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An energy formula for fully nonlinear degenerate parabolic equations in one spatial dimension

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Abstract

Energy (or Lyapunov) functions are used to prove stability of equilibria, or to indicate a gradient-like structure of a dynamical system. Matano constructed a Lyapunov function for quasilinear non-degenerate parabolic equations. We modify Matano's method to construct an energy formula for fully nonlinear degenerate parabolic equations. We provide several examples of formulae, and in particular, a new energy candidate for the porous medium equation.

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1 Main results

We consider the scalar fully nonlinear partial differential equation

$$f(x, u, u_x, u_{xx}, u_t) = 0, \tag{1.1}$$

for $x \in (0, 1)$ and $t > 0$ with appropriate initial data $u_0(x)$. Here indices abbreviate partial derivatives. We assume that $f \in C^2$ satisfies the following degenerate parabolic conditions

$$f_q \cdot f_r \leq 0, \quad \text{and} \quad f_r \neq 0, \tag{1.2}$$

for every argument $(x, u, p, q, r) := (x, u, u_x, u_{xx}, u_t) \in [0, 1] \times \mathbb{R}^4$. Conditions (1.2) imply that only the diffusion coefficient f_q may vanish, since $f_r \neq 0$ excludes

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time-evolution type degeneracies.[1] such as in *Trudinger's equation*, $(u^\alpha)_t = u_{xx}$, for $\alpha > 0$; see [45]. Without loss of generality, we consider $f_r < 0$ and thus $f_q \geq 0$. Indeed, if $f_r > 0$, then $f_{\tilde{r}} < 0$ for $\tilde{r} := -r$.

Moreover, in order to guarantee that the diffusion f_q degenerates in a meagre set, we also assume that the following set is of (Lebesgue) measure zero in $[0, 1] \times \mathbb{R}^2$,

$$\left\{(x, u, p) \in [0, 1] \times \mathbb{R}^2 \mid f_q(x, u, p, 0, 0) = 0\right\}. \tag{1.3}$$

In particular, the condition (1.3) prevents that $f_q(x, u, p, 0, 0) = 0$ for all $(x, u, p) \in [0, 1] \times \mathbb{R}^2$ and degeneracies of the same order as u_{xx}, such as the *dual porous medium equation*, $u_t = |u_{xx}|^{m-1}u_{xx}$, for $m > 1$, see [6, 49].

We consider (1.1) with two types of *separated boundary conditions* at $x = \iota \in \{0, 1\}$. For each boundary point $x = \iota$, separately, we either assume homogeneous Dirichlet boundary conditions or nonlinear boundary conditions of Robin type, respectively

$$u = 0, \tag{1.4a}$$

$$u_x = b^\iota(u). \tag{1.4b}$$

We assume $b^\iota \in C^1$. Neumann boundary conditions occur if $b^\iota(u) = 0$. See [1, 35] for abstract settings involving nonlinear boundary conditions of type (1.4b).

Equations (1.1)–(1.4) include classical examples, such as evolution involving p-laplacian diffusion, the porous medium equation or certain mean curvature flow. These classical equations with further nonlinear gradient-dependent forcing did not have any apparent variational structure, which we are now able to display. It is the scope of this paper to provide a unifying variational formulation to several degenerate fully nonlinear parabolic equations in one spatial dimension.

Below we construct a *Lyapunov function*

$$E := \int_0^1 L(x, u, u_x)\, dx \quad \text{such that} \quad \frac{dE}{dt} < 0 \tag{1.5}$$

along non-equilibrium solutions $u = u(t, x)$ of (1.1). Therefore, the time-dependent energy $t \mapsto E(u(t, .))$ decreases strictly, except at equilibria, i.e. $u_t \equiv 0$.

Before we present the main result, we rewrite the fully nonlinear equation (1.1) suitably, following the spirit of [31]. Then we modify of Matano's original idea in [39] in order to incorporate degeneracies of the PDE (1.1) for a Lyapunov function E as in (1.5).

Indeed, we split the Eq. (1.1) in order to emphasize the *degenerate diffusion*,

$$F(x, u, u_x, u_{xx}, u_t) = f_q(x, u, p, 0, 0)u_{xx}, \tag{1.6}$$

where $F(x, u, p, q, r) := -f(x, u, p, q, r) + f_q(x, u, p, 0, 0)u_{xx}$. The degeneracy conditions in Eq. (1.2) become

$$F_r(x, u, p, q, r) > 0, \quad \text{and} \quad f_q(x, u, p, 0, 0) \geq F_q(x, u, p, q, r). \tag{1.7}$$

[1] Note time-evolution degeneracies can be transformed into singular diffusion, see [49, Problem 3.6].

for every argument $(x, u, p, q, r) := (x, u, u_x, u_{xx}, u_t)$. Condition (1.3) implies that $F(x, u, p, q, r) = -f(x, u, p, q, r)$ for a set of measure zero.

Next, we split F into two parts: one which is independent of u_{xx} and u_t, whereas another that depends on them. First, we distinguish a term F^0 related to *reaction*, when $u_{xx} = u_t = 0$. Second we describe the *time evolution* term F^1, the only term that depends on u_t. Specifically, we define

$$\begin{aligned} F^0(x, u, p) &:= F(x, u, p, 0, 0), \\ F^1(x, u, p, q, r) &:= F(x, u, p, q, r) - F^0(x, u, p), \end{aligned} \tag{1.8}$$

where $F^0 \in C^2$ and $F^1 \in C^1$, since $f \in C^2$.

The parabolic equation (1.6) can be rewritten as

$$F^1(x, u, p, q, r) = f_q(x, u, p, 0, 0)u_{xx} - F^0(x, u, p). \tag{1.9}$$

The degeneracy conditions (1.2) incarnated in (1.7) imply that

$$F^1_r > 0 \quad \text{and} \quad f_q(x, u, p, 0, 0) \geq F^1_q(x, u, p, q, r). \tag{1.10}$$

for every $(x, u, p, q, r) := (x, u, u_x, u_{xx}, u_t)$.

The main modification of Matano's method is a different Ansatz for the function L in (1.5), yet to be found. Matano's Ansatz is $L_{pp} = \exp(g(x, u, p))$, which yields a first order PDE for the unknown $g(x, u, p)$ that can be solved by the method of characteristics. Instead, to accommodate degeneracies, we consider:

$$L_{pp} := f_q(x, u, p, 0, 0) \exp(g(x, u, p)), \tag{1.11}$$

for some function $g(x, u, p)$ to be found. Note (1.11) is not identically zero, since $f_q(x, u, p, 0, 0) \not\equiv 0$ due to (1.3). Moreover, whenever $f_q(x, u, p, 0, 0) = 0$, the Ansatz (1.11) implies that $L_{pp} \equiv 0$ for any bounded function $g(x, u, p)$. However, whenever $g(x, u, p)$ is unbounded, then the interplay between $f_q(x, u, p, 0, 0)$ and $g(x, u, p)$ plays a major role in the new Ansatz (1.11), and thus in the construction and regularity of the energies using the present method, in contrast to [31, 39].

To construct the unknown $g(x, u, p)$, we suppose that along the characteristic equations given by

$$\begin{aligned} \dot{x} &= f_q(x, u, p, 0, 0), \\ \dot{u} &= f_q(x, u, p, 0, 0)\, p, \\ \dot{p} &= F^0(x, u, p), \end{aligned} \tag{1.12}$$

there is a solution g of the following equation:

$$\dot{g} = -F^0_p(x, u, p) - f_{qx}(x, u, p, 0, 0) - f_{qu}(x, u, p, 0, 0)\, p. \tag{1.13}$$

Note that the characteristic equations (1.12) for degenerate PDEs is different from the one obtained by Matano in the non-degenerate case. Nevertheless, these equations can be transformed into each other by a suitable 'time' rescaling that absorbs $1/f_q(x, u, p, 0, 0) < \infty$ in case of non-degenerate equations. Moreover, global existence of the characteristic equations (1.12)–(1.13) might fail, in general.

Further issues arise when (1.9) is degenerate, i.e. when $f_q(x, u, p, 0, 0) = 0$ for some $(x, u, p) \in [0, 1] \times \mathbb{R}^2$, since at such degeneracy points the first two equations of the characteristics (1.12) are $\dot{x} = \dot{u} = 0$. On one hand, if $F^0(x, u, p) \neq 0$, then $\dot{p} \neq 0$, which triggers an eventual $f_q(x, u, p, 0, 0) \neq 0$, due to (1.3). On the other hand, if $F^0(x, u, p) = 0$, then $\dot{p} = 0$ and (1.12) encounters an equilibrium. Hence, the function $g(x, u, p)$ may not be constructed along solutions of (1.9). We call this the *obstacle problem*. Therefore, the obstacle problem is crucial to rigorously construct energies for degenerate equations using the present method. We explore some of these problems in the examples of Sect. 3.

Theorem 1.1 *Assume $f \in C^2$ satisfies (1.2). Suppose the characteristic equations (1.12)–(1.13) have global solutions and that L_{pp} in (1.11) is twice-integrable in p.*

Then there exists a Lagrange function $L = L(x, u, p)$ on bounded sets of $(u, p) \in \mathbb{R}^2$ such that $E := \int_0^1 L(x, u, u_x)\,dx$ is a Lyapunov function as in (1.5) for the Eq. (1.1). More precisely, bounded solutions $u(t, x)$ of (1.1) satisfy

$$\frac{dE}{dt} = -\int_0^1 \exp(g(x, u, u_x))F^1(x, u, u_x, u_{xx}, u_t) \cdot u_t\,dx, \tag{1.14}$$

where $g(\cdot)$ solves (1.13) and $F^1 \cdot u_t \geq 0$; the equality holds if, and only if, $u_t \equiv 0$.

For non-degenerate quasilinear equations, $f(x, u, p, q, r) = -r + a(x, u, p)q + h(x, u, p)$, where $a > 0$, a Lyapunov function E was constructed, independently, by Zelenyak [50] and Matano [39]. See also [19] for concise expositions of Matano's method. This method was extended to fully nonlinear non-degenerate parabolic equations, when $f_q \cdot f_r < 0$, in [31]. An analogous method for Jacobi systems, a spatially discrete variant, was developed in [20]. For an adaptation to diffusion with singular coefficients see [29].

We emphasize that the procedure to construct the energy function in (1.5) is formal. Once the Lagrange function L is obtained, one needs to verify various properties needed for a well-defined Lyapunov function, such as integrability, bounds, regularity, etc. For this reason, we call the formulae obtained using our method as *energy candidates*. Thus, the properties and applicability of each candidate still has to be dealt with on a case-by-case basis. In Sect. 3, we provide several examples of candidates. Note that even if a Lyapunov function is only well-defined for sufficiently regular initial data, one may obtain dynamical information on invariant subspaces of regular enough initial data, see [3, 4, 33, 41, 51]. In particular, non-degenerate equations possess enough regularity to produce a well-defined and regular energy. For a deeper regularity analysis of degenerate equations, see [7, 14, 15, 27, 44] and references therein. Similarly, energy candidates can potentially be used to obtain local energy estimates akin to [14].

We comment on modifications and possible applications of our result.

Note that our method can potentially treat singular diffusion, i.e., when $f_q(x, u, p, 0, 0)$ may be unbounded. An example is $u_t = (u^m u_x/|u_x|)_x$ for $m \geq 0$, which is called the *total variation flow* for $m = 0$, or the *heat equation in transparent media* for $m = 1$; see [22] and references therein. However, there are two delicate issues to obtain a Lyapunov function. First, the characteristic equations (1.12) may not have global solutions. Second, the Ansatz for L_{pp} defined in (1.11) might not be twice integrable (in p) in order to obtain a well-defined formula for L in (2.12). A further analysis of singular points must be pursued.

An alternative splitting of the fully nonlinear equation was pursued in [31], different than (1.9), yielding an energy that decays according to

$$\frac{dE}{dt} = -\int_0^1 L_{pp}\tilde{F}^1 u_t^2 \, dx \tag{1.15}$$

for some $\tilde{F}^1 > 0$. Instead of the decay in (1.14), one may also be able to obtain a Lyapunov function that decays according to (1.15), which extracts the L^2-gradient flow with weight $L_{pp}\tilde{F}^1 > 0$. However, we believe that these different splittings do not change the Lyapunov function itself, only the aesthetics of the abstract formulae.

A semiflow treatment of fully nonlinear degenerate equations (1.1) on an appropriate phase-space X has been lacking in its full generality, akin to the one for non-degenerate equations provided by [35]. We expect that additional growth conditions on f, similar to the non-degenerate case in [34, Proposition 3.5] and [46, Chapter 6, Sec. 5], imply that solutions of (1.1) are bounded, global and generate a dissipative semiflow. In particular, this would guarantee the global existence of the characteristics (1.12)–(1.13) after an appropriate cut-off of f outside a sufficiently large set, and thus the existence of a Lyapunov function E in such a bounded set. In more general settings, including solutions which blow-up, boundedness of E from below may fail. In fact, a delicate analysis of the characteristic equations (2.10) beyond such crude cut-off may be required in case of blow-up. For the non-existence of grow-up (i.e. infinite time blow-up) solutions using such Lyapunov functions, see [3, 4].

In addition, it would be desirable to extract dynamic information on the long-term behavior of solutions of (1.1). Indeed, under certain conditions on f that also guarantee asymptotic compactness of the semiflow, there should be a *global attractor* $\mathcal{A} \subset X$ as in the non-degenerate case in [23] or [28, Theorem 2.2]. For particular cases of degenerate type, see [9, 16]. Thus, as a consequence of the Lyapunov function (1.5), bounded trajectories should converge to (sets of) equilibria, according to the LaSalle invariance principle; see [24, Section 4.3] and [5, Chapter 5.7] for the non-degenerate case. However, the complete description of ω-limit sets is a delicate issue for the degenerate case. See [2, 8, 10, 18, 48] for specific degenerate cases, which are not in a fully nonlinear setting. In general, see [25, 38, 40, 42, 43] for a broad overview on the theory of strongly monotone semiflows, when convergence to the set of stationary solutions can be proved. Finally, the connection problem for the equations (1.1)–(1.2) that describes which equilibria are connected by means of a heteroclinic orbit remains open, see [30] and references therein for the non-denegerate case.

The remainder of the paper is organized as follows. In Sect. 2, we prove Theorem 1.1. In Sect. 3, we compute several significant examples of Lyapunov functions.

2 Proof

We recall the equation (1.9),

$$f_q(x,u,p,0,0)u_{xx} = F^0(x,u,p) + F^1(x,u,p,q,r), \tag{2.1}$$

with degenerate parabolicity conditions $F_r^1 > 0$ and $f_q \geq F_q^1$. See (1.10).

Differentiating the definition (1.5) of the Lyapunov function E with respect to time t along classical solutions $u(t,x)$ of (1.1), we obtain

$$\frac{dE}{dt} = \int_0^1 \left(L_u u_t + L_p u_{xt}\right)\,dx. \tag{2.2}$$

Here we used that $u_{xt} = p_t$. The Lagrange function L depends on $(x,u,p) = (x,u,u_x)$, only. It remains to determine L such that $dE/dt < 0$, except at equilibria. Integrating the term $L_p u_{xt}$ in (2.2) by parts, and carrying out the differentiation of L_p with respect to x, we obtain

$$\begin{aligned}\frac{dE}{dt} &= L_p u_t\Big|_{x=0}^{x=1} + \int_0^1 \left(L_u - \frac{d}{dx}L_p\right) u_t\,dx\\ &= L_p u_t\Big|_{x=0}^{x=1} + \int_0^1 \left(L_u - L_{px} - L_{pu}u_x - L_{pp}u_{xx}\right) u_t\,dx.\end{aligned} \tag{2.3}$$

At this point, Matano would plug in the non-degenerate PDE in u_{xx}. However, this can not be performed for degenerate equations, since we can not isolate u_{xx} in equation (2.1), as f_q may be zero. In order to remedy this, we modify Matano's original Ansatz, $L_{pp} = \exp(g(x,u,p))$, which would yield a first order PDE to be solved for $g(x,u,p)$. Instead, we consider the different Ansatz (1.11) for some function $g(x,u,p)$, yet to be found. Note (1.11) is not identically zero, since $f_q(x,u,p,0,0) \not\equiv 0$ due to (1.2). Thus

$$\frac{dE}{dt} = L_p u_t\Big|_0^1 + \int_0^1 \left(L_u - L_{px} - L_{pu}u_x - \exp(g) f_q u_{xx}\right) u_t\,dx. \tag{2.4}$$

We then substitute the PDE (1.1) recast in (2.1), to obtain

$$\frac{dE}{dt} = L_p u_t\Big|_0^1 + \int_0^1 (L_u - L_{px} - L_{pu}u_x - \exp(g)F^0)u_t\,dx - \int_0^1 \exp(g)F^1 u_t\,dx. \tag{2.5}$$

We seek to construct the Lagrange function L such that the boundary terms vanish, the parenthesis in the first integral (2.5) also vanishes, and satisfies the Ansatz (1.11) for some function $g(x, u, p)$. This yields a Lyapunov function such that

$$\frac{dE}{dt} = -\int_0^1 \exp(g)F^1 u_t\,dx. \tag{2.6}$$

Note $F^1 u_t \geq 0$, due the parabolicity condition $F^1_r > 0$. Next, we guarantee that there exists a function $g(x, u, p)$ such that

$$L_u - L_{px} - pL_{pu} - \exp(g)F^0 = 0, \tag{2.7}$$

for all $(x, u, p) \in [0, 1] \times \mathbb{R}^2$, and also $L_p u_t = 0$ on the boundaries $x = 0, 1$. Note that $(u, p) \in \mathbb{R}^2$ are real variables rather than solutions u, u_x of PDEs depending on (t, x). Differentiating (2.7) with respect to p, the terms L_{pu} cancel, yielding

$$L_{ppx} + pL_{ppu} + \exp(g)g_p F^0 = -\exp(g)F^0_p. \tag{2.8}$$

Rewriting (2.8) in terms of g, according to (1.11), amounts to the first order PDE,

$$f_q g_x + pf_q g_u + F^0 g_p = -F^0_p - f_{qx} - pf_{qu}. \tag{2.9}$$

The method of characteristics can solve (2.9): along solutions of the auxiliary ODEs

$$\begin{aligned}\dot{x} &= \frac{dx}{d\tau} = f_q(x, u, p, 0, 0),\\ \dot{u} &= \frac{du}{d\tau} = f_q(x, u, p, 0, 0)\,p,\\ \dot{p} &= \frac{dp}{d\tau} = F^0(x, u, p),\end{aligned} \tag{2.10}$$

the function g must satisfy

$$\dot{g} = \frac{dg}{d\tau} = -F^0_p(x, u, p) - f_{qx}(x, u, p, 0, 0) - f_{qu}(x, u, p, 0, 0)\,p, \tag{2.11}$$

with the initial condition $g(0, u_0, p_0)$, where $(u_0, p_0) := (u(0, 0), u_x(0, 0))$. Our differentiability assumptions on f imply $g \in C^0$, at least.

Without further assumptions on the nonlinearity f in (1.1), solutions to (2.10) may not exist on the whole required interval $x \in [0, 1]$. For this reason, we have assumed the global existence of solutions for the characteristic equations. Moreover, note that the global existence of the characteristics is not enough to guarantee the existence of a Lyapunov function, in general. Indeed, further complications occur when the diffusion degenerates (i.e. $f_q(x, u, p, 0, 0) = 0$), since $\dot{x} = \dot{u} = 0$ in (2.10). If $F^0(x, u, p) \neq 0$, then $\dot{p} \neq 0$ and this may trigger an eventual $f_q(x, u, p, 0, 0) \neq 0$. However, if $F^0(x, u, p) = 0$, then (2.10) encounters an equilibrium and the function $g(x, u, p)$ can not be constructed along solutions of (2.10). We call this the *obstacle problem*. Therefore, the obstacle problem is crucial to rigorously construct energies for degenerate equations using the present method. We explore some of these issues in the examples of Sect. 3.

After this construction, we now have to reverse gear and ascend from a function g satisfying (2.9) to a Lagrange function L satisfying (2.7). The general solution L of $L_{pp} = f_q \exp(g)$ can be obtained by integrating it twice with respect to p:

$$\begin{aligned} L(x, u, p) := &\int_0^p \int_0^{p_1} f_q(x, u, p_2, 0, 0) \exp(g(x, u, p_2))\, dp_2\, dp_1 \\ &+ L^0(x, u) + L^1(x, u)p. \end{aligned} \tag{2.12}$$

This solves (2.8). To ensure that L is also a solution of (2.7), we have to determine the integration "constants" L^0 and L^1, appropriately. Recall that (2.8) was obtained through differentiation of (2.7) with respect to p. Conversely, the left-hand side of (2.7) is therefore independent of p. Hence (2.7) is satisfied for all p, if it holds for some fixed value $p = p_* \in \mathbb{R}$.

Deriving (2.12) with respect to u and p yields

$$L_u = \int_0^p \int_0^{p_1} \left(f_{qu} + f_q g_u\right) \exp(g)\, dp_2\, dp_1 + L_u^0(x, u) + L_u^1(x, u)p, \tag{2.13a}$$

$$L_p = \int_0^p f_q \exp(g) dp_1 + L^1(x, u), \tag{2.13b}$$

where the integrand arguments are suppressed to alleviate the notation. Moreover, further differentiating L_p with respect to x and u produces

$$L_{px} = \int_0^p \left(f_{qx} + f_q g_x\right) \exp(g) dp_1 + L_x^1(x, u), \tag{2.14a}$$

$$L_{pu} = \int_0^p \left(f_{qu} + f_q g_u\right) \exp(g) dp_1 + L_u^1(x, u). \tag{2.14b}$$

Note that evaluating the Eq. (2.7) at p_* yields $L_u = L_{px} + p_* L_{pu} + \exp(g) F^0$. Substituting (2.13a) and (2.14), evaluated at p_*, and isolating L_u^0, we obtain that

$$\begin{aligned} L_u^0(x, u) = &L_x^1(x, u) + \exp(g(x, u, p_*)) F^0(x, u, p_*) \\ &+ \int_0^{p_*} \left[f_{qx} + f_q g_x + \left(f_{qu} + f_q g_u\right) p_*\right] \exp(g) dp_1 \\ &- \int_0^{p_*} \int_0^{p_1} \left(f_{qu} + f_q g_u\right) \exp(g)\, dp_2 dp_1. \end{aligned} \tag{2.15}$$

Note that the right hand side of (2.15) depends only on f, g, L^1 evaluated at p_*. Hence, we can integrate (2.15) with respect to u, which in turn yields the function $L^0(x,u)$ explicitly written as a function of (x,u) for any p_*. Mathematically, we achieve that

$$\begin{aligned} L^0(x,u) = \int_0^u \Bigg[& L_x^1(x,u_1) + \exp(g(x,u_1,p_*))F^0(x,u_1,p_*) \\ & + \int_0^{p_*} \left[f_{qx} + f_q g_x + \left(f_{qu} + f_q g_u\right) p_*\right] \exp(g) dp_1 \\ & - \int_0^{p_*} \int_0^{p_1} \left(f_{qu} + f_q g_u\right) \exp(g)\, dp_2 dp_1 \Bigg] du_1 + L^{00}(x). \end{aligned} \tag{2.16}$$

To complete the proof, it only remains to show that $L_p u_t$ vanishes at the boundaries $x = 0, 1$, which is done by appropriately constructing L^1. At any boundary of Dirichlet type (1.4a) this is trivial because $r = u_t = 0$. Thus we can either let $L^1 \equiv 0$, or

$$L^1(x,u) := -\int_0^{p_*} f_q(x,u,p,0,0) \exp(g(x,u,p)) dp, \tag{2.17}$$

which respectively yields that L_p is a finite value or zero, according to (2.13b). Note that the choice of L^1 influences the construction of L^0 in (2.16).

In the case of a nonlinear Robin boundary condition (1.4b) at only one boundary, either $x = 0$ or $x = 1$, we have to choose L such that $L_p(\iota, u, b^\iota(u)) = 0$. By our construction (2.12) of L, on behalf of (2.13b), this is equivalent to

$$L^1(\iota,u) := -\int_0^{b^\iota(u)} f_q(\iota,u,p,0,0) \exp(g(\iota,u,p))\, dp, \tag{2.18}$$

and we may choose L^1 to be independent of x.

For nonlinear Robin boundary conditions (1.4b) at both boundaries, $x = 0$ and $x = 1$, we define $L^1(\iota,u)$ as in (2.18) for $\iota = 0, 1$. Therefore, the linear interpolation $L^1(x,u) := (1-x)L^1(0,u) + xL^1(1,u)$ provides $L^1 \in C^1$ such that $L_p(\iota, u, b^\iota(u)) = 0$.

For example, if $p_* = 0$, the construction of L yields $L_p = L^1$, $L_{px} = L_x^1$ and $L_u = L_u^0$. Evaluating either (2.7) or (2.15) at $p_* = 0$ yields $L_u^0 = L_x^1 + \exp(g)F^0$. Integrating with respect to u, in agreement with (2.16), we can neglect an irrelevant additive constant $L^{00}(x)$ for E to obtain that

$$L^0(x,u) := \int_0^u \left[L_x^1(x,u_1) + \Big(\exp(g(x,u_1,p_*))F^0(x,u_1,p_*)\Big)\big|_{p_*=0}\right] du_1. \tag{2.19}$$

For $p_* = 0$, the choices of L^1, which depend on the boundary conditions, yield $L^1 \equiv 0$ for Dirichlet boundary conditions and (2.18) is unchanged for Robin boundary conditions.

Table 1 Comparison of known Lyapunov functions and the new candidate formulae for specific PDEs using Matano's method

Section	PDE	Energy
Section 3.1	$u_t = a(u_x)u_{xx} + h(u)$	Old : $\int_0^1 \left(\int_0^p \int_0^{p_1} a(p_2)dp_2dp_1 - \int_0^u h(u_1)du_1 \right) dx$ New : Same
Section 3.2	$u_t = \frac{1+u_x^2}{1-\left(1-\frac{u_x^2}{1+u_x^2}\right)u_{xx}}$	Old : $\lvert M_t^2\rvert^{1/2} \left(8\pi - \int_{M_t^2} H^2 d\mu_t \right)$, see [36], [37, Prop. 6.1] New : $\int_0^1 u_x \arctan(u_x) - \log(1+u_x^2) - u\, dx$
Section 3.3	$u_t = \left(\frac{u_x}{\sqrt{1+u_x^2}}\right)_x + u_x^n$	Old : Unknown New : $\int_0^1 \left(\int_0^p \int_0^{p_1} \frac{1}{\lvert p_2\rvert^n (1+p_2^2)^{\frac{3}{2}}} dp_2\, dp_1 \right) - u\, dx$ See Table 2 for $n = 1, 2, 3, 4, 5$
Section 3.4	$u_t = (\lvert u_x\rvert^{\rho-2}u_x)_x + u_x^n$	Old : $\rho = 2 :$ $\int_0^1 \left(\frac{\lvert u_x\rvert^{2-n}}{(2-n)(1-n)} - u \right) dx$, see [3, 33] $\rho \neq 2 :$ Unknown, see [4, 41] New: $n \neq \rho, \rho-1 :$ $\int_0^1 \frac{(\rho-1)}{(\rho-n)(\rho-n-1)} \lvert u_x\rvert^{\rho-n} - u\, dx$ $n = \rho - 1 :$ $\int_0^1 (\rho-1)\lvert u_x\rvert (\log\lvert u_x\rvert - 1) - u\, dx$ $n = \rho :$ $\int_0^1 (1-\rho)\log\lvert u_x\rvert - u\, dx$
Section 3.5	$u_t = (u^m)_{xx}$	Old : $\int_0^1 \frac{u^{m+1}}{m+1} dx$, see [2, 13, 49], [48, Eq. (2.7)] New : $\int_0^1 m u^{m-1} \lvert u_x\rvert (\log\lvert u_x\rvert - 1)\, dx$

3 Examples

We explicitly compute examples of energy candidates using the method in the previous section and compare them with well-known Lyapunov functions in the literature; see Table 1. For the sake of simplicity, we consider Dirichlet boundary conditions throughout the examples, which yield $L^1 \equiv 0$.

3.1 Gradient-degenerate quasilinear diffusion with nonlinear forcing

Consider the equation

$$u_t = a(u_x)u_{xx} + h(u), \tag{3.1}$$

with $a, h \in C^2$ such that $a(u_x) \geq 0$, where the equality only happens in a set of measure zero, due to (1.3). In the abstract setting in the previous section, we have that $f_q = a(p)$, $F^0 = -h(u)$ and $F^1 = u_t$.

Thus, the characteristic equations (2.10) are given by

$$\begin{aligned} \dot{x} &= a(p), \\ \dot{u} &= a(p)\, p, \\ \dot{p} &= -h(u), \end{aligned} \tag{3.2}$$

and g evolves according to (2.11), i.e.,

$$\dot{g} = 0. \tag{3.3}$$

Note that if $h(u) \neq 0$, then $\dot{p} \neq 0$ and the Eq. (3.2) does not encounter an equilibrium obstacle. However, if $h(u) = 0$ for some $u \in \mathbb{R}$, then there is a constant equilibrium of the PDE (3.1), which is also an equilibrium obstacle for the characteristic equations. However, in either case, note that $\dot{g} = 0$. Therefore $g(x(\tau), u(\tau), p(\tau))$ is a constant function along any solution of the (3.1), and therefore we obtain the trivial solution $g \equiv 0$ for the initial condition $g_0 := g(0, u(0), p(0)) = g(0, u_0, p_0) = 0$. Due to the Eq. (1.11), we obtain that $L_{pp} = a(p)$. Note that $L^1 \equiv 0$ due to Dirichlet boundary conditions, and $L^0 = -\int_0^u h(u_1)\, du_1$ due to the Eq. (2.19) for $p_* = p_0$. Hence (2.12) implies that

$$E = \int_0^1 \left(\int_0^p \int_0^{p_1} a(p_2)\, dp_2\, dp_1 - \int_0^u h(u_1)\, du_1 \right) dx, \tag{3.4}$$

which decays according to

$$\frac{dE}{dt} = -\int_0^1 u_t^2\, dx. \tag{3.5}$$

Note that for Robin boundary conditions, Eq. (2.18) yields the following term, $L^1(x, u) = -(1-x) \int_0^{b^0(u)} a(p_1)dp_1 - x \int_0^{b^1(u)} a(p_1)dp_1$, whereas Eq. (2.19) implies $L^0(x, u) = \int_0^u \left(\int_{b^1(u_1)}^{b^0(u_1)} a(p_1)dp_1 - h(u_1) \right) du_1$. Therefore, the energy candidate formula (3.4) can be modified accordingly. We reiterate that the rigorousness of this formula depends on the delicacy of solutions of the characteristic equations (3.2).

In particular, the ρ-Laplacian[2] equation occurs when $a(u_x) = (\rho - 1)|u_x|^{\rho-2}$ and $h \equiv 0$, and thus we recover its well-known energy $E = \int_0^1 |u_x|^\rho / \rho\, dx$. Also, the mean curvature flow for one dimensional graphs occurs when $a(u_x) = (1 + u_x^2)^{-3/2}$ and $h \equiv 0$, and thereby we also recover the energy $E = \int_0^1 \sqrt{1 + u_x^2}\, dx$. This energy accounts for the perimeter of the curve, which decreases under evolution of mean curvature according to (3.5). For a proof of infinite time blow-up (i.e. grow-up) for a mean curvature flow with a general Hamiltonian reaction of type $h(x, u)$, due to the existence of this well-known energy formula, see [11]. Note that the mean curvature flow only degenerates at infinity, i.e., when $|u_x| \to \infty$, and thus we expect that

[2] In the literature, this operator is called the p-Laplacian. However, in our notation $p := u_x$ and thus we replace the parameter p by ρ in the degenerate diffusion operator, i.e., $\partial_\rho u := (|u_x|^{\rho-2} u_x)_x$.

an appropriate compactification of the semiflow will be described by a degenerate equation at infinity, see [32].

The energy formula (3.4) is ubiquitous in the theory of degenerate parabolic PDEs in divergence form, whenever the degeneracy occurs in the gradient. It is a well-defined and sufficiently regular energy with several consequences which include well-posedness, regularity and dynamical properties. See, for example, [9, 14, 16].

3.2 Inverse mean curvature flow for certain graphs

Consider the equation

$$u_t = \frac{1+u_x^2}{1-\left(1-\frac{u_x^2}{1+u_x^2}\right)u_{xx}}. \tag{3.6}$$

This equation has been considered in higher dimensions in [36, Section 3], and we construct a different monotone quantity in comparison to [37, Proposition 6.1].

In this case, we have that

$$f_q = \frac{(1+u_x^2)^2}{(1+u_x^2-u_{xx})^2}, \quad F^0 = -(1+u_x^2), \quad F^1 = u_t + \frac{u_{xx}^2}{u_{xx}-(1+u_x^2)} \tag{3.7}$$

Note that this equation is not degenerate, since $f_q(x, u, p, 0, 0) = 1$, but it is singular whenever $u_{xx} = u_x^2 + 1$. The characteristic equations (2.10) is given by

$$\begin{aligned} \dot{x} &= 1, \\ \dot{u} &= p, \\ \dot{p} &= -(1+p^2), \end{aligned} \tag{3.8}$$

and (2.11) is given by

$$\dot{g} = 2p. \tag{3.9}$$

Since the Eq. (3.6) is non-degenerate, then $\dot{x} > 0$ and the characteristic equations (3.8) does not encounter an equilibrium obstacle. Thus, the global existence of characteristics is enough to pursue the construction in the previous section and guarantee the existence of a Lyapunov function. We can solve these equations explicitly:

$$p(\tau) = -\tan(\tau + \arctan(-p_0)) \tag{3.10}$$

and

$$g(\tau) = g_0 + 2\log\left(\frac{\cos(\tau + \arctan(-p_0))}{\cos(\arctan(-p_0))}\right). \tag{3.11}$$

Consequently,

$$
\begin{aligned}
g(p) &= g_0 + 2\log\left(\frac{\cos(\arctan(-p))}{\cos(\arctan(-p_0))}\right), \\
&= g_0 + \log\left(\frac{1+p_0^2}{1+p^2}\right)
\end{aligned}
\tag{3.12}
$$

Note that we obtain a finite value $g(p)$ for any $p \in \mathbb{R}$, including $p = 0$. Thus Eqs. (1.11), (2.19) with $p_* = 0$, and the Dirichlet boundary imply that

$$
L_{pp} = \exp(g_0)\frac{1+p_0^2}{1+p^2} \quad L^0 = -\exp(g_0)(1+p_0^2)u \quad \text{and} \quad L^1 = 0. \tag{3.13}
$$

Thus the energy (2.12), up to a multiplicative constant $\exp(g_0)(1+p_0^2)$, is given by

$$
E = \int_0^1 \Big(u_x \arctan(u_x) - \log(1+u_x^2) - u\Big)\,dx, \tag{3.14}
$$

which decays according to

$$
\frac{dE}{dt} = -\int_0^1 \frac{(2+u_x^2)u_x^2}{(1+u_x^2)^3} u_t^2\,dx, \tag{3.15}
$$

since F^1 given by (3.7) can be rewritten as $F^1 = \frac{(2+u_x^2)u_x^2}{(1+u_x^2)^2} u_t$ by substituting (3.6). Note that neither the energy formula (3.14), nor its decay in (3.15), possess singularities for bounded values $u, u_x \in \mathbb{R}$.

Note that for Robin boundary conditions, Eq. (2.18) yields the following term, $L^1(x,u) = -\exp(g_0)(1+p_0^2)\big[(1-x)\arctan(b^0(u)) + x\arctan(b^1(u))\big]$, whereas Eq. (2.19) implies $L^0(x,u) = \exp(g_0)(1+p_0^2)\int_0^u \arctan(b^0(u_1)) - \arctan(b^1(u_1))\,du_1$. Therefore the energy formula (3.14) can be modified accordingly.

We emphasize that, in this example, we have computed an energy formula for a fully nonlinear non-degenerate equation, where the method in [31] is not applicable. Indeed, the splitting of the PDE (3.6) according to [31] defines different functions F, F^0, F^1 than our present method, which yields a different splitting of the PDE in contrast to (1.9). In particular, the functions F, F^0, F^1 in [31] are not well-defined for this example. Roughly speaking, trying to isolate u_{xx} in (3.6) to define the function F in [31] yields an ill-defined vector field when $u_t = 0$. Therefore, the present example shows that our current method, which splits the PDE according to (1.9), overcomes certain problems arising even in the non-degenerate construction in [31].

3.3 Mean curvature flow with an external forcing

Consider the equation that describes the mean curvature flow for planar graphs with an external forcing given by u_x^n with $n \in \mathbb{N}$,

$$u_t = \left(\frac{u_x}{\sqrt{1+u_x^2}}\right)_x + u_x^n = \frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} + u_x^n. \tag{3.16}$$

In this case, we have that $f_q = (1+u_x^2)^{-3/2}$, $F^0 = -u_x^n$ and $F^1 = u_t$. Note that for solutions which do not blow-up in the gradient (e.g. differentiable solutions), $f_q > 0$ and thereby the Eq. (3.16) is non-degenerate. Hence the characteristic equations (2.10) are given by

$$\begin{aligned} \dot{x} &= \frac{1}{(1+p^2)^{\frac{3}{2}}}, \\ \dot{u} &= \frac{p}{(1+p^2)^{\frac{3}{2}}}, \\ \dot{p} &= -p^n, \end{aligned} \tag{3.17}$$

and (2.11) is given by

$$\dot{g} = np^{n-1}. \tag{3.18}$$

Since the Eq. (3.16) is non-degenerate, then $\dot{x} > 0$ and the characteristic equations (3.17) does not encounter an equilibrium obstacle for finite $p \in \mathbb{R}$. Note that if $p_0 = 0$, then $p(\tau) \equiv 0$ and $g(\tau) \equiv g_0$.[3] Moreover, if $p_0 > 0$, then $p(\tau)$ decreases to 0 as $\tau \to \infty$, however, if $p_0 < 0$, then $p(\tau)$ either decreases to 0 as $\tau \to \infty$ or blows up in finite time, respectively for n odd or even. In addition, if $p(\tau)$ blows up in finite time, note that the characteristic equations encounters a *singularity obstacle*, where $\dot{x} = \dot{u} = 0$, but $\dot{p} = \pm\infty$. Thus, for n odd, the global existence of characteristics is enough to pursue the construction in the previous section and guarantee the existence of a Lyapunov function, which is not the case for n even.

We can solve these equations explicitly,

$$p(\tau) = \begin{cases} p_0 e^{-\tau} & \text{for } n = 1 \\ \frac{p_0}{\left(1+(n-1)p_0^{n-1}\tau\right)^{\frac{1}{n-1}}} & \text{for } n > 1, \end{cases} \tag{3.19}$$

[3] For $p_0 = 0$ (i.e. $p \equiv 0$ and $g \equiv g_0$), we obtain that $L_{pp} = \exp(g_0)/(1+p^2)^{\frac{3}{2}}$ for all $n \in \mathbb{N}$. This yields $E = \int_0^1 \sqrt{1+u_x^2}\, dx$, up to a multiplicative constant $\exp(g_0)$, which is the perimeter of the curve; similar to the mean curvature flow with Hamiltonian forcing in Sect. 3.1.

and

$$g(\tau) = \begin{cases} \tau + g_0 & \text{for } n = 1 \\ \log\left(1 + (n-1)p_0^{n-1}\tau\right)^{\frac{n}{n-1}} + g_0 & \text{for } n > 1. \end{cases} \tag{3.20}$$

Consequently,

$$g(p) = \log\left(\frac{p_0}{p}\right)^n + g_0 \tag{3.21}$$

for all $n \in \mathbb{N}$. Therefore, the Eqs. (1.11), (2.19) with $p_* = p_0 \neq 0$, and Dirichlet boundary conditions yield

$$L_{pp} = \exp(g_0)|p_0|^n \frac{1}{|p|^n(1+p^2)^{\frac{3}{2}}} \quad L^0 = -\exp(g_0)|p_0|^n u \quad \text{and} \quad L^1 = 0. \tag{3.22}$$

Thus the energy (2.12), up to a multiplicative constant $\exp(g_0)|p_0|^n$, is formally given by

$$E = \int_0^1 \left(\int_0^p \int_0^{p_1} \frac{1}{|p_2|^n(1+p_2^2)^{\frac{3}{2}}} dp_2\, dp_1\right) - u\, dx, \tag{3.23}$$

which decays according to

$$\frac{dE}{dt} = -\int_0^1 \frac{u_t^2}{|u_x|^n}\, dx. \tag{3.24}$$

However, note that (3.22) may not be twice integrable for all $n \in \mathbb{N}$ and therefore the energy (3.23) may be ill-defined for some $n \in \mathbb{N}$. This is in contrast with the example in Sect. 3.4, which possess the same characteristic equation, but it has a different Ansatz for L_{pp}.

Since Eq. (3.16) is non-degenerate for bounded solutions, one can compute a Lyapunov function for classical bounded solutions following the original construction of Matano in [31, 39]. We now proceed with the original construction to compare with our present results. Indeed, instead of separating the Eq. (3.16) according to (1.9), which amounts to the characteristic equations (1.12), the characteristics in [31, 39] are given by

$$\begin{aligned} \dot{x} &= 1, \\ \dot{u} &= p, \\ \dot{p} &= -p^n(1+p^2)^{\frac{3}{2}}, \end{aligned} \tag{3.25}$$

whereas the unknown g should satisfy the follow equation, in contrast to (1.13),

$$\dot{g} = p^{n-1}[n + (n+3)p^2]\sqrt{1+p^2}. \tag{3.26}$$

Note that the evolution equation for p in (3.25) decouples from (x, u) similar to (3.17). Moreover, the global existence of the characteristics equation (3.25) still depends on the parity of n, i.e., global existence only occurs for odd n. However, a seemingly more complicated vector field appears in the right hand side. In order to obtain a simpler equation for p, which can be solved explicitly, we introduce a new time variable, $\tilde{\tau}$, such that $p' := dp/d\tilde{\tau} = (1+p^2)^{-3/2}\dot{p}$, which transforms the Eq. (3.25) into[4]

$$\begin{aligned} x' &= \frac{1}{(1+p^2)^{\frac{3}{2}}}, \\ u' &= \frac{p}{(1+p^2)^{\frac{3}{2}}}, \\ p' &= -p^n, \end{aligned} \tag{3.27}$$

whereas the unknown g satisfies

$$g' = \frac{p^{n-1}[n + (n+3)p^2]}{1+p^2}. \tag{3.28}$$

Similarly to (3.34), we can solve the relevant part of the Eq. (3.25) explicitly:

$$p(\tilde{\tau}) = \begin{cases} p_0 e^{-\tilde{\tau}} & \text{for } n = 1 \\ \frac{p_0}{\left(1+(n-1)p_0^{n-1}\tilde{\tau}\right)^{\frac{1}{n-1}}} & \text{for } n > 1, \end{cases} \tag{3.29}$$

and

$$g(\tilde{\tau}) = \begin{cases} g_0 + \log(1+p_0^2)^{\frac{3}{2}} + 4\tilde{\tau} + \log\left(\frac{e^{-3\tilde{\tau}}}{(1+p_0^2 e^{-2\tilde{\tau}})^{\frac{3}{2}}}\right) & \text{for } n = 1 \\ g_0 + \log(1+p_0^2)^{\frac{3}{2}} + \log\left(\frac{(1+\tilde{\tau}(n-1)p_0^{n-1})^{\frac{n+3}{n-1}}}{\left(p_0^2+(1+\tilde{\tau}(n-1)p_0^{n-1})^{\frac{2}{n-1}}\right)^{\frac{3}{2}}}\right) & \text{for } n > 1. \end{cases} \tag{3.30}$$

Consequently,

$$g(p) = g_0 + \log(1+p_0^2)^{\frac{3}{2}} + \log\left(\frac{p_0^n}{p^n(1+p^2)^{\frac{3}{2}}}\right). \tag{3.31}$$

[4] Note the characteristics (3.27) coincide with the one obtained through our construction, see (3.17). However, the equations for g given by (3.28) and (3.18) are different. This occurs since our Ansatz in (1.11) is different than Matano's, which is $L_{pp} = \exp(g)$.

Table 2 Explicit examples of the energy candidates (3.23) for the Eq. (3.16) for $n = 1, 2, 3, 4, 5$

n	Energy formula
1	$E = \int_0^1 -u_x \tanh^{-1}\left(\sqrt{1+u_x^2}\right) - u\, dx$
2	$E = \int_0^1 \tanh^{-1}\left(\sqrt{1+u_x^2}\right) - 2\sqrt{1+u_x^2} - u\, dx$
3	$E = \int_0^1 \frac{1}{\lvert u_x\rvert}\left(\sqrt{1+u_x^2} + 3u_x^2 \tanh^{-1}\left(\sqrt{1+u_x^2}\right)\right) - u\, dx$
4	$E = \int_0^1 \frac{(1+16u_x^2)\sqrt{1+u_x^2}}{u_x^2} - \frac{\tanh^{-1}\left(\sqrt{1+u_x^2}\right)}{2} - \frac{u}{3^{1/3}}\, dx.$
5	$E = \int_0^1 -\left(\frac{1}{u_x^2\lvert u_x\rvert}\left(\sqrt{1+u_x^2}(-2+19u_x^2) + 45u_x^4 \tanh^{-1}(\sqrt{1+u_x^2})\right) + u\right)\, dx$

Note that for $n = 1, 3, 5$, a Taylor series expansion nearby $u_x \approx 0$ yields that $u_x \tanh^{-1}(\sqrt{1+u_x^2}) \approx 0$, which yields a well-defined energy formula. However, for $n = 2, 4$, we obtain that $\tanh^{-1}(\sqrt{1+u_x^2}) = \infty$ for $u_x = 0$, which yields an ill-defined formula. These examples display the limitations of both Matano's and our methods

In turn, Matano's Ansatz yields

$$L_{pp} = \exp(g) = \exp(g_0)|p_0|^n(1+p_0^2)^{\frac{3}{2}} \frac{1}{|p|^n(1+p^2)^{\frac{3}{2}}}. \tag{3.32}$$

Therefore, we compare the resulting L_{pp} in (3.32) following Matano's construction to the resulting L_{pp} in (3.22) using our construction. Note these are equal, up to a multiplicative constant $(1 + p_0)^{3/2}$, and hence both our methods yield the same energy formulae. Therefore, the lack of integrability of L_{pp} for some n occurs both in our and Matano's methods. In particular, we have used the software *Maple* to formally compute the integrals (3.23) for $n = 1, 2, 3, 4, 5$, which is displayed in the Table 2.

Recall that the Eq. (3.16) is non-degenerate for bounded solutions, and therefore the characteristic equations (3.17) do not encounter the equilibrium obstacle. On one hand, if $p_0 \neq 0$ and n is odd, then solutions of the characteristics are global and a singularity is not reached in finite time. On the other hand, if $p_0 \leq 0$ and n is even, then the characteristics display finite time blow-up, which may suppress the well-definition of a Lyapunov function, as can be seen in the Table 2.

3.4 ρ-Laplacian diffusion with an external forcing

Consider the ρ-Laplacian equation with external forcing of type u_x^n with $n \in \mathbb{N}$,

$$u_t = \partial_\rho u + u_x^n = (\rho - 1)|u_x|^{\rho-2}u_{xx} + u_x^n, \tag{3.33}$$

where $\rho \geq 2$ and $n \geq 0$. In terms of the formulation in the previous section, we have that $f_q = (\rho - 1)|u_x|^{\rho-2}$, $F^0 = -u_x^n$, and $F^1 = u_t$.

Hence the characteristic equations (2.10) are given by

$$\begin{aligned} \dot{x} &= (\rho - 1)\,|p|^{\rho-2}, \\ \dot{u} &= (\rho - 1)\,|p|^{\rho-1}, \\ \dot{p} &= -p^n, \end{aligned} \tag{3.34}$$

and (2.11) becomes

$$\dot{g} = np^{n-1}. \tag{3.35}$$

Note that the Eq. (3.34) encounters the equilibrium obstacle. Indeed, whenever $p = 0$, we obtain that $\dot{x} = \dot{u} = \dot{p} = \dot{g} = 0$. Moreover, note that whenever $p = 0$ for some $x \in [0, 1]$, this amounts to $u_t = 0$ for such point $x \in [0, 1]$, due to (3.33). On one hand, if $p_0 = 0$, then we consider a constant $g \equiv g_0$, due to (3.35), which amounts to $L_{pp} = (\rho - 1)\exp(g_0)|p|^{\rho-2}$. On the other hand, if $p_0 > 0$, then one can solve the Eq. (3.34) and find g by the methods of characteristics, since $\dot{p} = -p^n$ implies that $p \to 0$ as $\tau \to \infty$, and therefore the equilibrium obstacle is not reached in finite time. For $p_0 \neq 0$, note that the relevant equations in (3.34) coincide with (3.34), we obtain the same solutions in (3.19) and (3.20). Hence the Eqs. (1.11), (2.19) with $p_* = p_0 \neq 0$, and Dirichlet boundary conditions yield

$$L_{pp} = (\rho - 1)\exp(g_0)|p_0|^n\,|p|^{\rho-n-2}, \quad L^0 = -\exp(g_0)|p_0|^n u, \quad \text{and} \quad L^1 = 0. \tag{3.36}$$

Hence the Lagrangian L can be obtained according to (2.12), yielding the following energy formula, up to a multiplicative constant $\exp(g_0)|p_0|^n$:

$$E = \begin{cases} \displaystyle\int_0^1 \frac{(\rho - 1)}{(\rho - n)(\rho - n - 1)}|u_x|^{\rho-n} - u\,dx & \text{for } n \neq \rho, \rho - 1, \\ \displaystyle\int_0^1 (\rho - 1)|u_x|\,(\log|u_x| - 1) - u\,dx & \text{for } n = \rho - 1, \\ \displaystyle\int_0^1 (1 - \rho)\log|u_x| - u\,dx & \text{for } n = \rho \end{cases} \tag{3.37}$$

which decays according to

$$\frac{dE}{dt} = -\int_0^1 \frac{u_t^2}{|u_x|^n}\,dx. \tag{3.38}$$

For $\rho = 2$ and $n > 2$, see [3] for the construction in case of a reaction term $|u_x|^n$. Moreover, for $\rho = 2$ and $n \in (0, 1)$, the same energy (3.37) with decay (3.38) was obtained in [33]. See both [3, 33] for a discussion on other values of n and in case of a signed reaction $a|u_x|^n$ for some $a \in \mathbb{R}$. For an interplay between a gradient and

Hamiltonian reaction, i.e. $h(u, u_x) = \epsilon(u^m)_x + u^n$, see [21]. For $\rho = 2$, the Eq. (3.33) is not degenerate and thus the Lyapunov function regular, as solutions of a strict parabolic equation are also regular for $t > 0$. However, for $\rho \neq 2$, instead of directly obtaining a Lyapunov function which is intrinsic for the degenerate equations, the authors in [4, 41] resourced to viscosity approximations and an associated approximating Lyapunov function. See [12] for a user's guide on viscosity solutions.

Since we have not proved any further regularity of the energy E, its derivative is also formal. For equilibria, $u_t \equiv 0$, the energy E in (3.37) is constant, and its derivative dE/dt given by (3.38) either vanishes or it attains the value $-\infty$ in case the integrand in (3.38) is not integrable, which means the derivative is not well defined. Similarly for time dependent solutions: either (3.38) is integrable yielding negative bounded values, or (3.38) is not integrable and thereby ill-defined. Thus, equilibria may be critical points of a non-differentiable energy. See [33], who mentions that (3.38) is singular and it is not clear how to give a meaning to it. However, [33, Proposition 9] provides a weaker result which is sufficient to obtain dynamical information. Indeed, note that one can still obtain dynamical information for continuous Lyapunov functions, see [24, Chapter 4] and [5, Chapter 5.7]. See also [3, 33, 41].

Note that the energy in this example remains true for $n < 0$, even though the reaction term in the vector field is singular when $u_x = 0$. In this case, the decay rate of the energy in (3.38) is bounded along bounded solutions of (3.33). Thus, our methods can be applied in certain cases of quenching phenomena, whenever the hypothesis (1.2) holds true and one can solve the characteristic equations. See [47] for an example of quenching in a fully nonlinear equation.

Note that our construction can be replicated for more general external forcing. For example, when $\rho = 2$ and the nonlinearity is of exponential type, see [51].

3.5 Porous medium equation

Consider the porous medium equation (PME) for $m \geq 1$,

$$u_t = (u^m)_{xx} = mu^{m-1}u_{xx} + m(m-1)u^{m-2}u_x^2. \tag{3.39}$$

Note this is a degenerate parabolic equation for non-negative solutions $u \geq 0$, only.[5] Instead of considering $(u^m)_{xx}$ as a nonlinear diffusion operator, we split it into two separate terms: a nonlinear degenerate diffusion, $mu^{m-1}u_{xx}$, and a nonlinear reaction, $m(m-1)u^{m-2}u_x^2$. Indeed, in the previous setting, $f_q = mu^{m-1}$, $F^0 = -m(m-1)u^{m-2}u_x^2$ and $F^1 = u_t$. Therefore the characteristic equations (2.10) are given by

$$\begin{aligned} \dot{x} &= mu^{m-1}, \\ \dot{u} &= mu^{m-1}p, \\ \dot{p} &= -m(m-1)u^{m-2}p^2, \end{aligned} \tag{3.40}$$

[5] The diffusion given by $(|u|^{m-1}u)_{xx} = m|u|^{m-1}u_{xx} + m(m-1)|u|^{m-3}uu_x^2$ is a natural extension that takes sign-changing solutions into account, which is thereby called the *signed PME* in [49]. For the sake of simplicity, we proceed with the non-signed PME in the main text.

and (2.11) reduces to

$$\dot{g} = m(m-1)u^{m-2}p. \tag{3.41}$$

Note that the Eq. (3.40) encounters the equilibrium obstacle. Indeed, whenever $u = 0$, we obtain that $\dot{x} = \dot{u} = \dot{p} = \dot{g} = 0$. Moreover, note that whenever $u = 0$ for some $x \in [0, 1]$, this amounts to $u_t = 0$ for such point $x \in [0, 1]$, due to (3.39). On one hand, if $u_0 = 0$ for $\tau \in (\tau_m, \tau_M)$, then we consider a constant $g \equiv g_0$ for $\tau \in (\tau_m, \tau_M)$, due to (3.41), which amounts to $L_{pp} = m\exp(g_0)u^{m-1}$. On the other hand, if $u_0 > 0$ for $\tau \in (\tau_m, \tau_M)$, then one can solve the Eq. (3.34) and find g by the methods of characteristics. For $u_0 \neq 0$, we introduce the variable $\tilde{\tau}$ such that $d\tilde{\tau}/d\tau = mu^{m-2}$, with notation $(.)' = d(.)/d\tilde{\tau}$, the characteristic equations become

$$\begin{aligned} x' &= u, \\ u' &= up, \\ p' &= -(m-1)p^2, \end{aligned} \tag{3.42}$$

and

$$g' = (m-1)p. \tag{3.43}$$

We can solve this explicitly, which yields

$$p(\tilde{\tau}) = \frac{1}{1/p_0 + (m-1)\tilde{\tau}}, \tag{3.44}$$

and

$$g(\tilde{\tau}) = g_0 + \log\left((m-1)p_0\tilde{\tau} + 1\right). \tag{3.45}$$

Hence

$$g(p) = g_0 + \log\left(\frac{p_0}{p}\right). \tag{3.46}$$

Thus Eqs. (1.11), (2.19) with $p_* = 0$, and the Dirichlet boundary imply that

$$L_{pp} = m\exp(g_0)\left|\frac{p_0}{p}\right|u^{m-1}, \quad L^0 = 0, \quad \text{and} \quad L^1 = 0. \tag{3.47}$$

Hence the Lagrangian L can be obtained according to (2.12), yielding the following energy candidate, up to a multiplicative constant $|p_0|\exp(g_0)$,

$$E = \int_0^1 mu^{m-1}|u_x|\left(\log|u_x| - 1\right)\,dx, \tag{3.48}$$

which decays according to

$$\frac{dE}{dt} = -\int_0^1 \frac{u_t^2}{|u_x|}\,dx. \tag{3.49}$$

Note that (3.48) is different from the usual energy given by $\tilde{E} = \int_0^1 \frac{u^{m+1}}{m+1}\,dx$ such that $\frac{d\tilde{E}}{dt} = -\int_0^1 [(u^m)_x]^2\,dx$. The energy E decays with respect to the L^2-norm of u_t with weight $1/|u_x|$, whereas $\tilde{E}$ decays with respect to the L^2-norm of $(u^m)_x$. Recall that the decay rate (3.49) is a formal computation and the energy E may not be differentiable. However, one is still able to infer dynamical properties from a continuous Lyapunov function; see the discussion after Eq. (3.38). Thus, the new energy E in (3.48) may be more suitable than $\tilde{E}$ to infer dynamical properties of the porous medium equation, such as [2, 13, 49], especially in case of further gradient-dependent forcing.

Yet another Lyapunov function of a rescaled porous medium equation was found in [48] for $m > 1$, along with the asymptotic classification of solutions in one dimension. Indeed, considering the following rescaling, $u(t,x) = t^{-\frac{1}{m-1}}\theta(\tau,x)$ where $t = \exp(\tau)$, the function θ satisfies the following PDE:

$$\theta_\tau = (\theta^m)_{xx} + \frac{1}{m-1}\theta. \tag{3.50}$$

This equation possess the following Lyapunov function,

$$V := \int_0^1 \frac{|(|\theta^{m-1}|\theta)_x|^2}{2} - \frac{m}{(m+1)(m-1)}|\theta|^{m+1}dx, \tag{3.51}$$

which decays according to

$$\frac{dV}{dt} = -m\int_0^1 |\theta|^{m-1}(\theta_\tau)^2 dx \le 0. \tag{3.52}$$

The Lyapunov function V decays in a similar manner as E, i.e., the energy V decreases except for $\theta \equiv 0$ and equilibria $\theta_\tau = 0$.

For an equivalence between the porous medium equation and the ρ-laplacian, see [26]. For the doubly nonlinear equation, which combines the diffusion of the porous medium and the ρ-laplacian, see [17, 27]. In particular, since we have obtained a new energy for the porous medium equation, we also expect to obtain a new formula for the doubly nonlinear equation and generalizations thereof.

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References

1. Amann, H.: Parabolic evolution equations and nonlinear boundary conditions. J. Differ. Equ. **7**(2), 201–269 (1988)
2. Aronson, D., Crandall, M., Peletier, L.: Stabilization of solutions of a degenerate nonlinear diffusion problem. Nonlinear Anal. T.M.A. **6**, 1001–1022 (1982)
3. Arrieta, J., Rodriguez-Bernal, A., Souplet, P.: Boundedness of global solutions for nonlinear parabolic equations involving gradient blow-up phenomena. Ann. Scuola Norm. Sup. Pisa **5**, 1–15 (2004)
4. Attouchi, A.: Boundedness of global solutions of a p-Laplacian evolution equation with a nonlinear gradient term. Asymp. Anal. **91**, 233–251 (2015)
5. Babin, A.V., Vishik, M.I.: Attractors of Evolution Equations. Elsevier Sc. (1992)
6. Bernis, F., Hulshof, J., Vázquez, J.L.: A very singular solution for the dual porous medium equation and the asymptotic behaviour of general solutions. J. Reine Angew. Math. (Crelle) **435**, 1–31 (1993)
7. Bögelein, V., Duzaar, F., Mingione, G.: The regularity of general parabolic systems with degenerate diffusion. Mem. American Math. Soc., vol. 221 (2013)
8. Bonforte, M., Figalli, A.: The Cauchy–Dirichlet problem for the fast diffusion equation on bounded domains (2023). arXiv:2308.08394
9. Carvalho, A.N., Cholewa, J., Dlotko, T.: Global attractors for problems with monotone operators. Boll. Uni. Matem. Ital. **8**, 693–706 (1999)
10. Carrillo, J.A., Jüngel, A., Markowich, P.A., Toscani, G., Unterreiter, A.: Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities. Monatshefte für Mathematik **133**, 1–82 (2001)
11. Chen, C.-N.: Infinite time blow-up of solutions to a nonlinear parabolic problem. J. Differ. Equ. **139**, 409–427 (1997)
12. Crandall, M., Ishii, H., Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. **27**, 1 (1992)
13. Diaz, G., Diaz, I.: Finite extinction time for a class of non-linear parabolic equations. Commun. PDE **4**, 1213–1331 (1979)
14. DiBenedetto, E.: Degenerate Parabolic Equations. Springer, New York (1993)
15. DiBenedetto, E., Urbano, J.M., Vespri, V.: Chapter 3: Current issues on singular and degenerate evolution equations. Handbook of Diff. Eq.: Evol. Eq., vol. 1, pp. 169 – 286 (2002)
16. Efendiev, M.: Attractors for degenerate parabolic type equations. Mathematical Surveys and Monographs, vol. 192. AMS (2013)
17. Esteban, J.R., Vazquez, J.L.: On the equation of turbulent filtration in one-dimensional porous media. Nonlinear Anal. T.M.A. **10**, 1303–1325 (1986)
18. Feireisl, E., Simondon, F.: Convergence for degenerate parabolic equations. J. Differ. Equ. **152**, 439–466 (1999)
19. Fiedler, B., Grotta-Ragazzo, C., Rocha, C.: An explicit Lyapunov function for reflection symmetric parabolic partial differential equations on the circle. Russ. Math. Surv. **69**, 27–42 (2014)
20. Fiedler, B., Gedeon, T.: A Lyapunov function for tridiagonal competitive-cooperative systems. SIAM J. Math. Anal. **30**(3), 469–478 (1999)
21. Fila, M., Sacks, P.: The transition from decay to blow-up in some reaction-diffusion-convection equations. In: Proc. 1st World Cong. Nonlin. Anal. '92 (ed. V. Lakshmikantham). De Gruyter, pp. 1303–1325 (1996)
22. Giacomelli, L., Moll, S., Petitta, F.: Nonlinear diffusion in transparent media: the resolvent equation. Adv. Calc. Var. **11**, 405–432 (2018)
23. Hale, J.K.: Asymptotic Behavior of Dissipative Systems. Math. Surv., vol. 25. AMS, Providence (1988)
24. Henry, D.: Geometric Theory of Semilinear Parabolic Equations. Springer, New York (1981)
25. Hirsch, M.: Stability and convergence in strongly monotone dynamical systems. J. Reine Angew. Math. (Crelle) **383**, 1–53 (1988)
26. Iagar, R.G., Sánchez, A., Vázquez, J.L.: Radial equivalence for the two basic nonlinear degenerate diffusion equations. J. Math. Pures Appl. **89**, 1–24 (2008)
27. Kalashnikov, A.: Some problems of the qualitative theory of nonlinear degenerate second-order parabolic equations. Russ. Math. Surv. **42**, 169–222 (1987)
28. Ladyzhenskaya, O.: Attractors for Semi-groups and Evolution Equations. Cambridge University Press, Cambridge (1991)

29. Lappicy, P.: Sturm attractors for quasilinear parabolic equations with singular coefficients. J. Dyn. Differ. Equ. **32**, 359–390 (2020)
30. Lappicy, P.: Sturm attractors for fully nonlinear parabolic equations. Rev. Mat. Complut. **36**, 725–747 (2023)
31. Lappicy, P., Fiedler, B.: A Lyapunov function for fully nonlinear parabolic equations in one spatial variable. São Paulo J. Math. Sci. **13**, 283–291 (2019)
32. Lappicy, P., Pimentel, J.: Unbounded Sturm attractors for quasilinear equations. arXiv:1809.08971
33. Laurençot, P.: Convergence to steady states for a one-dimensional viscous Hamilton–Jacobi equation with Dirichlet boundary conditions. Pac. J. Math. **230**, 347–364 (2007)
34. Lunardi, A.: On a class of fully nonlinear parabolic equations. Commun. PDE **16**, 145–172 (1991)
35. Lunardi, A.: Analytic Semigroups and Optimal Regularity in Parabolic Problems. Springer, Basel (1995)
36. Marquardt, T.: Inverse mean curvature flow for star-shaped hypersurfaces evolving in a cone. J. Geom. Anal. **23**, 1303–1313 (2013)
37. Marquardt, T.: Weak solutions of inverse mean curvature flow for hypersurfaces with boundary. J. Reine Angew. Math. **728**, 237–261 (2017)
38. Matano, H.: Strong comparison principle in nonlinear parabolic equations. In: Boccardo, L., Tesei, A. (eds.) Nonlinear Parabolic Equations: Qualitative Properties of Solutions, Pitman Res. Notes in Math., vol. 149, pp. 148–155 (1987)
39. Matano, H.: Asymptotic behavior of solutions of semilinear heat equations on S^1. In: Ni, W.-M., Peletier, L.A., Serrin, J. (eds.) Nonlinear Diffusion Equations and Their Equilibrium States II, pp. 139–162 (1988)
40. Poláčik, P.: Convergence in smooth strongly monotone flows defined by semilinear parabolic equations. J. Differ. Equ. **79**, 89–110 (1989)
41. Stinner, C.: Convergence to steady states in a viscous Hamilton-Jacobi equation with degenerate diffusion. J. Differ. Equ. **248**, 209–228 (2010)
42. Smith, H., Thieme, H.: Convergence for strongly order-preserving semiflows. SIAM J. Math. Anal. **22**, 1081–1101 (1991)
43. Smith, H.: Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems. AMS Math. Surveys and Monographs, vol. 41 (1995)
44. Teixeira, E., Urbano, J.M.: A geometric tangential approach to sharp regularity for degenerate evolution equations. Anal. PDE **7**, 733–744 (2014)
45. Trudinger, N.: Pointwise estimates and quasilinear parabolic equations. Commun. Pure Appl. Math. **XXI**, 205–226 (1968)
46. Uraltseva, N., Ladyzhenskaya, O., Solonnikov, V.A.: Linear and Quasi-linear Equations of Parabolic Type. American Mathematical Society (1968)
47. Vázquez, J.L., Gerbi, S., Galaktionov, V.A.: Quenching for a one-dimensional fully nonlinear parabolic equation in detonation theory. SIAM J. Appl. Math. **61**(4), 1253–1285 (2001)
48. Vázquez, J.L.: The Dirichlet problem for the porous medium equation in bounded domains. Asymptotic behavior. Monats. Math. **142**, 81–111 (2004)
49. Vázquez, J.L.: The Porous Medium Equation: Mathematical Theory. Oxford Mathematical Monographs (2006)
50. Zelenyak, T.I.: Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable. Differ. Uravn. **4**, 34–45 (1968)
51. Zhang, Z., Li, Y.: Boundedness of global solutions for a heat equation with exponential gradient source. Abstr. Appl. Anal. **4**, 1–10 (2012)