Picard theorems for moduli spaces of polarized varieties

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Abstract

As a result of our study of the hyperbolicity of the moduli space of polarized manifold, we give a general big Picard theorem for a holomorphic curve on a log-smooth pair (X, D) such that $W = X \setminus D$ admits a Finsler pseudometric that is strongly negatively curved when pulled back to the curve. We show, by some refinements of the classical Viehweg–Zuo construction, that this latter condition holds for the base space W, if nonsingular, of any algebraic family of polarized complex projective manifolds with semi-ample canonical bundles whose induced moduli map ϕ to the moduli space of such manifolds is generically finite and any ϕ-horizontal holomorphic curve in W. This yields the big Picard theorem for any holomorphic curves in the base space U of such an algebraic family by allowing this base space to be singular but with generically finite moduli map. An immediate and useful corollary is that any holomorphic map from an algebraic variety to such a base space U must be algebraic, i.e., the corresponding holomorphic family must be algebraic. We also show the related algebraic

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hyperbolicity property of such a base space U, which generalizes previous Arakelov inequalities and weak boundedness results for moduli stacks and offers, in addition to the Picard theorem above, another evidence in favor of the hyperbolic embeddability of such an U.

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Contents

1 Introduction

The paper of Viehweg and Zuo [49] pioneered the investigation of the analytic hyperbolicity of the moduli space of polarized varieties by showing that, for any algebraic family of canonically polarized complex manifolds with quasi-finite moduli map to the moduli space, the base space U of the family is Brody hyperbolic, i.e., U admits no non-constant holomorphic maps from $\mathbb{C}$.[1]

For compact spaces, Brody hyperbolicity is equivalent by Brody's reparametrization theorem to being hyperbolic in the sense of Kobayashi. However, most parameter spaces considered above are not compact. In the non-compact setting, the strongest notion of complex hyperbolicity (short of the existence of a hyperbolic compactifica-

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[1] The term "moduli space" used above should more properly be written as "moduli stack" though we will not belabour this point in this paper. Hence hyperbolicity of moduli space is taken to mean in the sense of stacks.

tion) is that of hyperbolic embeddability, meaning that there exists a compactification in which the space hyperbolically embeds.[2]

This paper offers two evidences of this hyperbolic embeddability in the moduli setting: we obtain for the said parameter spaces that (I) the Picard extension property of holomorphic maps from curves to them holds, generalizing the Brody hyperbolicity result of [49], and that (II) they are algebraically hyperbolic in the sense of Demailly, see Definition 1.5. In fact, we generalize these results, after a suitable recall and recasting of the construction in the above paper of Viehweg and Zuo, from the setting of canonically polarized manifolds to the setting of manifolds with semi-ample canonical bundle. It is fitting to remark here that hyperbolic embeddability generalizes Kobayashi hyperbolicity and that property (I) and (II) are implied by it though the converses corresponding to (I) and (II) are open, see e.g. [30, II§2], [27, Theorem (6.3.7)] and [37].

More precisely, consider an algebraic family of polarized complex projective manifolds with semi-ample canonical divisors and Hilbert polynomial h given by an equivalence class of pairs $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$, i.e. f is a proper and smooth morphism between algebraic varieties V and U with a relatively ample line bundle $\mathcal{L}$ over V having h as the fibrewise Hilbert polynomial (modulo the obvious equivalence). Suppose that the induced map from the parameter space U to the coarse moduli space M_h, also called the moduli map, is quasi-finite. Let $\bar{U}$ be an algebraic compactification of U, $C \subset \bar{C}$ an open subset of a complex curve with codimension one complement and $\gamma : C \to U$ a holomorphic map. We show that γ has a holomorphic extension $\bar{\gamma} : \bar{C} \to \bar{U}$ and, as a well-known consequence, that the result generalizes to the case when C is replaced by a complex space and holomorphic by the word meromorphic, obtaining in particular that γ is algebraic when C is an algebraic variety. Our proof of this proceeds in two major steps, comprising the two major parts of substance of the paper, the first extracts the necessary curvature conditions via a detailed recall and some necessary extensions of the Viehweg–Zuo paper [49] while the second uses Nevanlinna theory to prove our key technical theorem of independent interest, Theorem A. We offer two proofs of this technical theorem. The first offers a simplification and strengthening of some techniques developed by and used in Griffiths-King [19] to generalize a classical big Picard theorem in the setting of moduli spaces and follows a metric approach to Nevanlinna theory as espoused by Chern and Ahlfors. The second is inspired by the proof of the fundamental vanishing theorem of Siu-Yeung and Demailly ([11, 43]) for jet-differentials on holomorphic curves and proceeds by a reduction to the logarithmic derivative lemma. As a natural consequence of the first part of the paper, we show in the last section the algebraically hyperbolicity of U, generalizing previous Arakelov inequalities and weak boundedness results of moduli stacks (see Remark 5.1).

The Shafarevich problem from the 60's and its higher dimensional generalizations (cf. [29, 46] for an introduction), motivated the very extensive modern study of the

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[2] Hyperbolic embedding characterizes the Bailey–Borel compactifications of the moduli spaces of abelian varieties, a classical theorem of Borel. Since we are mainly interested in the more algebraic aspects of hyperbolicity in this paper, we do not give the definition of Kobayashi hyperbolicity, nor of hyperbolic embedding. The interested reader should consult standard references such as the books of Lang and of Kobayashi on complex hyperbolic geometry.

hyperbolicity properties of the moduli space of polarized manifolds with many major advances. We will only touch upon a couple of the recent advances that generalize the result of [49] below.

By refining a part of Viehweg–Zuo's construction of negatively curved pseudometrics, Popa–Taji–Wu in [38] proved the Brody hyperbolicity for the case of the moduli space of polarized manifold with nef and big canonical bundle. In [44] and [45], To and Yeung proved that the moduli spaces of canonically polarized and Ricci flat manifolds are Kobayashi hyperbolic by constructing negatively curved Finsler metrics on U. This is stronger than Brody hyperbolicity for a noncompact U, but still weaker than having a hyperbolic embedding (cf. [30, Chapter 2] or [27, Chapter 3, §3]). More recently, the first named author in [12] proved in addition that the moduli space of polarized varieties whose canonical divisors are semi ample and big is Kobayashi hyperbolic and that the moduli space without the bigness condition is Brody hyperbolic, both of which were conjectured by Viehweg and the fourth named author ([49, Question 0.2]). This was done via negatively curved Finsler metrics on U obtained by adding, at the end of the Viehweg–Zuo construction of Finsler pseudo-metrics in [49], two natural ingredients in the hyperbolic context: a pointwise argument plus a convex linear combination of components of the Viehweg–Zuo Finsler pseudo-metrics. This allowed the Viehweg–Zuo construction to extend to the semi-ample case and and is adapted here for some necessary improvements of the Viehweg–Zuo construction in this paper.

Another important motivation for our current investigation comes from the fact that many well-studied moduli spaces such as the moduli space of abelian varieties suitably rigidified and that of marked K3 surfaces are actually locally symmetric of non-compact type and such locally symmetric varieties admit the famous Baily–Borel compactifications (cf. [1]). Borel showed in [4] that, given such a locally symmetric variety, this compactification yields a hyperbolic embedding of the variety. In particular, the big Picard theorem holds on these moduli spaces.

Theorem 1.1 (Borel) *Let X be a torsion-free arithmetic quotient of a bounded symmetric domain. Denote by X^* the Baily-Borel compactification of X. Then X is hyperbolically embedded in X^*.*

For moduli spaces of canonically polarized varieties, the KSBA compactifications (see the survey [28]) are natural candidates for the hyperbolic embeddings and Borel's theorem leads us to pose:

Question 1.2 *Let $\bar{f} : X \to Y$ be a KSBA stable family over a projective variety Y. Denote by $U \subset Y$ the open subset over which f is smooth. Is U hyperbolically embedded in Y?*

Also inspired by this theorem, Javanpeykar and Kucharczyk in [24] formulated the following:

Definition 1.3 A finite type scheme X over $\mathbb{C}$ is *Borel hyperbolic* if, for every finite type reduced scheme S over $\mathbb{C}$, any holomorphic map from S to X is algebraic.

It is easy to see that Borel hyperbolicity implies Brody hyperbolicity and that hyperbolic embeddability implies Borel hyperbolicity. As observed in [19], Borel

hyperbolicity can be verified by restricting to the case S is one dimensional. The following is a natural question in our context:

Question 1.4 *Is every Brody hyperbolic moduli space of polarized manifolds Borel hyperbolic?*

Very recently, Bakker, Brunebarbe and Tsimerman have obtained some sweeping partial results in this direction, see [2, Corollary 7.1]. For a family with local Torelli injectivity, i.e. the period map is quasi-finite, the Borel hyperbolicity is a direct corollary of a conjecture of Griffiths. More precisely, let B be an algebraic variety with a polarized variation of Hodge structures (PVHS) and $\Phi : B \rightarrow \Gamma\backslash D$ the induced period map. Here D is the period domain (namely the classifying space of Hodge structures with fixed Hodge numbers) and Γ is the monodromy group of the PVHS on B. In [23], Griffiths conjectured that the image $\Phi(B) \subset \Gamma\backslash D$ is quasi-projective. Note that the quotient space $\Gamma\backslash D$ is in general a highly transcendental object. The paper [2] confirms this conjecture assuming that Γ is arithmetic as a corollary of its deep results on the o-minimal GAGA theorem. We remark that this conjecture with arbitrary monodromy group Γ was established when $\dim \Phi(B) = 1$ (cf. [6, 42]) and $\dim B = 2$ (cf. [18, Theorem 1.2.6]).

Nevertheless, families of polarized varieties where the local Torelli injectivity fails abounds.

1.1 The big Picard theorem and Borel hyperbolicity

Let $\bar{f} : X \rightarrow Y$ be an analytic family of projective manifolds over a projective base Y with degeneration locus $S \subset Y$. Hence $\bar{f}$ is a compactification of the smooth family $f : V \rightarrow U$ with $U = Y \setminus S$ and $V = \bar{f}^{-1}(U)$. Griffiths introduced in [20–22] the notion of polarized variation of Hodge structure on $U = Y \setminus S$. Schmid, Deligne and Cattani–Kaplan–Schmid ([8, 9, 39]) have studied the asymptotic behavior of the Hodge structures and the Hodge metric near the degeneration locus. Their results are of fundamental importance in the study of the geometry of families. Kawamata and Viehweg's positivity theorems on the direct image $f_* \omega_{X/Y}^\nu$ of powers of the relative dualizing sheaf are examples that play crucial roles in the investigation of the Iitaka conjecture. Another is Viehweg's work on constructing the moduli space of varieties with semi-ample dualizing sheaves.

As mentioned, the Torelli-type theorem fails in general for such a family. As a substitute, Viehweg and the fourth named author constructed in [49] a non-trivial comparison map between the usual Kodaira–Spencer map and the Kodaira–Spencer map on the Hodge bundles associated to some new family built from certain cyclic coverings of X. Consequently, using the semi-negativity of the kernels of the Kodaira–Spencer maps on the Hodge bundles (proven in [51]) and the positivity results on the direct image sheaves, the maximal non-zero iteration of Kodaira–Spencer map yields the "*bigness*" of the so-called *Viehweg–Zuo subsheaves* in symmetric powers of $\Omega_Y^1(\log S)$. These subsheaves give rise analytically to negatively curved complex Finsler pseudometrics on $U = Y \setminus S$.

In the paper [19], Griffiths and King studied the higher dimensional generalization of value distribution theory. With it, they obtained a Nevanlinna-theoretic proof of Borel's theorem via negative curvature (cf. [19, Corollary (9.22)]). Our negatively curved Finsler pseudometrics led us naturally to this approach of Griffiths-King in generalizing Borel's theorem, which we manage to simplify and strengthen to yield the first proof of the key technical theorem of this paper below.

Theorem A (Criterion for the big Picard theorem) *Let X be a projective manifold, ω a Kähler metric on X and D a simple normal crossing divisor on X. Let $\gamma : \mathbb{D}^* \to X \setminus D$ be a holomorphic map. Assume that there is a Finsler pseudometric h on $T_X(-\log D)$ (in the sense of Definition 2.14) such that $|\gamma'(z)|_h^2 \not\equiv 0$ and that the following inequality holds in the sense of currents*

$$\mathrm{dd}^c \log |\gamma'(z)|_h^2 \geq \gamma^* \omega. \tag{1.1}$$

Then γ extends to a holomorphic map $\bar{\gamma} : \mathbb{D} \to X$.

We give a second perhaps more modern proof of this theorem in Sect. 4, see also the remark just before that section. We mention that this criterion is also used by the first named author in [16] to prove the big Picard theorem for varieties having a quasi-finite period map. In addition, although the Nevanlinna theoretic tools involved are quite standard, our approaches in the two proofs of Theorem A are not and have further implications beyond the criterion proper. In particular, our proof can be modified to yield Corollary C below, omitted for simplicity as it follows from a deep theorem of Siu (the Borel hyperbolicity part there being a simple corollary of Theorem B).

As indicated above, the construction of [49] generalizes from the case of canonically polarized manifolds to the case of manifolds with semiample canonical bundle. This includes recent key observations from the paper of Popa–Taji–Wu [38] and from the first named author in [12] and is worked out in Sect. 2, where we construct the required metric h over a desingularization of the base space of the family, i.e., one that satisfies (1.1) of Theorem A. This yields our main theorem.

Theorem B [Big Picard theorem] *Let $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ be an algebraic family of polarized projective manifolds of Hilbert polynomial h and semi-ample canonical bundle. Suppose that the moduli map $U \to M_h$ from U to the coarse moduli space M_h is quasi-finite. Given a completion $\bar{U}$ of U, any holomorphic map $\gamma : \mathbb{D}^* \to U$ extends to a holomorphic map $\bar{\gamma} : \mathbb{D} \to \bar{U}$. In particular, any holomorphic map from an algebraic curve T to U is necessarily algebraic.*

Corollary C *Let $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ be as given in Theorem B. Let Y be a projective compactification of U. Then any holomorphic map $\gamma : \mathbb{D}^p \times (\mathbb{D}^*)^q \to U$ extends to a meromorphic map $\overline{\gamma} : \mathbb{D}^{p+q} \dashrightarrow Y$. In particular, U is Borel hyperbolic: Any holomorphic map from an algebraic variety T to U is necessarily algebraic; I.e., over an algebraic variety, any such holomorphic family of polarized projective manifolds is actually algebraic.*

Proof of part 1 of Corollary C By [41, Theorem 1], any meromorphic map to a compact Kähler manifold extends across a subvariety of codimension 2. As U embeds in

$\mathbb{C}P^N$, $\exists N$, it suffices to prove the extension property for a holomorphic map of the form $\gamma : \mathbb{D}^r \times \mathbb{D}^* \to U$. Now, the key result [41, p.442, (*)] of Siu states that γ extends to a meromorphic map $\overline{\gamma} : \mathbb{D}^{r+1} \dashrightarrow Y$ if for all z in a subset of $\mathbb{D}^r$ of nonzero Lebesque measure, the holomorphic map $\gamma|_{\{z\} \times \mathbb{D}^*} : \{z\} \times \mathbb{D}^* \to U$ can be extended to a holomorphic map from $\{z\} \times \mathbb{D}$ to Y. But this latter follows from Theorem B.

Note that, by the first part of Theorem B, respectively that of Corollary C, the holomorphic map $\gamma : T \to U$ extends to a meromorphic map between any of their projective compactifications, which is thus a rational map by Chow's theorem. The last parts of Theorem B and Corollary C follow.

In fact Theorem B can be modified to a Lang conjecture type statement: for a family of polarized manifolds $f : V \to U$ with maximal variation in moduli, i.e. with generically finite moduli map, there is a proper subvariety of the base U so that any punctured disk whose image is not contained in this proper subvariety satisfies the big Picard theorem. See Remark 3.5 for details.

1.2 Algebraic hyperbolicity

Algebraic hyperbolicity for a compact complex manifold X was introduced by Demailly in [10, Definition 2.2]. It is shown in [10, Theorem 2.1] that X is algebraic hyperbolic if it is Kobayashi hyperbolic. The notion of algebraic hyperbolicity was generalized to the case of smooth log-pairs (X, D) by Chen [7]. It naturally generalizes further to the case of arbitrary singular pairs of (reduced) projective varieties:

Definition 1.5 (*Algebraic hyperbolicity*) Let X be a projective variety and Δ an algebraic subset. For a reduced irreducible curve $C \subset X$ with $C \not\subset \Delta$ and $\nu : \tilde{C} \to C$ its normalization, let $i_X(C, \Delta)$ be the number of points in $\nu^{-1}(\Delta)$. The pair (X, Δ) is *algebraically hyperbolic* if there is a Kähler metric ω on X such that, for all curves $C \subset X$ as above,

$$2g(\tilde{C}) - 2 + i(C, \Delta) \geq \deg_\omega C := \int_C \omega. \tag{1.2}$$

It is easily seen, just as the fore-mentioned observation by Demailly in the case $\Delta = \emptyset$, that if $X \setminus \Delta$ is hyperbolically embedded into X, the pair (X, Δ) is algebraically hyperbolic (c.f. [37]).

Note that $2g(\tilde{C}) - 2 + i(C, \Delta)$ depends only on the complement $X \setminus \Delta$. Hence the above notion of hyperbolicity also makes sense for quasi-projective varieties: we say that a quasi-projective variety U is algebraically hyperbolic if it is birational to $Y \setminus D$ for a smooth log pair (Y, D) with a Kähler metric ω on Y satisfying the inequality (1.2) for all curves $C \subset Y$ such that $C \setminus D$ is finite over U.

The last main theorem in this paper is the algebraic hyperbolicity of the moduli spaces considered.

Theorem D (Algebraic hyperbolicity) *Let* $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ *be a polarized family as that given in Theorem B. Then the base U is algebraically hyperbolic.*

1.3 Outline

In Sect. 2, we explain in some detail the construction of Viehweg and the fourth named author in [48, 49] with some necessary additions in our more general setting. In Sects. 3 and 4, we give two distinct Nevanlinna theoretic proofs of Theorems A and B, the first being self-contained and based on the method of metric and curvature while the second is based on the lemma on logarithmic derivatives. Sect. 5 proves our theorem on algebraic hyperbolicity.

1.4 Notation

In general we follow the notations of [48, 49] on the construction of Higgs bundles. Let u,v be real non-negative valued functions on a set S. We write $u \gtrsim v$ if there exists a constant $c > 0$ such that $u(s) \geq c \cdot v(s)$ for all $s \in S$.

2 Recollections on the Viehweg–Zuo construction

In [48, 49], Viehweg and the fourth named author constructed two graded logarithmic Higgs bundles for any smooth family of projective manifolds $f_0 : V \to U$ in order to prove the hyperbolicity of the base U. We recall the construction with some simplifications and extensions.

We first note that f_0 is of maximal variation if $\mathrm{Var}(f_0) = \dim Y$, cf. [25, Section 1], where $\mathrm{Var}(f_0)$ is the dimension of the image of the moduli map. Therefore, if the base U of the original family is not smooth, which is the case considered in Sect. 1 in general, we can always pull back the family by a desingularization of the base to one of smooth base with maximal variation. Since all the arguments from Viehweg–Zuo allow for such birational modifications, we assume from now on unless stated otherwise that the base U of our family is non-singular, relaxing the quasi-finite hypothesis to that of maximal variation. We also assume, replacing U when necessary by the Zariski closure of the holomorphic curve C, that the holomorphic curve C under consideration is Zariski dense in U. Our proof of Theorem B then proceeds without loss of generality (see Remark 3.1).

2.1 Cyclic covering and the comparison map

Let $f : V \to U$ be a smooth algebraic family of polarized projective manifolds with semi-ample canonical bundles and with U nonsingular. Then there is a partial good compactification $\bar{f} : X \to Y$ of the this family, meaning by definition that:

1) X and Y are quasi-projective manifolds, and $U \subset Y$.
2) $S := Y \setminus U$ and $\Delta := (\bar{f}^*S)_{red}$ are normal crossing divisors.
3) $\bar{f}$ is a log smooth projective morphism between the log pairs (X, Δ) and (Y, S), and $\bar{f}^{-1}(U) \to U$ coincides with the original family $V \to U$.
4) Y has a non-singular projective compactification $\bar{Y}$ such that $\bar{Y} \setminus U$ is a normal crossing divisor and $\mathrm{codim}(\bar{Y} \setminus Y) \geq 2$.

We fix such a compactification and adopt the notations as given in this definition.

The strategy of Viehweg and Zuo [48, 49] is to exploit the Kawamata-Viehweg positivity result of the direct image sheaves, which they used to first construct cyclic coverings over the total space X that are natural with respect to the family. This construction is accomplished through:

Theorem 2.1 [[49, Corollary 4.3] or [48, Proposition 3.9]] *Denote by* $\mathcal{L} := \Omega^n_{X/Y}(\log \Delta)$ *the sheaf of top relative log differential forms* (n *is the relative dimension*). *Suppose that* $f = \bar{f}|_U$ *is of maximal variation. Then there exists an ample line bundle* A *on* $\bar{Y}$ *and an integer* $v \gg 1$ *such that* $\mathcal{L}^v \otimes f^* A^{-v}$ *is globally generated over* $V_0 := f^{-1}(U_0)$, *where* U_0 *is some open dense subset of* U.

It follows that the invertible sheaf $\mathcal{L}^v \otimes f^* A^{-v}$ has plenty of nontrivial sections for v large. A cyclic covering of X is thus obtained by taking the v-th roots of such a section s. We choose Z to be a desingularization of this cyclic covering and denote the induced morphisms by $\psi : Z \to X$ and $g : Z \to Y$. The new family g has in general a larger discriminant locus than S given by $S \cup T$ where T is the discriminant of the zero divisor $H = (s)$ over Y. Thus the restriction of g over $Y \setminus (S \cup T)$ is smooth, which we denote by $g_0 : Z_0 \to U_0$.

2.1.1 The Higgs bundle coming from the variation of Hodge structures

Consider the VHS on U_0 induced by the local system $\mathcal{V}_0 = R^n g_{0*} \mathbb{C}_{Z_0}$. By blowing up the closure $\bar{S} + \bar{T}$ of $S + T$ in $\bar{Y}$ if necessary and replacing $\bar{S} + \bar{T}$ by its preimage, we assume that $\bar{S} + \bar{T}$ is a simple normal crossing divisor. Deligne's quasi-canonical extension (cf. [39, §4]) then applies and we get a locally free sheaf $\mathcal{V}$ on $\bar{Y}$ extending $\mathcal{V}_0$ with the Gauss-Manin connection:

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{\bar{Y}}(\log (\bar{S} + \bar{T})).$$

By the nilpotent orbit theorem (in [39] and [8]), the Hodge filtration $\{\mathcal{F}^p_0\}$ of $\mathcal{V}_0$ extends as a filtration by subbundles $\{\mathcal{F}^p\}$ of $\mathcal{V}$ so that the associated Hodge bundle $E := \mathrm{Gr}_{\mathcal{F}^\bullet} \mathcal{V}$ is locally free on $\bar{Y}$. The induced Higgs map

$$\theta := \mathrm{Gr}_{\mathcal{F}^\bullet} \nabla : E \to E \otimes \Omega^1_{\bar{Y}}(\log (\bar{S} + \bar{T}))$$

has logarithmic poles along $\bar{S} + \bar{T}$. One can write the Hodge bundle summands as higher direct image sheaves of log forms if the divisor $\bar{S} + \bar{T}$ is smooth (cf. [50]). More precisely, we have

$$E^{p,q}|_{Y_0} \cong R^q g_* \Omega^p_{Z/Y}(\log \Pi)|_{Y_0}$$

for $Y_0 := Y \setminus \mathrm{Sing}(\bar{S} + \bar{T})$ and ($q := n - p$), where $\Pi := g^{-1}(S \cup T)$ (Π is assumed to be normal crossing after birational modification of Z). Clearly, $\mathrm{codim}(\bar{Y} \setminus Y_0) \geq 2$.

Remark 2.2 In the construction above we have already changed the birational model of U since we have to blow up T inside U. As mentioned, this is allowed in our application.

2.1.2 The Higgs bundle coming from deformation theory

The Hodge bundle (E, θ) so constructed above has in general extra, artificially introduced logarithmic poles along $T \cap U$. To study the hyperbolicity of the original base space U, we shall construct a Higgs bundle directly from the original family of maximal variation, whose Higgs map has logarithmic poles only along the boundary S. Recall that we have assumed that U is smooth.

As in [49] and [48], we shall use the tautological short exact sequences

$$0 \to f^*\Omega_Y^1(\log S) \otimes \Omega_{X/Y}^{p-1}(\log \Delta) \to \mathfrak{gr}(\Omega_X^p(\log \Delta)) \to \Omega_{X/Y}^p(\log \Delta) \to 0 \tag{2.1}$$

where

$$\mathfrak{gr}(\Omega_X^p(\log \Delta)) := \Omega_X^p(\log \Delta)/f^*\Omega_Y^2(\log S) \otimes \Omega_{X/Y}^{p-2}(\log \Delta).$$

Note that the short exact sequence can be established only when $f : (X, \Delta) \to (Y, S)$ is log smooth. Denote by $\mathcal{L} = \Omega_{X/Y}^n(\log \Delta)$ as before. We define on Y the reflexive Higgs sheaf

$$F_0^{p,q} := R^q f_*(\Omega_{X/Y}^p(\log \Delta) \otimes \mathcal{L}^{-1})/\text{torsion}$$

together with the edge morphisms

$$\tau_0^{p,q} : F_0^{p,q} \to F_0^{p-1,q+1} \otimes \Omega_Y^1(\log S)$$

induced by the exact sequence (2.1) tensored with $\mathcal{L}^{-1}$.

Remark 2.3 It is easy to see that $\tau_0^{n,0}|_U$ is nothing but the Kodaira–Spencer map of the family. So the Higgs maps $\tau_0^{p,q}$ can be regarded as the *generalized Kodaira–Spencer maps*.

We denote by $F^{p,q}$ the reflexive hull of $F_0^{p,q}$ on $\bar{Y}$. The Higgs maps $\tau_0^{p,q}$ extends automatically since $\text{codim}(\bar{Y} \setminus Y) \geq 2$. So we get the extended reflexive Higgs sheaf (F, τ), defined on $\bar{Y}$.

2.1.3 The comparison maps

In [48, 49] Viehweg and Zuo constructed the following comparison maps $\rho^{p,q}$, which connect (F, τ) and (E, θ).

Lemma 2.4 *Using the same notations as introduced above, let*

$$\iota : \Omega^1_{\bar{Y}}(\log \bar{S}) \to \Omega^1_{\bar{Y}}(\log (\bar{S} + \bar{T}))$$

be the natural inclusion. Then there exists morphisms $\rho^{p,q} : F^{p,q} \to A^{-1} \otimes E^{p,q}$ *such that the following diagram commutes.*

$$
\begin{array}{ccc}
F^{p,q} & \xrightarrow{\quad \tau^{p,q} \quad} & F^{p-1,q+1} \otimes \Omega^1_{\bar{Y}}(\log \bar{S}) \\
\downarrow{\scriptstyle \rho^{p,q}} & & \downarrow{\scriptstyle \rho^{p-1,q+1} \otimes \iota} \\
A^{-1} \otimes E^{p,q} & \xrightarrow{\quad \mathrm{id} \otimes \theta^{p,q} \quad} & A^{-1} \otimes E^{p-1,q+1} \otimes \Omega^1_{\bar{Y}}(\log (\bar{S} + \bar{T}))
\end{array}
\tag{2.2}
$$

Remark 2.5 *A priori*, our comparison map $\rho^{p,q}$ is defined only on Y_0. That is, it is a morphism between $F^{p,q}|_{Y_0} \cong R^q f_*(\Omega^p_{X/Y}(\log \Delta) \otimes \mathcal{L}^{-1})|_{Y_0}$/torsion and $E^{p,q}|_{Y_0} \cong R^q g_* \Omega^p_{Z/Y}(\log \Pi)|_{Y_0}$. But as $F^{p,q}$ is reflexive, $E^{p,q}$ locally free and $\mathrm{codim}(\bar{Y} \backslash Y_0) \geq 2$, each comparison map $\rho^{p,q}$ extends to $\bar{Y}$.

2.1.4 Injectivity of the comparison map

In order to use the comparison map to construct a negatively curved Finsler pseudo-metric, one needs some pointwise injectivity of the comparison map

$$(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0} : \mathcal{O}_{\bar{Y}} = F^{n,0} \to A^{-1} \otimes E^{n-1,1} \otimes \Omega^1_{\bar{Y}}(\log \bar{S})$$

on $U \setminus T$ as maps between vector bundles, as well as that of the induced map

$$\tau^1 : T_{\bar{Y}}(-\log \bar{S}) \to A^{-1} \otimes E^{n-1,1}. \tag{2.3}$$

Denote by $\rho^{p,q}_y$ the restriction of $\rho^{p,q}|_{U \setminus T}$, as a map between vector bundles, at a point $y \in U \backslash T$. In [49], the following fact on the injectivity of $\rho^{p,q}_y$ are obtained:

1) $\rho^{n,0}_y$ is injective for every $y \in U \setminus T$.
2) If the family is canonically polarized, then $\rho^{p,q}_y$ is an injection for each (p, q) with $p + q = n$ and for every $y \in U \setminus T$.

We remark that the injectivity of $\rho^{p,q}$ for all p, q follows from the Kodaira–Akizuki–Nakano vanishing theorem, but that the injectivity of $\rho^{n-1,1}$ (as first observed in [38]) only needs the Bogomolov-Sommese vanishing theorem. As the latter vanishing theorem holds for projective maniofolds of general type, and as the canonical bundle of such a manifold is semiample if and only if it is nef, one obtains:

Theorem 2.6 *Let* $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ *be a maximally varying family of polarized projective manifolds with nef and big canonical bundle. Then* $(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0}|_{U \setminus T}$, *as a map of vector bundles, is injective at every point in* $U \setminus T$ *where the Kodaira–Spencer map is injective.*

In our general setting of families of polarized manifold with semi-ample canonical divisors, we have the following key theorem of Viehweg–Zuo (cf. [48, Corollary 4.5]).

Theorem 2.7 (Viehweg–Zuo) *Let* $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ *be the polarized family as that in Theorem B. Then the map* $(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0}$ *along any algebraic curve* $\gamma : C \to U$ *does not vanish.*

We briefly outline the proof of this theorem, which used a global argument relying on the Griffiths curvature computation for Hodge metrics: Suppose that $(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0}$ vanishes along $\gamma(C)$. Then the image $\mathcal{O}_Y = F^{n,0} \subset E^{n,0} \otimes A^{-1}$ lies in $\mathrm{Ker}(\theta^{n-1,1}) \otimes A^{-1}$. Note that $\mathrm{Ker}(\theta^{n-1,1})$ is semi-negatively curved for the degenerated Hodge metric (cf. [51]), which essentially follows from the Griffiths curvature computation. By taking integration of the curvature form of Hodge metric restricted to $\mathcal{O}_Y$ one shows that the trivial line bundle is strictly negative because of the curvature decreasing property for holomorphic subbundles. This is of course a contradiction.

Very recently, the first named author observed that the argument of Viehweg–Zuo can be made pointwise, combining with a usual maximal principle argument. His argument runs as follows: instead of taking integration of the curvature form of Hodge metric, he evaluated the curvature form on a special point y_0 in U, where the norm function of the constant section of $\mathcal{O}_Y$ with respect to the Hodge metric of $E^{n,0} \otimes A^{-1}$ restricted to $\mathcal{O}_Y$ via $\rho^{n,0} : \mathcal{O}_Y \hookrightarrow E^{n,0} \otimes A^{-1}$ takes the maximal value, the metric being in fact a natural modification by Popa–Taji–Wu in [38] of the Viehweg–Zuo's metric on U to remove its singularities on T. This implies that the curvature form on $\mathcal{O}_Y$ is semi-positive at this point. On the other hand, if the map $(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0}$ at this specific point evaluated in some tangent vector vanishes, then the Griffiths curvature formula and the strict negativity of A^{-1} implies that the the curvature form on $\mathcal{O}_Y$ at y_0 is strictly negative along this tangent vector, which gives us a contradiction. We thus have:

Theorem 2.8 ([13]) *Let* $f : V \to U$ *be a family of polarized projective manifolds with semi-ample canonical sheaf over a nonsingular quasiprojective variety U. Assume that f is of maximal variation. Then the map τ^1 defined in (2.3) is a vector bundle injection on a Zariski open subset U° of U. In particular, the analytic version of Theorem 2.7 holds true; I.e., the map $(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0}$ does not vanish identically along any holomorphic curve $\gamma : C \to U$ with $\gamma(C) \cap U^\circ \neq \varnothing$.*

Conjecture 2.9 *With the assumption as that in Theorem 2.8, the map $(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0}$ is a vector bundle injection over the open subset of U with quasi-finite moduli map.*

Conjecture 2.9 has been verified for such a family with members of Kodaira dimension one in a joint paper of the third and fourth named authors with Xin Lu [31].

2.2 Iteration of generalized Kodaira–Spencer maps

We iterate the Higgs maps to get

$$\tau^{n-q+1,q-1} \circ \cdots \circ \tau^{n,0} : F^{n,0} \to F^{n-q,q} \otimes \bigotimes^{q} \Omega^1_{\bar{Y}}(\log \bar{S}).$$

Since the Higgs field τ satisfies $\tau \wedge \tau = 0$, this iterated composition factors through the map

$$\tau^q : F^{n,0} \to F^{n-q,q} \otimes Sym^q \, \Omega_{\bar{Y}}^1 (\log \bar{S}).$$

As $\mathcal{O}_{\bar{Y}}$ is a subsheaf of $F^{n,0}$, the following composition of maps makes sense

$$Sym^q \, T_{\bar{Y}}(-\log \bar{S}) \xrightarrow{\subset} F^{n,0} \otimes Sym^q \, T_{\bar{Y}}(-\log \bar{S}) \xrightarrow{\tau^q \otimes \mathrm{id}} F^{n-q,q} \otimes Sym^q \, \Omega_{\bar{Y}}^1 (\log \bar{S}) \otimes Sym^q \, T_{\bar{Y}}(-\log \bar{S})$$

$$\downarrow{\scriptstyle \mathrm{id} \otimes <,>}$$

$$F^{n-q,q}.$$

We will still denote this map as τ^q by abuse of notation.

Composing this τ^q with the comparison map $\rho^{n-q,q}$, we get the *iterated Kodaira–Spencer map*

$$Sym^q \, T_{\bar{Y}}(-\log \bar{S}) \xrightarrow{\tau^q} F^{n-q,q} \xrightarrow{\rho^{n-q,q}} A^{-1} \otimes E^{n-q,q}. \tag{2.4}$$

2.2.1 Maximal non-zero iteration

We define the *maximal non-zero iteration of Kodaira–Spencer map* to be the the m-th iterated Kodaira–Spencer map with $\rho^{n-m,m} \circ \tau^m (Sym^m \, T_{\bar{Y}}(-\log \bar{S})) \neq 0$ and with m being the largest number m_f satisfying this property, if it exists. More precisely,

$$Sym^q \, T_{\bar{Y}}(-\log \bar{S}) \xrightarrow{\rho^{n-m,m} \circ \tau^m \neq 0} A^{-1} \otimes E^{n-m,m}$$

$$\searrow{\scriptstyle =0} \qquad\qquad \downarrow{\scriptstyle \mathrm{id} \otimes \theta^{n-m,m}}$$

$$A^{-1} \otimes E^{n-m-1,m+1} \otimes \Omega_{\bar{Y}}^1 (\log (\bar{S} + \bar{T})).$$

Note that $\{0\} \neq \mathrm{Im}(\rho^{n-m,m} \circ \tau^m) \subset A^{-1} \otimes \mathrm{Ker}(\theta^{n-m,m})$. We call this number m_f the *maximal length of iteration* or the *maximal iteration length*.

Lemma 2.10 *Keeping the assumptions as 2.8, we have that $\rho^{n-1,1} \circ \tau^1$ is injective at the generic point of U, evaluated at each nonzero tangent vector.*

Proof This follows from Theorem 2.7 when the canonical divisor of a general fiber of the family is semi-ample and big and from Theorem 2.8 in general. □

Corollary 2.11 *Non-zero such iteration on Y exists with maximal iteration length $m_f \leq n$.*

Proof $\rho^{n-1,1} \circ \tau^1$ is non-zero by Lemma 2.10. The upper bound of m follows from $\theta^{0,n} = 0$. □

2.2.2 Maximal non-zero iteration of Kodaira–Spencer map along an analytic curve

Let $V \to U$ be the smooth family as that in Theorem 2.8. In our application we only need to consider an analytic map γ from a complex analytic curve C to the base manifold U so that $\gamma(C) \cap U^\circ \neq \varnothing$, where U° is the dense Zariski open set of U in Theorem 2.8. All the Higgs bundles can be pulled back to C, as well as the iteration process. We define the composition

$$\tau_\gamma^{p,q} : \gamma^* F^{p,q} \xrightarrow{\gamma^* \tau^{p,q}} \gamma^* F^{p-1,q+1} \otimes \gamma^* \Omega^1_{\bar{Y}}(\log \bar{S}) \xrightarrow{\mathrm{id} \otimes d\gamma} \gamma^* F^{p-1,q+1} \otimes \Omega^1_C$$

as the *Higgs map along* γ. We define θ_γ similarly. Then $(\gamma^* F, \tau_\gamma)$ and $(\gamma^* E, \theta_\gamma)$ are holomorphic Higgs bundles on the curve C. The iterated Kodaira–Spencer maps on C are also defined similarly:

$$T_C^{\otimes q} \xrightarrow{\tau_\gamma^q} \gamma^* F^{n-q,q} \xrightarrow{\gamma^* \rho^{n-q,q}} \gamma^* A^{-1} \otimes \gamma^* E^{n-q,q}.$$

Corollary 2.12 *Non-zero iteration of such maps along γ exists. I.e., it has length $m' \geq 1$.*

Proof Since one can always replace the base U by the Zariski closure of the image of $\gamma : C \to U$ and desingularize it, we may and do assume that $\gamma(C) \subset U$ is Zariski dense. Then from Lemma 2.10 we know that $\rho^{n-1,1} \circ \tau^1$ is injective at all the points of $\gamma(C)$ contained in a Zariski open subset of U, evaluated at all the tangent directions at those points. This implies that at least $\gamma^*(\rho^{n-1,1}) \circ \tau_\gamma^1(T_C)$ is non-zero. □

By its definition, the maximal non-zero iteration of Kodaira–Spencer map along γ of length m' has the properties: $\gamma^*(\rho^{n-m',m'} \circ \tau_\gamma^{m'})(T_C^{\otimes m'}) \neq 0$ and $\mathrm{Im}(\gamma^*(\rho^{n-m',m'} \circ \tau_\gamma^{m'})) \subset \gamma^* A^{-1} \otimes \mathrm{Ker}(\theta_\gamma)$. These properties are crucial for creating a negatively curved Finsler pseudometric along γ in Sect. 2.3.

Remark 2.13 One should be warned that the maximal length of iteration along γ could be *shorter* than the maximal length of iteration of the original family. This is because the iterated Kodaira–Spencer maps $Sym^q T_{\bar{Y}}(-\log \bar{S}) \xrightarrow{\rho^{n-q,q} \circ \tau^q} A^{-1} \otimes E^{n-q,q}$ with $q > 1$ are not injective in general.

2.3 The Finsler (pseudo)metric

Definition 2.14 (*Finsler metric*) Let E be a holomorphic vector bundle on a complex manifold X. A *Finsler pseudometric* on E is a *continuous* function $h : E \to [0, +\infty[$ such that

$$h(av) = |a| h(v)$$

for any $a \in \mathbb{C}$ and $v \in E$. We call h a Finsler metric if it is nondegenerate, i.e, if $h(v) = 0$ only when $v = 0$.
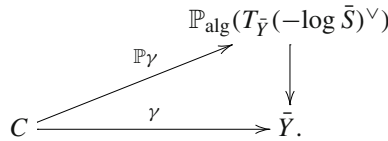
As in [49], we construct a Finsler pseudometric via the maximal iterated Kodaira–Spencer map

$$Sym^m \, T_{\bar{Y}}(-\log \bar{S}) \xrightarrow{\rho^{n-m,m} \circ \tau^m} A^{-1} \otimes E^{n-m,m}. \tag{2.5}$$

It is given as follows. Consider $g_{A^{-1}} \otimes g_{hod}$ on $A^{-1} \otimes E^{n-m,m}$, where g_A is the Fubini-Study metric of the ample line bundle A and g_{hod} is the Hodge metric on the Hodge bundle E. Pulling it back via $\rho^{n-m,m} \circ \tau^m$ and taking the m-th root, we get our desired Finsler pseudometric on $T_{\bar{Y}}(-\log \bar{S})$.

We use the notation $\mathbb{P}_{\mathrm{alg}}(T_{\bar{Y}}(-\log \bar{S})^\vee)$ to denote the *projective log tangent bundle* of $(\bar{Y}, \bar{S})$, and use $\mathcal{O}_{\mathbb{P}_{\mathrm{alg}}(T_{\bar{Y}}(-\log \bar{S})^\vee)}(-1)$ to denote the tautological line bundle. Then any Finsler (pseudo)metric h on $T_{\bar{Y}}(-\log \bar{S})$ is equivalent to a (semi)norm on the tautological line bundle $\mathcal{O}_{\mathbb{P}_{\mathrm{alg}}(T_{\bar{Y}}(-\log \bar{S})^\vee)}(-1)$, which we will denote as $|\cdot|_h$.

Now we consider a holomorphic map $\gamma : C \to U = \bar{Y} \backslash \bar{S}$ from a Riemann surface (for instance, $C = \mathbb{D}^*$ or the analytification of some smooth quasi-projective curve) to the base U. Then the tangent map of γ induces a holomorphic map from C to $\mathbb{P}_{\mathrm{alg}}(T_{\bar{Y}}(-\log \bar{S})^\vee)$, which we will denote as $\mathbb{P}\gamma$:

$$
\begin{array}{ccc}
 & & \mathbb{P}_{\mathrm{alg}}(T_{\bar{Y}}(-\log \bar{S})^\vee) \\
 & \nearrow^{\mathbb{P}\gamma} & \downarrow \\
C & \xrightarrow{\quad \gamma \quad} & \bar{Y}.
\end{array}
$$

Moreover, the tangent map $d\gamma : T_C \to \gamma^* T_{\bar{Y}}(-\log \bar{S})$ naturally lifts to

$$T_C \to (\mathbb{P}\gamma)^* \mathcal{O}_{\mathbb{P}_{\mathrm{alg}}(T_{\bar{Y}}(-\log \bar{S})^\vee)}(-1),$$

which we denote as $\mathbb{P}\gamma'$.

Therefore, one can use the map $\mathbb{P}\gamma'$ to pull back the (semi)norm $|\cdot|_h$ on $\mathcal{O}_{\mathbb{P}_{\mathrm{alg}}(T_{\bar{Y}}(-\log \bar{S})^\vee)}(-1)$ and obtain a (semi)norm on T_C. We will use the notation $(\mathbb{P}\gamma')^{-1}(|\cdot|_h \circ \gamma^*)$ to denote it.

2.3.1 Modification along the boundary

In fact, Viehweg and the fourth named author used some modified version of the Finsler pseudometric described above in order to have the right kind of curvature property. This method of modifying metrics on the log-tangent bundle appeared first in the second named author's thesis on extending meromorphic maps [33].

First we construct an auxiliary function associated to the boundary divisor $\bar{S}$. Denote by $\bar{S}_1, \ldots, \bar{S}_p$ the components of $\bar{S}$. Let L_i be the line bundle with section s_i such that $\bar{S}_i = \mathrm{div}(s_i)$. Equip each L_i with a smooth hermitian metric g_i. Let $l_i := -\log |s_i|^2_{g_i}$ and $l_S := l_1 l_2 \cdots l_p$. Recall that the Hodge metric g_{hod} has extra degeneration along $\bar{T}$ since the Hodge bundle (E, θ) has logarithmic poles along $\bar{T}$. To control the asymptotic behaviour of g_{hod} near $\bar{T}$, we construct another auxiliary function associated to

the divisor $\bar{T}$, in a similar manner as for l_S: Denote by $\bar{T}_1, \ldots, \bar{T}_q$ the non-singular components of $\bar{T}$. Let L_i' be the line bundle with section t_i such that $\bar{T}_i = \mathrm{div}(t_i)$. Equip each L_i' with a smooth hermitian metric g_i'. Let $l_i' := -\log |t_i|^2_{g_i'}$ and $l_T := l_1' l_2' \cdots l_q'$. By compactness of $\bar{Y}$, one can rescale g_i's and g_i''s to ensure that l_i's and l_i''s are positive functions on $\bar{Y}$.

Now for each positive integer α, we define a new singular hermitian metric $g_\alpha := g_A \cdot l_S^\alpha \cdot l_T^\alpha$ on A.

Before entering our setting of the Viehweg–Zuo construction, we remark that by suitably modifying a (pseudo)metric on the tautological line bundle of the projective log tangent bundle satisfying the hypothesis of Theorem A, one can already obtain the strongly negative curvature property that the holomorphic sectional curvature is bounded from above by a negative constant.

Proposition 2.15 *Let* (X, D), $\gamma : \mathbb{D}^* \to X \setminus D$ *be the same as in Theorem* A. *Let* h *be the Finsler pseudometric on* $T_X(-\log D)$ *(or equivalently, a semi-norm on* $\mathcal{O}_{\mathbb{P}_{alg}(T_X(-\log D)^\vee)}(-1)$*) satisfying the curvature inequality (1.1). Then there is a Finsler pseudometric* h_α *defined as* $h \cdot l_D^{-\alpha}$, *where* l_D *is the above auxiliary function of the boundary divisor* D *and* α *is some positive integer, such that its holomorphic sectional curvature is bounded from above by a negative constant, i.e.:*

$$-dd^c \log |\gamma'(z)|^2_{h_\alpha} \lesssim -\mu_\alpha$$

where μ_α *is the semi-positive* $(1, 1)$ *form associated with the possibly degenerate hermitian metric* $(\mathbb{P}\gamma')^{-1}(|\ |_{h_\alpha} \circ \gamma_*)^2 = ((\mathbb{P}\gamma')^{-1}h_\alpha)|\gamma_*|^2$ *on* $\mathbb{D}^*$ *and the inequality above holds in the sense of currents.*

Proof By direct computation we have

$$\begin{aligned}
dd^c \log |\gamma'(z)|^2_{h_\alpha} &= dd^c \log |\gamma'(z)|^2_h - \alpha \gamma^* \Sigma \tfrac{dd^c l_i}{l_i} + \tfrac{\sqrt{-1}}{2\pi} \alpha \gamma^* \Sigma \tfrac{\partial l_i \wedge \overline{\partial l_i}}{l_i^2} \\
&\geq dd^c \log |\gamma'(z)|^2_h - \alpha \gamma^* \Sigma \tfrac{dd^c l_i}{l_i} \geq \gamma^* \left(\omega_{FS} - \alpha \Sigma \tfrac{dd^c l_i}{l_i} \right).
\end{aligned} \tag{2.6}$$

Here we have used the inequality $dd^c \log |\gamma'(z)|^2_h \geq \gamma^* \omega_{FS}$. By the same argument in [33, §4, Proposition 1] (see also the proof of Lemma 7.1 in [49]), we can find a positive definite Hermitian form ω_α on $T_X(-\log D)$ such that

$$\omega_{FS} - \alpha \Sigma \frac{dd^c l_i}{l_i} \geq l_D^{-2} \cdot \omega_\alpha.$$

Note that $l_D^{-2} \cdot \omega_\alpha$ can also be regarded as a semi-norm on the dual of the tautological line bundle. So if we choose $\alpha > 2$, then $l_D^{-2} \cdot \omega_\alpha \gtrsim h_\alpha$ by the compactness of X. Therefore, we get the desired bound on the negative holomorphic sectional curvature $-dd^c \log |\gamma'(z)|^2_{h_\alpha} \lesssim -(\mathbb{P}\gamma')^{-1} h_\alpha$. $\qquad\square$

Now we come back to our setting of the Viehweg–Zuo construction. We first note that

Lemma 2.16 $\Theta(A, g_\alpha)$ *dominates the Kähler form* $\omega_{FS} := \Theta(A, g_A)$ *on* $\bar{Y}$ *as currents.*

Proof This follows from the computation

$$
\begin{aligned}
\Theta(A, g_\alpha) &= \Theta(A, g_A \cdot l_S^\alpha \cdot l_T^\alpha) \\
&= \Theta(A, g_A) - \alpha \Sigma \frac{\mathrm{dd}^c l_i}{l_i} - \alpha \Sigma \frac{\mathrm{dd}^c l_i'}{l_i'} + \frac{\sqrt{-1}}{2\pi} \alpha \Sigma \frac{\partial l_i \wedge \overline{\partial l_i}}{l_i^2} \\
&\quad + \frac{\sqrt{-1}}{2\pi} \alpha \Sigma \frac{\partial l_i' \wedge \overline{\partial l_i'}}{l_i'^2} \\
&\geq \Theta(A, g_A) - \alpha \Sigma \frac{\mathrm{dd}^c l_i}{l_i} - \alpha \Sigma \frac{\mathrm{dd}^c l_i'}{l_i'} \geq c \cdot \Theta(A, g_A),
\end{aligned}
$$

which holds for some positive constant c. Note that one can rescale g_i (respectively g_i') to make l_i (respectively l_i') sufficiently large and leave $\mathrm{dd}^c l_i$ (respectively $\mathrm{dd}^c l_i'$) unchanged. □

Remark 2.17 (1) The same computations as above show, by the hypothesis on h, that $dd^c \log |f'(z)|_{h_\alpha}^2$ dominates $dd^c \log |f'(z)|_h^2$ and $f^* \omega_{FS}$ as currents. (2) As we mentioned above, the second named author used this type of modification of metrics to prove his extension theorem (cf. [33, §4]). Later, Viehweg and the fourth named author applied it in [49, §7] to the Viehweg–Zuo metric of the family over U and obtained the curvature estimate in Lemma 2.16. Since the family concerned in [49] is canonically polarized, one can move the branch divisor of the cyclic covering such that the discriminant locus T intersects with the analytic curve $\gamma(C)$ only at the smooth part of T, and the intersection is transversal. Then, the monodromy of the pull-back local system around $\gamma^* T$ is finite, and the pull-back Hodge metric $\gamma^* g_{hod}$ is bounded (see Sect. 5 of [49] for details). Popa–Taji–Wu also observed that a similar modification along $\bar{T}$ applies without violating the curvature estimate and chose α sufficiently large such that the singular hermitian metric $g_\alpha^{-1} \otimes g_{hod}$ is bounded (in fact continuous) near $\bar{T}$ without the canonically polarized hypothesis (cf. [38, §3.1]). This means that the Finsler pseudometric on the log tangent bundle induced by $g_\alpha^{-1} \otimes g_{hod}$ is bounded, which is important for our curvature estimate. Note that here we use the property that the Hodge metrics have at most logarithmic growth along $\bar{S} + \bar{T}$, which is guaranteed by the study of the higher dimensional asymptotic behavior of the Hodge metric in [8].

2.3.2 The curvature inequalities

Now we consider an analytic map γ from a Riemann surface C (for instance, $C = \mathbb{D}^*$) to the base manifold U. Let m be the maximal length of iteration along γ.

It is very natural to use the hermitian metric $g_\alpha^{-1} \otimes g_{hod}$ and the iterated Kodaira–Spencer map to construct a Finsler pseudometric F_α on $T_{\bar{Y}}(-\log \bar{S})$:

$$
|v|_{F_\alpha}^2 := |\rho^{n-m,m} \circ \tau^m(v^{\otimes m})|_{g_\alpha^{-1} \otimes g_{hod}}^{2/m}, \quad \text{for } v \in T_{\bar{Y}}(-\log \bar{S}). \tag{2.7}
$$

We first state a curvature inequality associated to F_α *which validates the hypothesis in our criterion for big Picard theorem. This inequality is given, though not explicitly, in* [49, §7] *and holds in general once we have Theorem 2.8 in hand. We repeat the proof here only for consistency.*

Theorem 2.18 (The curvature inequality) *Let $V \to U$ be the same family as that in Theorem 2.8. Fix the analytic map $\gamma : C \to U$ so that $\gamma(C) \cap U^\circ \neq \varnothing$ where U° is the dense Zariski open set of U in Theorem 2.8 so that $\tau^1|_{U^\circ} : T_{U^\circ} \to A^{-1} \otimes E^{n-1,1}|_{U^\circ}$ is injective. Then for any positive integer α, the Finsler (pseudo)metric F_α constructed above satisfies the following curvature inequality*

$$\mathrm{dd}^c \log |\gamma'(z)|^2_{F_\alpha} \gtrsim \gamma^* \omega_{FS}.$$

Proof By the Poincaré-Lelong formula, we know that $\mathrm{dd}^c \log |\gamma'(z)|^2_{F_\alpha} = -\Theta(T_C, (\mathrm{d}\gamma)^* F_\alpha) + R \geq -\Theta(T_C, (\mathrm{d}\gamma)^* F_\alpha)$, where R is the ramification divisor of γ. Denote by N the saturation of the image of $\mathrm{d}\gamma : T_C \to \gamma^* T_Y(-\log S)$. Then we have the following curvature current estimate

$$
\begin{aligned}
\Theta(T_C, (\mathrm{d}\gamma)^* F_\alpha) &\leq \Theta(N, \gamma^* F_\alpha) = \frac{1}{m} \Theta(N^{\otimes m}, \gamma^* F_\alpha^{\otimes m}) \\
&\leq \frac{1}{m} \gamma^* \Theta(Sym^m T_Y(-\log S), (\rho^{n-m,m} \circ \tau^m)^*(g_\alpha^{-1} \otimes g_{hod}))|_{N^{\otimes m}} \\
&\leq \frac{1}{m} \gamma^* \Theta(A^{-1} \otimes E, g_\alpha^{-1} \otimes g_{hod})|_{\gamma^*(\rho^{n-m,m} \circ \tau^m)(N^{\otimes m})} \\
&= -\frac{1}{m} \gamma^* \Theta(A, g_\alpha) + \frac{1}{m} \gamma^* \Theta(E, g_{hod})|_{\gamma^* A \otimes \gamma^*(\rho^{n-m,m} \circ \tau^m)(N^{\otimes m})}.
\end{aligned}
\tag{2.8}
$$

Recall that m is the maximal length of the iteration along γ so that $\gamma^* A \otimes \gamma^*(\rho^{n-m,m} \circ \tau^m)(N^{\otimes m})$ lies in the kernel of θ_γ. Therefore, as the last term in the estimate (2.8) is semi-negative by the Griffiths curvature computation (see [39, Lemma (7.18)] or [51]), we have as currents that

$$\Theta(T_C, (\mathrm{d}\gamma)^* F_\alpha) \leq -\frac{1}{m} \gamma^* \Theta(A, g_\alpha).$$

Hence, we have

$$\mathrm{dd}^c \log |\gamma'(z)|^2_{F_\alpha} \geq \frac{1}{m} \gamma^* \Theta(A, g_\alpha) \gtrsim \gamma^* \omega_{FS},$$

the last inequality being given by Lemma 2.16. □

Using the curvature inequality in Theorem 2.18 and the estimate in Proposition 2.15, Viehweg and the fourth named author showed that F_α is strongly negatively curved along the analytic curve.

Theorem 2.19 (Viehweg–Zuo) *Keeping the assumptions on the family $V \to U$ the same as in Theorem 2.8. Fix the analytic map $\gamma : C \to U$. Then there exists a positive integer α (depending on the maximal length along γ) and $c_\alpha > 0$ (depending on α) such that the curvature of F_α satisfies*

$$K_{F_\alpha}(v) \leq -c_\alpha$$

for any nonzero tangent vector $v := \gamma'(z)$ of γ.

Remark 2.20 (1) Although $C = \mathbb{C}$ in [49], all the arguments above work for a general Riemann surface C, except the final step in [49, Lemma 7.9] where the Ahlfors-Schwarz lemma is used.

(2) Since $(d\gamma)^* F_\alpha$ is locally bounded on T_C by construction, the inequality above entails an inequality in the sense of currents just as that in Theorem 2.15.

Before entering the next section, we list two crucial points to the the arguments we present there:

- *logarithmic growth of the Hodge metric near boundary:* In fact this is the crucial point of Viehweg–Zuo's curvature estimates; and those estimates are crucial to our argument.
- *local boundedness of the Finsler pseudometric*: Used in defining the order function $T(r)$.

3 Big Picard theorem via negative curvature

In this section we shall prove Theorem A and Theorem B by using the negative curvature method inspired by the argument in §9 of Griffiths-King [19]. Now, we have two negatively curved Finsler pseudometrics: h_α of Proposition 2.15 and F_α of Theorem 2.19. Since h_α shares the same curvature properties as F_α thanks to Proposition 2.15, we only present the proof of Theorem B using F_α. The proof of Theorem A using h_α is verbatim.

We identify $\mathbb{D}^*$ with the inverted punctured unit disk $\mathbb{D}^\circ := \{z \in \mathbb{C} : |z| > 1\}$ in order to match the usual notations in Nevanlinna theory for entire curves. We set $\mathbb{D}_{r_0,r} := \{r_0 \leq |z| < r\} \subset \mathbb{C}^*$ and $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. Denote by $\gamma : \mathbb{D}^\circ \to U$ the analytic map in question. Then we want to show that γ extends over the point at infinity. We fix an $r_0 > 1$ from now on.

By the constructions in Subsect. 2.3, there is a Finsler pseudometric F_α on $T_Y(-\log S)$ (resp. h_α on $T_X(-\log D)$), or equivalently a semi-norm on the tautological line bundle $\mathcal{O}_{\mathbb{P}_{alg}(T_Y(-\log S)^\vee)}(-1)$, with the following inequalities of curvature currents

$$\gamma^* \omega_{FS} \lesssim dd^c \log (\mathbb{P}\gamma')^{-1} |~|_{F_\alpha}^2$$
$$\omega_\gamma \lesssim dd^c \log (\mathbb{P}\gamma')^{-1} |~|_{F_\alpha}^2,$$

by Theorems 2.18 and 2.19 (or by the hypothesis in Theorem A and Proposition 2.15) respectively. Here ω_γ is the semi-positve (1,1)-form associated to the semi-norm $(\mathbb{P}\gamma')^{-1}(\|\ \|_{F_\alpha} \circ \gamma_*)$ on $T_{\mathbb{D}^\circ}$.

Remark 3.1 For the construction of F_α, remember that one needs to change the birational model of U in the construction of those two Higgs bundles (F, τ) and (E, θ). In our application, we can always assume that the image of γ is Zariski dense by replacing $\bar{Y}$ by the Zariski closure of $\gamma(\mathbb{D}^\circ)$. Then the analytic map lifts to $\tilde{\gamma} : \mathbb{D}^\circ \to \tilde{U}$, where $\tilde{U}$ is the new birational model for the new zariski closure base space. Clearly, it suffices to prove the extension property for $\tilde{\gamma}$.

By the above argument, the following Nevanlinna characteristic (or order) functions

$$T_{\gamma^*\omega_{FS}}(r) := \int_{r_0}^r \frac{d\rho}{\rho} \int_{\mathbb{D}_{r_0,\rho}} \gamma^*\omega_{FS} \tag{3.1}$$

$$T_{\omega_\gamma}(r) := \int_{r_0}^r \frac{d\rho}{\rho} \int_{\mathbb{D}_{r_0,\rho}} \omega_\gamma$$

are both dominated (i.e. $\lesssim$) by

$$\int_{r_0}^r \frac{d\rho}{\rho} \int_{\mathbb{D}_{r_0,\rho}} dd^c \log (\mathbb{P}\gamma')^{-1}|\ |_{F_\alpha}^2 \leq \int_{r_0}^r \frac{d\rho}{\rho} \int_{\mathbb{D}_{r_0,\rho}} dd^c \log |\gamma'(z)|_{F_\alpha}^2. \tag{3.2}$$

It is elementary and classical that the asymptotic behavior of $T_{\gamma^*\omega_{FS}}(r)$ as $r \to \infty$ characterizes whether γ can be extended over ∞ (see e.g. [11, 2.11. cas local] or [36, Remark 4.7.4.(ii)]).

Lemma 3.2 $T_{\gamma^*\omega_{FS}}(r) = O(\log r)$ *if and only if* γ *can be extended holomorphically over* ∞. □

We will need the following Green–Jensen formula for the punctured disk.

Proposition 3.3 (Green–Jensen formula on punctured disk) *Let* ϕ *be function on* $\mathbb{D}^\circ$ *such that* ϕ *is differentiable outside a discrete set of points disjoint from* $\mathbb{D}_{r_0}$ *and* $dd^c\phi$ *exists as a current. Then for* $0 < r_0 < r$,

$$\int_{r_0}^r \frac{d\rho}{\rho} \int_{\mathbb{D}_{r_0,\rho}} dd^c\phi = \int_{\partial\mathbb{D}_r} \phi \cdot d^c \log |z| - \int_{\partial\mathbb{D}_{r_0}} \phi \cdot d^c \log |z| - \left(\log \frac{r}{r_0}\right) \cdot \int_{\partial\mathbb{D}_{r_0}} d^c\phi. \tag{3.3}$$

□

By using (3.3), we have for a fixed $r_0 > 1$ that,

$$\int_{r_0}^r \frac{d\rho}{\rho} \int_{\mathbb{D}_{r_0,\rho}} dd^c \log |\gamma'(z)|_{F_\alpha}^2 = \int_{\partial\mathbb{D}_r} \log |\gamma'(z)|_{F_\alpha}^2 \cdot d^c \log |z| + O(\log r). \tag{3.4}$$

We define the first term of the right hand side of (3.4) to be the *modified proximity function* $m_{\omega_\gamma}(r)$.

Lemma 3.4 *Denote by $\frac{d}{ds} := r \cdot \frac{d}{dr}$ the logarithmic derivative. Then*

$$m_{\omega_\gamma}(r) \le \log \frac{d^2 T_{\omega_\gamma}(r)}{ds^2} + O(\log r).$$

Proof Note that we can write $\omega_\gamma = |\gamma'(z)|^2_{F_\alpha} \cdot \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}$. Denote by $\xi := |\gamma'(z)|^2_{F_\alpha}$ for simplicity. By direct computation one finds that

$$\frac{1}{r} \frac{d}{dr}\left(r \frac{d}{dr} T_{\omega_\gamma}(r)\right) = \int_{\partial \mathbb{D}_r} \xi \cdot d^c \log |z|.$$

Using the concavity of the logarithmic function, we obtain

$$m_{\omega_\gamma}(r) = \int_{\partial \mathbb{D}_r} \log \xi \cdot d^c \log |z| \le \log \int_{\partial \mathbb{D}_r} \xi \cdot d^c \log |z|$$

$$= \log \left\{\frac{1}{r} \frac{d}{dr}\left(r \frac{d}{dr} T_{\omega_\gamma}(r)\right)\right\} = \log \left\{r^{-2} \cdot \frac{d^2}{ds^2} T_{\omega_\gamma}(r)\right\}$$

$$= -2\log r + \log \frac{d^2 T_{\omega_\gamma}(r)}{ds^2}.$$

$\square$

Applying the Calculus lemma (cf. Lemma 4.4) twice, we obtain for some $\epsilon, \delta > 0$ that

$$\frac{d^2 T_{\omega_\gamma}(r)}{ds^2} \le r^{2+\epsilon} \cdot T_{\omega_\gamma}(r)^{2+\delta} \ ||,$$

where $||$ means that the inequality holds for r outside a set of finite Lebesque measure. Thus

$$m_{\omega_\gamma}(r) \le (2+\delta) \cdot \log T_{\omega_\gamma}(r) + O(\log r) \ ||.$$

Combining (3.2) and (3.4), we get the inequalities

$$T_{\omega_\gamma}(r) \lesssim m_{\omega_\gamma}(r) + O(\log r) \lesssim \log T_{\omega_\gamma}(r) + O(\log r) \ ||;$$

$$T_{\gamma^* \omega_{FS}}(r) \lesssim m_{\omega_\gamma}(r) + O(\log r) \lesssim \log T_{\omega_\gamma}(r) + O(\log r) \ ||.$$

The first implies that $T_{\omega_\gamma}(r) = O(\log r)$, which then combines with the second to yield

$$T_{\gamma^* \omega_{FS}}(r) = O(\log r).$$

By Lemma 3.2, γ extends holomorphically over infinity and completes our proof of Theorem A.

Remark 3.5 One can see that the assumption on the family in Theorem B can be relaxed to a family of polarized manifolds with maximal variation of moduli if we require the image of the holomorphic map $\gamma : \mathbb{D}^* \to U$ to be not contained in a proper subvariety of U. We choose such a family whose base is nonsingular and a proper subvariety Z so that the map τ^1 defined in Subsect. 2.1.4 is injective outside Z. Since our $\gamma(\mathbb{D}^*)$, being zariski dense, is not contained in Z, the arguments in Sects. 2 and 3 go through, giving the extension of γ; see out setup just before Sect. 2.1. This completes the proof Theorem B.

Remark 3.6 It is a classical fact, up to adding the term $O(\log r)$, that the characteristic function

$$T_{(\mathbb{P}\gamma')^*\mathcal{O}(1)} := \int_2^r \frac{d\rho}{\rho} \int_{\mathbb{D}_\rho^\circ} dd^c \log (\mathbb{P}\gamma')^{-1} |\ |_{h_\alpha}^2$$

is independent of a *non-degenerate* Finsler metric h_α chosen on $\mathcal{O}(-1) = \mathcal{O}_{\mathbb{P}_{\text{alg}}(T_Y(-\log S)^\vee)}(-1)$. So the estimate above also gives us the so called tautological inequality in our case in the form:

$$T_{(\mathbb{P}\gamma')^*\mathcal{O}(1)}(r) = O(\log r). \tag{3.5}$$

In fact, the tautological inequality was first established by McQuillan for entire curves and requires no hypothesis on curvature, albeit with extra terms on the right side of (3.5). It generalizes easily to the case of punctured disks and we state a version where the holomorphic curve lies outside the boundary divisor (cf. [47, §29], which has a nice proof more amenable to arithmetic geometry).

Theorem 3.7 (Tautological Inequality for the punctured disk) *Let X be a projective manifold with a Fubini-Study metric ω_{FS} and let D be a simple normal crossing divisor on X. Let $\gamma : \mathbb{D}^\circ \to X \setminus D$ be a holomorphic map. Then $T_{(\mathbb{P}\gamma')^*O(1)}(r) \leq O(\log^+ T_{\gamma^*\omega_{FS}}(r)) + O(\log r) \|.$* $\qquad\square$

Our proof of Theorem A in the next section also proves this inequality, but more geometrically. Note that Theorem A can be derived easily from this ineqality and (3.5) by choosing a hermitian metric h dominating the possibly degenerate h_α. (The proximity function with respect to the former then dominates that of the latter and so, up to an $O(\log r)$ term, so does the characteristic functions.)

4 Big Picard theorem via the lemma on the logarthmic derivative

In a similar vein as that in establishing the fundamental vanishing theorem for (symmetric or jet) differentials, pioneered by Green-Griffiths [17] and completed by Siu-Yeung [43] and Demailly [11] via the logarithmic derivative lemma, we give another proof of Theorem A in this section. Theorem B is then an immediate corollary given our setup, as shown in Remark 3.5.

4.1 Preliminary in Nevanlinna theory

Let (X, ω) be a compact Kähler manifold. Consider a holomorphic map $\gamma : \mathbb{D}^* \to X$. We identify $\mathbb{D}^*$ with $\mathbb{D}^\circ := \{z \in \mathbb{C}; \ |z| > 1\}$ via $z \mapsto \frac{1}{z}$ as before. We can extend γ to a $\gamma : \mathbb{D}^* \cup \mathbb{D}^\circ \to X$ by setting $\gamma(z) = \gamma(\frac{1}{z})$ for $z \in \mathbb{D}^\circ$.

Fix any $r_0 > 1$. Recall that the *characteristic function* in (3.1) is defined by

$$T_{\gamma^*\omega}(r) := \int_{r_0}^r \frac{d\rho}{\rho} \int_{\mathbb{D}_{r_0,\rho}} \gamma^*\omega.$$

Let us first state a couple of useful inequalities.

Lemma 4.1 *Write* $\log^+ x := \max(\log x, 0)$. *Then*

$$\log^+\left(\sum_{i=1}^N x_i\right) \leq \sum_{i=1}^N \log^+ x_i + \log N, \quad \log^+\prod_{i=1}^N x_i \leq \sum_{i=1}^N \log^+ x_i \quad \text{for } x_i \geq 0.$$

(4.1)

□

The following lemma is well-known (see e.g. [11, Lemme 1.6]).

Lemma 4.2 *Let X be a projective manifold equipped with a hermitian metric ω and let $u : X \to \mathbb{P}^1$ be a rational function. Then for any holomorphic map $\gamma : \mathbb{D}^\circ \to X$, one has*

$$T_{(u \circ \gamma)^*\omega_{FS}}(r) \leq CT_{\gamma^*\omega}(r) + O(1) \tag{4.2}$$

where ω_{FS} is the Fubini-Study metric for $\mathbb{P}^1$. □

The following logarithmic derivative lemma for the punctured disk, see e.g. [35, Lemma 2.12], is crucial in our proof.

Lemma 4.3 *Let $u : \mathbb{D}^\circ \to \mathbb{P}^1$ be any meromorphic function. Then we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{u^{(1)}(re^{i\theta})}{u(re^{i\theta})}\right| d\theta \leq C(\log^+ T_{u^*\omega_{FS}}(r) + \log r) \ \|, \tag{4.3}$$

for some constant $C > 0$ which does not depend on r. Here the symbol $\|$ means that the inequality holds outside a Borel subset of $(r_0, +\infty)$ of finite Lebesgue measure. □

We also need an elementary lemma (due to E. Borel), called the calculus lemma.

Lemma 4.4 *Let $\phi(r) \geq 0(r \geq r_0 \geq 0)$ be a monotone increasing function. For every $\delta > 0$,*

$$\frac{d}{dr}\phi(r) \leq \phi(r)^{1+\delta} \ \|. \tag{4.4}$$

□

4.2 Criterion for big Picard theorem

Now we are ready to give a new proof of Theorem A.

Proof Via the isomorphism $\mathbb{D}^\circ \xrightarrow{\sim} \mathbb{D}^*$ by setting $z \mapsto \frac{1}{z}$ as before, we assume that $\gamma : \mathbb{D}^* \to X - D$ is a holomorphic map from $\mathbb{D}^\circ$ to $X - D$. We take a finite affine covering $\{U_\alpha\}_{\alpha \in I}$ of X and rational functions $(x_{\alpha 1}, \ldots, x_{\alpha n})$ on X which are holomorphic on U_α so that

$$dx_{\alpha 1} \wedge \cdots \wedge dx_{\alpha n} \neq 0 \text{ on } U_\alpha,$$
$$D \cap U_\alpha = (x_{\alpha, s(\alpha)+1} \cdots x_{\alpha n} = 0).$$

Hence

$$(e_{\alpha 1}, \ldots, e_{\alpha n}) := \left(\frac{\partial}{\partial x_{\alpha 1}}, \ldots, \frac{\partial}{\partial x_{\alpha s(\alpha)}}, x_{\alpha, s(\alpha)+1} \frac{\partial}{\partial x_{\alpha, s(\alpha)+1}}, \ldots, x_{\alpha n} \frac{\partial}{\partial x_{\alpha n}} \right) \quad (4.5)$$

is a basis for $T_X(-\log D)|_{U_\alpha}$. Write

$$(\gamma_{\alpha 1}(z), \ldots, \gamma_{\alpha n}(z)) := (x_{\alpha 1} \circ \gamma, \ldots, x_{\alpha n} \circ \gamma)$$

so that $\gamma_{\alpha j} : \mathbb{D}^\circ \to \mathbb{P}^1$ is a meromorphic function over $\mathbb{D}^\circ$ for any α and j. With respect to the local trivialization of $T_X(-\log D)$ induced by the basis (4.5), $\gamma'(z)$ can be written as

$$\gamma'(z) = \gamma'_{\alpha 1}(z)e_{\alpha 1} + \cdots + \gamma'_{\alpha s(\alpha)}(z)e_{\alpha s(\alpha)}$$
$$+ (\log \gamma_{\alpha, s(\alpha)+1})'(z)e_{\alpha, s(\alpha)+1} + \cdots + (\log \gamma_{\alpha n})'(z)e_{\alpha n}$$

over U_α. Let $\{\rho_\alpha\}_{\alpha \in I}$ be a partition of unity subordinated to $\{U_\alpha\}_{\alpha \in I}$.

Since h is Finsler pseudometric for $T_X(-\log D)$ which is continuous and locally bounded from above by Definition 2.14, and I is a finite set, there is a constant $C > 0$ so that

$$\rho_\alpha \circ \gamma \cdot |\gamma'(z)|_h^2 \leq C \left(\sum_{j=1}^{s(\alpha)} |\gamma'_{\alpha j}(z)|^2 + \sum_{i=s(\alpha)+1}^{n} |(\log \gamma_{\alpha i})'(z)|^2 \right) \quad \forall z \in \mathbb{D}^*$$

$$(4.6)$$

for any α. Hence

$$T_{\gamma^*\omega}(r) := \int_{r_0}^{r} \frac{d\rho}{\rho} \int_{\mathbb{D}_{r_0,\rho}} \gamma^*\omega \overset{(1.1)}{\leq} \int_{r_0}^{r} \frac{d\rho}{\rho} \int_{\mathbb{D}_{r_0,\rho}} dd^c \log |\gamma'|_h^2$$

$$\overset{(3.3)}{\leq} \frac{1}{2\pi} \int_0^{2\pi} \log |\gamma'(re^{i\theta})|_h d\theta + O(\log r)$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \sum_\alpha |\rho_\alpha \circ \gamma \cdot \gamma'(re^{i\theta})|_h d\theta + O(\log r)$$

$$\overset{(4.1)}{\leq} \sum_\alpha \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\rho_\alpha \circ \gamma \cdot \gamma'(re^{i\theta})|_h d\theta + O(\log r)$$

$$\overset{(4.6)+(4.1)}{\leq} \sum_\alpha \sum_{i=s(\alpha)+1}^n \frac{1}{2\pi} \int_0^{2\pi} \log^+ |(\log \gamma_{\alpha i})'(re^{i\theta})| d\theta$$

$$+ \sum_\alpha \sum_{j=1}^{s(\alpha)} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\rho_\alpha \circ \gamma \cdot \gamma'_{\alpha j}(re^{i\theta})| d\theta + O(\log r)$$

$$\overset{(4.3)}{\leq} C_1 \sum_\alpha \sum_{i=s(\alpha)+1}^n \left(\log^+ T_{\gamma_{\alpha i}, \omega_{FS}}(r) + \log r \right)$$

$$+ \sum_\alpha \sum_{j=1}^{s(\alpha)} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\gamma'_{\alpha j}(re^{i\theta})| d\theta + O(\log r) \quad \|$$

$$\overset{(4.2)}{\leq} C_2(\log^+ T_{\gamma^*\omega}(r) + \log r) + \sum_\alpha \sum_{j=1}^{s(\alpha)} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\gamma'_{\alpha j}(re^{i\theta})| d\theta \quad \|.$$

$$(4.7)$$

Here C_1 and C_2 are two positive constants which do not depend on r. $\qquad \square$

Claim *For any $\alpha \in I$, any $j \in \{1, \ldots, s(\alpha)\}$, one has*

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ |\rho_\alpha \circ \gamma \cdot \gamma'_{\alpha j}(re^{i\theta})| d\theta \leq C_3(\log^+ T_{\gamma^*\omega}(r) + \log r) + O(1) \quad \| \quad (4.8)$$

for a positive constant C_3 which does not depend on r.

Proof of Claim The proof of the claim is borrowed from [36, eq.(4.7.2)]. Pick $C > 0$ so that $\rho_\alpha^2 \sqrt{-1} dx_{\alpha j} \wedge d\bar{x}_{\alpha j} \leq C\omega$. Write $\gamma^*\omega := \sqrt{-1}B(t)dt \wedge d\bar{t}$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ |\rho_\alpha \circ \gamma \cdot \gamma'_{\alpha j}(re^{i\theta})| d\theta = \frac{1}{4\pi} \int_0^{2\pi} \log^+ (|\rho_\alpha^2 \circ \gamma| \cdot |\gamma'_{\alpha j}(re^{i\theta})|^2) d\theta$$

$$\leq \frac{1}{4\pi} \int_0^{2\pi} \log^+ B(re^{i\theta}) d\theta + O(1) \leq \frac{1}{4\pi} \int_0^{2\pi} \log(1 + B(re^{i\theta})) d\theta + O(1)$$

$$\leq \frac{1}{2}\log \left(1 + \frac{1}{2\pi} \int_0^{2\pi} B(re^{i\theta}) d\theta \right) + O(1)$$

$$= \frac{1}{2}\log \left(1 + \frac{1}{2\pi r} \frac{d}{dr} \int_{\mathbb{D}_{r_0,r}} rB dr d\theta \right) + O(1)$$

$$= \frac{1}{2}\log \left(1 + \frac{1}{2\pi r} \frac{d}{dr} \int_{\mathbb{D}_{r_0,r}} \gamma^*\omega \right) + O(1)$$

$$
\overset{(4.4)}{\leq} \frac{1}{2}\log\left(1 + \frac{1}{2\pi r}\left(\int_{\mathbb{D}_{r_0,r}}\gamma^*\omega\right)^{1+\delta}\right) + O(1) \quad \|
$$

$$
= \frac{1}{2}\log\left(1 + \frac{r^\delta}{2\pi}\left(\frac{d}{dr}T_{\gamma^*\omega}(r)\right)^{1+\delta}\right) + O(1) \quad \|
$$

$$
\overset{(4.4)}{\leq} \frac{1}{2}\log\left(1 + \frac{r^\delta}{2\pi}\left(T_{\gamma^*\omega}(r)\right)^{(1+\delta)^2}\right) + O(1) \quad \|
$$

$$
\leq 4\log^+ T_{\gamma^*\omega}(r) + \delta\log r + O(1) \quad \|.
$$

Here we pick $0 < \delta < 1$ and the last inequality follows. The claim is proved.

Putting (4.8) to (4.7), one obtains

$$
T_{\gamma^*\omega}(r) \leq C(\log^+ T_{\gamma^*\omega}(r) + \log r) + O(1) \quad \|
$$

for some positive constant C. Hence $T_{\gamma^*\omega}(r) = O(\log r)$. We apply Lemma 3.2 to conclude that γ extends to the ∞.

5 Algebraic hyperbolicity for moduli spaces of polarized manifolds

In this section we will prove the algebraic hyperbolicity of the moduli spaces studied above. The question was posed to the first named author by Erwan Rousseau in February 2019 at the CIRM.

Proof of Theorem D Since the moduli map of the polarized family $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ is quasi-finite, the family f is of maximal variation and we let $U^\circ \subset U$ be the Zariski open set given in Theorem 2.8. We first take a projective compactification Z_0 of the base U and let $D = Z_0 - U$. Let $Z_1, \ldots, Z_m$ be the irreducible components of $Z_0 - U^\circ$ whose intersections with U are nonempty. Let $\mu_i : X_i \to Z_i$ be a desingularization so that $D_i := \mu_i^{-1}(D)$ is of simple normal crossing. For each $U_i := X_i - D_i$, as the moduli map of the new family $(f_i : V \times_U U_i \to U_i, \mathcal{L}|_{V \times_U U_i})$ is generically finite, f_i is also of maximal variation and so yields a Zariski open subset $U_i^\circ \subset U_i$ by Theorem 2.8 that allows us to repeat for $i \geq 1$ the construction. Iterating using Theorem 2.8 and applying Theorem 2.18, we construct a set of log pairs $\{(X_j, D_j)\}_{j=0,\ldots,N}$ with the following properties.

(1) There are morphisms $\mu_i : X_i \to Z_0$ with $\mu_i^{-1}(D) = D_i$, so that each $\mu_i : X_i \to \mu_i(X_i)$ is a birational morphism.
(2) There are smooth Finsler pseudometrics $h_{i1}, \ldots, h_{in}$ for $T_{X_i}(-\log D_i)$.
(3) $\mu_i|_{U_i^\circ} : U_i^\circ \to \mu_i(U_i^\circ)$ is an isomorphism.
(4) There are smooth Kähler metrics $\omega_{i1}, \ldots, \omega_{in}$ on X_i such that for any curve $\gamma : C \to U_i$ with C an open set of $\mathbb{C}$ and $\gamma(C) \cap U_i^\circ \neq 0$, there exists some h_{ij} so that $|\gamma'(t)|^2_{h_{ij}} \not\equiv 0$, and

$$
dd^c\log|\gamma'|^2_{h_{ij}} \geq \gamma^*\omega_{ij}. \tag{5.1}
$$

(5) For any $i \in \{0, \ldots, N\}$, either $\mu_i(U_i) - \mu_i(U_i^\circ)$ is zero dimensional, or there exists $I \subset \{0, \ldots, N\}$ so that

$$\mu_i(U_i) - \mu_i(U_i^\circ) \subset \cup_{j \in I} \mu_j(X_j)$$

For any irreducible and reduced curve $C \subset Y$ with $C \not\subset D$. By the above construction, there is some log pair (X_i, D_i) so that $C \subset \mu_i(X_i)$ and $C \cap \mu_i(U_i^\circ) \neq \varnothing$. By (3), C is not contained in the exceptional locus of μ_i, and let $C_i \subset X_i$ be the strict transform of C under μ_i. Denote by $v_i : \tilde{C}_i \to C_i \subset X_i$ the normalization of C_i, and set $P_i := (\mu_i \circ v_i)^{-1}(D) = \mu_i^{-1}(D_i)$. Then one has

$$dv_i : T_{\tilde{C}_i}(-\log P_i) \to v_i^* T_{X_i}(-\log D_i).$$

By (4), there is a Finsler pseudometric h_{ij} for $T_{X_i}(-\log D_i)$ so that (5.1) holds. Consider $\tilde{h}_i := v_i^* h_{ij}$, which is a complex semi-norm over $T_{\tilde{C}_i}(-\log P_i)$. By (5.1), there is a Kähler metric ω_{ij} on X_i so that the curvature current satisfies

$$\frac{\sqrt{-1}}{2\pi} \Theta_{\tilde{h}_i^{-1}}(K_{\tilde{C}_i}(\log P_i)) \geq v_i^* \omega_{ij}$$

Since $\mu_i \circ v_i : \tilde{C}_i \to C$ is the normalization of C, one has

$$2g(\tilde{C}_i) - 2 + i(C, D) = \int_{\tilde{C}_i} \frac{\sqrt{-1}}{2\pi} \Theta_{\tilde{h}_i^{-1}}(K_{\tilde{C}_i}(\log P_i)) \geq \int_{\tilde{C}_i} v_i^* \omega_{ij}$$

Fix a Kähler metric ω_Y on Y. Then there is a constant $\varepsilon > 0$ so that $\omega_{ij} \geq \varepsilon \mu_i^* \omega_Y$ for any $i = 0, \ldots, N$ and $j = 1, \ldots, n$. We thus have

$$2g(\tilde{C}_i) - 2 + i(C, D) \geq \varepsilon \int_{\tilde{C}_i} \mu_i^* \omega_Y = \varepsilon \deg_{\omega_Y} C$$

This shows the algebraic hyperbolicity of the base U.

Remark 5.1 The algebraic hyperbolicity in Theorem D generalizes the Arakelov-type inequalities in [34] by Möller, Viehweg and the fourth named author, as well as the weak boundedness of moduli stacks of canonically polarized manifolds in [26] by Kovács-Lieblich. In [34, Theorem 0.3], the authors obtained Arakelov-type inequalities with *sharp bounds* for semistable families of projective manifolds with semi-ample canonical sheaf over $\mathbb{P}^1$. In [26, Definition 2.4], the authors introduced the notion of *weak boundedness* for quasi-projective varieties (which is weaker than the notion of algebraic hyperbolicity) and they proved that the moduli stacks of canonically polarized manifolds are weakly bounded.

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