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Volume asymptotics, Margulis function and rigidity beyond nonpositive curvature

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Abstract

In this article, we consider a closed rank one C^∞ Riemannian manifold M without focal points and its universal cover X. Let $b_t(x)$ be the Riemannian volume of the ball of radius $t > 0$ around $x \in X$, and h the topological entropy of the geodesic flow. We obtain the following Margulis-type asymptotic estimates

$$\lim_{t\to\infty} b_t(x)/\frac{e^{ht}}{h} = c(x)$$

for some continuous function $c : X \to \mathbb{R}$. We prove that the Margulis function $c(x)$ is in fact C^1. The result also holds for a class of manifolds without conjugate points, including all surfaces of genus at least 2 without conjugate points. If M is a rank one surface without focal points, we show that $c(x)$ is constant if and only if M has constant negative curvature. We also obtain a rigidity result related to the flip invariance of the Patterson–Sullivan measure. These rigidity results are new even in the nonpositive curvature case.

Contents

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1 Introduction

Consider a closed C^∞ Riemannian manifold (M, g) with negative sectional curvature everywhere. It is well known that the geodesic flow defined on the unit tangent bundle SM is an Anosov flow (cf. [1], [25, Section 17.6]). The ergodic theory of Anosov flows has many striking applications in the study of asymptotic geometry of the universal cover X of M. In his celebrated 1970 thesis [36, 37], Margulis obtained the following result:

$$\lim_{t\to\infty} b_t(x)/\frac{e^{ht}}{h} = c(x), \tag{1}$$

where $b_t(x)$ is the Riemannian volume of the ball of radius $t > 0$ around $x \in X$, h the topological entropy of the geodesic flow, and $c : X \to \mathbb{R}$ is a continuous function, which is called *Margulis function*.

The main tool in the proof of Margulis's theorem is the Bowen–Margulis measure, which is the unique measure of maximal entropy (MME for short) for the Anosov flow (cf. [5]). Margulis [37] gave an explicit construction of this measure, and showed that it is mixing, and the conditional measures on stable/unstable manifolds have the scaling property, (i.e., contract/expand with a uniform rate under the geodesic flow), and are invariant under unstable/stable holonomies. Margulis [37] then proved (1) using these ergodic properties of the Bowen–Margulis measure.

The ergodic theory of the geodesic flow on a closed rank one manifold of nonpositive curvature was developed by Pesin [38, 39] in 1970s. In this case the geodesic flow exhibits nonuniformly hyperbolic behavior (cf. [4]). In 1985, A. Katok [6] conjectured that such geodesic flow also admits a unique MME. In 1998, Katok's conjecture was settled by Knieper [27]. In his proof, Knieper used Patterson–Sullivan measures on the boundary at infinity of the universal cover of M to construct a MME (called *Knieper measure*), and showed that this measure is the unique MME. Knieper [26] used his measure to obtain the following asymptotic estimates: there exists $C > 0$ such that

$$\frac{1}{C} \leq b_t(x)/e^{ht} \leq C$$

for any $x \in X$. However, it is difficult to improve the above to the Margulis-type asymptotic estimates (1), see the remark after [28, Chapter 5, Theorem 3.1]. An unpublished preprint [22] also contains many inspiring ideas to this problem. Recently, a breakthrough was made by Link [32, Theorem C], where an asymptotic estimate for the orbit counting function is obtained for a CAT(0) space, and as a consequence (1) is established for rank one manifolds of nonpositive curvature.

A twin problem is the asymptotics of the number of free-homotopy classes of closed geodesics. Margulis [36, 37] proved that in the negative curvature case

$$\lim_{t\to\infty} \#P(t)/\frac{e^{ht}}{ht} = 1 \tag{2}$$

where $P(t)$ is the set of free-homotopy classes containing a closed geodesic with length at most t. Recently Ricks [41] proved (2) for rank one locally CAT(0) spaces, which include rank one manifolds of nonpositive curvature. Later, Climenhaga et al. [11] proved (2) for a class of manifolds (including all surfaces of genus at least 2) without conjugate points, and the author [45] proved (2) for rank one manifolds without focal points.

2 Statement of main results

In this paper, we first establish volume asymptotics (1) for rank one manifolds without focal points. Then we study properties of the Margulis function and obtain related rigidity results. The proof of (1) for a large class of manifolds without conjugate points is explained in the Appendix.

Suppose that (M, g) is a C^∞ closed n-dimensional Riemannian manifold, where g is a Riemannian metric. Let $\pi : SM \to M$ be the unit tangent bundle over M. For each $v \in S_pM$, we always denote by $c_v : \mathbb{R} \to M$ the unique geodesic on M satisfying the initial conditions $c_v(0) = p$ and $\dot{c}_v(0) = v$. The geodesic flow $\phi = (\phi^t)_{t\in\mathbb{R}}$ (generated by the Riemannian metric g) on SM is defined as:

$$\phi^t : SM \to SM, \quad (p, v) \mapsto (c_v(t), \dot{c}_v(t)), \qquad \forall\, t \in \mathbb{R}.$$

A vector field $J(t)$ along a geodesic $c : \mathbb{R} \to M$ is called a *Jacobi field* if it satisfies the *Jacobi equation*:

$$J'' + R(J, \dot{c})\dot{c} = 0$$

where R is the Riemannian curvature tensor and $'$ denotes the covariant derivative along c.

A Jacobi field $J(t)$ along a geodesic $c(t)$ is called *parallel* if $J'(t) = 0$ for all $t \in \mathbb{R}$. The notion of *rank* is defined as follows.

Definition 2.1 For each $v \in SM$, we define rank(v) to be the dimension of the vector space of parallel Jacobi fields along the geodesic c_v, and rank(M):=min{rank(v) : $v \in SM$}. For a geodesic c we define

$$\text{rank}(c) := \text{rank}(\dot{c}(t)), \quad \forall\, t \in \mathbb{R}.$$

Definition 2.2 Let c be a geodesic on (M, g).

(1) A pair of distinct points $p = c(t_1)$ and $q = c(t_2)$ are called *focal* if there is a Jacobi field J along c such that $J(t_1) = 0$, $J'(t_1) \neq 0$ and $\frac{d}{dt} |_{t=t_2} \|J(t)\|^2 = 0$;
(2) $p = c(t_1)$ and $q = c(t_2)$ are called *conjugate* if there is a nontrivial Jacobi field J along c such that $J(t_1) = 0 = J(t_2)$.

A compact Riemannian manifold (M, g) is called a manifold *without focal points/without conjugate points* if there is no focal points/conjugate points on any geodesic in (M, g).

By definition, if a manifold has no focal points then it has no conjugate points. All manifolds of nonpositive curvature always have no focal points.

2.1 Volume estimates of Margulis type

Let M be a rank one closed Riemannian manifold without focal points. Then SM splits into two invariant subsets under the geodesic flow: the regular set Reg $:= \{v \in SM : \text{rank}(v) = 1\}$, and the singular set Sing $:= SM \setminus$ Reg. The uniqueness of MME for geodesic flows on SM is obtained in [8, 9, 34].

We have the following Margulis-type asymptotic estimates:

Theorem A *Let M be a rank one closed Riemannian manifold without focal points, and X the universal cover of M. Then*

$$\lim_{t\to\infty} b_t(x)/\frac{e^{ht}}{h} = c(x),$$

where $b_t(x)$ is the Riemannian volume of the ball of radius $t > 0$ around $x \in X$, h the topological entropy of the geodesic flow, and $c : X \to \mathbb{R}$ is a continuous function.

In [32, Theorem C], Margulis-type asymptotic estimates for the orbit counting function was obtained for rank one CAT(0) spaces, including rank one manifolds of nonpositive curvature. Theorem A generalizes the formula from rank one manifolds of nonpositive curvature to those without focal points.

Link's proof is quite geometric, and based on Roblin's method in negative curvature [42]. Our proof in this paper is much different, using Margulis' approach which is more dynamical. We use the notion of local product flow box and apply π-convergence theorem introduced by Ricks [41] (see Sects. 3.4 and 4 below for more details). First we will establish the asymptotics formula for a pair of flow boxes (Sects. 5.1–5.4) using the mixing property of the unique MME and scaling property of Patterson–Sullivan measure. The asymptotics in (1) involves countably many pairs of flow boxes and an issue of nonuniformity arises. To overcome this difficulty, we will apply Knieper's results and techniques (Lemmas 5.13 and 5.14).

In [11, 41, 45], the authors also use flow boxes to calculate the asymptotic growth of the number of free homotopy classes of closed geodesics. Due to the equidistribution of closed geodesics, we only need consider one flow box and count the number of self-intersections of the box under the geodesic flow. The technical novelties in this paper is that we have to deal with countably many pairs of flow boxes and the essential difficulty caused by nonuniform hyperbolicity. Even for one pair of flow boxes, the calculations involved are more complicated. Moreover we also consider the intersection components of a flow box with a face under the geodesic flow (see Sect. 5.1 below), which makes the calculation more subtle.

The above method can also be adapted to a certain class of manifolds without conjugate points. In [10], the authors proved the uniqueness of MME for the *class* $\mathcal{H}$ of manifolds without conjugate points that satisfy:

(1) There exists a Riemannian metric g_0 on M for which all sectional curvatures are negative;
(2) The uniform visibility axiom (see Definition 3.1 below) is satisfied;
(3) The fundamental group $\pi_1(M)$ is residually finite: the intersection of its finite index subgroups is trivial;
(4) There exists $h_0 < h$ such that any ergodic invariant Borel probability measure μ on SM with entropy strictly greater than h_0 is almost expansive (cf. [11, Definition 2.8]) .

All surfaces of genus at least 2 without conjugate points belong to the class $\mathcal{H}$. The asymptotic formula of Margulis type for counting closed geodesic is obtained in [11].

Theorem A′ *Let M be a closed manifold without conjugate points belonging to the class $\mathcal{H}$, and X the universal cover of M. Then*

$$\lim_{t\to\infty} b_t(x)/\frac{e^{ht}}{h} = c(x),$$

where $b_t(x)$ is the Riemannian volume of the ball of radius $t > 0$ around $x \in X$, h the topological entropy of the geodesic flow, and $c : X \to \mathbb{R}$ is a continuous function.

We discuss the proof of Theorem A' in the Appendix.

2.2 Margulis function and rigidity

Let M be a rank one closed Riemannian manifold without focal points. The continuous function $c(x)$ is called *Margulis function*. It is easy to see that

$$\lim_{t\to\infty} s_t(x)/e^{ht} = c(x), \tag{3}$$

where $s_t(x)$ is the spherical volume of the sphere $S(x, t)$ around $x \in X$ of radius $t > 0$. It descends to a function on M, which we still denote by c.

We study some rigidity results related to Margulis function for manifolds without focal points, which are new even in the nonpositive curvature case.

In the negative curvature case, Katok conjectured that $c(x)$ is almost always not constant and not smooth. In [49] Yue answered Katok's conjecture. He studied the uniqueness of harmonic measures associated to the strong stable foliation of the geodesic flow and obtained some rigidity results involving the Margulis function. We extend Yue's results to rank one manifolds without focal points.

Theorem B *Let M be a rank one closed C^∞ Riemannian manifold without focal points, X the universal cover of M. Then*

(1) *The Margulis function c is a C^1 function.*
(2) *If $c(x) \equiv C$, then for any $x \in X$,*

$$h = \int_{\partial X} trU(x, \xi) d\tilde{\mu}_x(\xi)$$

where $U(x, \xi)$ and $trU(x, \xi)$ are the second fundamental form and the mean curvature of the horosphere $H_x(\xi)$ respectively, and $\tilde{\mu}_x$ is the normalized Patterson–Sullivan measure.

To the best of our knowledge, the uniqueness of harmonic measures in nonpositive curvature (and hence in the no focal points case) is not known.

Question 2.3 *For manifolds of nonpositive curvature, do we have a unique harmonic measure associated to the strong stable foliation of the geodesic flow? Do the leaves of the strong stable foliation have polynomial growth?*

Remark 2.4 For rank one manifolds without focal points, the uniqueness of harmonic measures associated to the weak stable foliation of the geodesic flow is proved in [31, Theorem 3.1].

If $\dim M = 2$, then $\mathrm{Vol}(B^s(x, r)) = 2r$ where $B^s(x, r)$ is any ball of radius r in a strong stable manifold. In this case, $B^s(x, r)$ is just a curve. Hence the leaves of the strong stable foliation have polynomial growth. Combining with a recent result in [13] on the unique ergodicity of the horocycle flow, we can show that there is a unique harmonic measure associated to the strong stable foliation for rank one surfaces without focal points. Then we can prove the following rigidity result.

Theorem C *Let M be a rank one closed Riemannian surface without focal points. Then $c(x) \equiv C$ if and only if M has constant negative curvature.*

Without the uniqueness of harmonic measures, we can still obtain some rigidity results in arbitrary dimension. The flip map $F : SM \to SM$ is defined as $F(v) := -v$. By the construction, the Knieper measure $\underline{m}$ is flip invariant. Consider the conditional measures $\{\bar{\mu}_x\}_{x\in M}$ of $\underline{m}$ with respect to the partition $SM = \cup_{x\in M} S_x M$. $\bar{\mu}_x$ can be identified as measures on ∂X, and it would be natural to consider if $\bar{\mu}_x$ and the normalized Patterson–Sullivan measures $\tilde{\mu}_x$ coincide. Yue [47, 48] obtained related rigidity results in negative curvature, which can be generalized to the no focal points case with the help of the Margulis function.

Theorem D *Let M be a rank one closed C^∞ Riemannian manifold without focal points. The conditional measures $\{\bar{\mu}_x\}_{x\in M}$ of the Knieper measure coincide almost everywhere with the normalized Patterson–Sullivan measures $\tilde{\mu}_x$ if and only if M is locally symmetric.*

If M is a rank one locally symmetric space, then obviously $c(x) \equiv C, x \in M$. It is conjectured in [49, p. 179] that in negative curvature $c(x) \equiv C, x \in M$ if and only if M is locally symmetric, and this is true in dimension two [49, Theorem 4.3]. Theorem C verifies the conjecture in dimension two for the more general case of no focal points. Theorem D gives a new characterization of rank one locally symmetric spaces among rank one closed manifolds without focal points.

3 Geometric and ergodic toolbox

We prepare some geometric and ergodic tools for rank one manifolds without focal points, which will be used in subsequent sections.

3.1 Boundary at infinity

Let M be a closed Riemannian manifold without focal points, and $\mathrm{pr} : X \to M$ the universal cover of M. Let $\Gamma \simeq \pi_1(M)$ be the group of deck transformations on X, so that each $\gamma \in \Gamma$ acts isometrically on X. Let $\mathcal{F}$ be a fundamental domain with respect to Γ. Denote by d both the distance functions on M and X induced by Riemannian metrics. The Sasaki metrics on SM and SX are also denoted by d if there is no confusion.

We still denote by $\mathrm{pr} : SX \to SM$ and $\gamma : SX \to SX$ the map on unit tangent bundles induced by pr and $\gamma \in \Gamma$. From now on, we use an underline to denote objects in M and SM, e.g. for a geodesic c in X and $v \in SX$, $\underline{c} := \mathrm{pr}c$, $\underline{v} := \mathrm{pr}v$ denote their projections to M and SM respectively.

We call two geodesics c_1 and c_2 on X *positively asymptotic* or just *asymptotic* if there is a positive number $C > 0$ such that $d(c_1(t), c_2(t)) \leq C, \quad \forall\, t \geq 0$. The relation of asymptoticity is an equivalence relation between geodesics on X. The class of geodesics that are asymptotic to a given geodesic c_v/c_{-v} is denoted by $c_v(+\infty)/c_v(-\infty)$ or v^+/v^- respectively. We call them *points at infinity*. Obviously, $c_v(-\infty) = c_{-v}(+\infty)$. We call the set ∂X of all points at infinity the *boundary at infinity*. If $\eta = v^+ \in \partial X$, we say v *points at* η.

We can define the visual topology on ∂X following [16, 17]. For each p, there is a bijection $f_p : S_pX \to \partial X$ defined by

$$f_p(v) = v^+, \quad v \in S_pX.$$

So for each $p \in M$, f_p induces a topology on ∂X from the usual topology on S_pX. The topology on ∂X induced by f_p is independent of $p \in X$, and is called the *visual topology* on ∂X.

Visual topology on ∂X and the manifold topology on X can be extended naturally to the so-called *cone topology* on $\overline{X} := X \cup \partial X$.

Under cone topology, $\overline{X}$ is homeomorphic to the closed unit ball in $\mathbb{R}^n$, and ∂X is homeomorphic to the unit sphere $\mathbb{S}^{n-1}$. For $x, y \in \overline{X}$, we denote by $c_{x,y}$ the geodesic connection x and y if it exists.

The *angle metric* on ∂X is defined as

$$\angle(\xi, \eta) := \sup_{x \in X} \angle_x(\xi, \eta), \quad \forall \xi, \eta \in \partial X,$$

where $\angle_x(\xi, \eta)$ denotes the angle between the unit tangent vectors at x of the geodesics $c_{x,\xi}$ and $c_{x,\eta}$. Then the angle metric defines a path metric d_T on ∂X, called the *Tits metric*. More precisely, for a continuous curve $c : [0, 1] \to \partial X$, define the length $L(c) := \sup \sum_{i=0}^{k-1} \angle(c(t_i), c(t_{i+1}))$ where the supremum is over all subdivisions $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ of $[0, 1]$. Then we can define the path metric $d_T(\xi, \eta) := \inf L(c)$, where the infimum is taken over all the continuous curves joining ξ and η. Clearly, $d_T \geq \angle$. The angle metric induces a topology on ∂X finer than the visual topology. Let c be a recurrent geodesic, not the boundary of a flat half plane, then by [45, Proposition 3.7], $d_T(c(-\infty), c(\infty)) > \pi$. If M has negative curvature everywhere, then $\angle(\xi, \eta) = \pi$ and hence $d_T(\xi, \eta) = \infty$ for any $\xi \neq \eta \in \partial X$. See [3, 45] for more information on Tits metric.

Definition 3.1 (Cf. [11, Definition 2.1]) A simply connected Riemannian manifold X is a *uniform visibility manifold* if for every $\epsilon > 0$ there exists $L = L(\epsilon) > 0$ such that whenever a geodesic $c : [a, b] \to X$ stays at distance at least L from some point $p \in X$, then the angle sustained by c at p is less than ϵ, that is,

$$\angle_p(c) := \sup_{a \leq s, t \leq b} \angle_p(c(s), c(t)) < \epsilon.$$

If M is a Riemannian manifold without conjugate points whose universal cover X is a uniform visibility manifold, then we say that M is a uniform visibility manifold.

Definition 3.2 (Cf. [11, Definition 2.2]) The manifold (M, g) has the *divergence property* if given any geodesics $c_1 \neq c_2$ with $c_1(0) = c_2(0)$ in the universal cover, we have $\lim_{t\to\infty} d(c_1(t), c_2(t)) = \infty$.

The uniform visibility property implies the divergence property. All manifolds without focal points have the divergence property.

3.2 Busemann function

For each pair of points $(p, q) \in X \times X$ and each point at infinity $\xi \in \partial X$, the *Busemann function based at* ξ *and normalized by* p is

$$b_\xi(q, p) := \lim_{t \to +\infty} \big(d(q, c_{p,\xi}(t)) - t\big),$$

where $c_{p,\xi}$ is the unique geodesic from p and pointing at ξ. The Busemann function $b_\xi(q, p)$ is well-defined since the function $t \mapsto d(q, c_{p,\xi}(t)) - t$ is bounded from above by $d(p, q)$, and decreasing in t (this can be checked by using the triangle inequality). Obviously, we have

$$|b_\xi(q, p)| \le d(q, p).$$

If $v \in S_p X$ points at $\xi \in \partial X$, we also write $b_v(q) := b_\xi(q, p)$.

The level sets of the Busemann function $b_\xi(q, p)$ are called the *horospheres* centered at ξ. The horosphere through p based at $\xi \in \partial X$, is denoted by $H_p(\xi)$. For more details of the Busemann functions and horospheres, please see [15, 43, 44].

According to [39, Theorem 6.1] and [43, Lemma 1.2], we have the following continuity property of Busemann functions.

Lemma 3.3 (Cf. [45, Corollary 2.7]) *The functions* $(v, q) \mapsto b_v(q)$ *and* $(\xi, p, q) \mapsto b_\xi(p, q)$ *are continuous on* $SX \times X$ *and* $\partial X \times X \times X$ *respectively.*

In fact, we have the following equicontinuity property of Busemann function $v \mapsto b_v(q)$.

Lemma 3.4 (Cf. [45, Lemma 2.9]) *Let* $p \in X$, $A \subset S_p X$ *be closed, and* $B \subset X$ *be such that* $A^+ := \{v^+ : v \in A\}$ *and* $B^\infty := \{\lim_n q_n \in \partial X : q_n \in B\}$ *are disjoint subsets of* ∂X. *Then the family of functions* $A \to \mathbb{R}$ *indexed by* B *and given by* $v \mapsto b_v(q)$ $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $\angle_p(v, w) < \delta$, *then* $|b_v(q) - b_w(q)| < \epsilon$ *for every* $q \in B$.

3.3 Patterson–Sullivan measure and Knieper measure

We will recall the construction of the Patterson–Sullivan measure and the Knieper measure, which are the main tools to the subsequent proofs.

Definition 3.5 Let X be a simply connected manifold without focal points and Γ a discrete subgroup of $\mathrm{Iso}(X)$, the group of isometries of X. For a given constant $r > 0$, a family of finite Borel measures $\{\mu_p\}_{p \in X}$ on ∂X is called an r-dimensional *Busemann density* if

(1) for any $p,\ q \in X$ and μ_p-a.e. $\xi \in \partial X$,

$$\frac{d\mu_q}{d\mu_p}(\xi) = e^{-r \cdot b_\xi(q,p)}$$

where $b_\xi(q, p)$ is the Busemann function;

(2) $\{\mu_p\}_{p\in X}$ is Γ-equivariant, i.e., for all Borel sets $A \subset \partial X$ and for any $\gamma \in \Gamma$, we have

$$\mu_{\gamma p}(\gamma A) = \mu_p(A).$$

Extending the techniques in [27] to manifolds without focal points, we constructed Busemann density via Patterson–Sullivan construction and showed in [34, Theorem B] that up to a multiplicative constant, the Busemann density is unique, i.e., the Patterson–Sullivan measure is the unique Busemann density.

The following Shadowing Lemma is one of the most crucial properties of the Patterson–Sullivan measure.

Lemma 3.6 (Shadowing Lemma, cf. [34, Proposition 15]) *Let $\{\mu_p\}_{p\in X}$ be the Patterson–Sullivan measure, which is the unique Busemann density with dimension h. Then there exists $R > 0$ such that for any $\rho \geq R$ and any $p, x \in X$ there is $b = b(\rho)$ with*

$$b^{-1}e^{-hd(p,x)} \leq \mu_p(\bar{f}_p(B(x, \rho))) \leq be^{-hd(p,x)}$$

where $\bar{f}_p(y) := c_{p,y}(+\infty)$ for any $y \in B(x, \rho)$.

Let $P : SX \to \partial X \times \partial X$ be the projection given by $P(v) = (v^-, v^+)$. Denote by $\mathcal{I}^P := P(SX) = \{P(v) \mid v \in SX\}$ the subset of pairs in $\partial X \times \partial X$ which can be connected by a geodesic. Note that the connecting geodesic may not be unique and moreover, not every pair $\xi \neq \eta$ in ∂X can be connected by a geodesic.

Fix a point $p \in X$, we can define a Γ-invariant measure $\overline{\mu}$ on $\mathcal{I}^P$ by the following formula:

$$d\overline{\mu}(\xi, \eta) := e^{h\cdot\beta_p(\xi,\eta)}d\mu_p(\xi)d\mu_p(\eta),$$

where $\beta_p(\xi, \eta) := -\{b_\xi(q, p) + b_\eta(q, p)\}$ is the Gromov product, and q is any point on a geodesic c connecting ξ and η. By [34, Propositions 6 and 7] (see also Proposition 3.7 below), $\overline{\mu}(\mathcal{I}^P) > 0$. It is easy to see that the function $\beta_p(\xi, \eta)$ does not depend on the choice of c and q. In geometric language, the Gromov product $\beta_p(\xi, \eta)$ is the length of the part of a geodesic c between the horospheres $H_\xi(p)$ and $H_\eta(p)$.

Then $\overline{\mu}$ induces a ϕ^t-invariant measure m on SX with

$$m(A) = \int_{\mathcal{I}^P} \mathrm{Vol}\{\pi(P^{-1}(\xi, \eta) \cap A)\}d\overline{\mu}(\xi, \eta),$$

for all Borel sets $A \subset SX$. Here $\pi : SX \to X$ is the standard projection map and Vol is the induced volume form on $\pi(P^{-1}(\xi, \eta))$. If there are more than one geodesics connecting ξ and η, then by the flat strip theorem, $\pi(P^{-1}(\xi, \eta))$ is exactly a k-flat submanifold connecting ξ and η for some $k \geq 2$, which consists of all the geodesics connecting ξ and η.

For any Borel set $A \subset SX$ and $t \in \mathbb{R}$, $\mathrm{Vol}\{\pi(P^{-1}(\xi,\eta) \cap \phi^t A)\} = \mathrm{Vol}\{\pi(P^{-1}(\xi,\eta) \cap A)\}$. Therefore m is ϕ^t-invariant. Moreover, Γ-invariance of $\overline{\mu}$ leads to the Γ-invariance of m. Thus m induced a ϕ^t-invariant measure $\underline{m}$ on SM which is determined by

$$m(A) = \int_{SM} \#(\mathrm{pr}^{-1}(\underline{v}) \cap A) d\underline{m}(\underline{v}).$$

It is proved in [34] that $\underline{m}$ is unique MME, which is called Knieper measure. Furthermore, $\underline{m}$ is proved to be mixing in [2, 33], Kolmogorov in [7, 9] and eventually Bernoulli in [7, 45].

3.4 Local product flow boxes

In this subsection, we fix a regular vector $v_0 \in SX$. Let $p := \pi(v_0)$, which will be the reference point in the following discussions. We also fix a scale $\epsilon \in (0, \min\{\frac{1}{8}, \frac{\mathrm{inj}(M)}{4}\})$.

The *Hopf map* $H : SX \to \partial X \times \partial X \times \mathbb{R}$ for $p \in X$ is defined as

$$H(v) := (v^-, v^+, s(v)), \quad \text{where } s(v) := b_{v^-}(\pi v, p).$$

From definition, we see $s(\phi^t v) = s(v) + t$ for any $v \in SX$ and $t \in \mathbb{R}$. s is continuous by Lemma 3.3.

Following [11, 45], we define for each $\theta > 0$ and $0 < \alpha < \frac{3}{2}\epsilon$,

$$\begin{aligned}
\mathbf{P}_\theta &:= \{w^- : w \in S_pX \text{ and } \angle_p(w, v_0) \le \theta\},\\
\mathbf{F}_\theta &:= \{w^+ : w \in S_pX \text{ and } \angle_p(w, v_0) \le \theta\},\\
B_\theta^\alpha &:= H^{-1}(\mathbf{P}_\theta \times \mathbf{F}_\theta \times [-\alpha, \alpha]).
\end{aligned}$$

B_θ^α is called a *flow box* with depth α. We will consider $\theta > 0$ small enough, which will be specified in the following.

The following lemma was crucial in constructing Knieper measure.

Proposition 3.7 [34, Propositions 6 and 7] *Let X be a simply connected manifold without focal points and $v_0 \in SX$ is regular. Then for any $\epsilon > 0$, there is an $\theta_1 > 0$ such that, for any $\xi \in \boldsymbol{P}_{\theta_1}$ and $\eta \in \boldsymbol{F}_{\theta_1}$, there is a unique geodesic $c_{\xi,\eta}$ connecting ξ and η, i.e., $c_{\xi,\eta}(-\infty) = \xi$ and $c_{\xi,\eta}(+\infty) = \eta$.*

Moreover, the geodesic $c_{\xi,\eta}$ is regular and $d(c_v(0), c_{\xi,\eta}) < \epsilon/10$.

Based on Proposition 3.7, we have the following result.

Lemma 3.8 [45, Lemma 2.13] *Let v_0, p, ϵ be as above and θ_1 be given in Proposition 3.7. Then for any $0 < \theta < \theta_1$,*

(1) *diam* $\pi H^{-1}(\boldsymbol{P}_\theta \times \boldsymbol{F}_\theta \times \{0\}) < \frac{\epsilon}{2}$;
(2) $H^{-1}(\boldsymbol{P}_\theta \times \boldsymbol{F}_\theta \times \{0\}) \subset SX$ *is compact*;

(3) *diam* $\pi B_\theta^\alpha < 4\epsilon$ *for any* $0 < \alpha \le \frac{3\epsilon}{2}$.

The following is a direct corollary of Lemma 3.4.

Corollary 3.9 *Given* $v_0, p, \epsilon > 0$ *as above, there exists* $\theta_2 > 0$ *such that for any* $0 < \theta < \theta_2$, *if* $\xi, \eta \in \boldsymbol{P}_\theta$ *and any* q *lying within* $\mathrm{diam}\mathcal{F} + 4\epsilon$ *of* $\pi H^{-1}(\boldsymbol{P}_\theta \times \boldsymbol{F}_\theta \times [0, \infty))$, *we have* $|b_\xi(q, p) - b_\eta(q, p)| < \epsilon^2$. *Similar result holds if the roles of* $\boldsymbol{P}_\theta$ *and* $\boldsymbol{F}_\theta$ *are reversed.*

Let $\theta_0 := \min\{\theta_1, \theta_2\}$, where θ_1 is given in Lemma 3.8, and θ_2 is given in Corollary 3.9. In the following, we always suppose that $0 < \theta < \theta_0$.

3.5 Regular partition-cover

Let us fix $x, y \in \mathcal{F} \subset X$, and $p = x$ the reference point. For each regular vector $w \in S_x X$, we can construct a local product flow box around w as in the last subsection. More precisely, consider the interior of $B_\theta^\alpha(w)$, $\mathrm{int}B_\theta^\alpha(w)$, which is an open neighborhood of w for some $\alpha > 0$ and $0 < \theta < \theta_0$ (here θ_0 depends on w). By second countability of $S_x X$, there exist countably many regular vectors $w_1, w_2, \ldots$ such that $S_x X \cap \mathrm{Reg} \subset \cup_{i=1}^\infty \mathrm{int}B_{\theta_i}^\alpha(w_i)$. Similarly, there exist countably many regular vectors $v_1, v_2, \ldots$ such that $S_y X \cap \mathrm{Reg} \subset \cup_{i=1}^\infty \mathrm{int}B_{\theta_i'}^\alpha(v_i)$. We note that the reference point p is always chosen to be x in the construction of all above flow boxes.

A *regular partition-cover* of $S_x X$ is a triple $(\{w_i\}, \{\mathrm{int}B_{\theta_i}^\alpha(w_i)\}, \{N_i\})$ where $\{N_i\}$ is a disjoint partition of $\mathrm{Reg} \cap S_x X$ and such that $N_i \subset \mathrm{int}B_{\theta_i}^\alpha(w_i)$ for each $i \in \mathbb{N}$. Similarly a regular partition-cover of $S_y X$ is a triple $(\{v_i\}, \{\mathrm{int}B_{\theta_i'}^\alpha(v_i)\}, \{V_i\})$ such that $\{V_i\}$ is a disjoint partition of $\mathrm{Reg} \cap S_y X$ and $V_i \subset \mathrm{int}B_{\theta_i'}^\alpha(v_i)$ for each $i \in \mathbb{N}$.

Recall the bijection $f_x : S_x X \to \partial X$ defined by $f_x(v) = v^+$, $v \in S_x X$. Let $\tilde{\mu}_x := (f_x^{-1})_* \mu_p$ which is a finite Borel measure on $S_x X$. Similarly, let $\tilde{\mu}_y := (f_y^{-1})_* \mu_p$ be a finite Borel measure supported on $S_y X$. The measures $\tilde{\mu}_x$ and $\tilde{\mu}_y$ will be used in Sect. 5.

The following result is essentially proved in [12, Proposition 2.4] in nonpositive curvature.

Lemma 3.10 *For any* $x \in X$, *we have* $\tilde{\mu}_x(\mathrm{Reg} \cap S_x X) = \tilde{\mu}_x(S_x X)$.

Proof Define $\mathrm{Sing}^+ := \{v^+ : v \in \mathrm{Sing}\}$. By the same proof of [26, Proposition 4.9], we see that Γ acts ergodically on ∂X. Since Sing^+ is Γ-invariant, either $\mu_x(\mathrm{Sing}^+) = \mu_x(\partial X)$ or $\mu_x(\mathrm{Sing}^+) = 0$ for any $x \in X$.

Let $\mathrm{Rec} \subset SM$ be the subset of vectors recurrent under the geodesic flow. Then its lift to SX, which is also denoted by Rec, has full m-measure. By [34], Reg also has full m-measure, and thus $\mathrm{Rec}\cap\mathrm{Reg}$ has full m-measure. Define $R := \{v^+ : v \in \mathrm{Rec}\cap\mathrm{Reg}\}$. By definition of the Knieper measure, we see that R has full μ_x-measure.

Let $v \in \mathrm{Rec} \cap \mathrm{Reg}$. By [34, Lemma 5.1], for every

$$w \in W^s(v) := \{w \in SX : w = -\mathrm{grad}\, b_\xi(q, \pi v), b_\xi(q, \pi v) = 0\},$$

i.e., every vector w on the strong stable horocycle manifold of v, we have $d(\phi^t v, \phi^t w) \to 0$ as $t \to +\infty$. Since v is recurrent and regular, then w is also regular. It follows that $R \cap \mathrm{Sing}^+ = \emptyset$ and so $\mu_x(\mathrm{Sing}^+) = 0$. The lemma then follows. □

From now on in Sects. 4 and 5, we will first consider a pair of V_i and N_j from the regular partition-covers of $S_x X$ and $S_y X$ respectively. At last, we will sum up our estimates over countably many such pairs from regular partition-covers.

4 Local uniform expansion

In this section, we use π-convergence theorem to illustrate local uniform expansion along unstable horospheres. As a consequence, we obtain estimates on the cardinality of certain subgroups of Γ.

Theorem 4.1 (Weak π-convergence theorem, [45, Theorem 3.9]) *Let X be a simply connected manifold without focal points, v_0, p, ϵ be fixed as in Sect. 3.4, and θ_1 be given in Proposition 3.7. Fix any $0 < \rho < \theta < \theta_1$.*

Suppose that $x \in X$, and $\gamma_i \in \Gamma$ such that $\gamma_i(x) \to p \in \boldsymbol{F}_\rho$ and $\gamma_i^{-1}(x) \to n \in \boldsymbol{P}_\rho$ as $i \to \infty$. Then for any open set U with $U \supset \boldsymbol{F}_\rho$, $\gamma_i(\boldsymbol{F}_\theta) \subset U$ for all i sufficiently large.

The above theorem is proved in [45] where one only needs to consider one flow box to get a closing lemma. However, in our setting we need consider two flow boxes each time, and so we need apply a modified version of the theorem. This will be explained later.

Consider a pair of V_i and N_j from the regular partition-covers of $S_x X$ and $S_y X$ respectively. For simplicity, we just denote $V := V_i$ and $N := N_j$. Then $N \subset \mathrm{int} B^{\alpha}_{\theta_j}(w_0)$ and $V \subset \mathrm{int} B^{\beta}_{\theta'_i}(v_0)$ for some $w_0 \in S_x X$ and $v_0 \in S_y X$. We also denote for $a > 0$,

$$B_a N := \{v \in S_x X : d(v, N) \le a\}$$
$$B_{-a} N := \{v \in N : B(v, a) \subset N\}.$$

Write $t_0 := s(v_0) = b_{v_0^-}(\pi v_0, p)$ where $p = x$. We denote the flow boxes by

$$N^\alpha := H^{-1}(N^- \times N^+ \times [-\alpha, \alpha]),$$
$$V^\beta := H^{-1}(V^- \times V^+ \times (t_0 + [-\beta, \beta])).$$

Notice that $N^\alpha \subset \mathrm{int} B^{\alpha}_{\theta_j}(w_0)$ and $V^\beta \subset \mathrm{int} B^{\beta}_{\theta'_i}(v_0)$. Given $\epsilon > 0$, we always consider $\frac{\epsilon^2}{100} \le \alpha, \beta \le \frac{3\epsilon}{2}$. By carefully adjusting the regular partition-covers, we can guarantee that

$$\mu_p(\partial V^+) = \mu_p(\partial V^-) = \mu_p(\partial N^+) = \mu_p(\partial N^-) = 0. \tag{4}$$

In the next section, we will count the number of elements in certain subsets of Γ. Let us collect the definitions here for convenience.

$$\begin{aligned}\Gamma(t,\alpha,\beta) &:= \{\gamma\in\Gamma : N^\alpha\cap\phi^{-t}\gamma V^\beta\neq\emptyset\},\\ \Gamma_{-\rho}(t,\alpha,\beta) &:= \{\gamma\in\Gamma : (B_{-\rho}N)^\alpha\cap\phi^{-t}\gamma(B_{-\rho}V)^\beta\neq\emptyset\},\\ \Gamma^* &:= \{\gamma\in\Gamma : \gamma V^+\subset N^+ \text{ and } \gamma^{-1}N^-\subset V^-\},\\ \Gamma^*(t,\alpha,\beta) &:= \Gamma^*\cap\Gamma(t,\alpha,\beta).\end{aligned}$$

Lemma 4.2 *For every $\rho>0$, there exists some $T_2>0$ such that for all $t\geq T_2$, we have $\Gamma_{-\rho}(t,\alpha,\beta)\subset\Gamma^*(t,\alpha,\beta)$.*

Proof Let U be the interior of N^+. Then $(B_{-\rho}N)^+\subset U$. We claim that there exists $T_2>0$ such that for all $t\geq T_2$ and $\gamma\in\Gamma$, if $(B_{-\rho}N)^\alpha\cap\phi^{-t}\gamma(B_{-\rho}V)^\beta\neq\emptyset$, then $\gamma V^+\subset U$.

Let us prove the claim. Assume not. Then for each i, there exist $t_i\to\infty$ and $\gamma_i\in\Gamma$ such that $v_i\in(B_{-\rho}N)^\alpha\cap\phi^{-t_i}\gamma_i(B_{-\rho}V)^\beta$, but $\gamma_i V^+\not\subseteq U$. Clearly, for any $x\in X$, $\gamma_i x$ goes to infinity. By passing to a subsequence, let us assume that $\gamma_i x\to\xi\in\partial X$.

By Lemma 3.8, $(B_{-\rho}N)^\alpha$ and $(B_{-\rho}V)^\beta$ are both compact. By passing to a subsequence, we may assume that $v_i\to v\in(B_{-\rho}N)^\alpha$ and $\gamma_i^{-1}\phi^{t_i}v_i\to w\in(B_{-\rho}V)^\beta$. Note that $\gamma_i\pi w\to\xi\in\partial X$. Since $d(\gamma_i w,\phi^{t_i}v_i)\to 0$, we have $\xi=\lim_i\pi\phi^{t_i}v_i\in(B_{-\rho}N)^+$.

We may assume that $\gamma_i^{-1}\pi v\to\eta\in\partial X$. Let $w_i=\gamma_i^{-1}\phi^{t_i}v_i\in(B_{-\rho}V)^\beta$. Then $d(\gamma_i^{-1}v,\phi^{-t_i}w_i)=d(\gamma_i^{-1}v,\gamma_i^{-1}v_i)\to 0$, and thus

$$d(\gamma_i^{-1}\pi v,\pi\phi^{-t_i}w_i)\to 0.$$

We then see that $\eta=\lim_i\pi\phi^{-t_i}w_i\in(B_{-\rho}V)^-$.

Now we have $\gamma_i\pi v\to\xi\in(B_{-\rho}N)^+$ and $\gamma_i^{-1}\pi v\to\eta\in(B_{-\rho}V)^-$. Now we apply a modified version of Theorem 4.1, with $\mathbf{P}_\rho$ replaced by $(B_{-\rho}V)^-$, $\mathbf{F}_\rho$ replaced by $(B_{-\rho}N)^+$, and $\mathbf{F}_\theta$ replaced by V^+. The proof of [45, Theorem 3.9] should be modified accordingly. Indeed, we observe that [45, Lemma 3.10] is still true with $\mathbf{F}_\theta$ replaced by V^+, since $\angle(n,c)\geq\pi$ for any $c\in V^+$ still holds.

So we have $\gamma_i V^+\subset U$ for all i sufficiently large. A contradiction and the claim follows.

By the claim, there exists some $T_2>0$ such that for all $t\geq T_2$ and $\gamma\in\Gamma_{-\rho}(t,\alpha,\beta)$, we have $\gamma V^+\subset U\subset N^+$.

Analogously, by reversing the roles of N^α and V^β, and the roles of γ and γ^{-1}, we can prove that $\gamma^{-1}N^-\subset V^-$. Thus $\gamma\in\Gamma^*$ and the proof of the lemma is completed. □

5 Using scaling and mixing

In this section, we prove Theorem A. First we use the scaling and mixing properties of Knieper measure m, to give an asymptotic estimates of $\#\Gamma^*(t,\epsilon^2,\beta)$ and $\#\Gamma(t,\epsilon^2,\beta)$.

5.1 Intersection components

Lemma 5.1 *We have* $N \subset N^{\epsilon^2}$, $V \subset V^{\epsilon^2}$.

Proof Let $w \in N$. By Corollary 3.9, we have

$$|s(w)| = |b_{w^-}(\pi w, p)| = |b_{w^-}(\pi w_0, p)| \leq |b_{w_0^-}(\pi w_0, p)| + \epsilon^2 = \epsilon^2.$$

By definition $w \in N^{\epsilon^2}$ and hence $N \subset N^{\epsilon^2}$. $V \subset V^{\epsilon^2}$ can be proved analogously. □

Lemma 5.2 *For any* $t > 0$ *and* $\gamma \in \Gamma$, *we have*

$$\{\gamma \in \Gamma : N^\epsilon \cap \phi^{-t}\gamma V \neq \emptyset\} \subset \{\gamma \in \Gamma : N^{\epsilon^2} \cap \phi^{-t}\gamma V^\epsilon \neq \emptyset\}.$$

Proof Let $\gamma \in \Gamma$ be such that $N^\epsilon \cap \phi^{-t}\gamma V \neq \emptyset$. If $v \in \phi^t\gamma^{-1}N^\epsilon \cap V$, then $w := \phi^{-t}\gamma v \in N^\epsilon$. So there exists $w' \in N^{\epsilon^2}$ such that $w = \phi^s w'$ where $|s| \leq \epsilon - \epsilon^2$. Then $\phi^{t+s}w' = \phi^t w = \gamma v$. So $\gamma^{-1}\phi^{t+s}w' = v \in V \subset V^{\epsilon^2}$ by Lemma 5.1. We have $\gamma^{-1}\phi^t w' \subset V^{|s|+\epsilon^2} \subset V^\epsilon$. So $N^{\epsilon^2} \cap \phi^{-t}\gamma V^\epsilon \neq \emptyset$ and the lemma follows. □

Lemma 5.3 *Let* $p \in X$, *then given any* $a > 0, \epsilon > 0$, *there exists* $T > 0$ *such that for any* $t \geq T$ *and* $v, w \in S_pX$, $d(\phi^t v, \phi^t w) \leq \epsilon$ *implies that* $\angle(v, w) < a$.

Proof In nonpositive curvature, the lemma is a consequence of the comparison theorem.

For rank one manifolds without focal points, assume the contrary. Then there exist $t_n \to \infty$ and $v_n, w_n \in S_pX$ such that

$$d(\phi^{t_n}v_n, \phi^{t_n}w_n) \leq \epsilon \quad \text{and} \quad \angle(v_n, w_n) \geq a.$$

By taking a subsequence, we can assume without loss of generality that $v_n \to v$, $w_n \to w$ for some $v, w \in S_pX$. Then $\angle(v, w) \geq a$. Take any $t > 0$. Choose n large enough such that $t_n > t$ and $d(\phi^t v, \phi^t w) \leq d(\phi^t v_n, \phi^t w_n) + \epsilon$. By monotonicity, $d(\phi^t v_n, \phi^t w_n) \leq d(\phi^{t_n}v_n, \phi^{t_n}w_n) \leq \epsilon$. It follows that $d(\phi^t v, \phi^t w) \leq 2\epsilon$ for any $t > 0$, which contradicts to the divergence property (see Definition 3.2). □

Lemma 5.4 *For any* $a > 0$, *there exists* $T_1 > 0$ *large enough such that for any* $t \geq T_1$,

$$\begin{aligned}&\{\gamma \in \Gamma : N^\alpha \cap \phi^{-t}\gamma V^\beta \neq \emptyset\}\\ &\quad\subset \{\gamma \in \Gamma : (B_aN)^{\alpha+\beta+\epsilon^2} \cap \phi^{-t}\gamma V \neq \emptyset\}.\end{aligned}$$

Proof Let $\gamma \in \Gamma$ such that $N^\alpha \cap \phi^{-t}\gamma V^\beta \neq \emptyset$. Then there exists $v \in \phi^t N^\alpha \cap \gamma V^\beta$. So $\gamma^{-1}v \in V^\beta$ and there exists $w \in V$ such that $w^- = \gamma^{-1}v^-$. So $|s(w)| \leq \epsilon^2$ since $V \subset V^{\epsilon^2}$ by Lemma 5.1. Moreover, there exists $w' \in W^u(w)$ (hence $s(w) = s(w')$) and $\phi^b w' = \gamma^{-1}v$ for some $b \in \mathbb{R}$. It follows that $|b| \leq \beta + \epsilon^2$ and $d(w, w') \leq 4\epsilon$ by Lemma 3.8. Then $\gamma w' \in W^u(\gamma w)$ with $d(\gamma w, \gamma w') \leq 4\epsilon$, $s(\gamma w) = s(\gamma w')$ and $\phi^b\gamma w' = v$. It follows that $\gamma w' \in \phi^t N^{\alpha+\beta+\epsilon^2}$. By Lemma 5.3, there exists

$T_1 > 0$ large enough such that for any $t \geq T_1$, then $d(\gamma w, \gamma w') \leq 4\epsilon$ implies that $\gamma w \in \phi^{[0,\infty]}(B_a N)$. We have that $\phi^{-t}\gamma w \in (B_a N)^{\alpha+\beta+\epsilon^2} \cap \phi^{-t}\gamma V$. The lemma follows. □

5.2 Depth of intersection

Given $\xi \in \partial X$ and $\gamma \in \Gamma$, define $b_\xi^\gamma := b_\xi(\gamma p, p)$.

Lemma 5.5 *Let $\xi, \eta \in N^-$, and $\gamma \in \Gamma(t, \alpha, \beta)$ with $t > 0$. Then $|b_\xi^\gamma - b_\eta^\gamma| < \epsilon^2$.*

Proof Let $\gamma \in \Gamma(t, \alpha, \beta)$, so there exists $v \in N^\alpha \cap \phi^{-t}\gamma V^\beta$. There is $q \in \pi V^\beta$ such that $\gamma q = \pi\phi^t v \in \pi H^{-1}(N^- \times N^+ \times [0, \infty))$. Since $p, y \in \mathcal{F}$, we have $d(\gamma p, \gamma q) = d(p, q) \leq \operatorname{diam}\mathcal{F} + 4\epsilon$ by Lemma 3.8. Thus by Corollary 3.9, $|b_\xi(\gamma p, p) - b_\eta(\gamma p, p)| < \epsilon^2$ for any $\xi, \eta \in N^-$. □

Lemma 5.6 *Given any $\gamma \in \Gamma^*(t, \alpha, \beta)$ and any $t \in \mathbb{R}$, we have*

$$N^{\epsilon^2} \cap \phi^{-t}\gamma V^\beta = \{w \in E^{-1}(N^- \times \gamma V^+) : \\ s(w) \in [-\epsilon^2, \epsilon^2] \cap (b_{w^-}^\gamma - t + t_0 + [-\beta, \beta])\}$$

where $t_0 = s(v_0)$.

Proof At first, we claim that $N^{\epsilon^2} \cap \phi^{-t}\gamma V^\beta \subset E^{-1}(N^- \times \gamma V^+)$. Indeed, let $v \in N^{\epsilon^2} \cap \phi^{-t}\gamma V^\beta$. Since $v \in N^{\epsilon^2}$, $v^- \in N^-$. On the other hand, since $v \in \phi^{-t}\gamma V^\beta$, we have $v^+ \in \gamma V^+$. This proves the claim.

Notice that as $\gamma \in \Gamma^*(t, \alpha, \beta)$, one has $\gamma V^+ \subset N^+$ and $\gamma^{-1}N^- \subset V^-$. Let $w \in E^{-1}(N^- \times \gamma V^+) \subset E^{-1}(N^- \times N^+)$. By definition of N^{ϵ^2}, $w \in N^{\epsilon^2}$ if and only if $s(w) \in [-\epsilon^2, \epsilon^2]$.

It remains to show that $w \in \phi^{-t}\gamma V^\beta$ if and only if $s(w) \in (b_{w^-}^\gamma - t + t_0 + [-\beta, \beta])$. To see this, note that

$$\begin{aligned}\gamma V^\beta &= \{\gamma v : v \in E^{-1}(V^- \times V^+) \text{ and } b_{v^-}(\pi v, p) \in t_0 + [-\beta, \beta]\} \\ &= \{w \in E^{-1}(\gamma V^- \times \gamma V^+) : b_{w^-}(\pi w, \gamma p) \in t_0 + [-\beta, \beta]\}.\end{aligned}$$

Since $s(\phi^t w) = s(w) + t$ and

$$b_{w^-}(\pi w, \gamma p) = b_{w^-}(\pi w, p) + b_{w^-}(p, \gamma p) = s(w) - b_{w^-}^\gamma,$$

we know $\phi^t w \in \gamma V^\beta$ if and only if $s(w) - b_{w^-}^\gamma + t \in t_0 + [-\beta, \beta]$. The lemma follows. □

The following lemma implies that the intersection components also have product structure.

Lemma 5.7 *If $\gamma \in \Gamma^*(t, \epsilon^2, \beta)$, then*

$$N^{\epsilon^2} \cap \phi^{-(t+4\epsilon^2)}\gamma V^{\beta+8\epsilon^2} \supset H^{-1}(N^- \times \gamma V^+ \times [-\epsilon^2, \epsilon^2]) := N^\gamma.$$

Proof Let $\gamma \in \Gamma^*(t, \epsilon^2, \beta)$, then $N^{\epsilon^2} \cap \phi^{-t}\gamma V^\beta \neq \emptyset$. By Lemma 5.6, there exists $\eta \in N^-$ such that

$$[-\epsilon^2, \epsilon^2] \cap (b_\eta^\gamma - t + t_0 + [-\beta, \beta]) \neq \emptyset.$$

It follows that $[-\epsilon^2, \epsilon^2] \subset (b_\eta^\gamma - t - 2\epsilon^2 + t_0 + [-\beta, \beta + 4\epsilon^2])$. Then by Lemma 5.5, for any $\xi \in N^-$ we have

$$[-\epsilon^2, \epsilon^2] \cap (b_\xi^\gamma - t - 2\epsilon^2 + t_0 + [-\beta, \beta + 4\epsilon^2]) \neq \emptyset,$$

which in turn implies that

$$[-\epsilon^2, \epsilon^2] \subset (b_\xi^\gamma - t - 4\epsilon^2 + t_0 + [-\beta, \beta + 8\epsilon^2]).$$

We are done by Lemma 5.6. □

5.3 Scaling and mixing calculation

We use the following notations in the asymptotic estimates.

$$\begin{aligned}
f(t) = e^{\pm C} g(t) &\Leftrightarrow e^{-C} g(t) \leq f(t) \leq e^C g(t) \text{ for all } t;\\
f(t) \lesssim g(t) &\Leftrightarrow \limsup_{t\to\infty} \frac{f(t)}{g(t)} \leq 1;\\
f(t) \gtrsim g(t) &\Leftrightarrow \liminf_{t\to\infty} \frac{f(t)}{g(t)} \geq 1;\\
f(t) \sim g(t) &\Leftrightarrow \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1;\\
f(t) \sim e^{\pm C} g(t) &\Leftrightarrow e^{-C} g(t) \lesssim f(t) \lesssim e^C g(t).
\end{aligned}$$

Lemma 5.8 *If* $\gamma \in \Gamma^*(t, \epsilon^2, \beta)$, *then*

$$\frac{m(N^\gamma)}{m(V^\beta)} = e^{\pm 26h\epsilon} e^{ht_0} e^{-ht} \frac{\epsilon^2 \mu_p(N^-)}{\beta \mu_p(V^-)}$$

where N^γ *is from Lemma* 5.7.

Proof The main work is to estimate $\beta_p(\xi, \gamma\eta)$ and $b_\eta(\gamma^{-1}p, p)$ for any $\xi \in N^-$ and $\eta \in V^+$.

Firstly, take q lying on the geodesic connecting ξ and $\gamma\eta$ such that $b_\xi(q, p) = 0$. Then

$$|\beta_p(\xi, \gamma\eta)| = |b_\xi(q, p) + b_{\gamma\eta}(q, p)| = |b_{\gamma\eta}(q, p)| \leq d(q, p) < 4\epsilon \tag{5}$$

where we used Lemma 3.8 in the last inequality.

Secondly, since $\gamma \in \Gamma^*(t, \epsilon^2, \beta)$, there exist $v \in V^\beta$ and $w \in N^{\epsilon^2}$ such that $w = \phi^{-t}\gamma v$. Take q' lying on the geodesic connecting $\gamma^{-1}\xi$ and η such that $b_{\gamma^{-1}\xi}(q', p) = t_0$. Then $q' \in V^\beta$. Define q'' to be the unique point in $\pi H^{-1}(N^- \times N^+ \times \{0\}) \cap c_{\xi,\gamma\eta}$. Then using Lemma 3.8,

$$d(q'', \gamma q') \le d(q'', \pi w) + d(\pi w, \pi\phi^t w) + d(\gamma\pi v, \gamma q') \le t + 8\epsilon$$
$$d(q'', \gamma q') \ge d(\pi w, \pi\phi^t w) - d(q'', \pi w) - d(\gamma\pi v, \gamma q') \ge t - 8\epsilon.$$

Noticing that $q'', \gamma q'$ lie on the geodesic $c_{\xi,\gamma\eta}$, we have

$$\begin{aligned} b_\eta(\gamma^{-1}p, p) &= b_{\gamma\eta}(p, \gamma p) = b_{\gamma\eta}(p, \gamma q') + b_{\gamma\eta}(\gamma q', \gamma p) \\ &\le b_{\gamma\eta}(q'', \gamma q') + d(q'', p) + b_\eta(q', p) \le t + 12\epsilon + b_\eta(q', p) \end{aligned}$$

and

$$\begin{aligned} b_\eta(\gamma^{-1}p, p) &= b_{\gamma\eta}(p, \gamma p) = b_{\gamma\eta}(p, \gamma q') + b_{\gamma\eta}(\gamma q', \gamma p) \\ &\ge b_{\gamma\eta}(q'', \gamma q') - d(q'', p) + b_\eta(q', p) \ge t - 12\epsilon + b_\eta(q', p). \end{aligned}$$

Thus we have

$$\begin{aligned} \frac{m(N^\gamma)}{m(V^\beta)} &= \frac{2\epsilon^2}{2\beta} \frac{\int_{N^-}\int_{V^+} e^{h\beta_p(\xi,\gamma\eta)} d\mu_p(\xi) d\mu_{\gamma^{-1}p}(\eta)}{\int_{V^-}\int_{V^+} e^{h\beta_p(\xi',\eta')} d\mu_p(\xi') d\mu_p(\eta')} \\ &= \frac{\epsilon^2}{\beta} e^{\pm 4h\epsilon} \frac{\int_{N^-}\int_{V^+} e^{-hb_\eta(\gamma^{-1}p,p)} d\mu_p(\xi) d\mu_p(\eta)}{\int_{V^-}\int_{V^+} e^{h\beta_p(\xi',\eta')} d\mu_p(\xi') d\mu_p(\eta')} \\ &= \frac{\epsilon^2}{\beta} e^{\pm 16h\epsilon} e^{-ht} \frac{\int_{N^-}\int_{V^+} e^{-hb_\eta(q',p)} d\mu_p(\xi) d\mu_p(\eta)}{\int_{V^-}\int_{V^+} e^{\pm 8h\epsilon} e^{-h(b_{\xi'}(q',p) + b_{\eta'}(q',p))} d\mu_p(\xi') d\mu_p(\eta')} \\ &= \frac{\epsilon^2}{\beta} e^{\pm 24h\epsilon} e^{-ht} e^{\pm 2h\epsilon^2} e^{ht_0} \frac{\mu_p(N^-)}{\mu_p(V^-)} \\ &= e^{\pm 26h\epsilon} e^{ht_0} e^{-ht} \frac{\epsilon^2 \mu_p(N^-)}{\beta \mu_p(V^-)}. \end{aligned}$$

where in the third equality we used the fact that $c_{\xi'\eta'}$ passes through a point in V^β, within a distance 4ϵ from q', and in the fourth equality we used Corollary 3.9 and $b_{\gamma^{-1}\xi}(q', p) = t_0$. □

Finally, we combine scaling and mixing properties of Knieper measure to obtain the following asymptotic estimates.

Proposition 5.9 *We have*

$$e^{-30h\epsilon} \lesssim \frac{\#\Gamma^*(t, \epsilon^2, \beta) e^{ht_0}}{\mu_p(V^-)\mu_p(N^+) e^{ht}} \frac{1}{2\beta} \lesssim e^{30h\epsilon}\left(1 + \frac{8\epsilon^2}{\beta}\right),$$

$$e^{-30h\epsilon} \lesssim \frac{\#\Gamma(t,\epsilon^2,\beta)e^{ht_0}}{\mu_p(V^-)\mu_p(N^+)e^{ht}}\frac{1}{2\beta} \lesssim e^{30h\epsilon}\left(1+\frac{8\epsilon^2}{\beta}\right).$$

Proof By Lemmas 4.2 and 5.7, for any $0<\rho<\theta$ and t large enough, we have

$$(B_{-\rho}\underline{N})^{\epsilon^2} \cap \phi^{-t}(B_{-\rho}\underline{V})^{\beta} \subset \bigcup_{\gamma\in\Gamma^*(t,\epsilon^2,\beta)} \underline{N}^{\gamma} \subset \underline{N}^{\epsilon^2} \cap \phi^{-(t+4\epsilon^2)}\underline{V}^{\beta+8\epsilon^2}.$$

By Lemma 5.8,

$$\frac{m(N^{\gamma})}{m(V^{\beta})} = e^{\pm 26h\epsilon}e^{ht_0}e^{-ht}\frac{\epsilon^2\mu_p(N^-)}{\beta\mu_p(V^-)}.$$

Estimating similarly to (5),

$$m(N^{\epsilon^2}) = 2\epsilon^2\int_{N^-}\int_{N^+} e^{h\beta_p(\xi,\eta)}d\mu_p(\xi)d\mu_p(\eta) = 2\epsilon^2 e^{\pm 4h\epsilon}\mu_p(N^-)\mu_p(N^+). \tag{6}$$

Thus we have

$$\begin{aligned} e^{-26h\epsilon}\underline{m}((B_{-\rho}\underline{N})^{\epsilon^2} \cap \phi^{-t}(B_{-\rho}\underline{V})^{\beta}) &\leq \#\Gamma^*(t,\epsilon^2,\beta)e^{ht_0}e^{-ht}\frac{\epsilon^2\mu_p(N^-)}{\beta\mu_p(V^-)}\underline{m}(\underline{V}^{\beta}) \\ &\leq e^{26h\epsilon}\underline{m}(\underline{N}^{\epsilon^2}\cap\phi^{-(t+4\epsilon^2)}\underline{V}^{\beta+8\epsilon^2}). \end{aligned}$$

Dividing by $\underline{m}(\underline{N}^{\epsilon^2})\underline{m}(\underline{V}^{\beta})$ and using mixing of $\underline{m}$, we get

$$\begin{aligned} e^{-26h\epsilon}\frac{m((B_{-\rho}N)^{\epsilon^2})m((B_{-\rho}V)^{\beta})}{m(N^{\epsilon^2})m(V^{\beta})} &\lesssim \frac{\#\Gamma^*(t,\epsilon^2,\beta)e^{ht_0}}{e^{ht}m(N^{\epsilon^2})}\frac{\epsilon^2\mu_p(N^-)}{\beta\mu_p(V^-)} \\ &\lesssim e^{26h\epsilon}\frac{m(V^{\beta+8\epsilon^2})}{m(V^{\beta})}. \end{aligned}$$

By (4), letting $\rho\to 0$, we obtain

$$e^{-26h\epsilon} \lesssim \frac{\#\Gamma^*(t,\epsilon^2,\beta)e^{ht_0}}{e^{ht}m(N^{\epsilon^2})}\frac{\epsilon^2\mu_p(N^-)}{\beta\mu_p(V^-)} \lesssim e^{26h\epsilon}\left(1+\frac{8\epsilon^2}{\beta}\right).$$

Thus by (6)

$$e^{-30h\epsilon} \lesssim \frac{\#\Gamma^*(t,\epsilon^2,\beta)e^{ht_0}}{\mu_p(V^-)\mu_p(N^+)e^{ht}}\frac{1}{2\beta} \lesssim e^{30h\epsilon}\left(1+\frac{8\epsilon^2}{\beta}\right). \tag{7}$$

To prove the second equation, we consider $\rho > 0$. Then by Lemma 4.2, $\Gamma^*(t,\epsilon^2,\beta) \subset \Gamma(t,\epsilon^2,\beta) \subset \Gamma^*_\rho(t,\epsilon^2,\beta)$. By (7),

$$\begin{aligned} e^{-30h\epsilon} &\lesssim \frac{\#\Gamma^*(t,\epsilon^2,\beta)e^{ht_0}}{\mu_p(V^-)\mu_p(N^+)e^{ht}}\frac{1}{2\beta} \le \frac{\#\Gamma(t,\epsilon^2,\beta)e^{ht_0}}{\mu_p(V^-)\mu_p(N^+)e^{ht}}\frac{1}{2\beta} \\ &\le \frac{\#\Gamma^*_\rho(t,\epsilon^2,\beta)e^{ht_0}}{\mu_p(V^-)\mu_p(N^+)e^{ht}}\frac{1}{2\beta} \\ &\lesssim e^{30h\epsilon}(1+\frac{8\epsilon^2}{\beta})\frac{\mu_p((B_\rho V)^-)\mu_p((B_\rho N)^+)}{\mu_p(V^-)\mu_p(N^+)}. \end{aligned}$$

Letting $\rho \searrow 0$ and by (4), we get the second equation in the proposition. □

5.4 Integration

Let $V \subset S_yX$, $N \subset S_xX$ be as above, and $0 \le a < b$. Let $n(a,b,V,N^0)$ denote the number of connected components at which $\phi^{[-b,-a]}\underline{V}$ intersects $\underline{N}^0$. $n_t(V^\beta,N^\alpha)$ (resp. $n_t(V,N^\alpha)$) denotes the number of connected components at which $\phi^{-t}\underline{V}^\beta$ (resp. $\phi^{-t}\underline{V}$) intersects $\underline{N}^\alpha$.

Lemma 5.10 *We have*

$$n_t(V,N^\epsilon) \lesssim e^{-ht_0}\mu_p(V^-)\mu_p(N^+)e^{ht}2\epsilon(1+8\epsilon)e^{30h\epsilon}.$$

Proof Setting $\alpha = \epsilon^2$ in Lemma 5.2,

$$n_t(V,N^\epsilon) \le n_t(V^\epsilon, N^{\epsilon^2}).$$

Setting $\beta = \epsilon$ in Proposition 5.9,

$$n_t(V^\epsilon,N^{\epsilon^2}) = \#\Gamma(t,\epsilon^2,\epsilon) \lesssim e^{-ht_0}\mu_p(V^-)\mu_p(N^+)e^{ht}2\epsilon(1+8\epsilon)e^{30h\epsilon}.$$

□

Lemma 5.11 *We have*

$$n_t(V,N^\epsilon) \gtrsim e^{-ht_0}\mu_p(V^-)\mu_p(N^+)e^{ht}2\epsilon(1-2\epsilon)e^{-30h\epsilon}.$$

Proof Setting $\alpha = \epsilon^2$, $\beta = \epsilon - 2\epsilon^2$ in Lemma 5.4, for any $a > 0$, we have

$$n_t(V,N^\epsilon) \ge n_t(V^{\epsilon-2\epsilon^2}, (B_{-a}N)^{\epsilon^2})$$

for any $t \ge T_1$ where T_1 is provided by Lemma 5.4. Setting $\beta = \epsilon - 2\epsilon^2$ in Lemma 5.9,

$$n_t(V^{\epsilon-2\epsilon^2}, (B_{-a}N)^{\epsilon^2}) \gtrsim e^{-ht_0}\mu_p(V^-)\mu_p((B_{-a}N)^+)e^{ht}2\epsilon(1-2\epsilon)e^{-30h\epsilon}.$$

Letting $a \to 0$, by (4) we obtain the conclusion of the lemma. □

Proposition 5.12 *There exists $Q > 0$ such that*

$$e^{-2Q\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)\frac{1}{h}e^{ht} \lesssim n(0, t, V, N^0)$$
$$\lesssim e^{2Q\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)\frac{1}{h}e^{ht}.$$

Proof It is clear that for any $b > 0$,

$$n(0, t, V, N^0) \sim n(b, t, V, N^0). \tag{8}$$

By Lemmas 5.10 and 5.11, we can choose b large enough such that

$$n(t-\epsilon, t+\epsilon, V, N^0) = n_t(V, N^\epsilon) = e^{\pm Q\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)e^{ht}2\epsilon$$

for some $Q > 2h$ large enough and for all $t \geq b$. Let $t_k = b + \epsilon + 2k\epsilon$, then

$$\begin{aligned}
n(b, t, V, N^0) &\leq \sum_{k=0}^{[\frac{t-b}{2\epsilon}]+1} n(t_k-\epsilon, t_k+\epsilon, V, N^0) \\
&\leq e^{Q\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)\sum_{k=0}^{[\frac{t-b}{2\epsilon}]+1} 2\epsilon e^{ht_k} \\
&\leq e^{Q\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)\int_{b-\epsilon}^{t+2\epsilon} e^{hs}ds \\
&= e^{Q\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)\frac{1}{h}(e^{h(t+2\epsilon)} - e^{h(b-\epsilon)}) \\
&\leq e^{2Q\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)\frac{1}{h}e^{ht}
\end{aligned}$$

and for $0 < r < 1$

$$\begin{aligned}
n(b, t, V, N^0) &\geq \sum_{k=0}^{[\frac{t-b}{2\epsilon}]-1} n(t_k - r\epsilon, t_k + r\epsilon, V, N^0) \\
&\geq e^{-Qr\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)\sum_{k=0}^{[\frac{t-b}{2\epsilon}]-1} 2r\epsilon e^{ht_k} \\
&\geq e^{-Qr\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)\int_{b+\epsilon}^{t-2\epsilon} re^{hs}ds
\end{aligned}$$

$$\geq e^{-Qr\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)\frac{r}{h}(e^{h(t-2\epsilon)}-e^{h(b+\epsilon)}).$$

Letting $r \to 1$, we get

$$n(b,t,V,N^0) \geq e^{-Q\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)\frac{1}{h}(e^{h(t-2\epsilon)}-e^{h(b+\epsilon)})$$
$$\gtrsim e^{-2Q\epsilon}e^{-ht_0}\mu_p(V^-)\mu_p(N^+)\frac{1}{h}e^{ht}.$$

The proposition then follows from (8). □

5.5 Summing over the regular partition-cover

Denote

$$a_t(x,y) := \#\{\gamma \in \Gamma : \gamma y \in B(x,t)\}$$

and

$$a_t^1(x,y) := \#\{\gamma \in \Gamma : \gamma y \in B(x,t), \text{ and } c_{x,\gamma y} \text{ is singular}\}.$$

It is easy to see that $b_t(x) = \int_{\mathcal{F}} a_t(x,y)d\text{Vol}(y)$. In the following, we give an asymptotic estimates of $a_t(x,y)$.

Lemma 5.13 *There exist $C > 0$ and $0 < h' < h$, such that for any $x, y \in \mathcal{F}$,*

$$a_t^1(x,y) \leq Ce^{h't}.$$

Proof Given any $\epsilon < \text{inj}M/5$ and $t > 0$, let $\gamma_1 \neq \gamma_2 \in \Gamma$ be such that $\gamma_1 y, \gamma_2 y \in B(x,t) \setminus B(x,t-\epsilon)$, and both $c_{x,\gamma_1 y}$ and $c_{x,\gamma_2 y}$ are singular. Then it is easy to see that $\dot{c}_{x,\gamma_1 y}$ and $\dot{c}_{x,\gamma_2 y}$ are (t,ϵ)-separated. By a result of Knieper [27, Theorem 1.1], the topological entropy of the singular set $h_{\text{top}}(\text{Sing})$ is strictly smaller than h. It follows that the number of $\gamma \in \Gamma$ as above is less than $C_1 e^{h't}$ for some $C_1 > 0$ and $h_{\text{top}}(\text{Sing}) < h' < h$.

Let $t_i = i\epsilon$, then

$$a_t^1(x,y) \leq \sum_{i=1}^{[t/\epsilon]+1} C'e^{h't_i} = \frac{C'}{\epsilon}\sum_{i=1}^{[t/\epsilon]+1} \epsilon e^{h't_i} \leq \frac{C'}{\epsilon}\int_0^{t+\epsilon} e^{h's}ds \leq Ce^{h't}$$

for some $C > 0$. □

Denote by

$$a(0,t,x,y,\cup_{j=n+1}^{\infty}N_j) := \#\{\gamma \in \Gamma : \gamma y \in \pi\phi^{[0,t]}\cup_{j=n+1}^{\infty}N_j\},$$

and similarly,

$$a(0,t,x,y,\cup_{i=m+1}^{\infty}V_i) := \#\{\gamma \in \Gamma : \gamma^{-1}x \in \pi\phi^{[-t,0]} \cup_{i=m+1}^{\infty} V_i\}.$$

Lemma 5.14 *There exists* $C > 0$ *such that*

$$\limsup_{t\to\infty} e^{-ht} a(0,t,x,y,\cup_{j=n+1}^{\infty}N_j) \le C \cdot \mu_p((\cup_{j=n+1}^{\infty}N_j)^+),$$

and

$$\limsup_{t\to\infty} e^{-ht} a(0,t,x,y,\cup_{i=m+1}^{\infty}V_i) \le C \cdot \mu_p((\cup_{i=m+1}^{\infty}V_i)^-).$$

Proof We start to estimate from above the spherical volume of $\pi\phi^t \cup_{j=n+1}^{\infty} N_j$ which is a subset of the sphere $S(x,t)$ of radius t around x.

Let $x_1,\dots,x_k$ be a maximal ρ-separated subset of $\pi\phi^t \cup_{j=n+1}^{\infty} N_j$, where $\rho \ge R$ from Shadowing Lemma 3.6. Then $B(x_i,\rho/2), i = 1,2,\dots,k$ are disjoint. By Lemma 5.3, for any $a > 0$ there exists $T_3 > 0$ such that if $t \ge T_3$, then $f_x B(x_i,\rho/2) \subset (B_a \cup_{j=n+1}^{\infty} N_j)^+$. By Shadowing Lemma 3.6 and the fact that $p = x$, we know $\mu_p(f_x B(x_i,\rho/2)) \ge b^{-1}e^{-ht}$ and hence

$$k \le be^{ht}\mu_p((B_a \cup_{j=n+1}^{\infty} N_j)^+).$$

From the uniformity of the geometry, there exists $l > 0$ such that $\mathrm{Vol}(B(x_i,\rho) \cap S(x,t)) \le l$ for each $1 \le i \le k$. So

$$\mathrm{Vol}\pi\phi^t \cup_{j=n+1}^{\infty} N_j \le lk \le lbe^{ht}\mu_p((B_a \cup_{j=n+1}^{\infty} N_j)^+).$$

Then there exists $C_1, C_2 > 0$ such that

$$\begin{aligned}\mathrm{Vol}\pi\phi^{[0,t]} \cup_{j=n+1}^{\infty} N_j &\le C_1 + \int_{T_3}^{t} \mathrm{Vol}\phi^s \cup_{j=n+1}^{\infty} N_j ds \\ &\le C_1 + C_2 e^{ht}\mu_p((B_a \cup_{j=n+1}^{\infty} N_j)^+). \end{aligned} \tag{9}$$

Note that C_2 is independent of T_3 and a.

Now since each $\gamma\mathcal{F}$ has equal finite diameter and volume, there exists $T_4 > 0$ such that

$$\begin{aligned} a(0,t,x,y,\cup_{j=n+1}^{\infty}N_j) &\le C_3 + C_4\mathrm{Vol}\pi\phi^{[T_4,t]} B_a \cup_{j=n+1}^{\infty} N_j \\ &\le C_5 + C_6 e^{ht}\mu_p((B_{2a} \cup_{j=n+1}^{\infty} N_j)^+) \end{aligned} \tag{10}$$

where we used (9) in the last inequality. Note that C_6 is independent of a. Thus

$$\limsup_{t\to\infty} e^{-ht} a(0,t,x,y,\cup_{j=n+1}^{\infty}N_j) \le C_6 \cdot \mu_p((B_{2a} \cup_{j=n+1}^{\infty} N_j)^+).$$

As $a > 0$ could be arbitrarily small and C_6 is independent of a, we get the first inequality in the lemma.

The second inequality can be proved analogously with minor modification: When applying Shadowing Lemma 3.6, we transfer from μ_p to μ_y by

$$\frac{d\mu_y}{d\mu_p}(\xi) = e^{-hb_\xi(y,p)} \leq e^{hd(y,p)} \leq e^{h\mathrm{diam}\mathcal{F}}$$

for any $\xi \in \partial X$. □

Proof of Theorem A Since the diameter of each flow box is no more than 4ϵ, we have

$$n(0, t, \cup V_i, \cup (N_j)^0) \leq a_{t+4\epsilon}(x, y) \tag{11}$$

and

$$a_{t-4\epsilon}(x, y) \leq a^1_{t-4\epsilon}(x, y) + a(0, t, x, y, \cup_{j=n+1}^{\infty} N_j) \\ + a(0, t, x, y, \cup_{i=m+1}^{\infty} V_i) + n(0, t, \cup_{i=1}^{m} V_i, \cup_{j=1}^{n} (N_j)^0). \tag{12}$$

For each V_i, denote by $t_0^i := b_{v_i^-}(\pi v_i, p)$ where $v_i \in V_i$. Recall that in Sect. 4 we suppressed i and write $t_0 = t_0^i$, since only one V_i is considered there. By Proposition 5.12, for any $m, n \in \mathbb{N}$

$$\begin{aligned}
&\liminf_{t\to\infty} e^{-ht} n(0, t, \cup V_i, \cup (N_j)^0) \\
&\geq \liminf_{t\to\infty} e^{-ht} \sum_{i=1}^{m} \sum_{j=1}^{n} n(0, t, V_i, (N_j)^0) \\
&\geq \sum_{i=1}^{m} \sum_{j=1}^{n} e^{-2Q\epsilon} e^{-hb_{v_i^-}(\pi v_i, p)} \mu_p((V_i)^-) \mu_p((N_j)^+) \frac{1}{h}.
\end{aligned}$$

Note that $v \mapsto b_{v^-}(\pi v, p)$ is a continuous function by Lemma 3.3. So if we choose a sequence of finer and finer regular partition-covers, and let $m, n \to \infty$ on the right hand,

$$\begin{aligned}
&\liminf_{t\to\infty} e^{-ht} n(0, t, \cup V_i, \cup (N_j)^0) \\
&\geq e^{-2Q\epsilon} \frac{1}{h} \int_{S_x M \cap \mathrm{Reg}} \int_{S_y M \cap \mathrm{Reg}} e^{-hb_{v^-}(\pi v, p)} d\tilde{\mu}_y(-v) d\tilde{\mu}_x(w).
\end{aligned}$$

Then by (11) we have

$$a_{t+4\epsilon}(x, y) \gtrsim e^{-2Q\epsilon} \frac{1}{h} e^{ht} \int_{S_x M \cap \mathrm{Reg}} \int_{S_y M \cap \mathrm{Reg}} e^{-hb_{v^-}(\pi v, p)} d\tilde{\mu}_y(-v) d\tilde{\mu}_x(w).$$

Replacing t by $t - 4\epsilon$, we have

$$a_t(x, y) \gtrsim e^{-2Q\epsilon} e^{-4\epsilon} \frac{1}{h} e^{ht} \int_{S_x M \cap \mathrm{Reg}} \int_{S_y M \cap \mathrm{Reg}} e^{-hb_{v^-}(\pi v, p)} d\tilde{\mu}_y(-v) d\tilde{\mu}_x(w). \tag{13}$$

On the other hand, by Proposition 5.12 and Corollary 3.9, for any $m, n \in \mathbb{N}$

$$\begin{aligned} &e^{2Q\epsilon} \frac{1}{h} \int_{S_x M \cap \mathrm{Reg}} \int_{S_y M \cap \mathrm{Reg}} e^{-hb_{v^-}(\pi v, p)} d\tilde{\mu}_y(-v) d\tilde{\mu}_x(w) \\ &\geq \sum_{i=1}^{m} \sum_{j=1}^{n} e^{2Q\epsilon} e^{-h(t_0^i + \epsilon^2 + 4\epsilon)} \mu_p((V_i)^-) \mu_p((N_j)^+) \frac{1}{h} \\ &\geq \limsup_{t \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} n(0, t, V_i, (N_j)^0) e^{-ht} e^{-h(\epsilon^2 + 4\epsilon)}. \end{aligned} \tag{14}$$

Combining with (12) and Lemmas 5.13, 5.14, one has

$$\begin{aligned} a_{t-4\epsilon}(x, y) \lesssim\ & C e^{h'(t-4\epsilon)} \\ &+ C e^{h(t-4\epsilon)} \cdot \mu_p((\cup_{j=n+1}^{\infty} N_j)^+) + C e^{h(t-4\epsilon)} \cdot \mu_p((\cup_{j=m+1}^{\infty} V_i)^+) \\ &+ e^{2Q\epsilon} e^{h(\epsilon^2 + 4\epsilon)} \frac{1}{h} e^{ht} \int_{S_x M \cap \mathrm{Reg}} \int_{S_y M \cap \mathrm{Reg}} e^{-hb_{v^-}(\pi v, p)} d\tilde{\mu}_y(-v) d\tilde{\mu}_x(w). \end{aligned}$$

Letting $m, n \to \infty$ and replacing $t - 4\epsilon$ by t, we have

$$\begin{aligned} &a_t(x, y) \\ &\lesssim e^{2Q\epsilon} e^{h(\epsilon^2 + 4\epsilon) + 4\epsilon} \frac{1}{h} e^{ht} \int_{S_x M \cap \mathrm{Reg}} \int_{S_y M \cap \mathrm{Reg}} e^{-hb_{v^-}(\pi v, p)} d\tilde{\mu}_y(-v) d\tilde{\mu}_x(w). \end{aligned} \tag{15}$$

Letting $\epsilon \to 0$ in (13) and (15) and recalling that $p = x$, we get

$$a_t(x, y) \sim \frac{1}{h} e^{ht} \cdot c(x, y)$$

where

$$c(x, y) := \int_{S_x X \cap \mathrm{Reg}} \int_{S_y X \cap \mathrm{Reg}} e^{-hb_{v^-}(\pi v, x)} d\tilde{\mu}_y(-v) d\tilde{\mu}_x(w).$$

By Lemma 3.10, in fact we have

$$c(x, y) = \int_{S_x X} \int_{S_y X} e^{-hb_{v^-}(\pi v, x)} d\tilde{\mu}_y(-v) d\tilde{\mu}_x(w).$$

It follows that $b_t(x) = \int_{\mathcal{F}} a_t(x, y)d\mathrm{Vol}(y) \sim \frac{1}{h}e^{ht}\int_{\mathcal{F}} c(x, y)d\mathrm{Vol}(y)$ by the dominated convergence Theorem. Indeed, by (12), Lemma 5.13, (10) and (14),

$$e^{-ht}a_t(x, y) \leq B_1 + B_2 c(x, y)$$

where the right hand side is integrable.

Define $c(x) := \int_{\mathcal{F}} c(x, y)d\mathrm{Vol}(y)$, we get $b_t(x) \sim c(x)\frac{e^{ht}}{h}$. Obviously, $c(\gamma x) = c(x)$ for any $\gamma \in \Gamma$. So c descends to a function from M to $\mathbb{R}$, which still denoted by c.

It remains to prove the continuity of c. Let $y_n \to x \in X$. For any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(x, y_n) < \epsilon$. Then

$$b_{t-\epsilon}(x) \leq b_t(y_n) \leq b_{t+\epsilon}(x).$$

We have

$$c(y_n) = \lim_{t\to\infty} \frac{b_t(y_n)}{e^{ht}/h} \leq \lim_{t\to\infty} \frac{b_{t+\epsilon}(x)}{e^{ht}/h} = c(x)e^{h\epsilon}$$

and

$$c(y_n) = \lim_{t\to\infty} \frac{b_t(y_n)}{e^{ht}/h} \geq \lim_{t\to\infty} \frac{b_{t-\epsilon}(x)}{e^{ht}/h} = c(x)e^{-h\epsilon}.$$

Thus $\lim_{n\to\infty} c(y_n) = c(x)$ and c is continuous. □

6 Properties of the Margulis function

We prove Theorem B in this section. Let M be a rank one closed C^∞ Riemannian manifold without focal points, X the universal cover of M. Firstly, we give an equivalent definition for the Patterson–Sullivan measure via the Margulis function.

Recall that $f_x : S_xX \to \partial X$, $f_x(v) = v^+$. Similarly, we define the canonical projection $f_x^R : S(x, R) \to \partial X$ by $f_x^R(y) = v_y^+$ where v_y is the unit normal vector of the sphere $S(x, R)$ at y. For any continuous function $\varphi : \partial X \to \mathbb{R}$, define a measure on ∂X by

$$\nu_x^R(\varphi) := \frac{1}{e^{hR}} \int_{S(x,R)} \varphi \circ f_x^R(y)d\mathrm{Vol}(y).$$

By Theorem A or (3), $\nu_x^R(\partial X)$ is uniformly bounded from above and below, and hence there exist limit measures of ν_x^R when $R \to \infty$. Take any limit measure ν_x. By Theorem A or (3), we see that

$$\nu_x(\partial X) = \lim_{R\to\infty} \frac{s_R(x)}{e^{hR}} = c(x).$$

Moreover, by definition one can check that

(1) For any $p,\ q \in X$ and ν_p-a.e. $\xi \in \partial X$,

$$\frac{d\nu_q}{d\nu_p}(\xi) = e^{-h\cdot b_\xi(q,p)}.$$

(2) $\{\nu_p\}_{p\in X}$ is Γ-equivariant, i.e., for every Borel set $A \subset \partial X$ and for any $\gamma \in \Gamma$, we have

$$\nu_{\gamma p}(\gamma A) = \nu_p(A).$$

(2) is obvious. Let us prove (1). Let $\rho > 0$ be arbitrarily small. Take a small compact neighborhood U_ξ of ξ in ∂X such that $|b_\xi(q,p) - b_{\xi'}(q,p)| < \rho$ for any $\xi' \in U_\xi$. Let $R > 0$ be large enough such that

(1) $|\frac{\mathrm{Vol}(A_R)}{e^{hR}} - \nu_p(U_\xi)| < \rho$ where $A_R = (f_p^R)^{-1}U_\xi$;
(2) $|d(q, c_{p,\xi'}(R)) - R - b_{\xi'}(q,p)| < \rho$ for any $\xi' \in U_\xi$.

Now we divide U_ξ into finitely many sufficiently small compact subsets $U_\xi^i \subset U_\xi$, $i = 1,\ldots,k$ such that the following holds. By enlarging R if necessary,

$$\left|\frac{\mathrm{Vol}(\bar{A}_R^i)}{\mathrm{Vol}(A_R^i)} - 1\right| < \rho$$

where $A_R^i = (f_p^R)^{-1}U_\xi^i$, $\bar{A}_R^i = (f_q^{d(q,c_{p,\xi_i}(R))})^{-1}U_\xi^i$ and $\xi_i \in U_\xi^i$. Then

$$\begin{aligned}
\nu_q(U_\xi) &= \sum_{i=1}^k \nu_q(U_\xi^i) \le \sum_{i=1}^k \nu_q^{d(q,c_{p,\xi_i}(R))}(U_\xi^i) + \rho \\
&\le \sum_{i=1}^k \frac{1}{e^{h(R+b_{\xi_i}(q,p)-\rho)}}\mathrm{Vol}(\bar{A}_R^i) + \rho \\
&\le \frac{1+\rho}{e^{h(R+b_\xi(q,p)-2\rho)}}\sum_{i=1}^k \mathrm{Vol}(A_R^i) + \rho \\
&\le \frac{1+\rho}{e^{h(b_\xi(q,p)-2\rho)}}\nu_p^R(U_\xi) + \rho \\
&\le \frac{1+\rho}{e^{h(b_\xi(q,p)-2\rho)}}(\nu_p(U_\xi) + \rho) + \rho.
\end{aligned}$$

As U_ξ shrinks to $\{\xi\}$, $\rho > 0$ could be arbitrarily small. So $\frac{d\nu_q}{d\nu_p}(\xi) \le e^{-h\cdot b_\xi(q,p)}$. By symmetry, we get $\frac{d\nu_q}{d\nu_p}(\xi) = e^{-h\cdot b_\xi(q,p)}$.

It follows that $\{\nu_x\}_{x\in X}$ is an h-dimensional Busemann density. By [34, Theorem B], there exists exactly one Busemann density up to a scalar constant, which is realized by the Patterson–Sullivan measure. Thus $\{\nu_x\}_{x\in X}$ coincide with $\{\mu_x\}_{x\in X}$ up to a scalar constant.

Proof of Theorem B Recall that $\bar{\mu}_x$ is the normalized Patterson–Sullivan measure. By the above discussion, particularly $\nu_x(\partial X) = c(x)$ and $\frac{d\nu_y}{d\nu_x}(\xi) = e^{-h\cdot b_\xi(y,x)}$, we have

$$\frac{d\bar{\mu}_y}{d\bar{\mu}_x}(\xi) = \frac{c(x)}{c(y)}e^{-h\cdot b_\xi(y,x)}.$$

Then $c(y) = c(x)\int_{\partial X} e^{-h\cdot b_\xi(y,x)}d\bar{\mu}_x(\xi)$.

Recall that $y \mapsto b_\xi(y,x)$ is C^2 [18, Theorem 2] (see also [31, Section 2.3]), and moreover both $\nabla_y b_\xi(y,x)$ and $\triangle_y b_\xi(y,x) = trU(y,\xi)$ depend continuously on ξ. It follows that c is C^1.

If c is constant, then $\int_{\partial X} e^{-h\cdot b_\xi(y,x)}d\bar{\mu}_x(\xi) \equiv 1$. Taking Laplacian with respect to y on both sides, we have

$$\int_{\partial X} h(h - trU(y,\xi))e^{-h\cdot b_\xi(y,x)}d\bar{\mu}_x(\xi) \equiv 0.$$

It follows that

$$h = \frac{\int_{\partial X} trU(y,\xi)e^{-h\cdot b_\xi(y,x)}d\bar{\mu}_x(\xi)}{\int_{\partial X} e^{-h\cdot b_\xi(y,x)}d\bar{\mu}_x(\xi)} = \int_{\partial X} trU(y,\xi)d\bar{\mu}_y(\xi)$$

for any $y \in X$. □

7 Rigidity in dimension two

7.1 Unique ergodicity of horocycle flow

Let M be a rank one closed Riemannian surface without focal points in this section. A horocyclic flow is a continuous flow h_s on SM whose orbits are horocycles, i.e., for $v \in SM$, $\{h_s v : s \in \mathbb{R}\} = W^s(v)$, where $W^s(v)$ is the strong stable horocycle manifold of v in SM.

Clotet [13] proved recently the unique ergodicity of the horocycle flow on a compact surface of genus at least 2 without conjugate points and with continuous Green bundle, see also [12] for the case of nonpositive curvature. We can focus on surfaces without focal points which will be used in our arguments later.

If M has constant negative curvature, Furstenberg [19] proved the unique ergodicity of the horocycle flow, which is extended to compact surfaces of variable negative curvature by Marcus [35]. To apply Marcus's method to surfaces without focal points, we need to define the horocycle flow using the so-called *Margulis parametrization*. Gelfert–Ruggiero [21] defined a quotient map $\chi : SM \to Z$ by an equivalence relation "collapsing" each flat strip to a single curve, which semiconjugates the geodesic flow on SM and a continuous flow on Z. They show that Z is a topological 3-manifold, and that the quotient flow is expansive, topologically mixing and has local product structure. In [13], the horocycle flow with Margulis parametrization h_s^M is defined on Z. Using Coudène's theorem [14], it showed in [13, Proposition 4.2] that the horocycle flow h_s^M on Z is uniquely ergodic, and the unique invariant measure is $\chi_* m$, the projection of Knieper measure m onto Z.

There is another natural parametrization of the horocycle flow on SM given by arc length of the horocycles, which clearly is well defined everywhere. It is called the *Lebesgue parametrization* and the Lebesgue horocycle flow is denoted by h_s^L. By constructing complete transversals to respective flows, it is showed in [13, Theorem 5.8] that there is a bijection between finite Borel measures invariant under h_s^M on Z and h_s^L on SM respectively.

By the above discussion, h_s^L is also uniquely ergodic [13, Theorem 5.10]. We denote by w^s the unique probability measure invariant under the *stable* horocycle flow h_s^L.

In [13], the subset of generalized rank one vectors is defined as

$$R_1 := \{v \in SX : G^u(v) \neq G^s(v)\}$$

where G^s and G^u are Green bundles. Clearly R_1 is nonempty and open. Let $v \in R_1$, $W^{wu}(v) := \{w \in SX : w^- = v^-\}$ the weak unstable manifold of $v \in SX$, and $W^u(v) := \{w \in W^{wu}(v) : b_{v^-}(\pi w, \pi v) = 0\}$ the strong unstable horocycle manifold of v. Put $W^{ws}(v) := -W^{wu}(-v)$ and $W^s(v) := -W^u(-v)$. Then $W^{wu}(v)$ contains a relatively compact neighborhood T of v in R_1, such that pr$T$ is a complete transversal to the (stable) horocycle flow h_s^L in the sense of [13, Defintion 5.5]. So locally for a subset E in a neighborhood of v,

$$w^s(E) = \int_{W^{wu}(v)} \int_{\mathbb{R}} 1_E(h_s^L(u)) ds d\mu_{W^{wu}(v)}(u)$$

where 1_E is the characteristic function of E, and $\mu_{W^{wu}(v)}$ is some Borel measure on $T \subset W^{wu}(v)$ which is in fact independent of the parametrization of the horocycle flow. Note that χ is a homeomorphism in a neighborhood of $v \in R_1$.

On the other hand, the unique invariant measure for h_s^M on Z is the projection of Knieper measure m, which can be expressed as

$$m(E) = \int_{\partial X} \int_{\partial X} \int_{\mathbb{R}} 1_E(\xi, \eta, t) e^{h \cdot \beta_x(\xi, \eta)} dt d\mu_x(\xi) d\mu_x(\eta)$$

since E contains no flat strips. Consider the canonical projection

$$P = P_v : W^u(v) \to \partial X, \quad P(w) = w^+,$$

then

$$\mu_{W^{wu}(v)}(A) = \int \int_{\mathbb{R}} 1_A(\phi^t u) dt d\mu_{W^u(v)}(u)$$

where $\mu_{W^u(v)}(B) := \int_{\partial X} 1_B(P_v^{-1}\eta)e^{-hb_\eta(\pi v,x)}d\mu_x(\eta)$ and A, B are in a neighborhood of $v \in R_1$.

Thus w^s is locally equivalent to the measure $ds \times dt \times d\mu_{W^u}$. If we disintegrate w^s along W^u foliation, the factor measure on a section $W^{ws}(v)$ is equivalent to $dsdt$, i.e., the Lebesgue measure Vol, and the conditional measures on the fiber $W^u(v)$ is equivalent to $P_v^{-1}\mu_x$.

7.2 Uniqueness of harmonic measure

We recall some facts from [49] on the ergodic properties of foliations. Let $\mathcal{G}$ be any foliation on a compact Riemannian manifold M. A probability measure ν on M is called *harmonic* with respect to $\mathcal{G}$ if $\int_M \triangle^L f d\nu = 0$ for any bounded measurable function f on M which is smooth in the leaf direction, where $\triangle^L$ denotes the Laplacian in the leaf direction.

A *holonomy invariant measure* of the foliation $\mathcal{G}$ is a family of measures defined on each transversal of $\mathcal{G}$, which is invariant under all the canonical homeomorphisms of the holonomy pseudogroup [40]. A measure is called *completely invariant* with respect to $\mathcal{G}$ if it disintegrates to a constant function times the Lebesgue measure on the leaf, and the factor measure is a holonomy invariant measure on a transversal. By [20], a completely invariant measure must be a harmonic measure.

Theorem 7.1 *Let M be a rank one closed Riemannian surface without focal points. Then there is precisely one harmonic probability measure with respect to the strong stable horocycle foliation.*

Proof If $\dim M = 2$, then the leaves of the strong stable horocycle foliation have polynomial volume growth. By [23], any harmonic measure must be completely invariant. By [13] or the previous subsection, there is a unique completely invariant measure w^s. As a consequence, w^s is the unique harmonic measure. □

7.3 Integral formulas for topological entropy

Recall that M is a rank one closed Riemannian surface without focal points. Using the measure w^s we can establish some formulas for topological entropy h of the geodesic flow.

Let $B^s(v, R)$ denote the ball centered at v of radius $R > 0$ inside $W^s(v)$. In fact, it is just a curve. By the uniqueness of harmonic measure w^s, we have

Lemma 7.2 (Cf. [49, Theorem 1.2]) *For any continuous $\varphi : SM \to \mathbb{R}$,*

$$\frac{1}{Vol(B^s(v,R))}\int_{B^s(v,R)} \varphi dVol(y) \to \int_{SM} \varphi dw^s$$

as $R \to \infty$ uniformly in $v \in SM$.

For continuous $\varphi : SM \to \mathbb{R}$, define $\varphi_x : X \to \mathbb{R}$ by $\varphi_x(y) = \varphi(v(y))$ where $v(y) \in SX$ is the unique vector such that $c_{v(y)}(0) = y$ and $c_{v(y)}(t) = x$ for some $t \geq 0$.

Based on Lemma 7.2, we get the following proposition. The proof is the same as the one before [49, Proposition 3.1] (see also [29, 30]), and hence will be skipped. The basic idea here is that horospheres in X can be approximated by geodesic spheres.

Proposition 7.3 *For any continuous $\varphi : SM \to \mathbb{R}$,*

$$\frac{1}{s_R(x)} \int_{S(x,R)} \varphi_x(y) dVol(y) \to \int_{SM} \varphi dw^s$$

as $R \to \infty$ uniformly in $x \in X$.

Theorem 7.4 *Let M be a rank one closed Riemannian surface without focal points. Then*

(1) $h = \int_{SM} trU(v) dw^s(v)$,
(2) $h^2 = \int_{SM} -tr\dot{U}(v) + (trU(v))^2 dw^s(v)$,
(3) $h^3 = \int_{SM} tr\ddot{U} - 3tr\dot{U}trU + (trU)^3 dw^s$,

where $U(v)$ and $trU(v)$ are the second fundamental form and the mean curvature of the horocycle $H_{\pi v}(v^+)$ at πv.

Proof Consider the following function

$$G_x(R) := \frac{s_R(x)}{e^{hR}} = \frac{1}{e^{hR}} \int_{S(x,R)} d\mathrm{Vol}(y).$$

Taking the derivatives, we have

$$\begin{aligned}
G_x'(R) &= -hG_x(R) + \frac{1}{e^{hR}} \int_{S(x,R)} trU_R(y) d\mathrm{Vol}(y), \\
G_x''(R) &= -h^2 G_x(R) - 2hG_x'(R) \\
&\quad + \frac{1}{e^{hR}} \int_{S(x,R)} -tr\dot{U}_R(y) + (trU_R(y))^2 d\mathrm{Vol}(y), \\
G_x'''(R) &= -h^3 G_x(R) - 3h^2 G_x'(R) - 3hG_x''(R) \\
&\quad + \frac{1}{e^{hR}} \int_{S(x,R)} tr\ddot{U}_R(y) - 3tr\dot{U}_R(y) trU_R(y) + (trU_R(y))^3 d\mathrm{Vol}(y),
\end{aligned}$$

where $U_R(y)$ and $trU_R(y)$ are the second fundamental form and the mean curvature of $S(x, R)$ at y.

Clearly, $trU_R(y) \to trU(v(y))$ as $R \to \infty$ uniformly. By Theorem A,

$$\lim_{R\to\infty} G_x(R) = \lim_{R\to\infty} \frac{s_R(x)}{e^{hR}} = c(x).$$

Combining with Proposition 7.3, we have

$$\lim_{R\to\infty} G_x'(R) = -hc(x) + c(x) \int_{SM} trU dw^s,$$

$$\lim_{R\to\infty} G''_x(R) = -h^2c(x) - 2h \lim_{R\to\infty} G'_x(R) + c(x)\int_{SM} -tr\dot{U} + (trU)^2 dw^s,$$
$$\lim_{R\to\infty} G'''_x(R) = -h^3c(x) - 3h^2 \lim_{R\to\infty} G'_x(R) - 3h \lim_{R\to\infty} G''_x(R)$$
$$+c(x)\int_{SM} tr\ddot{U} - 3tr\dot{U}trU + (trU)^3 dw^s. \tag{16}$$

Since $\lim_{R\to\infty} G'_x(R)$ exists and $\lim_{R\to\infty} G_x(R)$ is bounded, we have $\lim_{R\to\infty} G'_x(R) = 0$. Similarly, considering the second and third derivative, we have

$$\lim_{R\to\infty} G''_x(R) = \lim_{R\to\infty} G'''_x(R) = 0.$$

Plugging in (16), we have

(1) $h = \int_{SM} trU(v)dw^s(v)$,
(2) $h^2 = \int_{SM} -tr\dot{U}(v) + (trU(v))^2 dw^s(v)$,
(3) $h^3 = \int_{SM} tr\ddot{U} - 3tr\dot{U}trU + (trU)^3 dw^s$.

□

7.4 Rigidity

Recall that $\tilde{\mu}_x$ is a Borel measure on S_xX (hence descending to S_xM) induced by the Patterson–Sullivan measure μ_x and let us assume that it is normalized, by a slight abuse of the notation. We have the following characterization of w^s.

Proposition 7.5 *For any continuous $\varphi : SM \to \mathbb{R}$, we have*

$$C\int_{SM} \varphi dw^s = \int_M c(x)\int_{S_xM} \varphi d\tilde{\mu}_x(v) dVol(x)$$

where $C = \int_M c(x)dVol(x)$.

Proof The idea is to show the right hand side is a harmonic measure up to a normalization. Then the proposition follows from Theorem 7.1. The proof is completely parallel to that of [49, Proposition 4.1] (see also [30, 48]), and hence is omitted. □

Proof of Theorem C By Theorem 7.4,

$$h^2 = \int_{SM} -tr\dot{U}(v) + (trU(v))^2 dw^s(v).$$

By the Riccati equation, in dimension two we have

$$-\dot{U} + U^2 + K = 0$$

where K is the Gaussian curvature. Since now U is just a real number and hence $tr(U^2) = (trU)^2$, using Proposition 7.5 and Gauss-Bonnet formula we have

$$h^2 = \int_{SM} -K dw^s = \frac{1}{C}\int_M -c(x)K(x)d\mathrm{Vol}(x)$$
$$= \int -K d\mathrm{Vol}/\mathrm{Vol}(M) = -2\pi E/\mathrm{Vol}(M),$$

where E is the Euler characteristic of M. By Katok's result [24, Theorem B], $h^2 = -2\pi E/\mathrm{Vol}(M)$ if and only if M has constant negative curvature. □

8 Flip invariance of the Patterson–Sullivan measure

For each $x \in X$, denote by $\tilde{\mu}_x$ both the Borel probability measure on S_xX and ∂X given by the normalized Patterson–Sullivan measure. Define a measure w^s by

$$C\int_{SM} \varphi dw^s := \int_M c(x)\int_{S_xM} \varphi d\tilde{\mu}_x(v)d\mathrm{Vol}(x)$$

for any continuous $\varphi : SM \to \mathbb{R}$, where $C = \int_M c(x)d\mathrm{Vol}(x)$.

In view of the proof of Proposition 7.5, w^s is a harmonic measure associated to the strong stable foliation, though the uniqueness of harmonic measure is unknown in general. Without the uniqueness of harmonic measure, we can still obtain some rigidity results in this section.

Proposition 8.1 *For $\varphi \in C^1(SM)$, one has*

$$\int_{SM} \dot{\varphi} + (h - trU)\varphi dw^s = 0.$$

Proof Define a vector field on M by

$$Y(y) := \int_{S_yM} \varphi X(v)d\tilde{\mu}_y(v) = \int_{S_xM} \varphi X(v)e^{-hb_v(y)}d\tilde{\mu}_x(v)$$

where X is the geodesic spray. Since $\nabla b_v = -X$ and $divX = -trU$, one has

$$div|_{y=x}Y = \int_{S_xM} div|_{y=x}\varphi X(v)e^{-hb_v(y)}d\tilde{\mu}_x(v)$$
$$= \int_{S_xM} \dot{\varphi} + (h - trU)\varphi d\tilde{\mu}_x.$$

Integrating with respect to Vol on M and using Green's formula, we have $\int_{SM} \dot{\varphi} + (h - trU)\varphi dw^s = 0$. □

Proposition 8.2 *If w^s is ϕ^t-invariant, then M is locally symmetric.*

Proof If w^s is ϕ^t-invariant, by Proposition 8.1, we have

$$\int_{SM}(h - trU)\varphi dw^s = 0$$

for all $\varphi \in C^1(SM, \mathbb{R})$. It follows that $trU \equiv h$, i.e., M is asymptotically harmonic. By [50, Theorem 1.2] (see also [31, Proposition 2.2]), M is locally symmetric. □

For manifolds without focal points, not every pair of $\eta \neq \xi$ in ∂X can be connected by a geodesic. A point $\xi \in \partial X$ is called *hyperbolic* if for any $\eta \neq \xi$ in ∂X, there exists a rank one geodesic joining η to ξ. The set of hyperbolic points is dense in ∂X (see [31, Lemma 3.4]).

Lemma 8.3 *If for all $x \in M$, $\tilde{\mu}_x$ is flip invariant, then the Knieper measure $\underline{m}$ coincides with the Liouville measure Leb on SM.*

Proof First we lift every measure to the universal cover X and show that for all $x \in X$, $\frac{d\tilde{\mu}_x}{d\text{Leb}_x}$ is finite everywhere on S_xX. We still denote the measures $f_x\tilde{\mu}_x$ and $f_x\text{Leb}_x$ on ∂X by $\tilde{\mu}_x$ and Leb_x for simplicity.

Assume that there exists some $\xi \in \partial X$ such that

$$\limsup_{\epsilon \to 0} \frac{\tilde{\mu}_x(D_x(\xi, \epsilon))}{\text{Leb}_x(D_x(\xi, \epsilon))} = 0 \tag{17}$$

where $D_x(\xi, \epsilon) := \{\eta \in \partial X : \angle_x(\xi, \eta) \leq \epsilon\}$. Take $\epsilon > 0$. For any $\rho > 0$ small enough, choose a hyperbolic $\xi' \in \partial X$ close to ξ such that

$$D_x(\xi', (1-\rho)\epsilon) \subset D_x(\xi, \epsilon) \subset D_x(\xi', (1+\rho)\epsilon). \tag{18}$$

We can choose some constant $C_1 > 1$ independent of ϵ and ρ such that

$$\text{Leb}_x(D_x(\xi, \epsilon)) \leq \text{Leb}_x(D_x(\xi', (1+\rho)\epsilon)) \leq C_1\text{Leb}_x(D_x(\xi', (1-\rho)\epsilon)). \tag{19}$$

It follows from (17), (18) and (19) that

$$\frac{\tilde{\mu}_x(D_x(\xi', (1-\rho)\epsilon))}{\text{Leb}_x(D_x(\xi', (1-\rho)\epsilon))} \leq \frac{C_1\tilde{\mu}_x(D_x(\xi, \epsilon))}{\text{Leb}_x(D_x(\xi, \epsilon))}. \tag{20}$$

Then for any $\eta \in \partial X$, there exists a geodesic $c_{\xi'\eta}$ connecting ξ' and η. Take a point $y \in c_{\xi'\eta}$. Due to the flip invariance,

$$\frac{\tilde{\mu}_y(D_y(\xi', (1-\rho)\epsilon))}{\text{Leb}_y(D_y(\xi', (1-\rho)\epsilon))} = \frac{\tilde{\mu}_y(D_y(\eta, (1-\rho)\epsilon))}{\text{Leb}_y(D_y(\eta, (1-\rho)\epsilon))}.$$

Since the measures $\tilde{\mu}_y$ and $\tilde{\mu}_x$ (resp. Leb_y and Leb_x) are equivalent with positive Radon–Nikodym derivative, we have by (20),

$$\frac{\tilde{\mu}_y(D_x(\xi',(1-\rho)\epsilon))}{\mathrm{Leb}_y(D_x(\xi',(1-\rho)\epsilon))} \le \frac{C_2\tilde{\mu}_x(D_x(\xi,\epsilon))}{\mathrm{Leb}_x(D_x(\xi,\epsilon))}$$

for some $C_2 > 1$. Then by flip invariance,

$$\frac{\tilde{\mu}_y(U_\epsilon(\eta))}{\mathrm{Leb}_y(U_\epsilon(\eta))} \le \frac{C_2\tilde{\mu}_x(D_x(\xi,\epsilon))}{\mathrm{Leb}_x(D_x(\xi,\epsilon))}$$

where $U_\epsilon(\eta)$ is the image of $D_x(\xi',(1-\rho)\epsilon)$ under the flip map. Use again that $\tilde{\mu}_y$ and $\tilde{\mu}_x$ (resp. Leb_y and Leb_x) are equivalent with positive Radon–Nikodym derivative, we get for some $C_3 > 1$

$$\frac{\tilde{\mu}_x(U_\epsilon(\eta))}{\mathrm{Leb}_x(U_\epsilon(\eta))} \le \frac{C_3\tilde{\mu}_x(D_x(\xi,\epsilon))}{\mathrm{Leb}_x(D_x(\xi,\epsilon))}.$$

As $U_\epsilon(\eta)$ shrinks to $\{\eta\}$ as $\epsilon \to 0$, we see the Radon–Nikodym derivatives $\frac{d\tilde{\mu}_x}{d\mathrm{Leb}_x}$ is also zero at any $\eta \in \partial X$.

Similarly, if

$$\limsup_{\epsilon\to 0} \frac{\tilde{\mu}_x(D_x(\xi,\epsilon))}{\mathrm{Leb}_x(D_x(\xi,\epsilon))} = \infty$$

for some $\xi \in \partial X$, the Radon–Nikodym derivatives $\frac{d\tilde{\mu}_x}{d\mathrm{Leb}_x}$ is also infinity at any $\eta \in \partial X$.

Since both $\tilde{\mu}_x$ and Leb_x have finite total mass, their Radon–Nikodym derivatives must be finite somewhere and hence everywhere. Thus the Liouville measure Leb is equivalent to the Knieper measure. As the Knieper measure is ergodic, the two measures coincide. □

Lemma 8.4 *If for all $x \in M$, $\tilde{\mu}_x$ is flip invariant, then the Margulis function $c(x)$ is constant.*

Proof Any $\varphi \in C^2(M,\mathbb{R})$ can be lifted to a function on SM which we still denote by φ. Since any weak unstable manifold is diffeomorphic to X, we have $\triangle^{cs}\varphi = \triangle\varphi$ where $\triangle$ is the Laplacian along X and $\triangle^{cs}$ is the Laplacian along the weak stable foliation. By [46, Lemma 5.1], $\triangle^{cs}\varphi = \triangle^s\varphi + \ddot{\varphi} - trU\dot{\varphi}$. Then by definition of w^s and Proposition 8.1,

$$\begin{aligned}
&\int_M \triangle\varphi c(x) d\mathrm{Leb}\\
&= C\int \triangle\varphi dw^s\\
&= C\int_{SM} (\triangle^s\varphi + \ddot{\varphi} - trU\dot{\varphi}) dw^s
\end{aligned}$$

$$= C\left(\int_{SM} \triangle^s \varphi dw^s + \int_{SM} \ddot{\varphi} + (h - trU)\dot{\varphi} dw^s - \int_{SM} h\dot{\varphi} dw^s\right)$$
$$= -h\int_M c(x) d\text{Leb}(x) \int \dot{\varphi}(x,\xi) d\tilde{\mu}_x(\xi).$$

Since $d\tilde{\mu}_x(\xi) = d\tilde{\mu}_x(-\xi)$ and $\dot{\varphi}(x,\xi) = -\dot{\varphi}(x,-\xi)$, we have

$$\int_M \triangle\varphi c(x) d\text{Leb} = 0$$

for any $\varphi \in C^2(M,\mathbb{R})$. So $c(x)$ must be constant. □

Proof of Theorem D By the construction, the Knieper measure $\underline{m}$ is flip invariant. By the flip invariance of the partition $\{S_x M\}_{x\in M}$ and the uniqueness of conditional measures, we see that $\bar{\mu}_x$ is flip invariant for $\underline{m}$-a.e. $x \in M$. It follows that the normalized Patterson–Sullivan measures $\tilde{\mu}_x$ is flip invariant for $\underline{m}$-a.e. $x \in M$.

We claim that for all $x \in M$, $\tilde{\mu}_x$ is flip invariant. Indeed, note that for fixed x, the density

$$\frac{d\tilde{\mu}_y}{d\tilde{\mu}_x}(\xi) = \frac{c(x)}{c(y)} e^{-h\cdot b_\xi(y,x)},$$

is uniformly continuous in y. For each continuous function $\varphi : \partial X \to \mathbb{R}$, its geodesic reflection with respect to $z \in X$ is defined by $\varphi_z(\xi) := \varphi(c_{z,\xi}(-\infty))$. Let $x_k, x \in X$ and $x_k \to x$ as $k \to \infty$. Then by the above continuity,

$$\int_{\partial X} \varphi d\tilde{\mu}_x = \lim_{k\to\infty}\int_{\partial X} \varphi d\tilde{\mu}_{x_k} = \lim_{k\to\infty}\int_{\partial X} \varphi_{x_k} d\tilde{\mu}_{x_k} = \int_{\partial X} \varphi_x d\tilde{\mu}_x.$$

The claim follows.

By Lemma 8.3, the Knieper measure $\underline{m}$ coincides with the Liouville measure, and thus $\underline{m}$ projects to the Riemannian volume on M. By assumption, the conditional measures $\bar{\mu}_x$ coincides with $\tilde{\mu}$. Moreover, by Lemma 8.4, $c(x)$ is constant. Consequently, we see from definition that w^s coincides with the Knieper measure $\underline{m}$, and hence it is ϕ^t-invariant. By Proposition 8.2, M is locally symmetric. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix: Manifolds without conjugate points

In this appendix, we discuss the proof of Theorem A', which is an extension of Theorem A to manifolds without conjugate points belonging to the class $\mathcal{H}$. The proof is analogous to that of Theorem A with minor modifications. We skip the details and just sketch main steps where modifications are needed.

The local product flow boxes are constructed in [11, Section 3.2] near expansive vectors in the case of no conjugate points. We need modify the time interval from $[0, \alpha]$ to $[-\alpha, -\alpha]$, so that Lemma 5.1 still holds.

Corresponding versions of π-convergence Theorem 4.1 should be established. Nevertheless, we just need rephrase and reprove [11, Lemma 4.9] accordingly with minor modifications.

In both the proofs of Lemmas 5.4 and 5.14, Lemma 5.3 is used. For manifolds in class $\mathcal{H}$, it is a direct consequence of uniform visibility property. Indeed, if T is large enough, then by the triangle inequality, the geodesic connecting $\phi^t v$ and $\phi^t w$ stays at distance at least $L(a)$ from p. Thus $\angle(v, w) < a$. So we also have these lemmas in no conjugate points case.

Finally, let us comment on Lemma 5.13. In the case of no conjugate points, instead of singular vectors we need consider vectors which do not lie in a countable union of flow boxes near expansive vectors. More precisely, there exist countably many expansive vectors $w_1, w_2, \ldots$ such that $S_x X \cap \mathcal{E} \subset \cup_{i=1}^{\infty} \mathrm{int} B_{\theta_i}^{\alpha}(w_i)$, where $\mathcal{E}$ is the expansive set. See [10, (2.11)] for definition of expansive vectors and expansive set. The vectors outside of these flow boxes form a subset S which is closed and ϕ^t-invariant. Moreover, $S \cap \mathcal{E} = \emptyset$. Since the unique MME $\underline{m}$ gives full weight to $\mathcal{E}$ (cf. [10, Theorem 5.6]), we know $\underline{m}(S) = 0$. It follows that $h_{\mathrm{top}}(S) < h$ and thus Lemma 5.13 can be proved similarly.

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