A_{rs}^{22})\cos(\theta_r + \theta_s) + (A_{rs}^{11} + A_{rs}^{22})\cos(\theta_r - \theta_s) \\ &+ (A_{rs}^{12} + A_{rs}^{21})\sin(\theta_r + \theta_s) + (A_{rs}^{21} - A_{rs}^{12})\sin(\theta_r - \theta_s)]. \end{aligned}$$

Note that for all $1 \le r \le N, \theta_r \not\equiv 0$. So it follows from the Lemma 2.1 that $A_{rr}^{11} - A_{rr}^{22} = 0,\ A_{rr}^{12} + A_{rr}^{21} = 0$, which complete the proof. □

Since x is an isometric immersion, we have

$$\langle \frac{\partial x}{\partial u_k}, \frac{\partial x}{\partial u_l} \rangle = \delta_{kl}, \quad 1 \leq k, l \leq n, \tag{2}$$

where $u = (u_1, u_2, \cdots, u_n)$ is the coordinates of $\mathbb{R}^n$. Using the expression (1) of x, (2) can be rewritten as below:

$$\begin{aligned} &\sum_r \xi_{rk}\xi_{rl}\Theta_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A_{rr} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &\quad \Theta_r^t + \sum_{r<s} (\xi_{rk}\xi_{sl} + \xi_{sk}\xi_{rl})\Theta_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A_{rs} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Theta_s^t = \frac{\delta_{kl}}{4\pi^2}, \end{aligned} \tag{3}$$

where ξ_{rk} is the k-th coordinates of ξ_r, and we have used the fact that

$$\Theta_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A_{rs} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Theta_s^t = \Theta_s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A_{sr} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Theta_r^t,$$

which can be verified easily. It follows from Lemma 2.2 that the first term in the left hand side of the Eq. (3) is constant. Hence for all $1 \leq k, l \leq n$, we have

$$\begin{aligned} 0 = &\sum_{r<s} (\xi_{rk}\xi_{sl} + \xi_{sk}\xi_{rl})[(A_{rs}^{11} - A_{rs}^{22}) \cos(\theta_r + \theta_s) + (A_{rs}^{11} + A_{rs}^{22}) \cos(\theta_r - \theta_s) \\ &+ (A_{rs}^{12} + A_{rs}^{21}) \sin(\theta_r + \theta_s) + (A_{rs}^{21} - A_{rs}^{12}) \sin(\theta_r - \theta_s)]. \end{aligned} \tag{4}$$

Define $\mathcal{E} = \{\xi_r \pm \xi_s, 1 \leq r < s \leq n\}$, we call the set of pairs

$$\{(\xi_{r_1}, \xi_{s_1}), \ldots, (\xi_{r_p}, \xi_{s_p}), (\xi_{r_{p+1}}, -\xi_{s_{p+1}}), \ldots, (\xi_{r_q}, -\xi_{s_q})\}$$

η-set if

$$\xi_{r_1} + \xi_{s_1} = \cdots = \xi_{r_p} + \xi_{s_p} = \xi_{r_{p+1}} - \xi_{s_{p+1}} = \cdots = \xi_{r_q} - \xi_{s_q} = \eta \in \mathcal{E}.$$

Using $|\xi_{r_j}| = |\xi_{s_j}|$ we have

$$\langle \eta, \eta \rangle = |\xi_{r_j}|^2 + |\xi_{s_j}|^2 \pm 2\langle \xi_{r_j}, \xi_{s_j} \rangle = 2\langle \xi_{r_j}, \eta \rangle > 0, \quad 1 \leq j \leq q.$$

It is straightforward to verify that $\pm\xi_{r_1}, \ldots, \pm\xi_{r_q}, \pm\xi_{s_1}, \ldots, \pm\xi_{s_q}$ are distinct with each other.

Denote by $\xi_{r_j} \odot \xi_{s_j}$ the symmetric product of ξ_{r_j} and ξ_{s_j}, we have the following lemma.

Lemma 2.3 *Let $x : T^n = \mathbb{R}^n/\Lambda_n \longrightarrow \mathbb{S}^m$ be a linearly full minimal flat torus. If for any $\eta \in \mathcal{E}$, the η-set forms a linearly independent set of symmetric products $\{\xi_{r_1} \odot \xi_{s_1}, \xi_{r_2} \odot \xi_{s_2}, \cdots, \xi_{r_q} \odot \xi_{s_q}\}$, then m is odd and x is homogeneous.*

Proof Set

$$t_{r_a s_a} \triangleq A^{11}_{r_a s_a} - A^{22}_{r_a s_a}, \; 1 \le a \le p, \quad t_{r_b s_b} \triangleq A^{11}_{r_b s_b} + A^{22}_{r_b s_b}, \; p+1 \le b \le q.$$

By the linear independence of trigonometric functions, it follows from (4) that $t_{r_1 s_1}, \ldots, t_{r_q s_q}$ satisfy the following linear equations:

$$\begin{aligned}&(\xi_{r_1 j}\xi_{s_1 k} + \xi_{r_1 k}\xi_{s_1 j})t_{r_1 s_1}\\ &\quad+(\xi_{r_2 j}\xi_{s_2 k} + \xi_{r_2 k}\xi_{s_2 j})t_{r_2 s_2}\\ &\quad+\cdots+(\xi_{r_q j}\xi_{s_q k} + \xi_{r_q k}\xi_{s_q j})t_{r_q s_q} = 0, \quad 1 \le j, k \le n.\end{aligned} \tag{5}$$

Note that (5) is equivalent to

$$t_{r_1 s_1}\xi_{r_1} \odot \xi_{s_1} + t_{r_2 s_2}\xi_{r_2} \odot \xi_{s_2} + \cdots + t_{r_q s_q}\xi_{r_q} \odot \xi_{s_q} = 0.$$

It implies $t_{r_j s_j} = 0$ for all $1 \le j \le q$, since these symmetric products are linearly independent. Similarly, the other coefficients also vanish in (4) and we have $A_{rs} = 0 \; (r \ne s)$.

By embedding $\mathbb{S}^m$ into $\mathbb{S}^{2N-1}$, we can assume the immersion x has the form as given in (1). From Lemma 2.2 and $A_{rs} = 0 \; (r \ne s)$, the matrix A in the expression (1) satisfies

$$AA^t = \operatorname{diag}\Big\{a_1, a_1, a_2, a_2, \ldots, a_N, a_N\Big\},$$

with

$$a_1 + a_2 + \cdots + a_N = 1, \;\; 0 \le a_1 \le a_2 \le \cdots \le a_N, \;\; 1 \le j \le N.$$

Consider the QR decomposition of A^t, there exists an upper triangular matrix L such that

$$L^t L = \operatorname{diag}\Big\{a_1, a_1, a_2, a_2, \ldots, a_N, a_N\Big\},$$

which implies L must be diagonal, i.e.,

$$L = \operatorname{diag}\Big\{\sqrt{a_1}, \sqrt{a_1}, \sqrt{a_2}, \sqrt{a_2}, \ldots\ldots, \sqrt{a_N}, \sqrt{a_N}\Big\}.$$

Hence, up to an orthogonal transformation, x can be expressed as below:

$$x = \big(\Theta_1 \; \Theta_2 \; \cdots \; \Theta_N\big) \operatorname{diag}\Big\{\sqrt{a_1}, \sqrt{a_1}, \sqrt{a_2}, \sqrt{a_2}, \ldots, \sqrt{a_N}, \sqrt{a_N}\Big\}.$$

Since x is linearly full, we can obtain that m is odd, and x is the orbit of a torus group acting at

$$(\sqrt{a_1}, \sqrt{a_1}, \sqrt{a_2}, \sqrt{a_2}, \ldots, \sqrt{a_{\frac{m+1}{2}}}, \sqrt{a_{\frac{m+1}{2}}}).$$

□

Definition 2.4 A finite set V of rank k in $\mathbb{R}^n$ is called unimodular, if all the k-dimensional parallelepipeds spanned in V have a same volume.

Remark 2.5 Suppose V is a finite set of rank k and satisfies the unimodular condition, then for $\{v_1, v_2, \ldots, v_l\}$ and $\{w_1, w_2, \ldots, w_l\}$, which are two arbitrary collections of l vectors in V with $l \leq k$, there must have

$$v_1 \wedge v_2 \wedge \cdots \wedge v_l = \pm w_1 \wedge w_2 \wedge \cdots \wedge w_l$$

when $\mathrm{Span}_{\mathbb{R}}\{v_1, v_2, \ldots, v_l\} = \mathrm{Span}_{\mathbb{R}}\{w_1, w_2, \ldots, w_l\}$.

Lemma 2.6 *If $\{\xi_1, \xi_2, \ldots, \xi_N\}$ satisfies the unimodular condition, then for any $\eta \in \mathcal{E}$, the η-set forms a linearly independent set $\{\xi_{r_1} \odot \xi_{s_1}, \xi_{r_2} \odot \xi_{s_2}, \ldots, \xi_{r_q} \odot \xi_{s_q}\}$.*

Proof We claim that $\eta, \xi_{r_1}, \xi_{r_2}, \ldots, \xi_{r_q}$ are linearly independent. Then by extending $\{\eta, \xi_{r_1}, \xi_{r_2}, \ldots, \xi_{r_q}\}$ to a basis of $\mathbb{R}^n$, it is easy to see that

$$\xi_{r_1} \odot \xi_{r_1}, \ \ \xi_{r_2} \odot \xi_{r_2}, \ \ \ldots, \ \ \xi_{r_q} \odot \xi_{r_q}, \ \ \xi_{r_1} \odot \eta, \ \ \xi_{r_2} \odot \eta, \ \ \ldots, \ \ \xi_{r_q} \odot \eta$$

are linearly independent. Combining this with $\xi_{s_j} = \pm(\eta - \xi_{r_j})$, we can derive that

$$\xi_{r_1} \odot \xi_{s_1}, \ \ \xi_{r_2} \odot \xi_{s_2}, \ \ \ldots, \ \ \xi_{r_q} \odot \xi_{s_q}$$

are linearly independent.

Now it suffices to prove the above claim. We prove it by contradictions. First we extend η to a maximal linearly independent subset in $\{\eta, \xi_{r_1}, \xi_{r_2}, \ldots, \xi_{r_q}\}$, which can be assumed to be $\{\eta, \xi_{r_1}, \xi_{r_2}, \ldots, \xi_{r_{t-1}}\}$ with $t < q+1$. Then $\eta, \xi_{r_1}, \xi_{r_2}, \ldots, \xi_{r_{t-1}}, \xi_{r_t}$ must be linearly dependent. It is left to discuss the following two cases.

The first case is that $\xi_{r_1}, \xi_{r_2}, \ldots, \xi_{r_{t-1}}, \xi_{r_t}$ are linearly dependent.

Let

$$\xi_{r_t} = c_1\xi_{r_1} + c_2\xi_{r_2} + \cdots\cdots + c_{t-1}\xi_{r_{t-1}}. \tag{6}$$

It follows from the unimodular condition that these coefficients all take values in $\{0, \pm 1\}$. Taking inner product of (6) with η, we can obtain

$$\sum_{j=1}^{t-1} c_j = 1. \tag{7}$$

Note that

$$\xi_{s_t} \wedge \xi_{r_1} \wedge \cdots \wedge \xi_{r_{t-1}} = \pm\,\eta \wedge \xi_{r_1} \wedge \cdots \wedge \xi_{r_{t-1}} \neq 0.$$

For any $1 \leq j \leq t-1$, using $\xi_{s_j} = \pm(\eta - \xi_{r_j})$, we have

$$\xi_{s_t} \wedge \xi_{r_1} \wedge \cdots \wedge \xi_{r_{j-1}} \wedge \xi_{s_j} \wedge \xi_{r_{j+1}} \wedge \cdots \wedge \xi_{r_{t-1}} = \pm(c_j - 1)\xi_{s_t} \wedge \xi_{r_1} \wedge \cdots \wedge \xi_{r_{t-1}}.$$

From the unimodular condition again we get

$$c_j - 1 \in \{0, \pm 1\}, \quad 1 \leq j \leq t-1,$$

which implies $c_j \in \{0, 1\}$ for all $1 \leq j \leq t-1$. Combining this with (7), we derive that only one c_j is nonzero and equals 1. It follows that $\xi_{r_t} = \xi_{r_j}$, contradicting with the fact that $\pm\xi_{r_1}, \ldots, \pm\xi_{r_q}, \pm\xi_{s_1}, \ldots, \pm\xi_{s_q}$ are distinct with each other.

The second case is that $\xi_{r_1}, \xi_{r_2}, \ldots, \xi_{r_t}$ are linearly independent. Then we can assume

$$\eta = c_1\xi_{r_1} + c_2\xi_{r_2} + \cdots\cdots + c_t\xi_{r_t}. \tag{8}$$

Taking inner product of (8) with η, we can obtain

$$\sum_{j=1}^{t} c_j = 2. \tag{9}$$

On the other hand, using $\xi_{s_j} = \pm(\eta - \xi_{r_j})$, we have

$$\xi_{s_j} = \pm\left(c_1\xi_{r_1} + \cdots + c_{j-1}\xi_{r_{j-1}} + (c_j - 1)\xi_{r_j} + c_{j+1}\xi_{r_{j+1}} + \cdots + c_t\xi_{r_t}\right), \quad 1 \leq j \leq t.$$

It follows from the unimodular condition that

$$c_j, c_j - 1 \in \{0, \pm 1\}, \quad 1 \leq j \leq t,$$

which implies $c_j \in \{0, 1\}$ for all $1 \leq j \leq t$. Combining this with (9), we derive that only two c_j are nonzero and equal 1, which can be assumed to be c_1 and c_2. It follows that $\eta = \xi_{r_1} + \xi_{r_2}$. So we have $\xi_{r_2} = \pm\xi_{s_1}$ also contradicting to the fact that $\pm\xi_{r_1}, \ldots, \pm\xi_{r_q}, \pm\xi_{s_1}, \ldots, \pm\xi_{s_q}$ are distinct. □

By Lemmas 2.3 and 2.6 we immediately obtain the following proposition.

Proposition 2.7 *Let $x : T^n = \mathbb{R}^n/\Lambda_n \longrightarrow \mathbb{S}^m$ be a linearly full minimal flat torus. If $\{\xi_1, \xi_2, \ldots, \xi_N\}$ satisfies the unimodular condition, then m is odd and x is homogeneous.*

Remark 2.8 In Sect. 6, we will show that all λ_1-minimal flat tori of dimension no more than 4 are homogeneous. In fact all of them satisfy the unimodular condition with only one exception.

3 Variational characterizations of homogeneous minimal flat tori

In this section, we give two variational characterizations for homogeneous minimal flat n-tori in spheres, from which two construction approaches can be derived. The construction of λ_1-minimal flat n-tori is also discussed.

Let $x : T^n = \mathbb{R}^n/\Lambda_n \to \mathbb{S}^m$ be a homogeneous minimal isometric immersion, with the metric given by

$$\frac{4\pi^2}{n}(du_1^2 + du_2^2 + \cdots + du_n^2).$$

Suppose $\{\xi_j\}_{j=1}^N$ are all lattice vectors (up to ± 1) in Λ_n^* of length 1. From the homogeneity of x, we can assume $m = 2N - 1$ and write x as follows:

$$x = (c_1 e^{i\theta_1}, c_2 e^{i\theta_2}, \ldots\ldots, c_N e^{i\theta_N}), \tag{10}$$

where $\theta_j = 2\pi\langle \xi_j, u\rangle,\ 1 \le j \le N$. Then we have

$$\begin{pmatrix}\xi_1^t & \xi_2^t & \cdots & \xi_N^t\end{pmatrix} \begin{pmatrix} c_1^2 & & & \\ & c_2^2 & & \\ & & \ddots & \\ & & & c_N^2 \end{pmatrix} \begin{pmatrix}\xi_1 \\ \xi_2 \\ \vdots \\ \xi_N\end{pmatrix} = \frac{1}{n} I_n. \tag{11}$$

Assume $\{\eta_1, \eta_2, \ldots, \eta_n\}$ is a generator of Λ_n^*. Then there exist integers a_{j_k} such that

$$\xi_j = a_{j_1}\eta_1 + a_{j_2}\eta_2 + \cdots\cdots + a_{j_n}\eta_n, \quad 1 \le j \le N.$$

We denote

$$A_j = (a_{j_1}, a_{j_2}, \ldots\ldots, a_{j_n}), \quad Y^t = (A_1^t, \cdots, A_N^t), \quad 1 \le j \le N,$$

and use Y also representing the set of its row vectors if no confusion caused.

Set

$$Q = \begin{pmatrix}\eta_1 \\ \eta_2 \\ \vdots \\ \eta_n\end{pmatrix} \begin{pmatrix}\eta_1^t & \eta_2^t & \cdots & \eta_n^t\end{pmatrix}.$$

It is a positive-definite matrix, called Gram matrix of Λ_n^*. It is well-known that the volume of x equals $\frac{2^n \pi^n}{\sqrt{n^n \det(Q)}}$.

Remark 3.1 The minimal immersion x given in (10) is uniquely determined by the following data set

$$\{Y,\ Q,\ (c_1^2, c_2^2, \cdots, c_N^2)\},$$

which is called the ***matrix data*** of x, and will be used to present examples in Sect. 4.

It is obvious that the condition $|\xi_j| = 1$ is equivalent to

$$A_j Q A_j^t = 1, \tag{12}$$

i.e., A_j lies on the hyper-ellipsoid $\mathcal{Q}$ determined by

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} Q \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 1.$$

Furthermore, it is straightforward to verify that the flat condition (11) is equivalent to

$$c_1^2 A_1^t A_1 + c_2^2 A_2^t A_2 + \cdots\cdots + c_N^2 A_N^t A_N = \frac{1}{n} Q^{-1}. \tag{13}$$

3.1 Two variational characterizations

Consider the space $S(n)$ of $n \times n$ symmetric matrices over $\mathbb{R}$, which is a $\frac{n(n+1)}{2}$-dimensional Euclidean space endowed with the inner product:

$$\langle S_1, S_2 \rangle = \mathrm{tr}(S_1 S_2), \quad S_1, S_2 \in S(n).$$

For any given vector $v \in \mathbb{R}^n$, $\Pi_a(v) := \{M | \langle v^t v, M \rangle = vMv^t = a, a \in \mathbb{R}\}$ is an affine hyperplane dividing $S(n)$ into two half spaces:

$$S_a^+(v) = \{M | vMv^t \geq a\}, \quad S_a^-(v) = \{M | vMv^t \leq a\}.$$

Let Σ be the set of semi-positive definite matrices. Then it is well known that $\Sigma = \cap_{v \in \mathbb{R}^n} S_0^+(v)$ is a convex cone in $S(n)$ with the set of positive definite matrices as its interior, which is denoted by Σ_+. By our definition, we have $Q^{-1} \in \Sigma_+$, and $A_j^t A_j \in \Sigma, 1 \leq j \leq N$.

Given a subset $X \subset \mathbb{Z}^n$, let C_X be the convex hull spanned by $A^t A$ for all $A \in X$, and V_X be the affine subspace $\cap_{A \in X} \Pi_1(A) \subset S(n)$. Moreover, we also consider the smooth linear submanifold $W_X = V_X \cap \Sigma_+$ (see Fig. 1), whose geometric meaning is the set of all hyper-ellipsoids passing through X.

With respect to these notations, the condition (12) is equivalent to $Q \in W_Y$, and (13) is equivalent to $\frac{Q^{-1}}{n} \in C_Y$.

Theorem 3.2 *Let $x : T^n = \mathbb{R}^n / \Lambda_n \to \mathbb{S}^{2N-1}$ be an isometric homogeneous immersion. If x is minimal, then Q is a critical point of the determinant function restricted on W_Y, where Y is the set of integer vectors determined by x, and Q the Gram matrix of Λ_n^* under some chosen generator.*

Conversely, given a set X of integer vectors, if Q is a critical point of the determinant function restricted on W_X and $\frac{Q^{-1}}{n}$ lies in the convex hull C_X, then the torus $\mathbb{R}^n / \Lambda_n$

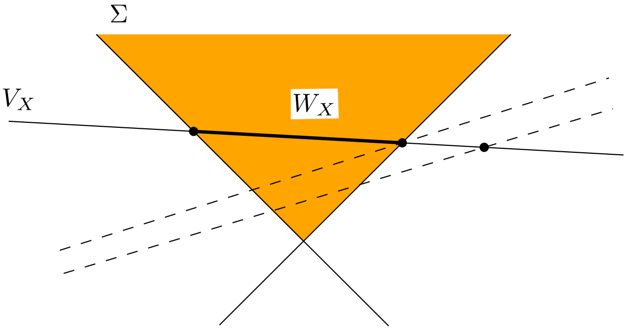


Fig. 1 W_X in Σ_+

determined by Q^{-1} *(as the Gram matrix of* Λ_n*) admits an isometric minimal immersion in* $\mathbb{S}^{2N-1}$.

Proof Let $\gamma(t)$ be a smooth curve in W_Y passing through Q at $t = 0$. Set $f(t) \triangleq \det \gamma(t)$, it is easy to verify that

$$f^{'}(0) = f(0)\langle Q^{-1}, \gamma'(0)\rangle.$$

As an isometric homogeneous immersion, x is minimal if and only if $Q \in W_Y$, $Q^{-1} \in C_Y$. The conclusion follows from that $\mathrm{Span}\{A^t A \mid A \in Y\}$ is the normal space of W_Y at $\gamma(0) = Q$. □

Remark 3.3 We point out that if Q is a critical point of the determinant function restricted on W_X, then $\frac{Q^{-1}}{n}$ lies in $\mathrm{Span}\{A^t A \mid A \in X\}$ automatically. In fact let $Q + tS$ be an arbitrary segment in W_X, then it follows from $\langle Q^{-1}, S\rangle$=0 that

$$\langle \frac{Q^{-1}}{n}, Q + tS\rangle = 1,$$

which implies $\frac{Q^{-1}}{n} \in \mathrm{Span}\{A^t A \mid A \in X\}$.

As a result, if all the vectors A_i in X can be arranged as row vectors to form a block diagonal matrix, then the critical point Q is also block diagonal.

Although the following conclusion is well known, we give a proof here for completeness.

Lemma 3.4 *The function* $\ln \circ \det$ *restricted on* Σ_+ *is strictly concave.*

Proof Given a positive definite matrix $P \in \Sigma_+$, Let $\gamma(t)$ be a smooth curve in Σ_+ with $\gamma'(0) = S \in S(n)\backslash\{0\}$. Let $f(t) = \ln \circ \det(\gamma(t))$, then we have $f'(t) = \mathrm{tr}\left(\gamma(t)^{-1}S\right)$ and

$$f''(0) = -\,\mathrm{tr}\left(P^{-1}SP^{-1}S\right) = -\,\mathrm{tr}\,(HSHHSH) = -|HSH|^2 < 0,$$

where H is a square root of the positive definite matrix P^{-1} (i.e., $H^2 = P^{-1}$), and we have used the fact that $S \in S(n)$. □

Therefore, for any given subset $X \subset \mathbb{Z}^n$, if $W_X \neq \emptyset$, the determinant function det has only one critical point in W_X, which is the maximal point. This leads to the following corollary about uniqueness.

Corollary 3.5 *Let $x : T^n = \mathbb{R}^n/\Lambda_n \to \mathbb{S}^{2N-1}$ and $\tilde{x} : \widetilde{T}^n = \mathbb{R}^n/\widetilde{\Lambda}_n \to \mathbb{S}^{2N-1}$ be two isometric homogeneous minimal immersions, if after choosing suitable generators respectively, x and $\tilde{x}$ share the same subset $Y \subset \mathbb{Z}^n$, then T^n and $\widetilde{T}^n$ are isometric. Moreover, x is congruent to $\tilde{x}$ if $\{A^tA \mid A \in Y\}$ are linearly independent.*

It seems that we can use the following approach to construct a homogeneous minimal immersion of flat torus. Choose a subset $X \subset \mathbb{Z}^n$, determine whether W_X is not empty, and then discriminate whether the maximal point Q of det restricted on W_X lies in the the convex hull C_X.

Note that in general the dimension of V_X is $n(n+1)/2 - \sharp(X)$, which implies W_X could be empty if X involves too many vectors. Even if V_X exists, it also could have no intersection with Σ_+. For example, we will get a degenerated matrix when

$$X = \begin{pmatrix} 1 & & 1 & 1 \\ & 1 & 1 & 1 & \\ & & 1 & 1 & 1 \end{pmatrix}, \quad \text{for which} \quad Q = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}.$$

It seems to be a challenge to obtain a general method by which one can efficiently construct the desirable integer set X.

Next we give an alternative method to determine homogeneous minimal immersions of flat tori, by use of a variational characterization of Q^{-1} (comparing Theorem 3.2).

Theorem 3.6 *Let $x : T^n = \mathbb{R}^n/\Lambda_n \to \mathbb{S}^{2N-1}$ be a homogeneous minimal flat torus, where N is the half dimension of the eigenspace of T^n corresponding to n. If x is minimal and linearly full, then $\frac{Q^{-1}}{n}$ lies in the interior of C_Y, and is a critical point of the determinant function restricted on C_Y.*

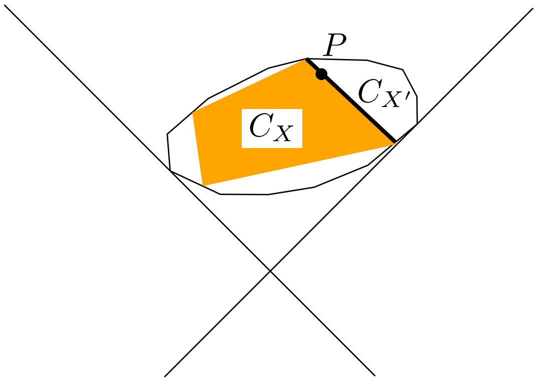
Conversely, given a finite set X of integer vectors such that $C_X \cap \Sigma_+ \neq \emptyset$, if $P \in \overset{\circ}{C}_X$ is a critical point of the determinant function restricted on C_X, then the torus $\mathbb{R}^n/\Lambda_n$ determined by nP (as the Gram matrix of Λ_n) admits an isometric minimal immersion in some $\mathbb{S}^{2N-1}$.

Proof For the first part of this theorem, $\frac{Q^{-1}}{n} \in C_Y$ just follows from the flat condition (13). To prove that $\frac{Q^{-1}}{n}$ is a critical point of det restricted on C_Y, we only need to check whether the derivatives of $\det|_{C_Y}$ at this point equal zero, which can be computed directly and are omitted here.

For the converse part, suppose $X = \{A_1, A_2, \cdots, A_k\}$, and $P = \sum_{j=1}^k y_j A_j^t A_j$. It follows from $P \in \overset{\circ}{C}_X$ that $y_j > 0$ for all $1 \le j \le k$. We only need to prove that $\frac{P^{-1}}{n} \in W_X$, i.e.,

$$\langle \frac{P^{-1}}{n}, A_j^t A_j \rangle = 1, \quad 1 \le j \le k. \tag{14}$$

Fig. 2 C_X and its faces



For any given $2 \leq j \leq k$, consider the line segment

$$\gamma_j(t) = P + t(A_j^t A_j - A_1^t A_1).$$

It is obvious that for sufficient small t, $\gamma_j(t)$ lies in C_X. So P is a critical point of $\det(\gamma_j(t))$, which implies that $\langle P^{-1}, \gamma_j'(0)\rangle = 0$. Therefore we have

$$\langle P^{-1}, A_j^t A_j\rangle = \langle P^{-1}, A_1^t A_1\rangle, \quad 2 \leq j \leq k.$$

It follows from $\sum_{j=1}^k y_j = 1$ that

$$n = \langle P^{-1}, P\rangle = \sum_{j=1}^{k} y_j \langle P^{-1}, A_j^t A_j\rangle = \Big(\sum_{j=1}^{k} y_j\Big)\langle P^{-1}, A_1^t A_1\rangle = \langle P^{-1}, A_1^t A_1\rangle,$$

which completes the proof. □

Remark 3.7 The above theorem provides another approach to construct minimal flat tori. One can choose a finite set X of integer vectors such that $C_X \cap \Sigma_+ \neq \varnothing$ (see Fig. 2), and calculate the the maximal point P of det on C_X. If $P \in \overset{\circ}{C}_X$, then the matrix data $\{\frac{P^{-1}}{n}, X\}$ provides a minimal flat n-torus. Otherwise, let $C_{X'}$ be the face of C_X so that $P \in \overset{\circ}{C}_{X'}$ (such face could have high codimension and the existance is due to the compactness of C_X), then $\{\frac{P^{-1}}{n}, X'\}$ provides a minimal flat n-torus.

This remark can be seen as a generalization of Bryant's characterization to minimal flat 2-tori in spheres, see Proposition 3.3 in [2].

3.2 Homogeneous λ_1-minimal flat tori

To determine homogeneous λ_1-minimal immersions of flat tori, one has to solve the following problem.

Problem *Firstly how to choose a subset $X \subset \mathbb{Z}^n$, such that in W_X there is a maximal point Q of* det*, and $Q^{-1}/n \in \overset{\circ}{C}_X$. Then how to discriminate in the interior of the hyper-ellipsoid determined by Q, whether there exist other nonzero integer vectors, i.e., to discriminate whether $vQv^t \geq 1$ hold for all $v \in \mathbb{Z}^n \backslash \{0\}$.*

In Σ_+, the infinite constraints

$$vMv^t \geq 1 \, for \;\; all \; v \in \mathbb{Z}^n \backslash \{0\} \tag{15}$$

define a non-compact convex domain Ω.

So X can determine a homogeneous λ_1-minimal immersion of some flat torus if and only if $Q \in \partial\Omega$. Given a positive definite matrix Q, it seems that to verify whether $Q \in \partial\Omega$, i.e., satisfying (15)*, infers an infinite process. However, due to Minkowski's reduction theory of lattices (see [30] for reference), only finite inequalities need to be checked. This can also be seen from the following simple theorem.*

Theorem 3.8 *For a given $n \times n$ positive definite matrix Q with diagonal entries* 1*, there exist n integers $a_k(Q) > 0$ $(1 \leq k \leq n)$ such that $vQv^t \geq 1$ for all $v \in \mathbb{Z}^n \backslash \{0\}$ if and only if it holds for all integer vectors in*

$$S \triangleq \left\{ v = (v_1, \ldots, v_n) \in \mathbb{Z}^n \,\Big|\, |v_k| < a_k(Q), 1 \leq k \leq n \right\}.$$

Proof Let $\{\xi_i\}$ be a basis of $\mathbb{R}^n$ with Gram matrix Q, Π_k the $(n-1)$-dimensional subspace spanned by $\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n$. So the distance of ξ_k to Π_k is a fixed value $d_k > 0$, which can be determined from the entries a_{ij} of Q by using the least square method, such as

$$d_n^2 = |\xi_n|^2 - |\xi_n^\top|^2 = 1 - |\xi_n^\top|^2 = 1 - (a_{1n}, \ldots, a_{(n-1)n})(a_{ij})^{-1}_{1 \leq i,j \leq n-1}(a_{1n}, \ldots, a_{(n-1)n})^t, \tag{16}$$

where $\xi_n^\top$ is the orthogonal projection of ξ_n into Π_n.

Suppose $a_k(Q)$ is the integer such that $a_k(Q) - 1 \leq d_k^{-1} < a_k(Q)$, then we will get $|\alpha| \geq |c_k| d_k > 1$ for any $\alpha = c_1\xi_1 + \cdots + c_n\xi_n$ when some $|c_k| \geq a_k(Q)$. □

Although the criterion given in the above theorem only infers finite steps, it still requests a lot of computations. For $n \leq 6$, we give an alternative criterion which comes from Minkowski's reduction theory (see [29]). For the case $n > 6$, so far we do not find such an explicit description in the literature. Applied to our case, the criterion is stated as below.

Theorem 3.9 *Suppose Q is an $n \times n$ positive definite matrix with diagonal entries* 1*, and $n \leq 6$. Then $vQv^t \geq 1$ holds for all $v \in \mathbb{Z}^n \backslash \{0\}$, if and only if, it holds for those integer vectors v, whose entries take values from the first n integers in each rows below* *with arbitrary sign ("+" or "−"):*

$$\begin{array}{cccccc}
1&1&0&0&0&0\\
1&1&1&0&0&0\\
1&1&1&1&0&0\\
1&1&1&1&1&0\\
1&1&1&1&2&0\\
1&1&1&1&1&1\\
1&1&1&1&1&2\\
1&1&1&1&2&2\\
1&1&1&1&2&3.
\end{array}$$

4 Examples of λ_1-minimal flat tori of dimension 3 and 4

In this section, we present a series of λ_1-minimal immersions for flat tori $T^3 = \mathbb{R}^3/\Lambda_3$ and $T^4 = \mathbb{R}^4/\Lambda_4$, which are all homogeneous. Instead of giving the explicit petty immersions, we use the matrix data

$$\{Y,\ Q,\ (c_1^2, c_2^2, \cdots, c_N^2)\}$$

introduced in the last section (see Remark 3.1). Recall that Q is a Gram matrix of the lattice Λ_n^* under some chosen generator, which can determine a flat torus $T^n = \mathbb{R}^n/\Lambda_n$; N denotes the number of distinct shortest lattice vectors of Λ_n^* up to ± 1, i.e., the half dimension of the first eigenspace of flat torus $T^n = \mathbb{R}^n/\Lambda_n$; Y is an $n \times N$ integer matrix describing the linear relations of those shortest lattice vectors; c_i^2 involves the information of immersion.

First of all, given two λ_1-minimal tori $f_i : T^{n_i} \to \mathbb{S}^{m_i}\,(i = 1, 2)$, one can construct a new λ_1-minimal $(n_1 + n_2)$-torus by the following direct product (see [3, 33, 35])

$$f = \left(\sqrt{\frac{n_1}{n_1 + n_2}}f_1, \sqrt{\frac{n_2}{n_1 + n_2}}f_2\right) : T^{n_1} \times T^{n_2} \to \mathbb{S}^{m_1+m_2+1}. \tag{17}$$

In the sequel, λ_1-minimal torus constructed in such way is called *reducible*, and those non-product ones are called *irreducible*.

Remark 4.1 A reducible λ_1-minimal flat torus constructed from two homogeneous and λ_1-minimal tori is also homogeneous. For such torus, by definition, suitable generator of the corresponding dual lattice can be chosen such that the matrix Y is block diagonal. Then it follows from Remark 3.3 that the Gram matrix Q is also block diagonal. Conversely, a λ_1-minimal flat torus given by diagonal data set is a reducible λ_1-minimal flat torus.

Example 4.2 As stated in the introduction, the Clifford torus and equilateral torus are two (only two) λ_1-minimal flat 2-torus, whose matrix data are

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (c_1^2, c_2^2) = \left(\frac{1}{2}, \frac{1}{2}\right),$$

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad (c_1^2, c_2^2, c_3^2) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Taking one from these two tori, considering the direct product (as in (17)) of it with the unite circle in $\mathbb{R}^2$, we can obtain two examples of λ_1-minimal flat 3-torus in $\mathbb{S}^5$ and $\mathbb{S}^7$, whose volume is respectively $\frac{8\sqrt{3}}{9}\pi^3$ and $\frac{16}{9}\pi^3$. Their matrix data can be stated as follows:

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (c_1^2, c_2^2, c_3^2) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right),$$

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (c_1^2, c_2^2, c_3^2, c_4^2) = \left(\frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{3}\right).$$

Next we give some data with neither Q nor Y being block diagonal, which are obtained originally by applying our variational characterization, where some tedious but routine computation is involved, and we omit it here. Alternatively, one can verify directly that all these data fit (12) and (13), and satisfy the criterion given in Theorem 3.9, hence provide irreducible examples of λ_1-minimal flat tori.

Example 4.3 N=4. The following matrix data

$$Q = \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$
$$(c_1^2, c_2^2, c_3^2, c_4^2, c_5^2, c_6^2) = \left(\frac{1}{6}, \frac{1}{6}, \frac{5}{48}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}\right)$$

gives a λ_1-minimal flat 3-torus in $\mathbb{S}^7$, which is irreducible, linearly full and has volume $2\pi^3$.

Example 4.4 N=5. The following matrix data

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$
$$(c_1^2, c_2^2, c_3^2, c_4^2, c_5^2) = \left(\frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{2}{9}\right)$$

gives a λ_1-minimal flat 3-torus in $\mathbb{S}^9$, which is irreducible, linearly full and has volume $\frac{32\sqrt{3}}{27}\pi^3$.

Example 4.5 N=6. The following matrix data

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix},\ Y^t = \begin{pmatrix} 1\,0\,1\,0\,0\,1 \\ 0\,1\,1\,0\,1\,1 \\ 0\,0\,0\,1\,1\,1 \end{pmatrix}$$
$$,\ (c_1^2,\ c_2^2,\ c_3^2,\ c_4^2,\ c_5^2,\ c_6^2) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right).$$

gives a λ_1-minimal flat 3-torus in $\mathbb{S}^{11}$, which is irreducible, linearly full and has volume $\frac{8\sqrt{6}}{9}\pi^3$.

We will prove in the next section that Examples 4.2 to 4.5 enumerate all examples of λ_1-minimal flat 3-tori in spheres.

Example 4.6 Taking the direct products (as in (17)) of two λ_1-minimal flat 2-tori, one can obtain three reducible λ_1-minimal flat 4-tori in $\mathbb{S}^7$, $\mathbb{S}^9$ and $\mathbb{S}^{11}$, they are all linearly full with volumes π^4, $\frac{3\sqrt{3}}{4}\pi^4$ and $\frac{4}{3}\pi^4$, respectively.

Taking the direct product (as in (17)) of one λ_1-minimal flat 3-torus given in Examples 4.3 to 4.5 with the unite circle in $\mathbb{R}^2$, we can obtain another three reducible λ_1-minimal flat 4-tori in $\mathbb{S}^9$, $\mathbb{S}^{11}$ and $\mathbb{S}^{13}$, they are all linearly full with volumes $\frac{2\sqrt{3}}{3}\pi^4$, $\frac{4}{3}\pi^4$ and $\sqrt{2}\pi^4$, respectively.

For brevity, we omit the data set of these 6 reducible λ_1-minimal flat 4-tori.

Example 4.7 $N = 5$. The following matrix data

$$Q = \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1\,0\,0\,0\,1 \\ 0\,1\,0\,0\,1 \\ 0\,0\,1\,0\,1 \\ 0\,0\,0\,1\,1 \end{pmatrix},$$
$$(c_1^2,\ c_2^2,\ c_3^2,\ c_4^2,\ c_5^2) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$$

gives a λ_1-minimal flat 4-torus in $\mathbb{S}^9$, which is irreducible, linearly full and has volume $\frac{16\sqrt{5}}{25}\pi^4$.

Example 4.8 $N = 6$. The following matrix data

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{2} & 1 & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & 1 & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1\,0\,1\,0\,0\,1 \\ 0\,1\,1\,0\,0\,1 \\ 0\,0\,0\,1\,0\,1 \\ 0\,0\,0\,0\,1\,1 \end{pmatrix},$$

$$(c_1^2, c_2^2, c_3^2, c_4^2, c_5^2, c_6^2) = \left(\frac{1}{6}, \frac{1}{6}, \frac{5}{48}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}\right)$$

gives a λ_1-minimal flat 4-torus in $\mathbb{S}^{11}$, which is irreducible, linearly full and has volume $\frac{3}{2}\pi^4$.

Example 4.9 $N = 7$. The following matrix data

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1\,0\,1\,0\,1\,0\,1 \\ 0\,1\,1\,0\,1\,0\,1 \\ 0\,0\,0\,1\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1\,1 \end{pmatrix},$$

$$(c_1^2, c_2^2, c_3^2, c_4^2, c_5^2, c_6^2, c_7^2) = \left(\frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

gives a λ_1-minimal flat 4-torus in $\mathbb{S}^{13}$, which is irreducible, and has volume $\frac{8\sqrt{3}}{9}\pi^4$. Note that it is not linearly full in $\mathbb{S}^{13}$ but in $\mathbb{S}^{11}$. As far as we know, this is the first explicit example of λ_1-minimal immersion in the literature that does not span the whole first eigenspace. Note that branched minimal surfaces with extra eigenfunctions have been studied by Ejiri and Kotani in [7, 8].

Example 4.10 $N = 7$. The following matrix data

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{2} & 1 & -\frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{4} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1\,0\,1\,0\,1\,0\,1 \\ 0\,1\,1\,0\,1\,0\,1 \\ 0\,0\,0\,1\,1\,0\,1 \\ 0\,0\,0\,0\,0\,1\,1 \end{pmatrix},$$

$$(c_1^2, c_2^2, c_3^2, c_4^2, c_5^2, c_6^2, c_7^2) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}\right)$$

gives a λ_1-minimal flat 4-torus in $\mathbb{S}^{13}$, which is irreducible, linearly full and has volume $\frac{8\sqrt{3}}{9}\pi^4$.

Example 4.11 $N = 7$. The following matrix data

$$Q = \begin{pmatrix} 1 & \frac{\sqrt{13}-7}{12} & \frac{\sqrt{13}-7}{12} & \frac{4-\sqrt{13}}{6} \\ \frac{\sqrt{13}-7}{12} & 1 & \frac{1-\sqrt{13}}{6} & \frac{\sqrt{13}-7}{12} \\ \frac{\sqrt{13}-7}{12} & \frac{1-\sqrt{13}}{6} & 1 & \frac{\sqrt{13}-7}{12} \\ \frac{\sqrt{13}-4}{6} & \frac{\sqrt{13}-7}{12} & \frac{\sqrt{13}-7}{12} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1\,0\,0\,1\,0\,0\,1 \\ 0\,1\,0\,1\,0\,1\,1 \\ 0\,0\,1\,1\,0\,1\,1 \\ 0\,0\,0\,0\,1\,1\,1 \end{pmatrix},$$

$$(c_1^2, c_2^2, c_3^2, c_4^2, c_5^2, c_6^2, c_7^2)$$

$$= \left(\frac{\sqrt{13}-2}{12}, \frac{5-\sqrt{13}}{8}, \frac{5-\sqrt{13}}{8}, \frac{\sqrt{13}-2}{12}, \frac{\sqrt{13}-2}{12}, \frac{\sqrt{13}-2}{12}, \frac{5-\sqrt{13}}{12} \right)$$

gives a λ_1-minimal flat 4-torus in $\mathbb{S}^{13}$, which is irreducible, linearly full and has volume $\frac{\sqrt{26\sqrt{13}-70}}{3}\pi^4$.

As far as we know, this is the first example of λ_1-minimal immersion in the literature whose Gram matrix are not rational, comparing to that minimal 2-tori always have rational Gram matrices up to a rescaling(see [2]).

Example 4.12 $N = 8$. The following matrix data

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & -\frac{1}{4} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1\,0\,1\,0\,0\,1\,0\,1 \\ 0\,1\,1\,0\,1\,1\,0\,1 \\ 0\,0\,0\,1\,1\,1\,0\,1 \\ 0\,0\,0\,0\,0\,0\,1\,1 \end{pmatrix},$$

$$(c_1^2,\, c_2^2,\, c_3^2,\, c_4^2,\, c_5^2,\, c_6^2,\, c_7^2,\, c_8^2) = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{24}, \frac{1}{6}, \frac{1}{6} \right)$$

gives a λ_1-minimal flat 4-torus in $\mathbb{S}^{15}$, which is irreducible, linearly full and has volume $\frac{2\sqrt{6}}{3}\pi^4$.

Example 4.13 $N = 8$. The following matrix data

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2}-1 & 1-\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2}-\frac{\sqrt{3}}{2} & \sqrt{3}-2 \\ \frac{\sqrt{3}}{2}-1 & \frac{1}{2}-\frac{\sqrt{3}}{2} & 1 & \frac{1}{2}-\frac{\sqrt{3}}{2} \\ 1-\frac{\sqrt{3}}{2} & \sqrt{3}-2 & \frac{1}{2}-\frac{\sqrt{3}}{2} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1\,0\,1\,0\,1\,0\,0\,1 \\ 0\,1\,1\,0\,1\,0\,1\,1 \\ 0\,0\,0\,1\,1\,0\,1\,1 \\ 0\,0\,0\,0\,0\,1\,1\,1 \end{pmatrix},$$

$$(c_1^2,\, c_2^2,\, c_3^2,\, c_4^2,\, c_5^2,\, c_6^2,\, c_7^2,\, c_8^2)$$
$$= \left(\frac{3-\sqrt{3}}{12}, \frac{\sqrt{3}}{12}, \frac{3-\sqrt{3}}{12}, \frac{\sqrt{3}}{12}, \frac{3-\sqrt{3}}{12}, \frac{\sqrt{3}}{12}, \frac{\sqrt{3}}{12}, \frac{3-\sqrt{3}}{12} \right)$$

gives a λ_1-minimal flat 4-torus in $\mathbb{S}^{15}$, which is irreducible, linearly full and has volume $\frac{\sqrt{12+8\sqrt{3}}}{3}\pi^4$. This is another example of λ_1-minimal immersion whose Gram matrix are not rational.

Example 4.14 $N = 9$. The following matrix data

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{6} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{6} \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{6} & -\frac{1}{2} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1\,0\,1\,0\,0\,1\,0\,0\,1 \\ 0\,1\,1\,0\,1\,1\,0\,0\,1 \\ 0\,0\,0\,1\,1\,1\,0\,1\,1 \\ 0\,0\,0\,0\,0\,0\,1\,1\,1 \end{pmatrix},$$

$$
\begin{aligned}
&(c_1^2,\ c_2^2,\ c_3^2,\ c_4^2,\ c_5^2,\ c_6^2,\ c_7^2,\ c_8^2,\ c_9^2) \\
&= \left(\frac{1}{8},\ \frac{1}{8},\ \frac{1}{12},\ \frac{1}{12},\ \frac{1}{8},\ \frac{1}{12},\ \frac{1}{8},\ \frac{1}{8},\ \frac{1}{8}\right)
\end{aligned}
$$

gives a λ_1-minimal flat 4-torus in $\mathbb{S}^{17}$, which is irreducible, linearly full and has volume $\frac{16}{9}\pi^4$.

Example 4.15 $N = 9$. The following matrix data

$$
Q = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1\,0\,1\,0\,1\,0\,0\,1\,0 \\ 0\,1\,1\,0\,1\,0\,1\,1\,1 \\ 0\,0\,0\,1\,1\,0\,1\,1\,0 \\ 0\,0\,0\,0\,0\,1\,1\,1\,1 \end{pmatrix},
$$

$$
\begin{aligned}
&(c_1^2,\ c_2^2,\ c_3^2,\ c_4^2,\ c_5^2,\ c_6^2,\ c_7^2,\ c_8^2,\ c_9^2) \\
&= \left(\frac{1}{9},\ \frac{1}{9},\ \frac{1}{9},\ \frac{1}{9},\ \frac{1}{9},\ \frac{1}{9},\ \frac{1}{9},\ \frac{1}{9},\ \frac{1}{9}\right)
\end{aligned}
$$

gives a λ_1-minimal flat 4-torus in $\mathbb{S}^{17}$, which is irreducible, linearly full and has volume $\sqrt{3}\pi^4$.

Example 4.16 $N = 10$. The following matrix data

$$
Q = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}, \quad Y^t = \begin{pmatrix} 1\,0\,1\,0\,0\,1\,0\,0\,0\,1 \\ 0\,1\,1\,0\,1\,1\,0\,0\,1\,1 \\ 0\,0\,0\,1\,1\,1\,0\,1\,1\,1 \\ 0\,0\,0\,0\,0\,0\,1\,1\,1\,1 \end{pmatrix},
$$

$$
\begin{aligned}
&(c_1^2,\ c_2^2,\ c_3^2,\ c_4^2,\ c_5^2,\ c_6^2,\ c_7^2,\ c_8^2,\ c_9^2,\ c_{10}^2) \\
&= \left(\frac{1}{10},\ \frac{1}{10},\ \frac{1}{10},\ \frac{1}{10},\ \frac{1}{10},\ \frac{1}{10},\ \frac{1}{10},\ \frac{1}{10},\ \frac{1}{10},\ \frac{1}{10}\right)
\end{aligned}
$$

gives a λ_1-minimal flat 4-torus in $\mathbb{S}^{19}$, which is irreducible, linearly full and has volume $\frac{4}{\sqrt{5}}\pi^4$.

We will prove in the next section that Examples 4.6 to 4.16, and Example 1.1 enumerate all examples of λ_1-minimal flat 4-tori in spheres.

5 Shortest vectors in lattices

As mentioned in the introduction, the classification of λ_1-minimal tori of dimension 3 and 4 in spheres relies on deep investigation of shortest lattice vectors for lattices of rank 3 and 4. This section is devoted to the discussion of some related properties of lattices, which are of independent interest.

Let Λ_n^* be a lattice of rank n, N be the number of distinct shortest lattice vectors of Λ_n^* up to ± 1, and Ξ be the set of these N shortest vectors, which are denoted by

$$\xi_1, \xi_2, \dots, \xi_N.$$

For simplicity, in the sequel, these shortest vectors are assumed to be of unit length.

Definition 5.1 A set of $k+1$ vectors in $\mathbb{R}^n$ is called a ***generic*** $(k+1)$-***tuple*** if it is of rank k and any k vectors in it are linearly independent.

Definition 5.2 A lattice Λ_n^* with $N \geq n$ is called ***s-reducible*** if there is a non-trivial decomposition $\mathbb{R}^n = V_1 \oplus V_2$ such that

$$\Xi = (\Xi \cap V_1) \cup (\Xi \cap V_2).$$

It will be called *s-irreducible* otherwise.

It is easy to see that if $N = n$, or $N = n+1$ and Ξ is not a generic $(n+1)$-tuple, or $\mathrm{rank}(\Xi) < n$, then Λ_n^* is s-reducible.

Lemma 5.3 *For an s-irreducible lattice Λ_n^* of rank $n \geq 3$, there always exists a generic k-tuple in Ξ for some $k \geq 4$.*

Proof Let $\{\xi_1, \dots, \xi_n\} \subset \Xi$ be of rank n. Suppose the opposite that there are only generic 3-tuples in Ξ. Then for any $n+1 \leq i \leq N$, when we write ξ_i as a linear combination of these generator vectors, there are exactly two non-zero coefficients. From the s-irreducible assumption we know $N \geq n+2$. We can assume $\xi_{n+1} = a_1\xi_1 + a_2\xi_2$ with $a_1 a_2 \neq 0$. Consider two subspaces $V_1 \triangleq \mathrm{Span}\{\xi_3, \dots, \xi_n\}$ and $V_2 \triangleq \mathrm{Span}\{\xi_1, \xi_2\}$, it follows that there exists at least one vector, say ξ_{n+2}, belonging to neither V_1 nor V_2.

We assume $\xi_{n+2} = b_1\xi_1 + b_2\xi_3$ with $b_1 b_2 \neq 0$. Then it is straightforward to verify that $\{\xi_2, \xi_3, \xi_{n+1}, \xi_{n+2}\}$ forms a generic 4-tuple, which gives us a contradiction. □

Definition 5.4 A lattice Λ_n^* is called *prime* if this lattice can be generated by any n linearly independent vectors in Ξ.

Remark 5.5 By definition, for a prime lattice, Ξ always satisfies the unimodular condition defined in Definition 2.4.

The following conclusion is easy to obtain.

Lemma 5.6 *Let Λ_n^* be a prime lattice generated by $\{\alpha_1, \dots, \alpha_n\} \subset \Xi$. Then for any $\xi \in \Xi$, we have*

$$\xi = a_1\alpha_1 + \cdots + a_n\alpha_n, \quad a_i \in \{0, \pm 1\}.$$

For the lattices of rank no more than 4, we obtain a sufficient condition to discriminate whether it is prime. The covering radius in lattice theory will be used, which is defined for a lattice Λ_n^* as follows:

$$\mu(\Lambda_n^*) \triangleq \inf \left\{ r \,\middle|\, \mathbb{R}^n = \cup_{p\in\Lambda_n^*} B^n(p, r) \right\},$$

where $B^n(p, r)$ is the ball centered at p with radius r. Any ball with radius larger than $\mu(\Lambda_n^*)$ must contain a point of Λ_n^*. The following well-known estimate (see [14]) will be used to prove our main theorem in this section.

Lemma 5.7 *Let Λ_n^* be a lattice of rank n, if* $\operatorname{rank}(\Xi) = n$ *and the shortest vectors of Λ_n^* are of unit length,*
then
$\mu(\Lambda_n^*) \le \frac{\sqrt{n}}{2}$.

Theorem 5.8 *Suppose Λ_n^* is a lattice of rank n with* $\operatorname{rank}(\Xi)$*=n. If $n \le 4$, then either Λ_n^* is prime, or it can be generated by the row vectors of*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{18}$$

Proof It is obviously true for $n = 1$. Suppose it is true for $n = k - 1$ (≤ 3), we will show that in the case of $n = k$ the conclusion is also true. Let $\{\xi_1, \xi_2, \cdots, \xi_k\}$ be any given linearly independent shortest vectors in Λ_k^*. Let Λ_{k-1}^* be the sublattice of rank $k-1$ containing $\{\xi_2, \cdots, \xi_k\}$. Then it follows from the inductive hypothesis that there exists a generator $\{\eta_1, \xi_2, \cdots, \xi_k\}$ of Λ_k^* such that

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{pmatrix},$$

where $a_1, a_2, \ldots, a_k \in \mathbb{Z}$, we can assume they are all non-negative and $a_1 > 0$ by changing directions of these vectors. Since $\eta_1 + c\xi_2, \xi_2, \ldots, \xi_k$ ($c \in \mathbb{Z}$) is still a generator of Λ_k^*, we get

$$\xi_1 = a_1(\eta_1 + c\xi_2) + (a_2 - ca_1)\xi_2 + \cdots + a_k\xi_k.$$

So suitable value c will make $a_1 > a_2 - ca_1 \ge 0$. Therefore, we may assume $a_1 > a_i \ge 0$ for $2 \le i \le k$.

If $a_1 = 1$ then $a_i = 0$ for $2 \le i \le k$ and we derive that $\xi_1, \xi_2, \ldots, \xi_k$ is a generator of Λ_k^*. If $a_1 \ge 2$, let $\eta_1 = \eta_1^{\perp} + \eta_1^{\top}$, where $\top$ (w.r.t. $\perp$) denotes the orthogonal projection onto $\Pi_0 \triangleq \mathrm{Span}_{\mathbb{R}}\{\xi_2, \ldots, \xi_k\}$ (w.r.t. the normal space of Π_0). So $\xi_1 = a_1\eta_1^{\perp} + \xi_1^{\top}$ and

$$0 < |\eta_1^{\perp}|^2 = \frac{|\xi_1|^2 - |\xi_1^{\top}|^2}{a_1^2} = \frac{1 - |\xi_1^{\top}|^2}{a_1^2} \leq \frac{1}{4}.$$

It follows that the intersection of k-ball $B^k(0, 1)$ with the hyperplane $\Pi \triangleq \eta_1^{\perp} + \Pi_0$ is a $(k-1)$-ball $B^{k-1}(\eta_1^{\perp}, r)$ with

$$r^2 = \frac{a_1^2 - 1}{a_1^2} + \frac{|\xi_1^{\top}|^2}{a_1^2} \geq \frac{3}{4}.$$

When $k < 4$, we can see r is larger than the covering radius of Λ_{k-1}^*. Therefore, in $B^{k-1}(\eta_1^{\perp}, r) \subset B^k(0, 1)$, there exists at least one point of $\Lambda_k^* \cap \Pi$. This contradicts with our assumption that 1 is the shortest length of Λ_k^*. Similarly, in the case of $k = 4$, if $r^2 > \frac{3}{4}$, we can also obtain a contradiction.

Therefore, when $k = 4$, we have $r^2 = \frac{3}{4}$. This yields $a_1 = 2$ and $\xi_1^{\top} = 0$. Moreover, $a_2 = a_3 = a_4 = 1$. Otherwise, say $a_4 = 0$, then

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \xi_2 \\ \xi_3 \end{pmatrix},$$

which implies in the sublattice $\mathrm{Span}_{\mathbb{Z}}\{\eta_1, \xi_2, \xi_3\}$, the shortest vectors ξ_1, ξ_2, ξ_3 can not form a generator, which contradicts with the induction hypothesis.

Note that $\xi_1^{\top} = 0$ means ξ_1 is orthogonal to other ξ_i. It follows from $2\eta_1 = \xi_1 - \xi_2 - \xi_3 - \xi_4$ and $|\eta_1| \geq 1$ that $|\xi_2 + \xi_3 + \xi_4| \geq \sqrt{3}$. Moreover, using

$$\left| \frac{\xi_1 + \xi_2 - \xi_3 - \xi_4}{2} \right| = |\eta_1 + \xi_2| \geq 1,$$

we have $|\xi_2 - \xi_3 - \xi_4| \geq \sqrt{3}$. Similarly, $|\xi_2 - \xi_3 + \xi_4| \geq \sqrt{3}$ and $|\xi_2 + \xi_3 - \xi_4| \geq \sqrt{3}$. Combining these with the following identities,

$$\begin{aligned} &|\xi_2 + \xi_3 + \xi_4|^2 + |\xi_2 - \xi_3 - \xi_4|^2 + |\xi_2 - \xi_3 + \xi_4|^2 + |\xi_2 + \xi_3 - \xi_4|^2 \\ &\quad = 4(|\xi_2|^2 + |\xi_3|^2 + |\xi_4|^2) = 12, \end{aligned}$$

we can derive that

$$|\xi_2 + \xi_3 + \xi_4|^2 = |\xi_2 - \xi_3 - \xi_4|^2 = |\xi_2 - \xi_3 + \xi_4|^2 = |\xi_2 + \xi_3 - \xi_4|^2 = 3$$

which implies $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ forms an orthonormal basis of $\mathbb{R}^4$ and we complete the proof of this theorem. □

Given a lattice Λ_n^* of rank n, to investigate the set Ξ constituted by all shortest lattice vectors up to ± 1, we can consider the intersections of it with all sublattices of rank $n-1$. Let

$$m(\Xi) \triangleq \max \left\{ \sharp(\Xi \cap \tilde{\Lambda}) \,\middle|\, \tilde{\Lambda} \subset \Lambda_n^* \text{ is a sublattice of rank } n-1 \right\}, \tag{19}$$

and Ξ' be one of the intersection attaining $m(\Xi)$. Note that there may be more than one such intersections.

In the rest part of this section, we always assume Λ_n^* is a prime lattice of rank n. For such lattice, we have $m(\Xi) \geq n-1$. Let Ξ' be a chosen intersection attaining $m(\Xi)$, we assume that

$$\Xi' = \{\xi_1, \xi_2, \ldots, \xi_{m(\Xi)}\},$$

and Λ_n^* is generated by $\{\xi_1, \xi_2, \ldots, \xi_{n-1}, \xi_{m(\Xi)+1}\}$. Then there exist N integer vectors

$$\{A_i = (a_{i1}, a_{i2}, \ldots, a_{in})\}_{i=1}^N \subset \mathbb{Z}^n,$$

such that

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n-1} \\ \xi_{m(\Xi)+1} \end{pmatrix}. \tag{20}$$

Obviously, $\{A_1, A_2, \ldots, A_{n-1}, A_{m(\Xi)+1}\}$ is the standard basis of $\mathbb{R}^n$. It follows from the prime assumption and Lemma 5.6 that

$$a_{ji} \in \{0, \pm 1\}, \quad 1 \leq j \leq N, 1 \leq i \leq n.$$

Moreover, by changing the direction of some vectors in Ξ if necessary, we can assume the last coordinate of A_i equals 0 for $1 \leq i \leq m(\Xi)$, and 1 for $m(\Xi)+1 \leq i \leq N$.

We will still abuse the notation Y to denote either the set constituted by A_i, or the matrix constructed by A_i as in (20). Note that the prime assumption on $\Xi \subset \Lambda_n^*$ is equivalent to saying that any n linearly independent vectors in Y form a generator of $\mathbb{Z}^n$. For applications in the next section, we conclude some further properties of prime lattices in the following four lemmas.

Lemma 5.9 *For a prime lattice, in terms of matrix, all minors of Y^t can only take values in* $\{0, \pm 1\}$.

Proof Suppose the opposite that there is a nonzero minor involving the i_1th, i_2th, $\ldots$, i_kth rows and j_1th, j_2th, $\ldots$, j_kth columns that is not equal to ± 1, then $\mathbb{Z}^n$ can not be generated by $\{A_{j_1}, A_{j_2}, \ldots, A_{j_k}\}$ and $\{A_{i_{k+1}}, A_{i_{k+2}}, \ldots, A_{i_n}\}$, since

$$|A_{j_1} \wedge \cdots \wedge A_{j_k} \wedge A_{i_{k+1}} \wedge \cdots \wedge A_{i_n}| > 1,$$

where $\{i_{k+1}, \ldots, i_n\}$ is the complementary set of $\{i_1, \ldots, i_k\}$ in $\{1, 2, \ldots, n-1, m(\Xi)+1\}$. This gives a contradiction with our prime assumption. □

Lemma 5.10 *For any given $1 \leq i \leq n$ and $m(\Xi)+1 \leq j, k \leq N$, we have $a_{ji}a_{ki} \geq 0$, where a_{ji} is the ith coordinate of A_j.*

Proof Assume $a_{ji}a_{ki} < 0$. Then from $a_{jn} = a_{kn} = 1$ we get the minor $\begin{vmatrix} a_{ji} & a_{ki} \\ 1 & 1 \end{vmatrix} = \pm 2$, which is a contradiction with our prime assumption. □

For a subset $I \subset \{1, 2, \ldots, n\}$, we say a subset $X \subset \mathbb{Z}^n$ has a *partial order according to* I, if for any given two vectors $\xi, \eta \in X$, their ith coordinates ξ_i, η_i and jth coordinates ξ_j, η_j satisfy

$$(\xi_i - \eta_i)(\xi_j - \eta_j) \geq 0, \quad i, j \in I.$$

Lemma 5.11 *If there is a vector $A_j \in Y$ such that three coordinates $\{a_{ji_1}, a_{ji_2}, a_{ji_3}\}$ of A_j satisfy*

$$a_{ji_1}a_{ji_2} = 1, \quad a_{ji_3} = 0,$$

then for any $1 \leq r \leq N$, we have

$$a_{ri_1}a_{ri_2} \geq 0,$$

and the subset $Y_{i_3} \triangleq \{A_k \in Y \mid a_{ki_3} = 1\}$ has a partial order according to $\{i_1, i_2\}$, so does the subset $\widehat{Y}_{i_3} \triangleq \{A_k \in Y \mid a_{ki_3} = -1\}$.

Proof Similarly as in the proof of Lemma 5.10, by considering the minor $\begin{vmatrix} a_{ji_1} & a_{ri_1} \\ a_{ji_2} & a_{ri_2} \end{vmatrix}$, we can derive the first conclusion.

Given two arbitrary vectors $A_k, A_l \in Y_{i_3}$, we have $a_{ki_3} = a_{li_3} = 1$, which implies $a_{ki_1}a_{li_1} \geq 0$ and $a_{ki_2}a_{li_2} \geq 0$ by considering the minors $\begin{vmatrix} a_{ki_1} & a_{li_1} \\ 1 & 1 \end{vmatrix}$ and $\begin{vmatrix} a_{ki_2} & a_{li_2} \\ 1 & 1 \end{vmatrix}$, respectively. Therefore, $a_{ki_1} - a_{li_1}$ and $a_{ki_2} - a_{li_2}$ all take values in $\{0, \pm 1\}$. Consider the minor

$$\begin{vmatrix} a_{ji_1} & a_{ki_1} & a_{li_1} \\ a_{ji_2} & a_{ki_2} & a_{li_2} \\ 0 & 1 & 1 \end{vmatrix} = \pm[(a_{ki_1} - a_{li_1}) - (a_{ki_2} - a_{li_2})].$$

It follows from the prime assumption that $(a_{ki_1} - a_{li_1})(a_{ki_2} - a_{li_2}) \geq 0$. Similar discussion can be applied to $\widehat{Y}_{i_3}$. □

Lemma 5.12 *If $A_1 + A_2 + \cdots + A_{n-1} + A_{m(\Xi)+1}$ belongs to Y, then all the coordinates of any $A_j \in Y$ must be either ≥ 0 or ≤ 0. Especially, after changing the directions of some vectors, all entries of Y take values in $\{0, 1\}$.*

Proof Let $A_0 = A_1 + A_2 + \cdots + A_{n-1} + A_{m(\Xi)+1}$. If there is a vector $A_j \in Y$ which doesn't satisfy the conclusion, then one can find a 2-minor in A_0 and A_j taking values other than $\{0, \pm 1\}$. This gives a contradiction with our prime assumption. □

The following definition will also be used in the next section.

Definition 5.13 A vector in $\mathbb{Z}^n$ is called k-null, if it has exactly $n - k$ nonzero coordinates. Under some fixed generator of Λ_n^*, a lattice vector in Λ_n^* is called k-null if its coordinate vector is k-null.

6 Classification of conformally flat and λ_1-minimal 3-tori and 4-tori

It follows from the works of Montiel–Ros [23] and El Soufi–Ilias [10] that for each conformal structure on compact manifold $(M^n, [g_0])$, there exists at most one metric $g \in [g_0]$ so that (M^n, g) can be minimally immersed into a sphere by the first eigenfunctions. Moreover, if (M^n, g_0) is homogeneous, then such λ_1-minimal metric must be g_0 itself (up to a constant dilation). Note that the flat torus T^n is homogeneous, so we only need to classify all non-congruent λ_1-minimal immersions of falt 3-tori and 4-tori in spheres.

Let $x : T^n = \mathbb{R}^n/\Lambda_n \to \mathbb{S}^{2N-1}$ be a λ_1-minimal flat torus. Here we do not assume it is linearly full, and denote by N the number of distinct shortest lattice vectors of Λ_n^* up to ± 1, i.e., the half dimension of the first eigenspace of flat torus $T^n = \mathbb{R}^n/\Lambda_n$. Without loss of generality, we assume this shortest length is 1. It is well known that $N \geq n$ (see Corollary 3.4 in [5]). Let Ξ be the set of these N shortest vectors, which are denoted by $\xi_1, \xi_2, \ldots, \xi_N$. Then according to Remark 4.1, x is reducible if and only if Λ_n^* is s-reducible defined as in Sect. 5.

6.1 Classification of conformally flat 3-tori

Theorem 6.1 *Up to congruence, there are five distinct λ_1-minimal immersions of conformally flat 3-tori in spheres. Two of them are reducible ones given in Example 4.2, others are irreducible ones given in Examples 4.3 to 4.5. They are all listed in the Table 1.*

Proof For λ_1-minimal flat 3-torus, it follows from Theorem 5.8 that the dual lattice Λ_3^* is always prime. Hence by Remark 5.5 and Proposition 2.7, the λ_1-minimal immersion x is homogeneous. Then it follows from Corollary 3.5 that we only need to prove the corresponding integer set Y of x is exactly that given in Examples 4.3 to 4.5, for which $\{A^t A \mid A \in Y\}$ is of rank N can be easily checked.

Suppose x is irreducible. It follows from Lemma 5.3 that there exists a generic 4-tuple in Ξ, which is assumed to be $\{\xi_1, \xi_2, \xi_3, \xi_4\}$. Moreover, we choose $\{\xi_1, \xi_2, \xi_3\}$ to be a generator of Λ_3^* so that $\xi_4 = \xi_1 + \xi_2 + \xi_3$. Then Lemma 5.12 implies that all coordinates of lattice vectors in Ξ can be assumed to take values in $\{0, 1\}$. Therefore $\Xi \setminus \{\xi_1, \xi_2, \xi_3, \xi_4\}$ is constituted by 1-null lattice vectors, the number of which is no more than 3. On the other hand, from

$$(\xi_1 + \xi_2) \wedge (\xi_2 + \xi_3) \wedge (\xi_1 + \xi_3) = 2\xi_1 \wedge \xi_2 \wedge \xi_3,$$

we know this number can not be 3, which implies $N \leq 6$. If $N = 4$, we obtain Example 4.3. If $N = 5$, we obtain Example 4.4 by making a permutation to $\{\xi_1, \xi_2, \xi_3\}$ such that the only 1-null lattice vector in Ξ is $\xi_1 + \xi_2$. Similarly, we arrive at Example 4.5 for $N = 6$. □

6.2 Classification of conformally flat 4-tori

For λ_1-minimal flat 4-torus, it follows from Theorem 5.8 that the dual lattice Λ_4^* is prime with only one exception, which is generated by the row vectors of (18). We will firstly discuss the λ_1-minimal immersion of such exceptional torus.

Lemma 6.2 *The λ_1-minimal isometric immersion of such exceptional torus is homogeneous.*

Proof It follows from (18) that the shortest vectors of Λ_4^* are composed of the following column vectors ξ_i and $-\xi_i$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1 & 1/2 & -1/2 & -1/2 & -1/2 & -1/2 & 1/2 & 1/2 & -1/2 \end{pmatrix}, \tag{21}$$

which can be divided into three blocks: I_4, S, S^t. It is obvious that S is an orthogonal matrix and $S^3 = I_4$.

It suffices to show that the symmetric products from any η-set are linearly independent, where the conclusion arises according to Lemma 2.7.

Suppose $\eta = \xi_{r_1} \pm \xi_{s_1}$. Either $\{\xi_{r_1}, \xi_{s_1}\}$ comes from the same block which can be assumed I_4 and thus $|\eta|^2 = 2$, or from different blocks which can be assumed I_4 and S so that $|\eta|^2 = 1$ or 3. Here we have used the symmetry induced by S.

When $|\eta|^2 = 3$, we may assume the first coordinate of η being $3/2$. Then it is easy to verify that there is no other possibilities for $\{\pm\xi_{r_i}, \pm\xi_{s_j}\}$ (note that these vectors have to be distinct). Clearly, $\xi_{r_1} \odot \xi_{s_1}$ is linearly independent.

When $|\eta|^2 = 2$, the other pairs $\{\xi_{r_i}, \xi_{s_i}\}$ must also come from a same block, for ξ_{r_i} and ξ_{s_i} have to be orthogonal. We may assume $\eta = (1, 1, 0, 0)$ such that all pairs $\{\xi_{r_i}, \xi_{s_i}\}$ are given as follows,

$$\eta = \xi_1 + \xi_2 = \xi_5 + \xi_6 = \xi_9 + \xi_{10}.$$

Then by direct computation, we obtain that $\xi_1 \odot \xi_2$, $\xi_5 \odot \xi_6$ and $\xi_9 \odot \xi_{10}$ are linearly independent.

When $|\eta|^2 = 1$, η is one of the shortest vectors. We may assume $\eta = \xi_5$ such that all pairs $\{\xi_{r_i}, \xi_{s_i}\}$ are given as follows,

$$\eta = \xi_1 - \xi_{12} = \xi_2 + \xi_{11} = \xi_3 + \xi_{10} = \xi_4 + \xi_9.$$

Then it is straightforward to verify that $\xi_1 \odot \xi_{12}$, $\xi_2 \odot \xi_{11}$, $\xi_3 \odot \xi_{10}$ and $\xi_4 \odot \xi_9$ are linearly independent. □

Proposition 6.3 *For such exceptional torus, there is altogether a 2-parameter family of λ_1-minimal isometric immersions in $\mathbb{S}^{23}$ up to congruence, given as follows (see also Example 1.1):*

$$\begin{aligned}(&a_1 e^{i\pi(u_1+u_2+u_3+u_4)}, a_1 e^{i\pi(u_1+u_2-u_3-u_4)}, a_1 e^{i\pi(u_1-u_2+u_3-u_4)}, a_1 e^{i\pi(-u_1+u_2+u_3-u_4)},\\ &a_2 e^{i\pi(u_1+u_2+u_3-u_4)}, a_2 e^{i\pi(u_1+u_2-u_3+u_4)}, a_2 e^{i\pi(u_1-u_2+u_3+u_4)}, a_2 e^{i\pi(u_1-u_2-u_3-u_4)},\\ &a_3 e^{2i\pi u_1}, a_3 e^{2i\pi u_2}, a_3 e^{2i\pi u_3}, a_3 e^{2i\pi u_4}),\end{aligned} \tag{22}$$

where $0 \le a_1 \le a_2 \le a_3$ and $a_1^2 + a_2^2 + a_3^2 = \frac{1}{4}$.

Proof From (21) one can see that Λ_4^* can be generated by $-\xi_1, -\xi_2, -\xi_3, \xi_5$ whose Gram matrix is

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}. \tag{23}$$

By a straightforward computation, we can derive that the matrix Y characterizing all shortest lattice vectors up to ± 1 is given by

$$Y = \begin{pmatrix} 1\,1\,0\,0\,1\,0\,0\,1\,1\,0\,0\,1 \\ 1\,0\,1\,0\,0\,1\,0\,1\,0\,1\,0\,1 \\ 0\,1\,1\,0\,0\,0\,1\,1\,0\,0\,1\,1 \\ 1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,2 \end{pmatrix}^t.$$

We point out that the order of A_i in Y does not coincide with that of ξ in (21). Moreover, for any $c_1^2 + c_2^2 \le 1/4$,

$$\left(c_2^2,\ c_2^2,\ c_2^2,\ c_2^2,\ \frac{1}{4}-c_1^2-c_2^2,\ \frac{1}{4}-c_1^2-c_2^2,\ \frac{1}{4}-c_1^2-c_2^2,\ \frac{1}{4}-c_1^2-c_2^2,\ c_1^2\ c_1^2,\ c_1^2,\ c_1^2\right) \tag{24}$$

defines a λ_1-minimal isometric immersion. In fact, these enumerate all possibilities of the λ_1-minimal isometric immersion, since the rank of $\{A^t A \mid A \in Y\}$ is 10, which can be verified directly. Define $a_1 = \sqrt{c_2^2}$, $a_2 = \sqrt{\frac{1}{4} - c_1^2 - c_2^2}$, $a_3 = \sqrt{c_1^2}$, then these immersions can be written down explicitly as (22).

Next, we discuss the congruence of these immersions. Note that an ambient congruence induces an isometry on the flat torus $T^n = \mathbb{R}^n/\Lambda_n$. It is well known that there are two kinds of isometries on T^n. One is produced by the translations on $\mathbb{R}^n$.

It is obvious that the ambient congruence corresponding to such isometry can not transform one immersion of the form (22) to another. The other is induced from the orthogonal transformations on $\mathbb{R}^n$ which preserve the lattice Λ_n, and then Λ_n^*. Since such orthogonal transformations preserve the lengths and angles of lattice vectors, they induce perturbations between the sets $\{\pm I_4\}$, $\{\pm S\}$ and $\{\pm S^2\}$, which further induce perturbations on $\{a_1, a_2, a_3\}$. □

Remark 6.4 The immersion given in (21) can be seen as a twist product:

$$x(\mathbf{u}) \triangleq \Big(a_1 f(\mathbf{u}S),\ a_2 f(\mathbf{u}S^2),\ a_3 f(\mathbf{u})\Big) \in \mathbb{S}^7(2a_1) \times \mathbb{S}^7(2a_2) \times \mathbb{S}^7(2a_3) \subset \mathbb{S}^{23},$$

where $\mathbf{u} = (u_1, u_2, u_3, u_4)$, $f(\mathbf{u}) = (e^{2i\pi u_1}, e^{2i\pi u_2}, e^{2i\pi u_3}, e^{2i\pi u_4})$, and S is the following orthogonal matrix of order 3:

$$\begin{pmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \end{pmatrix}.$$

The flat torus involved in $x(\mathbf{u})$ is

$$\mathbb{R}^4/\mathrm{Span}_{\mathbb{Z}}\{e_1 - e_4, e_2 - e_4, e_3 - e_4, 2e_4\},$$

with the volume $2\pi^4$. Note that when $a_3 = \frac{1}{2}$, $x(\mathbf{u})$ reduces to the double covering of the Clifford 4-torus with the underlying flat torus

$$\mathbb{R}^4/\mathrm{Span}_{\mathbb{Z}}\{e_1, e_2, e_3, e_4\}.$$

Theorem 6.5 *Up to congruence, Examples 4.6 to 4.16, and Proposition 6.3 exhaust all λ_1-minimal immersions of conformally flat 4-tori in spheres. They are all listed in the Table 2.*

Proof The exceptional case has been discussed in Proposition 6.3. Next, we assume Λ_4^* is prime. It follows from Remark 5.5 and Proposition 2.7 that the λ_1-minimal immersion x is homogeneous. Combining this with Corollary 3.5, we only need to prove the corresponding integer set Y of x is exactly that given in Example 4.6 $\sim$ Example 4.16, for which $\{A^t A \mid A \in Y\}$ is of rank N can be easily checked.

Note that all reducible ones have be given in Example 4.6. We will complete the classification of irreducible ones through the following lemmas and propositions involving prime lattices of rank 4. □

In the rest discussion, when some generator of Λ_4^* is chosen, we will identify the lattice vectors with their coordinates with respect to the given generator such that all vectors belong to $\mathbb{Z}^4$.

Lemma 6.6 *Suppose Λ_4^* is an s-irreducible prime lattice of rank 4, if there is no generic 5-tuple in Ξ, then $N \leq 7$ and Y takes the form as given in Example 4.9.*

Proof It follows from Lemma 5.3 that there always exists a generic 4-tuple in Ξ. Set $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ to be a generator of Λ_4^* such that the generic 4-tuple is given by

$$\{\eta_1, \eta_2, \eta_3, \eta_5 \triangleq \eta_1 + \eta_2 + \eta_3\}.$$

Using Lemma 5.12, we can assume that all the coordinates of lattice vectors in $\Xi \cap \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ take values in $\{0, 1\}$.

Since Λ_4^* is s-irreducible, there is at least one vector in $\Xi \backslash \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ other than η_4. Let η be such vector which can not be 2-null. Otherwise, $\eta = \eta_j \pm \eta_4$ for some $1 \leq j \leq 3$ and then $\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta\}\backslash\{\eta_j\}$ is a generic 5-tuple. So we can assume η is $(a_1, a_2, 0, 1)$ by the symmetry of η_1, η_2, η_3. It follows from Lemma 5.11 that $a_1 a_2 > 0$. Therefore, after changing the direction of η and η_4 if necessary we can assume $a_1 = a_2 = 1$. Moreover, using Lemma 5.10 and the partial order according to $\{1, 2, 3\}$ (see Lemma 5.11), we know there exists no other 1-null vectors in $\Xi \backslash \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$, which means $\Xi \backslash \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\} = \{\eta_4, \eta\}$.

If $\eta_1 + \eta_3$ (resp. $\eta_2 + \eta_3$) belongs to Ξ, then $\{\eta_4, \eta, \eta_5\}$ together with $\{\eta_1 + \eta_3, \eta_1\}$ (resp. $\{\eta_2 + \eta_3, \eta_2\}$) constitute a generic 5-tuple. Therefore, besides $\{\eta_1, \eta_2, \eta_3, \eta_5\}$, $\eta_1 + \eta_2$ is the only vector may appear in Ξ from $\mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$, which completes the proof of this lemma. □

Lemma 6.7 *Suppose* Ξ *contains a generic* 5*-tuple* X. *Then we can choose* Ξ' *such that* $\sharp(\Xi' \cap X) = 3$.

Proof We assume $X = \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$ and $\eta_5 = \eta_1 + \eta_2 + \eta_3 + \eta_4$. Choose $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ to be a generator of Λ_4^*, then by Lemma 5.12 we have all coordinates of vectors in Ξ taking values in $\{0, 1\}$. Since $\mathrm{rank}(\Xi') = 3$, it suffices to prove that we can choose Ξ' such that $\sharp(\Xi' \cap X) \geq 3$.

If Ξ' contains only 0-null or 3-null vectors (i.e. η_i) then $\Xi = \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$ and $m(\Xi) = 3$. For $m(\Xi) > 3$, Ξ' contains at least a 1-null or 2-null vector. Note that by rechoosing a generator of Λ_4^* in X, a given 2-null vector (such as $\eta_1 + \eta_2$) can be transformed to a 1-null vector (such as $\eta_1 + \eta_2 + \eta_3$ by choosing $\{\eta_5, -\eta_3, -\eta_4, -\eta_1\}$ as a new generator). Therefore, without loss of generality, we assume there is a 1-null vector $\eta \triangleq \eta_1 + \eta_2 + \eta_3 \in \Xi'$.

It is easy to see that if $\sharp(\Xi' \cap \{\eta_1, \eta_2, \eta_3\}) = 2$ then $\Xi' = \Xi \cap \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$.

If $\sharp(\Xi' \cap \{\eta_1, \eta_2, \eta_3\}) = 1$, we can assume it is η_1. Since $\sharp(\Xi \cap \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}) \geq 4$ and $\eta_1, \eta \in \Xi'$, we know $\sharp(\Xi' \backslash \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}) \geq 2$ in which all the vectors have coordinates 1 with respect to η_4. It follows from Lemma 5.11 that the existence of $\eta = \eta_1 + \eta_2 + \eta_3$ implies the vectors in $\Xi' \setminus \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ obey the partial order according to $\{1, 2, 3\}$, which means there is at most one k-null vector in $\Xi' \setminus \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ for every $0 \leq k \leq 3$. When $\Xi' \setminus \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ is composed of exactly two vectors, we get $\Xi \cap \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ also attaining $m(\Xi)$ so that it can be chosen as the Ξ' we desired. Or else, Ξ' must contain η_4 or η_5. Since $\eta_5 - \eta_4 = \eta \in \Xi'$, we have Ξ' must contain both η_4 and η_5 simultaneously, which implies $\eta_1, \eta_4, \eta_5 \in \Xi'$. The third one in $\Xi' \setminus \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ must be $\eta_1 + \eta_4$ if it's 2-null or $\eta_2 + \eta_3 + \eta_4$ if it's 1-null. Otherwise, η_2 or η_3 will be contained in Ξ'. Moreover, only one of these two vectors could appear in Ξ' for they violate the partial order according to $\{1, 2, 3\}$. This is obviously a case of $N = 7$.

If $\Xi' \cap \{\eta_1, \eta_2, \eta_3\} = \emptyset$ then $\Xi' \cap \text{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\} = \{\eta\}$. Similarly, we have $\sharp(\Xi' \backslash \text{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}) \geq 3$ and all the vectors in $\Xi' \setminus \text{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ admit at most one k-null vector for every $0 \leq k \leq 3$. Therefore, $\Xi' \cap \{\eta_4, \eta_5\} \neq \emptyset$ and thus $\eta_4, \eta_5 \in \Xi'$. Note that there are still others left in $\Xi' \setminus \text{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$. However, any 1-null (resp. 2-null) minus η_5 (resp. η_4) will yield $\Xi' \cap \{\eta_1, \eta_2, \eta_3\} \neq \emptyset$. Hence we finish the proof by this contradiction. □

Remark 6.8 From the proof of above two lemmas one can see that for $N \geq 8$, we can always choose a generic 5-tuple and a generator of Λ_4^* as in Lemma 6.7 such that $\Xi' = \Xi \cap \text{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ and $\eta \in \Xi'$.

Proposition 6.9 *If $N \leq 7$ and there exists a generic 5-tuple in Ξ, then we can find a generator of Λ_4^* such that Y takes the form as given in Examples* 4.7, 4.8, 4.10 *and* 4.11.

Proof Suppose $\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$ is a generic 5-tuple in Ξ, with $\eta_5 = \eta_1 + \eta_2 + \eta_3 + \eta_4$. When $N = 5$, it is easy to see that after choosing $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ to be a generator of Λ_4^*, we obtain Y as given in Example 4.7.

When $N \geq 6$, similarly as discussed in the proof of Lemma 6.7, we can assume

$$\eta_1, \eta_2, \eta_3, \eta_1 + \eta_2 + \eta_3, \eta_4, \eta_5 \in \Xi.$$

For the case of $N = 6$, by choosing $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ to be a generator of Λ_4^*, we arrive at Example 4.8. For the case of $N = 7$, if the remainder lattice vector lies on $\text{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$, then we obtain Y as given in Example 4.10 after a permutation in $\{\eta_1, \eta_2, \eta_3\}$. Now we assume the remainder lies on $\Xi \backslash \text{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$. Either it is 1-null, which can be assumed $\eta_2 + \eta_3 + \eta_4$ without loss of generality, so that we arrive at Example 4.11. Or it is 2-null, which can be assumed $\eta_3 + \eta_4$ without loss of generality. Then choosing $\{-(\eta_1 + \eta_2 + \eta_3), \eta_5, -(\eta_3 + \eta_4), -\eta_2\}$ as a generator of Λ_4^*, we also obtain Y as in Example 4.11. □

Proposition 6.10 *If $N \geq 8$, then there exist a generator of Λ_4^* such that Y takes the form as given in Examples* 4.12 *to* 4.16.

Proof It follows from Remark 6.8 that there exist a generic 5-tuple $\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$ and Ξ' such that

$$\{\eta_1, \eta_2, \eta_3, \eta \triangleq \eta_1 + \eta_2 + \eta_3\} \in \Xi', \quad \eta_5 = \eta_1 + \eta_2 + \eta_3 + \eta_4.$$

By choosing $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ as a generator of Λ_4^*, it follows from Lemma 5.12 that all coordinates of vectors in Y can only take values in $\{0, 1\}$.

If $m(\Xi) = 4$, the vectors in Ξ other than $\eta_1, \eta_2, \eta_3, \eta, \eta_4, \eta_5$ don't lie on $\text{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$. As in the proof of Lemma 6.7, they can be assumed $\eta_3 + \eta_4$ and $\eta_2 + \eta_3 + \eta_4$ due to the partial order according to $\{1, 2, 3\}$ (see Lemma 5.11). It means $\sharp(\Xi \cap \text{Span}_{\mathbb{Z}}\{\eta_2, \eta_3, \eta_4\}) = 5$ against $m(\Xi) = 4$. So $m(\Xi) \geq 5$.

When $m(\Xi) = 5$, we can assume $\eta_1 + \eta_2 \in \Xi'$ after making a permutation to $\{\eta_1, \eta_2, \eta_3\}$. Note that $\eta, \eta_5 \in \text{Span}_{\mathbb{Z}}\{\eta_1 + \eta_2, \eta_3, \eta_4\}$, neither $\eta_3 + \eta_4$ nor $\eta_1 + \eta_2 + \eta_4$

appears in Ξ which will make $\sharp(\Xi \cap \mathrm{Span}_{\mathbb{Z}}\{\eta_1 + \eta_2, \eta_3, \eta_4\}) \geq 6 > m(\Xi)$. Due to the partial order according to $\{1, 2, 3\}$ and the symmetry of η_1 and η_2, there is at most one 1-null vector in $\Xi \setminus \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ which can be assumed $\eta_2 + \eta_3 + \eta_4$, and at most one 2-null vector in $\Xi \setminus \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ which can be assumed $\eta_2 + \eta_4$. So $N \leq 9$. If $N = 9$, both $\eta_2 + \eta_3 + \eta_4$ and $\eta_2 + \eta_4$ exist and Y takes the form as given in Example 4.15. If $N = 8$ and only $\eta_2 + \eta_3 + \eta_4$ exists, then we arrive at Example 4.13. If $N = 8$ and only $\eta_2 + \eta_4$ exists, then after choosing $\{-\eta_2, -\eta_1, -\eta_3, \eta_5\}$ as a new generator, we can also obtain Y as given in Example 4.13.

When $m(\Xi) = 6$, up to a permutation of $\{\eta_1, \eta_2, \eta_3\}$, we can assume $\eta_1 + \eta_2, \eta_2 + \eta_3 \in \Xi'$. From

$$
\begin{aligned}
(\eta_1 + \eta_3) \wedge (\eta_1 + \eta_2) \wedge (\eta_2 + \eta_3) \wedge \eta_4 &= (\eta_1 + \eta_3 + \eta_4) \wedge (\eta_1 + \eta_2) \\
&\quad \wedge (\eta_2 + \eta_3) \wedge \eta_4 \\
&= (\eta_2 + \eta_4) \wedge (\eta_2 + \eta_3) \wedge (\eta_1 + \eta_2) \wedge \eta_5 = 2\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \eta_4,
\end{aligned}
$$

we know none of $\eta_1 + \eta_3$, $\eta_1 + \eta_3 + \eta_4$ or $\eta_2 + \eta_4$ can be contained in Ξ. Combining this with the partial order according to $\{1, 2, 3\}$ and the symmetry of η_1 and η_3, there is at most one 1-null vector in $\Xi \setminus \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ which can be assumed $\eta_2 + \eta_3 + \eta_4$, and at most one 2-null vector in $\Xi \setminus \mathrm{Span}_{\mathbb{Z}}\{\eta_1, \eta_2, \eta_3\}$ which can be assumed $\eta_3 + \eta_4$. So $N \leq 10$. If $N = 8$ then Y takes the form as given in Example 4.12. If $N = 10$, both $\eta_2 + \eta_3 + \eta_4$ and $\eta_3 + \eta_4$ exist, which leads to Example 4.16. If $N = 9$ and only $\eta_3 + \eta_4$ exists, then we obtain Y as given in Example 4.14. If $N = 9$ and only $\eta_2 + \eta_3 + \eta_4$ exists, then after choosing $\{-\eta_3, -\eta_2, -\eta_1, \eta_5\}$ as a new generator, we have also Y taking the form as given in Example 4.14.

The non-existence of $\eta_1 + \eta_3$ implies $m(\Xi) < 7$ and we finish the proof. □

Remark 6.11 Through the discussion in this section, we can obtain that in a prime lattice of rank $n \leq 4$, there exists at most $\frac{n(n+1)}{2}$ distinct lattice vectors of shortest length up to ± 1.

7 On λ_1-minimal flat tori of higher dimension

Similarly as in Sect. 4, we can construct many examples of higher dimensional λ_1-minimal flat tori in spheres. For simplicity, we choose a certain class to introduce in this section.

Consider the set $X_n \subset \mathbb{Z}^n$ given by the column vectors of

$$
X_n^t = \begin{pmatrix}
1 & & 1 & & & 1 & & & & 1 \\
& 1 & 1 & & 1 & 1 & & & & 1 \\
& & & 1 & 1 & 1 & & & & 1 \\
& & & & & & \ddots & & & \vdots \\
& & & & & & & 1 & \cdots & 1 \\
& & & & & & 1 & 1 & \cdots & 1
\end{pmatrix}_{n \times \frac{n(n+1)}{2}}.
$$

We call X_n the *ladder set*. It is straightforward to verify that W_{X_n} is a singleton set, only contains

$$Q_n = \begin{pmatrix} 1 & -\frac{1}{2} & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & 1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1 & -\frac{1}{2} \\ & & & & -\frac{1}{2} & 1 \end{pmatrix}.$$

By direct computation, we can obtain that

$$Q_n^{-1} = \frac{2}{n+1} \begin{pmatrix} n & n-1 & n-2 & \cdots & 3 & 2 & 1 \\ n-1 & (n-1)2 & (n-2)2 & \cdots & 3\cdot 2 & 2\cdot 2 & 2 \\ n-2 & (n-2)2 & (n-3)3 & \cdots & 3\cdot 3 & 3\cdot 2 & 3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 3 & 3\cdot 2 & 3\cdot 3 & \cdots & (n-3)3 & (n-3)2 & n-2 \\ 2 & 2\cdot 2 & 3\cdot 2 & \cdots & (n-2)2 & (n-1)2 & n-1 \\ 1 & 2 & 3 & \cdots & n-2 & n-1 & n \end{pmatrix}, \tag{25}$$

and

$$Q_n^{-1} = \frac{2}{n+1}\left(A_1^t A_1 + A_2^t A_2 + \cdots + A^t_{\frac{n(n+1)}{2}} A_{\frac{n(n+1)}{2}}\right). \tag{26}$$

Moreover, for any $(a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n \backslash 0$, we have

$$\begin{aligned}(a_1, a_2, \cdots, a_n) Q_n (a_1, a_2, \cdots, a_n)^t &= \sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n-1} a_i a_{i+1} \\ = \sum_{i=1}^{n-1} \frac{(a_i - a_{i+1})^2}{2} + \frac{a_1^2 + a_n^2}{2} \geq 1, \end{aligned} \tag{27}$$

with the equality holding if and only if $(a_1, a_2, \cdots, a_n)$ comes from X_n.

Proposition 7.1 *The matrix data set* $\{X_n,\ Q_n,\ \frac{2}{n(n+1)}(1, 1, \cdots, 1)\}$ *provides a* λ_1-*minimal flat torus in* $\mathbb{S}^{n(n+1)-1}$, *which is linearly full and has volume* $\frac{(2\sqrt{2})^n}{\sqrt{n+1}(\sqrt{n})^n}\pi^n$.

One can see that when we take $n = 2, 3, 4$, we obtain the equilateral torus, Examples 4.5, and 4.16, respectively.

Next, for a given $1 \leq k \leq n$, we consider the integer set $X_{n,k}$ obtained by removing the last $n-k$ rows from the ladder set X_n, i.e.,

$$X_{n,k}^t = \begin{pmatrix} 1 & 1 & & 1 & & & & \\ & 1 & 1 & & 1 & 1 & & \\ & & & 1 & 1 & 1 & & 1 \\ & & & & & \ddots & & \vdots \\ & & & & & & 1 \cdots & 1 \\ & & & & & 1 & 1 \cdots & 1 \end{pmatrix}_{n\times(\frac{n(n-1)}{2}+k)} . \tag{28}$$

Remark 7.2 Note that when $k = n$, $X_{n,k} = X_n$, which can automatically determine a λ_1-minimal flat n-torus as discussed above. When $k = 1$, $X_{n,k}$ is a block diagonal matrix, from which a reducible λ_1-minimal flat n-torus can be obtained.

Next let us consider the case of $2 \leq k \leq n-1$. Such kind of $X_{n,k}$ is called the *faulted ladder set*.

Lemma 7.3 *Define*

$$Q_{n,k} \triangleq Q_n + \frac{1}{2k}(E_{n,n-k} + E_{n-k,n}),$$

where $E_{i,j} = e_i^t e_j$ *and* e_i *is the* i*-th row of* I_n*, then* $Q_{n,k} \in W_{X_{n,k}}$*, and*

$$Q_{n,k}^{-1} = R_n + S_k - T_k,$$

with

$$R_n = \begin{pmatrix} Q_{n-1}^{-1} & \\ & 0 \end{pmatrix}, \quad S_k = \begin{pmatrix} O & \\ & Q_k^{-1} \end{pmatrix}, \quad T_k = \begin{pmatrix} O & & \\ & Q_{k-1}^{-1} & \\ & & 0 \end{pmatrix}.$$

Proof The first conclusion is easy to be verified, we only prove the second one.
Write

$$Q_n^{-1} = \frac{2}{n+1}\left(\sum_{1\leq i<j\leq n} i(n+1-j)\left(E_{i,j}+E_{j,i}\right) + \sum_{1\leq i\leq n} i(n+1-i)E_{i,i}\right),$$

It follows that

$$R_n = \frac{2}{n}\left(\sum_{1\leq i<j\leq n-1} i(n-j)\left(E_{i,j}+E_{j,i}\right) + \sum_{1\leq i\leq n-1} i(n-i)E_{i,i}\right), \tag{29}$$

$$S_k = \frac{2}{k+1}\left(\sum_{1\leq i<j\leq k} i(k+1-i)\left(E_{n-k+i,n-k+j}+E_{n-k+j,n-k+i}\right)\right.$$

$$+\sum_{1\le i\le k} i(k+1-i)E_{n-k+i,n-k+i}\Bigg), \tag{30}$$

$$T_k = \frac{2}{k}\Bigg(\sum_{1\le i<j\le k-1} i(k-i)\left(E_{n-k+i,n-k+j}+E_{n-k+j,n-k+i}\right)$$

$$+\sum_{1\le i\le k-1} i(k-i)E_{n-k+i,n-k+i}\Bigg). \tag{31}$$

Note that we can also express $Q_{n,k}$ as follows:

$$\begin{aligned} Q_{n,k} &= \begin{pmatrix} Q_{n-1} & \\ & 1 \end{pmatrix} - \frac{1}{2}(E_{n,n-1}+E_{n-1,n}) + \frac{1}{2k}(E_{n,n-k}+E_{n-k,n}), \\ &= \begin{pmatrix} Q_{n-k} & \\ & Q_k \end{pmatrix} - \frac{1}{2}(E_{n-k+1,n-k}+E_{n-k,n-k+1}) + \frac{1}{2k}(E_{n,n-k}+E_{n-k,n}) \\ &= \begin{pmatrix} Q_{n-k} & & \\ & Q_{k-1} & \\ & & 1 \end{pmatrix} - \frac{1}{2}(E_{n,n-1}+E_{n-1,n}) - \frac{1}{2}(E_{n-k+1,n-k} \\ &\quad + E_{n-k,n-k+1}) + \frac{1}{2k}(E_{n,n-k}+E_{n-k,n}). \end{aligned} \tag{32}$$

Combining the fact that

$$E_{i,j}E_{k,l} = \delta_{j,k}E_{i,l}, \quad 1\le i,j,k,l\le n$$

with (29) $\sim$ (32), we can obtain that

$$\begin{aligned} Q_{n,k}R_n &= \begin{pmatrix} I_{n-1} & \\ & 0 \end{pmatrix} + \sum_{n-k<i\le n-1} \frac{n-k-i}{k} E_{n,i}, \\ Q_{n,k}S_k &= \begin{pmatrix} 0 & \\ & I_k \end{pmatrix} - \sum_{1\le i\le k} \frac{k-i}{k} E_{n-k,n-k+i}, \\ Q_{n,k}T_k &= \begin{pmatrix} 0 & & \\ & I_{k-1} & \\ & & 0 \end{pmatrix} - \sum_{1\le i\le k-1} \frac{i}{k} E_{n,n-k+i} - \sum_{1\le i\le k-1} \frac{k-i}{k} E_{n-k,n-k+i}, \end{aligned}$$

from which the conclusion follows. □

It follows from (26) that

$$R_n = \frac{2}{n}\left(A_1^t A_1 + \cdots + A^t_{\frac{n(n-1)}{2}} A_{\frac{n(n-1)}{2}}\right) = \frac{2}{n}\sum_{j=1}^{n-1}\sum_{i=1}^{j} A^t_{\frac{j(j+1)}{2}+i} A_{\frac{j(j+1)}{2}+i},$$

$$S_k = \frac{2}{k+1}\sum_{j=n-k}^{n-1}\sum_{i=1}^{j+1+k-n} A^t_{\frac{j(j+1)}{2}+i} A_{\frac{j(j+1)}{2}+i}, \quad T_k = \frac{2}{k}\sum_{j=n-k}^{n-2}\sum_{i=1}^{j+1+k-n} A^t_{\frac{j(j+1)}{2}+i} A_{\frac{j(j+1)}{2}+i},$$

where A^t_i is the i-th column vector of $X^t_{n,k}$. Note that in $Q^{-1}_{n,k}$, the coefficient of $A^t_{\frac{j(j+1)}{2}+i} A_{\frac{j(j+1)}{2}+i}$ is

$$\frac{2}{n} + \frac{2}{k+1} - \frac{2}{k} = \frac{2(k^2+k-n)}{nk(k+1)}$$

for $n-k \le j \le n-2$ and $1 \le i \le j+1+k-n$; the other coefficients are $\frac{2}{n} > 0$ or $\frac{2}{k+1} > 0$.

Proposition 7.4 *Suppose* $2 \le k \le n-1$.

(1) *When* $k^2+k > n$, *the faulted ladder set* $X^t_{n,k}$ *given in* (28) *can determine a linearly full and* λ_1*-minimal flat* n*-torus in* $\mathbb{S}^{n(n-1)+2k-1}$.

(2) *When* $k^2+k = n$, *the faulted ladder set* $X^t_{n,k}$ *given in* (28) *can determine a linearly full and* λ_1*-minimal flat torus in a sphere of dimension* $n(n-1)-k^2+3k-1$, *with* $k(k-1)$ *dimensional eigenfunctions redundant.*

Proof We assume Λ^*_n is the lattice determined by $Q_{n,k}$ ($1 \le k \le n$), with $\{\xi_1, \cdots, \xi_n\}$ being a generator. Let Λ_n be the dual lattice. Define $\left(c_1^2, c_2^2, \cdots\cdots, c^2_{\frac{n(n+1)}{2}+k}\right)$ as follows:

$$c^2_{\frac{j(j+1)}{2}+i} = \begin{cases} \dfrac{2(k^2+k-n)}{nk(k+1)}, & n-k \le j \le n-2 \text{ and } 1 \le i \le j+1+k-n; \\ \dfrac{2}{n}, & j < n-k \text{ and } 1 \le i \le j; \\ \dfrac{2}{k+1}, & \text{others}. \end{cases}$$

Since the matrix data set $\left\{X_{n,k}, Q_{n,k}, \left(c_1^2, c_2^2, \cdots\cdots, c^2_{\frac{n(n+1)}{2}+k}\right)\right\}$ satisfy (12) and (13), they can determine an isometric minimal immersion of $T^n = \mathbb{R}^n/\Lambda_n$ in $\mathbb{S}^{n(n-1)+2k-1}$.

We are left to show 1 is the shortest length in Λ^*_n, and all the lattice vectors of this length having coordinates vectors as given in $X_{n,k}$ up to the sign.

Consider the sublattice generated by $\{\xi_1, \cdots, \xi_{n-1}\}$. It is obvious that it takes Q_{n-1} as the Gram matrix. Using (27), we know 1 is the shortest length in this sublattice. Next, we will use Theorem 3.8 to show this is also true for Λ^*_n.

Firstly, we calculate the distance from ξ_n to the hyperplane $\mathrm{Span}_{\mathbb{R}}\{\xi_1, \ldots, \xi_{n-1}\}$. Define $v \triangleq (v_1, \ldots, v_{n-1})$, where $v_{n-k} = 1/(2k)$, $v_{n-1} = -1/2$ and the others are 0. By (16) we get

$$d_n^2 = 1 - vQ^{-1}_{n-1}v^t = 1 - \frac{k-1}{2k} = \frac{k+1}{2k} > \frac{1}{4},$$

which implies for any vector $\xi = a_1\xi_1 + \cdots + a_n\xi_n \in \Lambda_n^*$, if $|a_n| > 1$ then $|\xi| > 1$. When $a_n = 0$, as a lattice vector in the sublattice, it is obvious that $|\xi| \geq 1$, with equality holding if and only if $\pm(a_1.a_2, \ldots, a_{n-1})$ belongs to X_{n-1}, which can be embedded into $X_{n,k}$. So we only need to discuss the case of $a_n = \pm 1$. By considering $-\xi$ if necessary, we can assume $a_n = 1$. Then we have

$$\begin{aligned}|\xi|^2 - 1 &= a_1^2 + \cdots + a_{n-1}^2 - a_1a_2 - \cdots - a_{n-2}a_{n-1} - a_{n-1} + \frac{1}{k}a_{n-k}\\ &= \sum_{i=1}^{n-1}\frac{1}{2}(a_i - a_{i+1})^2 + \frac{1}{2}\left(a_1^2 - 1\right) + \frac{1}{k}a_{n-k},\end{aligned}$$

which is greater than 0 when $a_{n-k} > 0$. For the case of $a_{n-k} = 0$, it is obvious that $|\xi|^2 - 1 \geq 0$, and the equality holds if and only if $a_1 = \cdots = a_{p-1} = 0$ and $a_p = \cdots = a_n = 1$ for some $p > n - k$, which exactly corresponds to a certain row of $X_{n,k}$. When $a_{n-k} < 0$, using Cauchy inequality, we have

$$\sum_{i=n-k}^{n-1}(a_i - a_{i+1})^2 \geq \frac{1}{k}\left(\sum_{i=n-k}^{n-1}(a_i - a_{i+1})\right)^2 = \frac{1}{k}(1 - a_{n-k})^2 > -\frac{2}{k}a_{n-k},$$

from which it follows that $|\xi|^2 - 1 > 0$, and we finish the proof. □

Remark 7.5 The λ_1-minimal flat torus in this section has volume $\frac{\sqrt{k}(2\sqrt{2})^n}{\sqrt{k+1}(\sqrt{n})^{n+1}}\pi^n$ $(1 \leq k \leq n)$. It comes from the fact that $\det Q_{n,k} = \frac{n(k+1)}{2^n k}$ which again indicates $Q_{n,k}$ is positive-definite. One can easily check that Example 4.4, 4.12 and 4.14 exactly correspond to the case $(n, k) = (3, 2)$, $(4, 2)$ and $(4, 3)$, respectively.

8 Berger's problem on conformally flat 3-tori and 4-tori

As recalled in the introduction, on n-tori ($n \geq 3$), there is no solution to the Berger's problem, i.e., one can not expect a uniform upper bound for $\mathcal{L}(g)$ among all smooth Riemannian metrics. However, if restricted to the flat metric, or a certain class of some conformally flat metrics, we can solve the Berger's problem for $n \leq 4$. That is, we will prove Theorems 2 and 3 given in the introduction.

Note that finding the upper bound for $\lambda_1(g)V(g)^{\frac{2}{n}}$ among all flat n-tori, is equivalent to find the upper bound for volumes of all flat n-tori with $4\pi^2$ as the first eigenvalue, which can be done by calculating the minimum of the determinant of those lattices (of rank n) with 1 as the shortest length. Since a shortest lattice vector can always be extended to be the first vector of a generator, in terms of the Gram matrix, we only need to calculate the minimum of det on the following set:

$$\Omega_1 \triangleq \left\{Q \in \Sigma_+ \,\middle|\, Q_{11} = 1, vQv^t \geq 1 \text{ forall } v \in \mathbb{Z}^n\backslash\{0\}\right\},$$

where Q_{ij} is the entry of Q. We still use the notation Ξ to denote the set of shortest lattice vectors up to ± 1 in a given lattice Λ_n^*.

Lemma 8.1 *For the lattice Λ_n^* determined by $Q \in \Omega_1$, if* $\operatorname{rank}(\Xi) < n$*, then Q is not a minimal point of* det *on Ω_1.*

Proof We assume Ξ is contained in the $(n-1)$-dimensional sublattice Λ_{n-1} with a generator $\{\xi_1, \cdots, \xi_{n-1}\}$ and α is a vector in $\Lambda_n^* \backslash \Lambda_{n-1}$ such that $\{\xi_1, \cdots, \xi_{n-1}, \alpha\}$ is a generator of Λ_n^*.

Set $\alpha = \alpha^\top + \alpha^\perp$, in which $\alpha^\top$ is the orthogonal projection of α onto $\operatorname{Span}\{\xi_1, \cdots, \xi_{n-1}\}$. Let $\Lambda_{n,t}^*$ be a new lattice generated by $\{\xi_1, \cdots, \xi_{n-1}, \alpha - t\alpha^\perp\}$ for $0 \le t < 1$, $\Pi_{k,t}$ be the affine hyperplane defined by $\sum_{i=1}^{n-1} c_i \xi_i + k(\alpha - t\alpha^\perp)$ $(c_i \in \mathbb{R})$ for any $k \in \mathbb{Z}$.

Note that there exists an integer $K > 0$ such that $\Pi_{k,0} \cap \overline{B(0,1)} = \emptyset$ if and only if $|k| > K$. By continuity, there is $0 < t_0 < 1$ such that for any $0 \le t \le t_0$, $\Pi_{k,t} \cap \overline{B(0,1)} = \emptyset$ if and only if $|k| > K$.

As for $0 < |k| \le K$, it is easy to see that for any vector of $\Lambda_{n,t}^* \cap \Pi_{k,t}$, its orthogonal projection onto $\operatorname{Span}\{\xi_1, \cdots, \xi_{n-1}\}$ does not depend on t, while $\Pi_{k,t} \cap \overline{B(0,1)}$ is a family of co-centered $(n-1)$-dimensional closed balls expanding as t goes from 0 to t_0. Given the fact that all the vectors of $\Lambda_{n,0}^* \cap \Pi_{k,0}$ are located outside of $\Pi_{k,0} \cap \overline{B(0,1)}$, there exists $t_k > 0$ such that all the vectors of $\Lambda_{n,t}^* \cap \Pi_{k,t}$ are still located outside of $\Pi_{k,t} \cap \overline{B(0,1)}$ for any $0 \le t \le t_k$.

Set $\delta = \min\{t_0, \cdots, t_K\}$. Then the lattice $\Lambda_{n,\delta}^*$ admits the same shortest vectors as Λ_n^* while the Gram matrix Q_δ satisfying $\det Q_\delta = (\det(\xi_1, \cdots, \xi_{n-1}, \alpha - \delta\alpha^\perp))^2 = (1-\delta)^2 \det Q$. $\square$

Due to Theorem 5.8 and (23), for a lattice Λ_n^* with $\operatorname{rank}(\Xi) = n \le 4$, we can always choose a generator of Λ_n^* such that the Gram matrix takes 1 as its diagonal entries. Set

$$\Omega_2 \triangleq \{Q \in \Sigma_+ \,\big|\, Q_{ii} = 1 \text{ for all } 1 \le i \le n, \text{ and } vQv^t \ge 1 \text{ for all } v \in \mathbb{Z}^n \backslash \{0\}\}. \tag{33}$$

Then according to Lemma 8.1, we have the minimum of det on Ω_1 can only be attained on Ω_2.

Lemma 8.2 *Ω_2 is convex and compact.*

Proof The convexity of Ω_2 is obvious. To prove the compactness, we only need to prove that Ω_2 is both bounded and closed.

Suppose $Q \in \Omega_2$. It follows from $\langle Q, Q \rangle = \operatorname{tr} Q^2 < (\operatorname{tr} Q)^2 = n^2$ that Ω_2 is bounded.

Next, we prove that Ω_2 is closed. Suppose $\{Q_n\}$ is a convergent sequence in Ω_2, whose limit is denoted by Q_0. We denote by D_{nk} (resp. D_{0k}) the leading principal minor of order k for Q_n (resp. Q_0). Note that for all $1 \le k \le n$, the submatrix corresponding to D_{nk} is also positive definite. It can determine a sublattice of rank k, which also takes 1 as the shortest length. It follows from the Theorem 13 in [30] (a corollary of the Minkowski's first theorem) that

$$\sqrt{D_{nk}} \geq \frac{V(\mathbb{S}^k)}{2^k},$$

where $V(\mathbb{S}^k)$ is the volume of the standard round k-sphere. By continuity, we have $\sqrt{D_{0k}} \geq \frac{V(\mathbb{S}^k)}{2^k} > 0$ for all $1 \leq k \leq n$, which implies $Q_0 \in \Sigma_+$. In the mean time, it is obvious that the diagonal entries of Q_0 are all 1 and $vQ_0v^t \geq 1$ for all $v \in \mathbb{Z}^n \backslash \{0\}$. So we get $Q_0 \in \Omega_2$, and thus Ω_2 is closed. □

Lemma 8.3 Ω_2 *is a convex polytope.*

Proof We denote by $\{0, \pm 1\}^n$ the set of integer vectors whose coordinates take values only in $\{0, \pm 1\}$. Define

$$\Omega_3 \triangleq \left\{ Q \in S(n) \,\middle|\, Q_{ii} = 1 \text{ for all } 1 \leq i \leq n, \text{ and } vQv^t \geq 1 \text{ for all nonzero } v \in \{0, \pm 1\}^n \right\}. \tag{34}$$

For simplicity, we only prove the case of $n \leq 4$. To prove the higher dimensional case, one only need to replace the integer set $\{0, \pm 1\}^n$ in Ω_3 by the finite set of integer vectors appearing in Minkowski's reduction theory (see [30]). It follows from Theorem 3.9 that $\Omega_2 = \Omega_3 \cap \Sigma_+$.

We claim that $\Omega_2 = \Omega_3$. For this, let us consider the topology on Ω_3 induced from the ambient space $S(n)$. It is easy to show that Ω_3 is convex, which implies Ω_3 is connected. Seeing Ω_2 as a subset in Ω_3, we can firstly derive that it is closed by Lemma 8.2. On the other hand, note that Σ_+ is open in $S(n)$, it follows from $\Omega_2 = \Omega_3 \cap \Sigma_+$ that Ω_2 is also open in Ω_3. Then the claim follows from the fact that $\Omega_2 \neq \emptyset$.

It is not hard to see that the constraints $vQv^t \geq 1$ for all nonzero $v \in \{0, \pm 1\}^n$ define a polyhedron in $S(n)$ by the intersection of finite half-spaces, and Ω_3 is one of its facets. Combining this and Lemma 8.2, we derive that $\Omega_2 = \Omega_3$ is a convex polytope. □

According to the concavity of $\ln \circ \det$ on Σ_+ introduced in Lemma 3.4, one can easily obtain the next conclusion.

Lemma 8.4 *The minimum of* det *on* Ω_2 *is attained at some vertex of* Ω_2.

Lemma 8.5 *Suppose* Q *is a vertex of* Ω_2*, then* $\sharp(\Xi) \geq \frac{n(n+1)}{2}$.

Proof Since Q is a vertex and $\dim S(n) = \frac{n(n+1)}{2}$, it satisfies the constraints given in (34), among which there should be a system of linear equations with rank $\frac{n(n+1)}{2}$. As a result, there should be at least $\frac{n(n+1)}{2}$ integer vectors such that $vQv^t = 1$, from which the conclusion follows. □

Proposition 8.6 *Suppose* $n \leq 4$*, then every vertex of* Ω_2 *can determine a* λ_1*-minimal flat* n*-torus in some sphere. Up to congruence, these* λ_1*-minimal flat* n*-tori are*
(1) *the equilaterial* 2*-torus in* $\mathbb{S}^5$ *given in Example 4.2 for* $n = 2$*;*
(2) *the* λ_1*-minimal flat* 3*-torus in* $\mathbb{S}^{11}$ *given in Example 4.5 for* $n = 3$*;*

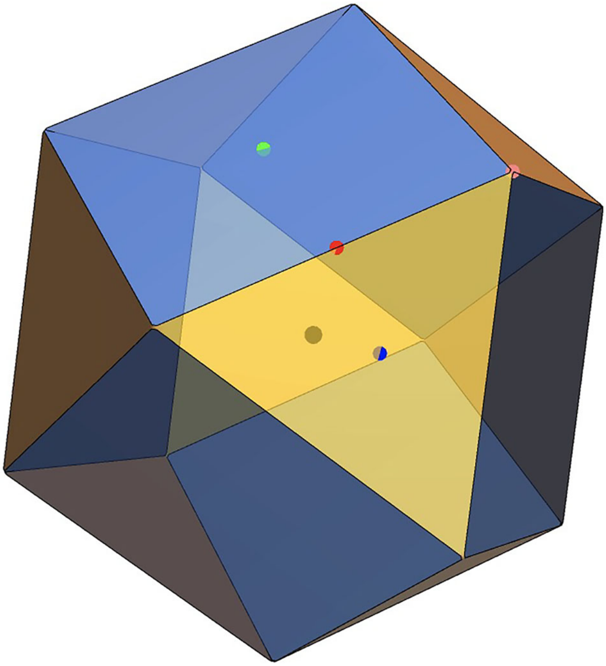


Fig. 3 Ω_2 and the distribution of all λ_1-minimal flat 3-tori

(3) *the* λ_1*-minimal flat* 4*-torus in* $\mathbb{S}^{19}$ *given in Example* 4.16*, or those* λ_1*-minimal flat* 4*-tori given in Example* 1.1 *for* $n = 4$.

Proof Firstly, for these examples, using their matrix data, one can check directly that the Gram matrix Q belongs to Ω_2, and the rank of $\{A^t A | A \in Y\}$ is exactly $\frac{n(n+1)}{2}$. Therefore they all correspond to the vertices of Ω_2.

Suppose Q is a vertex of Ω_2. If the lattice determined by Q is not prime, then we arrive at the only exceptional torus, and the conclusion follows from Proposition 6.3.

Next, we assume the lattice determined by Q is prime. Let X be the set of integer vectors corresponding to Ξ. Combining Remark 6.11 and Lemma 8.5, we can derive that $\sharp(\Xi) = \frac{n(n+1)}{2}$. Furthermore, it follows from the discussion in Sect. 6 that X is exactly the ladder set X_n up to a unimodular transformation in $SL(n, \mathbb{Z})$. Note that W_{X_n} is a singleton set. So Q is congruent to Q_n by a unimodular transformation in $SL(n, \mathbb{Z})$, which completes the proof of this proposition. □

Remark 8.7 It follows from Theorem 3.2, Remark 3.3 that for every facets of Ω_2, in general one can determine a unique flat torus which admitting the λ_1-minimal immersion in spheres. Conversely, when $n \leq 4$, every λ_1-minimal flat torus can determined a Gram matrix Q in Ω_2.

For instance, when $n = 3$, Ω_2 is constituted by matrices $\begin{pmatrix} 1 & u & v \\ u & 1 & w \\ v & w & 1 \end{pmatrix}$, with u, v, w all taking values in $[-\frac{1}{2}, \frac{1}{2}]$, and satisfying

$$-1 \leq u+v+w \leq 1, \quad -1 \leq u+v-w \leq 1, \quad -1 \leq u-v+w \leq 1 \quad -1 \leq u-v-w \leq 1.$$

It is straightforward to verify that there is a correspondence between the λ_1-minimal flat 3-tori and the barycenters of facets of Ω_2. In Fig. 3, it shows the two reducible ones described in Example 4.2 respectively correspond to the gray point (center of the body) and the green point (center of the square), and those points painted by blue (center of the triangle), red (center of the edge) and pink (the vertex) corresponds to Examples 4.3 to 4.5, respectively.

Combining Lemma 8.4 with Proposition 8.6, we can directly calculate the minimum of det on Ω_2 (hence on Ω_1), from which Theorem 2 follows.

For a given flat metric g_0 on n-torus T^n, it was proved by El Soufi and Ilias in [11] that g_0 maximizes $\mathcal{L}(g)$ in $[g_0]$, if the first eigenspace of g_0 is of dimension no less than $2n$. Combining this with Theorem 2, we can obtain Theorem 3.

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Data availability Data available within this article.

Declarations

Conflict of interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

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