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Classification of solutions to $-\Delta u = u^{-\gamma}$ in the half-space

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Abstract
We provide a classification result for positive solutions to $-\Delta u = \frac{1}{u^\gamma}$ in the half space, under zero Dirichlet boundary condition.

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1 Introduction

We deal with the classification of positive solutions to the singular problem

$$\begin{cases} -\Delta u = \dfrac{1}{u^\gamma} & \text{in } \mathbb{R}^N_+ \\ u > 0 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{$\mathcal{P}_\gamma$}$$

where $N \geq 1$, $\gamma > 1$, $x \in \mathbb{R}^N_+$ is represented by $x = (x', x_N)$, $x' \in \mathbb{R}^{N-1}$ and $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N : x_N > 0\}$. This is a captivating problem itself but it also arises in the study of limiting scaling arguments at the boundary, in bounded domains, for solutions to

$$-\Delta u = \frac{1}{u^\gamma} + f(u)$$

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since the term $1/u^\gamma$ is the leading part near zero, see e.g. [2, 4]. Although the expert reader may guess that the decreasing nature of the nonlinearity is favourable for the application of maximum and comparison principles, we stress that here, solutions are not in the right Sobolev space in order to do that (in particular for $\gamma > 1$). This causes a deep and challenging issue that we approach introducing a technique based on precise asymptotic estimates. The underlying idea is that the equation is almost harmonic far from the boundary once we deduce precise estimates.
Taking into account the nature of the problem, it follows that, the natural assumption that we shall adopt in all the paper is

$$u \in C^2(\mathbb{R}^N_+) \cap C^0(\overline{\mathbb{R}^N_+}).$$

Note that the continuity up to the boundary of the solutions can be proved as in [3]. Therefore the equation is understood in the classic meaning in the interior of the domain, or in the variational meaning as in the following:

$$\int_{\mathbb{R}^N_+} \nabla u \cdot \nabla \varphi = \int_{\mathbb{R}^N_+} \frac{1}{u^\gamma} \varphi \qquad \forall \varphi \in C^1_c(\mathbb{R}^N_+). \tag{1}$$

We will classify all the locally bounded solutions according to the following hypothesis
(hp) There exists $\bar{\lambda} > 0$ such that u is bounded on the set $\Sigma_{\bar{\lambda}}$. We set $\theta \in \mathbb{R}$ such that

$$\theta := \sup_{\Sigma_{\bar{\lambda}}} u(x).$$

where the strip $\Sigma_{\bar{\lambda}}$ is defined in Sect. 2. Our main result is the following

Theorem 1 *Let u be a solution of ($\mathcal{P}_\gamma$) fulfilling* (**hp**). *Then*

$$u(x) = u(x', x_N) = u(x_N).$$

Consequently either

$$u(t) = \frac{(\gamma+1)^{\frac{2}{\gamma+1}}}{(2\gamma-2)^{\frac{1}{\gamma+1}}} t^{\frac{2}{\gamma+1}}$$

or

$$u(t) = \lambda^{-\frac{2}{\gamma+1}} v(\lambda t) \quad \lambda > 0,$$

where $v(t) \in C^2(\mathbb{R}_+) \cap C(\overline{\mathbb{R}_+})$ is the unique solution to

$$\begin{cases} -v'' = \dfrac{1}{v^\gamma} & t > 0 \\ v(t) > 0 & t > 0 \\ v(0) = 0 & \lim_{t \to +\infty} v'(t) = 1. \end{cases} \tag{2}$$

The starting crucial issue in our proof is the accurate study of the asymptotic behavior of the solutions up to the boundary as well as at infinity. Actually we shall show that every solution has at most linear growth far from the boundary, which is a sharp estimate. This analysis will allow us to exploit a celebrated result of Berestycki, Caffarelli and Nirenberg [1] to deduce that the solutions exhibit 1-D symmetry. Then, taking into account the non standard nature of the equation arising from the singular term, we will carry out a ODE analysis to complete our proof.

The paper is organized as follows: in Sect. 2 we provide the proofs of the asymptotic analysis and exploit it to prove the 1-D result. In Sect. 3 we carry out the ODE analysis. We conclude in Sect. 4 with the proof of our main result.

2 Asymptotic analysis and 1-D symmetry

In all the paper we shall use the notation given in the following

Definition 2 Given $0 < a < b$, we define the strip $\Sigma_{(a,b)}$ as the set given by

$$\Sigma_{(a,b)} := \{x \in \mathbb{R}^N_+ : a < x_N < b\}. \tag{3}$$

We also set $\Sigma_{(0,b)} := \Sigma_b$.

In all this section we will use some ODEs arguments that are actually contained in the more general analysis of Sect. 3. We start proving

Lemma 3 *Under the assumption* (**hp**), *it follows that*

$$u(x) \le C x_N^{\frac{2}{\gamma+1}} \qquad \text{in } \Sigma_{\bar{\lambda}},$$

with $C = C(\gamma, \theta)$ a positive constant.

Proof Let us consider the 1-D solution $w(x_N)$ of

$$\begin{cases} -w'' = \dfrac{1}{w^\gamma} & \text{in } \mathbb{R}^+ \\ w(0) = 0 & \\ w > 0 & \text{in } \mathbb{R}^+, \end{cases}$$

given in (25). Note that $w_\beta = \beta w$ solves $-\Delta w_\beta = \beta^{\gamma+1}/w_\beta^\gamma$. Therefore, for $\beta > 1$, we have

$$\begin{cases} -\Delta w_\beta > \dfrac{1}{w_\beta} & \text{in } \mathbb{R}^+_N \\ w_\beta > 0 & \text{in } \mathbb{R}^N_+ \\ w_\beta = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{4}$$

Now, since

$$w_\beta = \beta w(\bar{\lambda}) \quad \text{on } \{x \in \mathbb{R}^N_+ : x_N = \bar{\lambda}\}, \tag{5}$$

we take β large so that, for $\beta \geq \theta / w(\bar{\lambda})$ we deduce

$$w_\beta \geq \beta w(\bar{\lambda}) \geq \theta.$$

Consequently u, w_β are well ordered on the boundary of the strip $\Sigma_{\bar{\lambda}}$ (see (3)), namely

$$u \leq w_\beta \quad \text{on } \partial \Sigma_{\bar{\lambda}}. \tag{6}$$

In order to prove a comparison principle, we have to take into account that we are working in the unbounded domain $\mathbb{R}^N_+$ and both u, w_β lose regularity at the boundary of the half-space $\mathbb{R}^N_+$. For this reason we start defining $\phi_R(x') : \mathbb{R}^{N-1} \to \mathbb{R}$ such that

$$\begin{cases} \phi_R(x') = 1 & \text{in } B'_R(0) \\ \phi_R(x') = 0 & \text{in } \mathbb{R}^{N-1} \setminus B'_{2R}(0) \\ |\nabla \phi_R(x')| \leq \dfrac{C}{R} & \text{in } \mathbb{R}^{N-1}, \end{cases} \tag{7}$$

where we recall that a point $x \in \mathbb{R}^{\mathbb{N}}_+$ is denoted by $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and where

$$B'_R(0) := \{x' \in \mathbb{R}^{N-1} : |x'| < R\}.$$

Moreover, let us define the translated function (indeed still a supersolution to (4))

$$w_{\beta,\varepsilon} = w_\beta(x_N + \varepsilon)$$

and let φ_R defined as

$$\varphi_R = (u - w_{\beta,\varepsilon})^+ \phi_R^2.$$

One can check, using a suitable argument based on the continuity of u and $w_{\beta,\varepsilon}$, that φ_R is indeed a suitable function test to both problems ($\mathcal{P}_\gamma$) and (4). Let us also define the cylinder

$$C(R) := \left\{ \Sigma_{\bar{\lambda}} \cap \overline{\{B'_R(0) \times \mathbb{R}\}} \right\}.$$

Then using φ_R in the weak formulations satisfied by u and by w_ε, we obtain

$$\int_{C(2R)} (\nabla u, \nabla (u - w_{\beta,\varepsilon})^+) \phi_R^2 \, dx + 2 \int_{C(2R)} (\nabla u, \nabla \phi_R) \phi_R (u - w_{\beta,\varepsilon})^+ \, dx$$
$$= \int_{C(2R)} \frac{1}{u^\gamma} (u - w_{\beta,\varepsilon})^+ \phi_R^2 \, dx$$

and

$$\int_{C(2R)} (\nabla w_{\beta,\varepsilon}, \nabla(u-w_{\beta,\varepsilon})^+)\phi_R^2\,dx + 2\int_{C(2R)} (\nabla w_{\beta,\varepsilon}, \nabla\phi_R)\phi_R(u-w_{\beta,\varepsilon})^+\,dx$$
$$\geq \int_{C(2R)} \frac{1}{w_{\beta,\varepsilon}^{\gamma}}(u-w_{\beta,\varepsilon})^+\phi_R^2\,dx.$$

Subtracting the last inequalities we obtain

$$\begin{aligned}
&\int_{C(2R)} |\nabla(u-w_{\beta,\varepsilon})^+|^2\phi_R^2\,dx \\
&\leq 2\int_{C(2R)} |\nabla(u-w_{\beta,\varepsilon})^+||\nabla\phi_R|\phi_R(u-w_{\beta,\varepsilon})^+\,dx \\
&\quad + \int_{C(2R)} \left(\frac{1}{u^{\gamma}} - \frac{1}{w_{\beta,\varepsilon}^{\gamma}}\right)(u-w_{\beta,\varepsilon})^+\phi_R^2\,dx \\
&\leq 2\int_{C(2R)} |\nabla(u-w_{\beta,\varepsilon})^+||\nabla\phi_R|\phi_R(u-w_{\beta,\varepsilon})^+\,dx \\
&\quad + \int_{C(2R)} \left(\frac{1}{u^{\gamma}} - \frac{1}{w_{\beta,\varepsilon}^{\gamma}}\right)\frac{[(u-w_{\beta,\varepsilon})^+]^2}{(u-w_{\beta,\varepsilon})}\phi_R^2\,dx.
\end{aligned} \tag{8}$$

We observe that there exists a positive constant $\eta = \eta(\gamma, \theta)$ such that

$$\left(\frac{1}{u^{\gamma}} - \frac{1}{w_{\beta,\varepsilon}^{\gamma}}\right)\frac{1}{u-w_{\beta,\varepsilon}} \leq -\eta < 0,$$

since u is bounded on $\Sigma_{\bar{\lambda}}$ by (**hp**). Moreover using Young inequality and (7), we also deduce that

$$\begin{aligned}
&\int_{C(2R)} |\nabla(u-w_{\beta,\varepsilon})^+||\nabla\phi_R|\phi_R(u-w_{\beta,\varepsilon})^+\,dx \\
&\quad \leq \delta\int_{C(2R)} |\nabla(u-w_{\beta,\varepsilon})^+|^2\phi_R^2\,dx + \frac{C(\delta)}{R^2}\int_{C(2R)} [(u-w_{\beta,\varepsilon})^+]^2\,dx.
\end{aligned} \tag{9}$$

Therefore, using (9) in (8), we get

$$\begin{aligned}
&\int_{C(2R)} |\nabla(u-w_{\beta,\varepsilon})^+|^2\phi_R^2\,dx \\
&\quad \leq 2\delta\int_{C(2R)} |\nabla(u-w_{\beta,\varepsilon})^+|^2\phi_R^2\,dx + \frac{2C(\delta)}{R^2}\int_{C(2R)} [(u-w_{\beta,\varepsilon})^+]^2\,dx \\
&\qquad -2\eta\int_{C(2R)} [(u-w_{\beta,\varepsilon})^+]^2\,dx.
\end{aligned}$$

For δ small fixed we deduce that

$$(1-2\delta)\int_{C(2R)}|\nabla(u-w_{\beta,\varepsilon})^+|^2\phi_R^2\,dx \le \frac{2C(\delta)}{R^2}\int_{C(2R)}[(u-w_{\beta,\varepsilon})^+]^2\,dx \\ -2\eta\int_{C(2R)}[(u-w_{\beta,\varepsilon})^+]^2\,dx. \quad (10)$$

For R large we have that $C(\delta)/R^2 < \eta$ and therefore

$$\int_{C(2R)}|\nabla(u-w_{\beta,\varepsilon})^+|^2\phi_R^2\,dx \le 0.$$

By Fatou's Lemma for $R \to +\infty$, we obtain

$$\int_{\Sigma_{\bar\lambda}}|\nabla(u-w_{\beta,\varepsilon})^+|^2\,dx \le 0.$$

Exploiting (6) we deduce that actually

$$u \le w_{\beta,\varepsilon}, \quad \text{for all } \varepsilon > 0.$$

Finally, by continuity, we have that $u \le w_\beta$. Recalling (5) and (26) we get thesis. □

Without assuming any a priori assumption, we prove the following

Lemma 4 *There exists a constant $C = C(\gamma)$ such that*

$$u \ge C x_N^{\frac{2}{\gamma+1}} \quad \text{in } \mathbb{R}^N_+.$$

Proof Let us consider the first eigenfunction $\varphi_1 \in C^2(\overline{B_1(0)})$ solution to

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 & \text{in } B_1(0) \\ \varphi_1 > 0 & \text{in } B_1(0) \\ \varphi_1 = 0 & \text{on } \partial B_1(0). \end{cases} \quad (11)$$

Setting

$$w = C\varphi_1^{\frac{2}{\gamma+1}},$$

with $C > 0$ to be chosen, by a straightforward computations

$$\Delta w = \frac{2C(1-\gamma)}{(1+\gamma)^2}\varphi_1^{-\frac{2\gamma}{1+\gamma}}|\nabla\varphi_1|^2 + \frac{2C}{1+\gamma}\varphi_1^{\frac{1-\gamma}{1+\gamma}}\Delta\varphi_1.$$

Using (11), we obtain

$$
\begin{aligned}
-\Delta w &= \frac{2C(\gamma-1)}{(1+\gamma)^2}\varphi_1^{-\frac{2\gamma}{1+\gamma}}|\nabla\varphi_1|^2 + \frac{2C\lambda_1}{1+\gamma}\varphi_1^{\frac{2}{1+\gamma}} \\
&= \frac{1}{C^\gamma\varphi_1^{\frac{2\gamma}{\gamma+1}}}\left(\frac{2C^{\gamma+1}(\gamma-1)}{(\gamma+1)^2}|\nabla\varphi_1|^2 + \frac{2\lambda_1 C^{\gamma+1}}{\gamma+1}\varphi_1^2\right) \\
&:= \frac{\alpha(x)}{w^\gamma} \quad \text{in } B_1(0).
\end{aligned}
$$

For $C = C(\gamma)$ small enough, we get $\alpha(x) < 1$ and therefore w is a subsolution to $-\Delta w = w^{-\gamma}$ in $B_1(0)$.

Let now $x_0 = (x_0', x_{0,N}) \in \mathbb{R}^N_+$ and set

$$
w_{x_0,R} = R^{\frac{2}{\gamma+1}} w\left(\frac{x-x_0}{R}\right) \quad \text{in } B_R(x_0),
$$

where $R = x_{0,N}$. We have

$$
\begin{aligned}
-\Delta w_{x_0,R} &= -R^{-\frac{2\gamma}{\gamma+1}}\Delta w\left(\frac{x-x_0}{R}\right) \\
&\le \frac{1}{R^{\frac{2\gamma}{\gamma+1}} w^\gamma\left(\frac{x-x_0}{R}\right)} = \frac{1}{w^\gamma_{x_0,R}} \quad \text{in } B_R(x_0).
\end{aligned} \tag{12}
$$

Let u be a solution to $(\mathcal{P}_\gamma)$; we observe that

$$
\begin{cases}
-\Delta u = \frac{1}{u^\gamma} & \text{in } B_R(x_0) \\
-\Delta w_{x_0,R} \le \frac{1}{w^\gamma_{x_0,R}} & \text{in } B_R(x_0).
\end{cases} \tag{13}
$$

For $\varepsilon > 0$, we can use

$$
(w_{x_0,R} - u - \varepsilon)^+
$$

as a test function in (13) obtaining that

$$
w_{x_0,R} \le u + \varepsilon, \quad \text{for all } \varepsilon > 0.
$$

Then $u \ge w_{x_0,R}$ in $B_R(x_0)$, hence

$$
u(x_0) = R^{\frac{2}{\gamma+1}} w(0) = C(R\varphi_1(0))^{\frac{2}{\gamma+1}}.
$$

Recalling that $R = x_{0,N}$, since x_0 is arbitrary, we obtain the thesis. □

Proposition 5 *Under the assumption* **(hp)***, there exists a positive constant* $C = C(\gamma, \theta, N)$ *such that*

$$u(x) \leq C x_N$$

in the set $\mathbb{R}^N_+ \setminus \Sigma_{\bar{\lambda}}$.

Proof In what follows, without loss of generality, from (**hp**), using the natural scaling for the problem ($\mathcal{P}_\gamma$)

$$u_{\frac{\bar{\lambda}}{2}}(x) = \left(\frac{\bar{\lambda}}{2}\right)^{-\frac{2}{\gamma+1}} u\left(\frac{\bar{\lambda}}{2}x\right), \tag{14}$$

we may assume that our solution u is indeed bounded in the strip $\Sigma_{(0,2)}$.
Let $x_0 \in \mathbb{R}^+_N$, $x_0 = (x_0'; x_{0,N})$, with $x_{0,N} > 2$ and let R>0, such that

$$x_{0,N} = 4R. \tag{15}$$

Let $u_R(x) = u(x_0 + R(x - x_0))$; then

$$-\Delta u_R = R^2 \frac{1}{u_R^{\gamma+1}} u_R \quad \text{in } B_4(x_0), \tag{16}$$

$u_R > 0$ in $B_4(x_0)$. Since in Lemma 4 we showed that

$$u \geq C x_N^{\frac{2}{\gamma+1}},$$

in the whole $\mathbb{R}^N_+$, we infer that

$$u_R^{\gamma+1}(x) \geq 4C^{\gamma+1} R^2 \quad \text{in } B_2(x_0),$$

where C is the positive constant given in Lemma 4. Therefore

$$c(x) := \frac{R^2}{u_R^{\gamma+1}} \leq \frac{R^2}{4C^{\gamma+1} R^2} \quad \text{in } B_2(x_0).$$

We point out that, from the arbitrariness of x_0, we deduce that

$$c(x) \leq C(\gamma) \quad \text{in } \Sigma_{(\frac{5}{2}, \frac{11}{2})}.$$

Consequently from (16), we deduce that

$$-\Delta u_R = c(x) u_R \quad \text{in } B_2(x_0).$$

By Harnack inequality [5, Theorem 8.20] we have that

$$\sup_{B_1(x_0)} u_R \le C_H \inf_{B_1(x_0)} u_R, \tag{17}$$

where $C_H = C_H(\gamma, N)$. Now let us consider, for $N \ge 3$, the fundamental solution of the Laplace operator. So let us define

$$v_{c,k} = c\left(\frac{1}{|x-x_0|^{N-2}} + k\right)$$

that fulfills

$$\Delta v_{c,k} = 0 \quad \text{in } \mathbb{R}^N \backslash \{x_0\},$$

for all $c, k \in \mathbb{R}$. Exploiting (17) with $u_0 = u(x_0)$, we infer that

$$u_0 \le \sup_{B_R(x_0)} u = \sup_{B_1(x_0)} u_R \le C_H \inf_{B_1(x_0)} u_R = C_H \inf_{B_R(x_0)} u \le C_H u(x),$$

hence

$$u(x) \ge C_H^{-1} u_0 \quad \text{on } \partial B_R(x_0).$$

We new choose c amd k such that

$$\begin{cases} v_{c,k} = C_H^{-1} u_0 & \text{on } \partial B_{2R}(x_0) \\ v_{c,k} = 0 & \text{on } \partial B_{4R}(x_0). \end{cases} \tag{18}$$

Direct computation shows that the system (18) holds for

$$c = \frac{C_H^{-1} u_0 (4R)^{N-2}}{2^{N-2}-1} := \tilde{c}_N u_0 R^{N-2} \quad \text{and} \quad k = -\frac{1}{(4R)^{N-2}}, \tag{19}$$

with $\tilde{c}_N = C_H^{-1} 4^{N-2}/(2^{N-2}-1)$. Summarizing we have that

$$\begin{cases} -\Delta u = \dfrac{1}{u^\gamma} \ge 0 & \text{in } B_{4R}(x_0)\backslash B_{2R}(x_0) \\ -\Delta v_{c,k} = 0 & \text{in } B_{4R}(x_0)\backslash B_{2R}(x_0) \\ u, v_{c,k} > 0 & \text{in } B_{4R}(x_0)\backslash B_{2R}(x_0). \end{cases} \tag{20}$$

Using $(v_{c,k} - u - \varepsilon)^+$, for $\varepsilon > 0$, as test function in (20) (see also (18)), we get

$$\int_{B_{4R}(x_0)\backslash B_{2R}(x_0)} |\nabla (v_{c,k} - u - \varepsilon)^+|^2 \, dx \le 0,$$

namely $v_{c,k} \leq u + \varepsilon$, for all $\varepsilon > 0$. Therefore

$$u(x) \geq v_{c,k} \quad \text{in } B_{4R}(x_0) \setminus B_{2R}(x_0). \tag{21}$$

Therefore

$$\begin{aligned} u(x_0', 1) &\geq v_{c,k}(x_0', 1) \\ &= c\left(\frac{1}{|(x_0', 1) - (x_0', x_{0,N})|^{N-2}} + k\right) = c\left(\frac{1}{|1 - 4R|^{N-2}} + k\right) \\ &= \tilde{c}_N u_0 R^{N-2}\left(\frac{1}{(4R-1)^{N-2}} - \frac{1}{(4R)^{N-2}}\right), \end{aligned}$$

where in the last line we used (19). Finally by Lagrange theorem ve have

$$u(x_0', 1) \geq \tilde{c}_N \frac{(N-2)u_0}{4^{N-1}R}.$$

Therefore, since $u \in L^\infty(\Sigma_{(0,2)})$, we deduce

$$u(x_0) = u_0 \leq CR,$$

for some constant $C = C(\gamma, \bar{\lambda}, \theta, N)$ that does not depend on R. Since x_0 is arbitrary we obtain that

$$u(x) \leq CR \quad \text{in } \{x \in \mathbb{R}^N_+ : x_N > 2\}.$$

Scaling back, using (14) and (15) we obtain the thesis for $N \geq 3$. The case $N = 2$ follows repeating the same argument but replacing the fundamental solutions with the logarithmic one. □

It is straightforward to deduce the following

Corollary 6 *Under the assumption* **(hp)**, *u has linear growth, namely there exits $c_1, c_2 > 0$ depending on γ, θ, N such that*

$$u(x) \leq c_1 + c_2 x_N.$$

Proposition 7 *Under the assumption* **(hp)**, *there exists $C = C(\gamma, \theta, N)$ such that the following hold*

$$\begin{aligned} &(i) \quad |\nabla u| \leq C x_N^{\frac{1-\gamma}{\gamma+1}} \quad \textit{in } \Sigma_{\bar{\lambda}}, \\ &\textit{and} \\ &(ii) \ |\nabla u| \leq C \quad \textit{in } \mathbb{R}^N_+ \setminus \Sigma_{\bar{\lambda}}. \end{aligned}$$

Proof Let us start noticing that, without loss of generality, we may and do assume that the solution is bounded in the strip $\Sigma_{2\bar{\lambda}}$. Let now $P \in \Sigma_{\bar{\lambda}}$, with $P = (x', x_N)$. Set $R = x_N$ and let us define

$$u_R(x) = R^{-\frac{2}{\gamma+1}} u(Rx) \quad \text{in } B_{\frac{1}{2}}\left(\frac{P}{R}\right).$$

Then u_R satisfies

$$-\Delta u_R = \frac{1}{u_R^\gamma} \quad \text{in } B_{\frac{1}{2}}\left(\frac{P}{R}\right).$$

Exploiting Lemma 4, we deduce that

$$\frac{1}{u_R^\gamma} = \left(\frac{R^{\frac{2}{\gamma+1}}}{u(Rx)}\right)^\gamma \leq \frac{4^{\frac{\gamma}{\gamma+1}}}{C^\gamma}\left(\frac{R^{\frac{2}{\gamma+1}}}{R^{\frac{2}{\gamma+1}}}\right)^\gamma := C,$$

with $C = C(\gamma)$, i.e. $1/u_\delta^\gamma \in L^\infty(B_{1/2}(P/R))$. On the other hand Lemma 3 we also get

$$u_R(x) = R^{-\frac{2}{\gamma+1}} u(Rx) \leq C,$$

where $C = C(\gamma, \bar{\lambda})$. By regularity estimates, see e.g. [5, Theorem 3.9]

$$|\nabla u_R(x)| \leq C(\gamma, N) \quad \text{in } B_{\frac{1}{4}}\left(\frac{P}{R}\right).$$

Consequently we deduce

$$|\nabla u(Rx)| \leq C R^{\frac{1-\gamma}{\gamma+1}} \quad \text{in } B_{\frac{1}{4}}\left(\frac{P}{R}\right)$$

and hence

$$|\nabla u(x)| \leq C R^{\frac{1-\gamma}{\gamma+1}} \quad \text{in } B_{\frac{R}{4}}(P),$$

thus proving (i). Arguing now in the same way, let us define

$$u_R(x) = \frac{u(Rx)}{R} \quad \text{in } B_{\frac{1}{2}}\left(\frac{P}{R}\right).$$

By Proposition 5 we have that $u_R \leq C(\gamma, \theta, N)$ and it satisfies

$$-\Delta u_R = \frac{R^2}{R^{\gamma+1}} \frac{1}{u_R^\gamma} := h \quad \text{in } B_{\frac{1}{2}}\left(\frac{P}{R}\right),$$

where $h(x) \leq C(\gamma)$ in $B_{1/2}(P/R)$ (see Lemma 4). By regularity estimates $|\nabla u_R| \leq C$ in $B_{1/4}(P/R)$ and therefore $|\nabla u(x)| \leq C$ in $B_{R/4}(P)$. □

We are now ready to prove the $1-D$ symmetry result.

Theorem 8 *Let u be a solution to $(\mathcal{P}_\gamma)$. Under the assumption* (**hp**)

$$u(x) = u(x', x_N) = u(x_N) \quad in\ \mathbb{R}^N_+.$$

Proof Let $\tau, \sigma \in \mathbb{R}$, with $\sigma > 0$, chosen opportunely later. Define

$$u_{\tau,\sigma} = u(x + \tau e_i + \sigma e_N),$$

and for $i = 1, \ldots, N-1$. Obviously $-\Delta u_{\tau,\sigma} = 1/u^\gamma_{\tau,\sigma}$ in $\mathbb{R}^N_+$. Setting $z := u - u_{\tau,\sigma}$, we get

$$-\Delta z = \frac{1}{u^\gamma} - \frac{1}{u^\gamma_{\tau,\sigma}} \quad \text{in } \mathbb{R}^N_+. \tag{22}$$

In the following we use [1, Lemma 2.1]. From Lemmas 3 and 4 we infer that there exist constants C_1, C_2 such that

$$C_1 x_N^{\frac{2}{\gamma+1}} \leq u(x) \leq C_2 x_N^{\frac{2}{\gamma+1}} \quad \text{in } \Sigma_{\bar{\lambda}}. \tag{23}$$

For $\sigma > 0$, by (23) there exists $\rho > 0$ and $\hat{\lambda} < \bar{\lambda}$ (actually think to $\hat{\lambda} \approx 0$) such that $u < \rho$ in $\Sigma_{\hat{\lambda}}$ and $u_{\tau,\sigma} > 2\rho$ in $\Sigma_{\hat{\lambda}}$, for all $\tau \in \mathbb{R}$. Defining the strip $D := \mathbb{R}^N_+ \backslash \Sigma_{\hat{\lambda}}$, $z \leq 0$ on ∂D holds. Moreover using Lagrange theorem jointly to Proposition 7, we also get that z is bounded in $\overline{D}$.
Setting

$$c(x) := \left(\frac{1}{u^\gamma} - \frac{1}{u^\gamma_{\tau,\sigma}}\right) \frac{1}{u - u_{\tau,\sigma}},$$

we observe that $c(x)$ is continuous in $\overline{D}$ (indeed $u, u_{\tau,\sigma} \geq c > 0$ in D, see (23)) and $c(x) \leq 0$ (in D) by its own definition. By (22) applying [1, Lemma 2.1] to the problem

$$\begin{cases} \Delta z + c(x) z \geq 0 & \text{in } D \\ z \leq 0 & \text{on } \partial D, \end{cases}$$

we obtain $z := u - u_{\tau,\sigma} \leq 0$ in D. We point out that already in $\Sigma_{\hat{\lambda}} \cup \{x_n = \hat{\lambda}\}$, we have $u - u_{\tau,\sigma} \leq 0$. Hence $u \leq u_{\tau,\sigma}$ in $\mathbb{R}^N_+$.
Letting $\sigma \to 0$ we obtain

$$u \leq u_\tau \quad \text{for all } \tau \in \mathbb{R}.$$

By the arbitrariness of τ we deduce that $u = u(x_N)$. □

3 ODE analysis

We start with the study of the one dimensional problem. We consider the following

$$\begin{cases} -u'' = \dfrac{1}{u^\gamma} & \text{in } \mathbb{R}_+ \\ u(t) > 0 & \text{in } \mathbb{R}_+ \\ u(0) = 0. \end{cases} \tag{24}$$

It is straighforward to verify that the function

$$u(t) = C_\gamma \, t^{\frac{2}{\gamma+1}} \tag{25}$$

where

$$C_\gamma = \frac{(\gamma+1)^{\frac{2}{\gamma+1}}}{(2\gamma-2)^{\frac{1}{\gamma+1}}}, \tag{26}$$

is a solution of (24).

A scaling argument. Let $v \in C^2(\mathbb{R}_+) \cap C(\overline{\mathbb{R}_+})$ be a solution of problem (24). Let

$$\sigma(t) = v_{\alpha,\lambda}(t) := \lambda^\alpha v(\lambda t), \tag{27}$$

for a given $\lambda > 0$ and $\alpha \in \mathbb{R}$. Then $\sigma(0) = 0$, $\sigma(t) > 0$ and for $t > 0$

$$\sigma''(t) = -\frac{1}{\sigma(t)^\gamma} \lambda^{\alpha(1+\gamma)+2}.$$

Choosing $\alpha = -2/(1+\gamma)$, then σ satisfies (24) too. A similar computation showed that the same scaling works in the main problem $(\mathcal{P}_\gamma)$.

Let us define, by means of (25), the function

$$w(t) := C_\gamma \left(t + t^{\frac{2}{\gamma+1}} \right) = C_\gamma t + u(t)$$

and notice that, since $u(t) < w(t)$ for $t > 0$,

$$w''(t) = u''(t) = -u(t)^{-\gamma} < -w(t)^{-\gamma}.$$

Since $w(0) = u(0) = 0$, $w(t) > 0$ and

$$-w''(t) \geq \frac{1}{w(t)^\gamma} \quad t > 0,$$

then w is a supersolution for problem (24). Moreover $w'(t) \to C_\gamma$ as $t \to +\infty$.

Taking into account the supersolution w, let us fix $t_0 > 0$ and consider the following problem

$$\begin{cases} -v'' = \dfrac{1}{v^\gamma} & t > t_0 \\ v(t_0) > w(t_0) \\ v'(t_0) > w'(t_0). \end{cases} \tag{28}$$

Proposition 9 *Each solution of problem* (28) *is such that* $v(t) > w(t)$ *for* $t \geq t_0$ *and there exists (finite)* $\lim_{t\to\infty} v'(t) \geq C_\gamma$.

Proof A unique local solution for problem (28) there exists; indeed, it can be proved, that the solution is defined in the whole $[t_0, +\infty)$ since it is concave. Moreover

$$\left(v'(t_0) - w'(t_0)\right)' \geq \frac{1}{w(t_0)^\gamma} - \frac{1}{v(t_0)^\gamma}.$$

Since $\left(v'(t_0) - w'(t_0)\right)' > 0$, there exists $\delta > 0$ such that for all $t \in [t_0, t_0 + \delta)$, $\left(v'(t) - w'(t)\right)' > 0$. Actually $\left(v'(t) - w'(t)\right)' > 0$ for each $t > t_0$; if not, denoting by $\tau := \sup\{t > t_0 : \left(v'(t) - w'(t)\right)' > 0\}$, it follows that

$$0 = (v'(\tau) - w'(\tau))' \geq \frac{1}{w(\tau)^\gamma} - \frac{1}{v(\tau)^\gamma},$$

hence

$$\frac{1}{v(\tau)^\gamma} \geq \frac{1}{w(\tau)^\gamma}. \tag{29}$$

Since $(v' - w')$ is continuous and (strictly) increasing on the interval $[t_0, \tau)$, and $v(t_0) > w(t_0)$, therefore $v(\tau) > w(\tau)$. This contradict (29). As a consequence, $v(t) > w(t)$ for $t \geq t_0$.

The solution of (28) is positive on $[t_0, +\infty)$, therefore $v''(t)$ is negative on the same interval. This implies that $v'(t)$ is decreasing and its limit there exists for $t \to \infty$. Thus $\lim_{t\to\infty} v'(t) \geq \lim_{t\to\infty} w'(t) = C_\gamma$. □

Lemma 10 *For any* $L \in \mathbb{R}_+$, *there exists a solution* $\tilde{v}$ *for the problem*

$$\begin{cases} -v'' = \dfrac{1}{v^\gamma} & t > 0 \\ v(t) > 0 & t > 0 \\ v(0) = 0 & \lim_{t\to+\infty} v'(t) = L. \end{cases} \tag{30}$$

Proof We start proving that, choosing $v'(t_0) > v(t_0)/t_0$ in (28), then there exists $\tau_0 \in (0, t_0]$ such that $v(t)$ given in Proposition 9, can be extended as a solution of

$$\begin{cases} -v'' = \dfrac{1}{v^\gamma} & t > \tau_0 \\ v > 0 & t > \tau_0 \\ v(\tau_0) = 0. \end{cases}$$

Indeed each extension of $v(t)$ for $t < t_0$ is such that $v''(t) \le 0$ and therefore the graph of $v(t)$ lies below to the tangent line to $v(t)$ in $(t_0, v(t_0))$. Since $v'(t_0) > v(t_0)/t_0$, then a such $\tau_0 > 0$ exists.

Let $v_0(t)$ be a such solution, let us define $\tilde{v}(t) := v_0(t + \tau_0)$. Then $\tilde{v}(0) = 0$ and verifies (30). □

Theorem 11 *Let $M > 0$ be fixed. Then there exists a solution to*

$$\begin{cases} -w'' = \dfrac{1}{w^\gamma} & t > 0 \\ w(t) > 0 & t > 0 \\ w(0) = 0 & \lim_{t\to+\infty} w'(t) = M, \end{cases} \tag{31}$$

and the solution is unique.

Proof Let v be a solution of problem (30). Let

$$\lambda := \left(\frac{M}{L}\right)^{\frac{\gamma+1}{\gamma-1}},$$

where $L := \lim_{t\to+\infty} v'(t)$. By the scaling (27), we have

$$w(t) = \lambda^{-\frac{2}{\gamma+1}} v(\lambda t) = \left(\frac{M}{L}\right)^{-\frac{2}{\gamma-1}} v\left(\left(\frac{M}{L}\right)^{\frac{\gamma+1}{\gamma-1}} t\right)$$

is a solution of (31) and since $v'(t) \to L$ as $t \to +\infty$, $w'(t) \to M$ as $t \to +\infty$.

About the uniqueness, let us consider (by contradiction) w_1, w_2 two different solutions of (31). At first, let us assume that there exists $t_0 > 0$, the smallest value for which $w_1(t_0) = w_2(t_0)$. Taking into account the initial condition $w_1(0) = w_2(0) = 0$ and that w_1, w_2 are continuous, by the weak comparison principle it follows that $w_1(t) = w_2(t)$ on the interval $[0, t_0]$. Indeed, let us suppose without loss of generality, that $w_1 \le w_2$ in $[0, t_0]$; for any $\varepsilon > 0$, let $\varphi := (w_2 - w_1 - \varepsilon)$ be a test function for problems (30) and (31). So we have

$$\int_0^{t_0} |\nabla(w_2 - w_1 - \varepsilon)|^2 dx = \int_0^{t_0} \left(\frac{1}{w_2^\gamma} - \frac{1}{w_1^\gamma}\right)(w_2 - w_1 - \varepsilon)\, dx \le 0.$$

Then $w_1 = w_2 + \varepsilon$ in $[0, t_0]$ for all $\varepsilon > 0$, therefore $w_1 = w_2$ in $[0, t_0]$. As a rule $w_1(t_0) = w_2(t_0)$ and $w_1'(t_0) = w_2'(t_0)$ then $w_1 = w_2$ in $\mathbb{R}^+$ by uniqueness for ODEs (note that $w_1, w_2 > 0$ in $\mathbb{R}^+$ so that $-w'' = w^{-\gamma}$ is a regular ODE). Consequently, different solutions w_1 and w_2 do not cross.

From now on we may assume that $w_1 < w_2$ for all $t \in \mathbb{R}^+$. Notice that,

$$(w_1' - w_2')' = \frac{1}{w_2^\gamma} - \frac{1}{w_1^\gamma} < 0 \quad \text{in} \ \mathbb{R}^+.$$

Since $w_1'(t), w_2'(t) \to M$ as $t \to +\infty$, then $w_1'(t) - w_2'(t) > 0$ for all $t \in \mathbb{R}^+$ namely $w_1 - w_2$ should be increasing in $\mathbb{R}^+$ causing $w_1 = w_2$ in $\mathbb{R}^+$. □

4 Conclusion: proof of Theorem 1

Once that Theorem 8 is in force and therefore we know that

$$u(x) = u(x_N),$$

we get that u is a positive solution to

$$-u'' = \frac{1}{u^\gamma} \quad \text{in} \ \mathbb{R}^+,$$

with $u(0) = 0$. Therefore the ODEs analysis of Sect. 3 allows us to conclude that, either the solution is given by (25) or has linear growth and is completely classified by Theorem 11, taking into account the scaling in (27).

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Declarations

Conflict of interest The authors declare that they have no competing interest.

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