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The Hadamard formula for nonlocal eigenvalue problems

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Abstract

In this paper we obtain a Hadamard type formulae for simple and multiple eigenvalues for a class of nonlocal eigenvalue problems. The cases that we consider include among others, the classical nonlocal problems with Dirichlet and Neumann conditions. The Hadamard formula is computed allowing domain perturbations given by embeddings of n-dimensional Riemannian manifolds (possibly with boundary) of finite volume.

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1 Introduction

There are many works in the literature which connect the shape of a region to the eigenvalues and eigenfunctions of a given operator. In this context, the rate of change of simple eigenvalues plays an essential role and it has been studied since the pioneering work of Hadamard [12] who in 1908 first computed the domain derivative of a simple eigenvalue of the bi-Laplacian under Dirichlet boundary condition.

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Since then, the Hadamard formula has been generalized in a number of significant ways. Such generalizations include the use of Neumann and Robin boundary conditions, multiple eigenvalues, and second order variations for a large class of differential and integral operators. Among many references, we cite the monographs [13, 14, 24] and the recent works [10, 11, 18, 19, 21], all of them concerned with boundary perturbation problems to differential equations and their applications to eigenvalue problems.

In this work, we study a class of nonlocal eigenvalue problems with non-singular kernels on a n-dimensional Riemannian manifold $(\mathcal{M}, g)$ of finite volume. More precisely, we consider an operator $\mathcal{B}_{\mathcal{M}} : L^2(\mathcal{M}) \mapsto L^2(\mathcal{M})$ of the form

$$\mathcal{B}_{\mathcal{M}} u(x) = a_{\mathcal{M}}(x)u(x) - \int_{\mathcal{M}} J(x, y)u(y)\, dv_g(y), \quad x \in \mathcal{M}. \tag{1.1}$$

According to [20], nonlocal diffusion equations governed by these operators were used in early population genetics models, see for instance [7]. In Ecology, Othmer et al. [22] were the first authors to introduce a jump process to model the dispersion of individuals, which later, was generalized by Hutson et al. [16] associating the kernel of the nonlocal operator to a radial probability density. The prototype of the nonlocal equation is given by considering $a_{\mathcal{M}}(x) \equiv 1$. For instance, if one takes $\mathcal{M} = \Omega \subset \mathbb{R}^n$ a domain, $J(x, y) = J(|x - y|)$ for some nonnegative $J \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ with $J(0) > 0$ and $\int_{\mathbb{R}^n} J(|z|)dz = 1$, and assumes $u(x) \equiv 0$ in $\mathbb{R}^n \backslash \Omega$, the operator $\mathcal{B}_\Omega$ becomes

$$\mathcal{B}_\Omega u(x) = \int_{\mathbb{R}^n} J(|x - y|)(u(x) - u(y))\, dy, \quad x \in \Omega.$$

In the context of population models, the set $\Omega^c = \mathbb{R}^n \backslash \Omega$ represents a hostile surrounding, since the particles (whose density is set by u) die when they land in Ω^c. As observed, for instance in [1, 8], this is a nonlocal analog to the Laplace operator with Dirichlet boundary condition in bounded domains of $\mathbb{R}^n$.

On the other hand, if we take $a_\Omega(x) = \int_\Omega J(|x - y|)\, dy$, we get a nonlocal analog to the Laplacian with Neumann boundary condition

$$\mathcal{B}_\Omega u(x) = \int_\Omega J(|x - y|)(u(x) - u(y))\, dy, \quad x \in \Omega.$$

In this case the particle can just jump inside of Ω living in an isolated surrounding. As expected, under this Neumann condition, constants are eigenfunctions whenever one takes $\lambda(\Omega) = 0$.

Actually, several continuous models for species and human mobility have been proposed using such nonlocal equations, in order to look for more realistic dispersion models [2, 4, 5, 26]. Besides the applied models with such kernels, the mathematical interest is mainly due to the fact that, in general, there is no regularizing effect and therefore no general compactness tools are available.

In this paper, we obtain a Hadamard type formula for simple eigenvalues of the operator (1.1), under certain conditions for the kernel J. These conditions will be discussed in detail in Sect. 2 and we denote then by (H). Then our first result reads as follows.

Theorem 1.1 *Let λ_0 be a simple eigenvalue of $\mathcal{B}_{\mathcal{M}}$ with corresponding normalized eigenfunction u_0 and $J \in \mathcal{C}^1(\mathcal{N} \times \mathcal{N}, \mathbb{R})$ satisfying* (H). *Also, let us assume that $\Phi : \mathrm{Diff}^1(\mathcal{M}) \mapsto \mathcal{C}^1(\mathcal{M})$ given by $\Phi(h)(x) = (h^* a_{\mathcal{M}_h})(x)$, $x \in \mathcal{M}$, is differentiable as a map defined between Banach spaces.*

Then, there is a neighbourhood $\mathcal{O}$ of the inclusion $i_{\mathcal{M}} \in \mathrm{Diff}^1(\mathcal{M})$, and $\mathcal{C}^1$-functions u_h and λ_h from $\mathcal{O}$ into $L^2(\mathcal{M})$ and $\mathbb{R}$ respectively satisfying for all $h \in \mathcal{O}$ that

$$h^* \mathcal{B}_{h(\mathcal{M})} h^{*-1} u_h(x) = \lambda_h u_h(x), \quad x \in \mathcal{M}, \tag{1.2}$$

with $u_h \in \mathcal{C}^1(\mathcal{M})$. Moreover, λ_h is a simple eigenvalue with $(\lambda_{i_{\mathcal{M}}}, u_{i_{\mathcal{M}}}) = (\lambda_0, u_0)$ and the domain derivative

$$\begin{aligned} \frac{\partial \lambda}{\partial h}(i_{\mathcal{M}})V = &\int_{\mathcal{M}} u_0^2(w)\, D_t^T \left(h^* a_{\mathcal{M}_h}\right)\big|_{t=0} dv_g(w) \\ &- \int_{\partial\mathcal{M}} (a_{\mathcal{M}}(s) - \lambda_0)\, u_0^2(s)\, \langle V^T, N\rangle(s)\, dS \\ &- \int_{\mathcal{M}} (a_{\mathcal{M}} - \lambda_0) u_0^2(w) \langle \vec{H}, V^{\perp}\rangle dv_g(w) \\ &- \int_{\mathcal{M}} u_0^2(w) \langle \nabla_w a_{\mathcal{M}}, V^{\perp}\rangle dv_g(w), \end{aligned} \tag{1.3}$$

for all $V \in \mathcal{X}^1(\mathcal{N})$ where $\mathcal{X}^1(\mathcal{N})$ denotes the set of $\mathcal{C}^1$ vector fields on $\mathcal{N}$ and $D_t^T f = \frac{\partial f}{\partial t} - \langle V^T, \nabla f\rangle$, V^T is the component of V tangential to $\mathcal{M}$ and h^* is the composition map set by the embedding h. Note that at the boundary the tangent space of $\mathcal{M}$ splits into vectors that are tangential to $\partial\mathcal{M}$ (and to $\mathcal{M}$) and one vector that is normal to $\partial\mathcal{M}$ (and tangential to $\mathcal{M}$). Then $N \in T(\mathcal{M})$ denotes this unitary normal vector that is normal to $\partial\mathcal{M}$. $\vec{H}$ is the mean curvature vector associated to $\mathcal{M}$ and $V^{\perp}$ is the component of V normal to $\mathcal{M}$.

As a corollary of this result we obtain

Corollary 1.1 *Let u_h be the family of eigenfunctions associated with the operator $\mathcal{B}_{h(\mathcal{M})}$ and eigenvalues λ_h given by Theorem 1.1.*

Then, the derivative of u_h at $h = i_{\mathcal{M}}$ and $V \in \mathcal{X}^1(\mathcal{N})$ is the unique solution of

$$(\lambda_0 - \mathcal{B}_{\mathcal{M}})w = f_V$$

where $f_V \in L^2(\mathcal{M})$ is the function given by

$$\begin{aligned} f_V = &-\frac{\partial \lambda}{\partial h}(i_{\mathcal{M}})V\, u_0 + \frac{\partial}{\partial t}\left(h^* a_{h(t,\mathcal{M})}\right)\Big|_{t=0} u_0 + \left[\mathcal{J}_{\mathcal{M}}, \langle V^T, \nabla(\cdot)\rangle\right] u_0 \\ &- \int_{\partial\mathcal{M}} J(y,w) u_0(w) \langle V^T, N\rangle\, dS(w) \end{aligned}$$

$$-\int_{\mathcal{M}} J(y,w)u_0\langle \vec{H}, V^{\perp}\rangle dv_g(w)$$
$$-\int_{\mathcal{M}} u_0(w)\,\langle \nabla_w(J(y,w)), V^{\perp}\rangle dv_g(w)$$

with $\frac{\partial\lambda}{\partial h}(i_{\mathcal{M}})V$ *given by* (1.3), $\mathcal{J}$ *is the non-local term in* $\mathcal{B}_{h(\mathcal{M})}$ *(see* (2.7) *for a precise definition) and* $[\cdot,\cdot]$ *denotes the commutator.*

We also analyze the case of eigenvalues of higher multiplicity, obtaining the following result.

Theorem 1.2 *Let* λ_0 *be an eigenvalue of multiplicity* m *of the operator* $\mathcal{B}_{\mathcal{M}}$ *with* $m > 1$. *Assume the non-singular kernel* $J : \mathcal{N}\times\mathcal{N} \mapsto \mathbb{R}$ *is an analytic function satisfying condition* (H), $h(t,x) = x + tV(x)$ *for all* $t \in \mathbb{R}$ *and* $x \in \mathcal{M}$ *for some* $V \in \mathcal{X}^1(\mathcal{N})$. *In addition assume that* $\Phi : \mathbb{R} \mapsto \mathcal{C}^1(\mathcal{M})$, *given by* $\Phi(t) = \left(h(t,\cdot)^* a_{\mathcal{M}_{h(t,\cdot)}}\right)$ *is an analytic map.*

Then, if $\lambda(t)$ *is one of the curves given by Lemma* 4.1, *we have that* $\dot{\lambda} = \frac{\partial\lambda}{\partial t}(0)$ *is an eigenvalue of the symmetric matrix* $B = (B_{ki})_{k,i=1}^m$ *defined by*

$$\begin{aligned} B_{ki} = &\int_{\mathcal{M}} D_t^T\left(h(t,\cdot)^* a_{h(t,\mathcal{M})}\right)\Big|_{t=0}\phi_k\phi_i\,dv_g - \int_{\partial\mathcal{M}}(a_{\mathcal{M}}-\lambda_0)\,\phi_i\phi_k\,\langle V^T, N\rangle dS \\ &- \int_{\mathcal{M}}(a_{\mathcal{M}}-\lambda_0)\,\phi_k\phi_i\,\langle \vec{H}, V^{\perp}\rangle\,dv_g - \int_{\mathcal{M}}\phi_i\phi_k\langle\nabla_y a_{\mathcal{M}}, V^{\perp}\rangle\,dv_g(y) \end{aligned} \quad (1.4)$$

where N *is the unitary normal to* $\partial\mathcal{M}$, $D_t^T f = \frac{\partial f}{\partial t} - \langle V^T, \nabla f\rangle$, V^T *is the component of* V *tangential to* $\mathcal{M}$, $\vec{H}$ *is the mean curvature vector of* $\mathcal{M}$ *and* $\{\phi_1,\ldots,\phi_m\}$ *is an orthonormal basis for the eigenspace associated to* λ_0.

Notice that in the simplest case where $\mathcal{M}$ is an open domain of $\mathbb{R}^{n+1}$ (with co-dimension 0) we have $V^{\perp} = 0$, and then, $V = V^T$. Hence, by Theorem 1.1, the Hadamard formula for the simple eigenvalues of the Dirichlet problem (set by $a_{\mathcal{M}}(x) \equiv 1$ at (1.1)) becomes

$$\frac{\partial\lambda}{\partial h}(i_{\mathcal{M}})V = -\int_{\partial\mathcal{M}}(1-\lambda_0)\,u_0^2(s)\,\langle V, N\rangle\,dS(s)$$

which agrees with the previous results obtained in [3, 9]. More generally, if $\lambda(t)$ is the analytic curve of eigenvalues given by Theorem 1.2 for the same Dirichlet problem with $\lambda_0 = \lambda(0)$ being an m-fold eigenvalue, then, the rate $\frac{d\lambda}{dt}(0)$ is an eigenvalue of the matrix $B = (B_{ki})_{k,i=1}^m$ with

$$B_{ki} = -\int_{\partial\mathcal{M}}(1-\lambda_0)\,\langle V, N\rangle\phi_k(x)\phi_i(x)dS(x).$$

Other examples with additional geometric features can be found at the end of Sect. 3. One of them is if we consider $\mathcal{M} = \mathbb{S}^n$ and $\mathcal{N} = \mathbb{R}^{n+1}$ taking $a \equiv 0$, then this example has higher co-dimension, $V^{\perp}$ is not necessarily equal to 0, $\vec{H}(p) = \frac{n}{R} p$ and $\frac{d\lambda}{dt}(0)$ is an eigenvalue of the matrix $B = (B_{ki})_{k,i=1}^{m}$ with

$$B_{ki} = \frac{n\lambda_0}{R} \int_{\mathbb{S}^n} \phi_k(w)\phi_i(w)\langle w, V^{\perp}\rangle \, dv_g(w).$$

The organization of the paper is as follows. In Sect. 2 we discuss the set-up of our problem, including the assumptions on our operators and preliminary results. The Hadamard formulae for simple and multiple eigenvalues are computed respectively in Sects. 3 and 4, in both cases using the approach developed in [14] to deal with boundary perturbation problems. Finally, in Sect. 4.1, we discuss an application of our results to the generic simplicity of the eigenvalues for the Dirichlet problem set by (1.1) in open bounded sets of $\mathbb{R}^n$.

2 Our nonlocal eigenvalue problem

Throughout this paper we consider an n-dimensional Riemannian manifold $(\mathcal{M}, g)$ (possibly with boundary) of finite volume and we set the following nonlocal eigenvalue problem

$$a_{\mathcal{M}}(x)u(x) - \int_{\mathcal{M}} J(x, y)u(y)dv_g(y) = \lambda(\mathcal{M})\, u(x), \quad x \in \mathcal{M} \tag{2.5}$$

for some unknown value $\lambda(\mathcal{M})$ where $a_{\mathcal{M}} : \overline{\mathcal{M}} \mapsto \mathbb{R}$ is assumed to be a continuous function, and J is a non-singular kernel satisfying

(H) $J \in \mathcal{C}(\mathcal{M} \times \mathcal{M}, \mathbb{R})$ is a nonnegative, symmetric function $(J(x, y) = J(y, x))$ with $J(x, x) > 0$.

We also assume that $\int_{\mathcal{M}} J(x, y)dv_g(y) < \infty$.

Remark 2.1 Here, $dv_g(y)$ refers to the measure on the manifold, which in coordinates is equivalent to $\sqrt{g}(y)dy$ and g is the determinant of the matrix g_{ij}. The measure induced at the boundary of our manifold will be denoted by dS.

Notice that analyzing the spectral properties of (2.5) is equivalent to study the spectrum of the linear operator $\mathcal{B}_{\mathcal{M}} : L^2(\mathcal{M}) \mapsto L^2(\mathcal{M})$ given by

$$\mathcal{B}_{\mathcal{M}}u(x) = a_{\mathcal{M}}(x)u(x) - \int_{\mathcal{M}} J(x, y)u(y)\, dv_g(y), \quad x \in \mathcal{M}. \tag{2.6}$$

See that $\mathcal{B}_{\mathcal{M}}$ is the difference of the multiplication operator $a_{\mathcal{M}}$, which maps $u(x) \mapsto a_{\mathcal{M}}(x)u(x)$, and the integral operator $\mathcal{J}_{\mathcal{M}} : L^2(\mathcal{M}) \mapsto L^2(\mathcal{M})$ given by

$$\mathcal{J}_{\mathcal{M}}u(x) = \int_{\mathcal{M}} J(x, y)u(y)dv_g(y), \quad x \in \mathcal{M} \tag{2.7}$$

which is self-adjoint and compact by [25, Propositions 3.5 and 3.7] (since $\mathcal{M}$ is a measurable metric space.)

The spectrum of $\mathcal{B}_{\mathcal{M}}$

It is known from [25, Theorem 3.24] (see also [20, Theorem 2.2] for open bounded sets $\mathcal{M} = \Omega \subset \mathbb{R}^n$) that the spectrum set $\sigma(\mathcal{B}_{\mathcal{M}})$ of $\mathcal{B}_{\mathcal{M}}$ satisfies

$$\sigma(\mathcal{B}_{\mathcal{M}}) = \mathrm{R}(a_{\mathcal{M}} I) \cup \{\lambda_n(\mathcal{M})\}_{n=0}^{l} \tag{2.8}$$

for some $l \in \{0, 1, \ldots, \infty\}$ where $\mathrm{R}(a_{\mathcal{M}} I)$ denotes de range of the map $a_{\mathcal{M}} I$ and $\lambda_n(\mathcal{M})$ are the eigenvalues of $\mathcal{B}_{\mathcal{M}}$ with finite multiplicity. Also, the essential spectrum of $\mathcal{B}_{\mathcal{M}}$ is given by

$$\sigma_{ess}(\mathcal{B}_{\mathcal{M}}) = [m, M]$$

where

$$m = \min_{x \in \overline{\mathcal{M}}} a_{\mathcal{M}}(x) \quad \text{and} \quad M = \max_{x \in \overline{\mathcal{M}}} a_{\mathcal{M}}(x).$$

As a consequence of the characterization (2.8), we notice that the eigenfunctions of $\mathcal{B}_{\mathcal{M}}$ possess the same regularity of the functions J and $a_{\mathcal{M}}$. In fact, for all $x \in \mathcal{M}$, one has

$$\mathcal{B}_{\mathcal{M}} u(x) = \lambda(\mathcal{M}) u(x) \iff (a_{\mathcal{M}}(x) - \lambda(\mathcal{M})) u(x) = \int_{\mathcal{M}} J(x, y) u(y)\, dv_g(y). \tag{2.9}$$

On the other hand, the convolution-type operator $(J * u)(x) = \int_{\mathcal{M}} J(x, y) u(y)\, dv_g(y) \in \mathcal{C}^k(\overline{\mathcal{M}})$ whenever $J(\cdot, y) \in \mathcal{C}^k(\mathcal{M})$ for every $y \in \mathcal{M}$ and $u \in L^1(\mathcal{M})$. Therefore, if $\lambda(\mathcal{M})$ is an eigenvalue of $\mathcal{B}_{\mathcal{M}}$ with corresponding eigenfunction u, we obtain from (2.8) that $\lambda(\mathcal{M}) \in [m, M]^c$ implying that $a_{\mathcal{M}} - \lambda(\mathcal{M}) \neq 0$ in $\mathcal{M}$. Consequently, we get from (2.9) that

$$u \in \mathcal{C}^k(\overline{\mathcal{M}}) \quad \text{whenever} \quad J(\cdot, y) \text{ and } a_{\mathcal{M}} \text{ are } \mathcal{C}^k\text{-functions for every fixed } y \in \mathcal{M}$$

for $k = 0, 1, 2 \ldots$. In particular, u is analytic whenever $J(\cdot, y)$ and $a_{\mathcal{M}}$ are analytical for all $y \in \mathcal{M}$.

Under appropriate conditions, the existence of the principal eigenvalue of $\mathcal{B}_{\mathcal{M}}$ is guaranteed by [20, Theorem 2.1]. Recall that the principal eigenvalue of a linear and bounded operator is the minimum of the real part of the spectrum which is simple, isolated and it is associated with a continuous and strictly positive eigenfunction.

3 Hadamard formula for simple eigenvalues

In this section we prove Theorem 1.1 and Corollary 1.1. Specifically, we perturb simple eigenvalues of the operator $\mathcal{B}_{\mathcal{M}}$ computing derivatives with respect to several kinds of variations of the manifold $\mathcal{M}$. In the particular case of $\mathcal{M} = \Omega \subset \mathbb{R}^n$ a domain, our approach agrees with the one introduced in [14] for perturbing a fixed domain Ω by diffeomorphisms. As a consequence, we extend the expressions obtained for the domain derivative for simple eigenvalues given in [3, 9].

Let $(\mathcal{M}, g_{\mathcal{M}})$ and $(\mathcal{N}, g_{\mathcal{N}})$ be $\mathcal{C}^1$-regular manifolds ($\mathcal{M}$, possibly with boundary). Assume in addition that $\overline{\mathcal{M}}$ is compact. If $h : \mathcal{M} \mapsto \mathcal{N}$ is a $\mathcal{C}^1$ embedding, i.e., a diffeomorphism to its image, we set the composition map h^* (also called the pull-back) by

$$h^* \tilde{u}(x) = (\tilde{u} \circ h)(x), \quad x \in \mathcal{M},$$

when $\tilde{u}$ is any given function defined on $h(\mathcal{M})$. The metric on $\mathcal{N}$ induces the pullback metric on $\mathcal{M}$ through h as follows: for $u, v \in T_{h^{-1}(x)}\mathcal{M}$ we have $h^* g_{\mathcal{N}}(u, v) = g_{\mathcal{N}}(dh_x(u), dh_x(v))$. It is not difficult to see that $h^* : L^2(h(\mathcal{M}), g_{\mathcal{N}}) \mapsto L^2(\mathcal{M}, h^* g_{\mathcal{N}})$ is an isomorphism with inverse $(h^*)^{-1} = (h^{-1})^*$.

We assume that $\mathcal{N}$ has a Riemannian metric $g_{\mathcal{N}}$ and we denote by $g_h = h^* g_{\mathcal{N}}$ the metric on $\mathcal{M}$ induced by the embedding h. For instance, if $\mathcal{M} = \Omega \subset \mathbb{R}^n = \mathcal{N}$ then the metric $h^* g_{\mathcal{N}}$ is given by $g_{ij} = \frac{\partial h}{\partial x_i} \cdot \frac{\partial h}{\partial x_j}$ and in particular, if $h = id_{\mathbb{R}^n}$ in the interior of Ω the metrics of $\mathcal{M}$ and $\mathcal{N}$ agree.

In general, for any embedding h we can consider the operator

$$\begin{aligned} \left(\mathcal{B}_{h(\mathcal{M})} \tilde{u}\right)(y) &= (a_{h(\mathcal{M})} \circ h)(x)(\tilde{u} \circ h)(x) - \left(\mathcal{J}_{h(\mathcal{M})} \tilde{u}\right)(h(x)) \\ &= (h^* a_{h(\mathcal{M})})(x)\,(h^* \tilde{u})(x) - \left(h^* \mathcal{J}_{h(\mathcal{M})} \tilde{u}\right)(x) \end{aligned} \tag{3.10}$$

if $y = h(x)$ for $x \in \mathcal{M}$, where $a_{h(\mathcal{M})} : h(\mathcal{M}) \mapsto \mathbb{R}$ is assumed to be a continuous function in $\overline{h(\mathcal{M})}$ for any isomorphism h. Notice that $\mathcal{B}_{h(\mathcal{M})} : L^2(h(\mathcal{M}), g_{\mathcal{N}}) \mapsto L^2(h(\mathcal{M}), g_{\mathcal{N}})$ is a self-adjoint operator for any h as is the operator $\mathcal{J}_{h(\mathcal{M})}$.

On the other hand, we can use the pull-back operator h^* to consider

$$h^* \mathcal{B}_{h(\mathcal{M})} h^{*-1} : L^2(\mathcal{M}) \mapsto L^2(\mathcal{M})$$

defined by $h^* \mathcal{B}_{h(\mathcal{M})} h^{*-1} u(x) = \mathcal{B}_{h(M)}\,(u \circ h)\left(h^{-1}(x)\right)$. Hence,

$$h^* \mathcal{B}_{h(\mathcal{M})} h^{*-1} u(x) = (h^* a_{h(\mathcal{M})})(x)\, u(x) - \left(h^* \mathcal{J}_{h(\mathcal{M})} h^{*-1} u\right)(x), \quad \forall x \in \mathcal{M}. \tag{3.11}$$

As it is known, expressions (3.10) and (3.11) are the customary way to describe deformations or motions of regions. Equation (3.10) is called the Eulerian description, and (3.11) the Lagrangian one. The latter is written in fixed coordinates while the Eulerian is not.

Due to (3.10) and (3.11), it is easy to see that

$$h^*\mathcal{B}_{h(\mathcal{M})}h^{*-1}u(x) = \mathcal{B}_{h(\mathcal{M})}\tilde{u}(y) \quad \text{and} \quad h^*\mathcal{J}_{h(\mathcal{M})}h^{*-1}u(x) = \mathcal{J}_{h(\mathcal{M})}\tilde{u}(y) \tag{3.12}$$

whenever $y = h(x)$ and $\tilde{u}(y) = (u \circ h^{-1})(y) = h^{*-1}u(y)$ for $y \in h(\mathcal{M})$.

Moreover, we have $\mathcal{B}_{h(\mathcal{M})}\tilde{u}(y) = \lambda\tilde{u}(y)$ for $y \in h(\mathcal{M})$ and some value λ, if and only if,

$$h^*\mathcal{B}_{h(\mathcal{M})}h^{*-1}u(x) = \lambda u(x), \quad \forall x \in \mathcal{M},$$

with $\tilde{u}(y) = h^{*-1}u(y)$. Hence, λ is an eigenvalue of multiplicity $m \in \mathbb{N}$ of the operator $\mathcal{B}_{h(\mathcal{M})}$, if and only if, is an eigenvalue of multiplicity m of $h^*\mathcal{B}_{h(\mathcal{M})}h^{*-1}$. As $\mathcal{B}_{h(\mathcal{M})}$ is a self-adjoint operator for any embedding h, we obtain that the spectrum of $h^*\mathcal{B}_{h(\mathcal{M})}h^{*-1}$ is also a subset of the real line. We have the following.

Proposition 3.1 *Let $h : \mathcal{M} \mapsto \mathcal{N}$ be an embedding. Then, $\sigma\left(h^*\mathcal{B}_{h(\mathcal{M})}h^{*-1}\right) = \sigma\left(\mathcal{B}_{h(\mathcal{M})}\right) \subset \mathbb{R}$ where $\sigma\left(\mathcal{B}_{h(\mathcal{M})}\right)$ is given by (2.8). More precisely, $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{B}_{h(\mathcal{M})}$ with multiplicity $m \in \mathbb{N}$, if and only if, is an eigenvalue of $h^*\mathcal{B}_{h(\mathcal{M})}h^{*-1}$ with multiplicity m. Also,*

$$\sigma_{ess}\left(h^*\mathcal{B}_{h(\mathcal{M})}h^{*-1}\right) = \sigma_{ess}\left(\mathcal{B}_{h(\mathcal{M})}\right).$$

Proof As $\mathcal{B}_{h(\mathcal{M})}$ is a self-adjoint operator in $L^2(h(\mathcal{M}), g_h)$, we have that $\sigma\left(\mathcal{B}_{h(\mathcal{M})}\right) \subset \mathbb{R}$. It follows from the relationship (3.12) that a value λ is an eigenvalue of $\mathcal{B}_{h(\mathcal{M})}$ with multiplicity $m \in \mathbb{N}$, if and only if, is an eigenvalue of $h^*\mathcal{B}_{h(\mathcal{M})}h^{*-1}$ with the same multiplicity. Thus, $\sigma\left(h^*\mathcal{B}_{h(\mathcal{M})}h^{*-1}\right) = \sigma\left(\mathcal{B}_{h(\mathcal{M})}\right) \subset \mathbb{R}$ with $\sigma\left(\mathcal{B}_{h(\mathcal{M}}\right)$ given by (2.8).

Now, it follows from (2.8) and expressions (2.6) and (3.11) that

$$\sigma_{ess}\left(\mathcal{B}_{h(\mathcal{M})}\right) = [m_h, M_h] \quad \text{and} \quad \sigma_{ess}\left(h^*\mathcal{B}_{h(\mathcal{M})}h^{*-1}\right) = [m_{h^*}, M_{h^*}]$$

where

$$m_h = \min_{y \in h(\mathcal{M})} a_{h(\mathcal{M})}(y), \quad M_h = \max_{y \in h(\mathcal{M})} a_{h(\mathcal{M})}(y)$$

and

$$m_{h^*} = \min_{x \in \mathcal{M}} a_{h(\mathcal{M})}(h(x)), \quad M_{h^*} = \max_{x \in \mathcal{M}} a_{h(\mathcal{M})}(h(x)).$$

As $m_h = m_{h^*}$ and $M_h = M_{h^*}$, the proof is complete. □

Remark 3.1 Notice that Proposition 3.1 guarantees that the essential spectrum of $\mathcal{B}_{h(\mathcal{M})}$ does not change under perturbations given by embeddings $h : \mathcal{M} \mapsto \mathcal{N}$.

From now on, we consider a family of embeddings $h : [0, T] \times \mathcal{M} \mapsto \mathcal{N}$ that depends on a parameter t. We denote the perturbed domain $h(t, \mathcal{M})$ by $\mathcal{M}_t$ in order to simplify the notation. We study the differentiability of simple eigenvalues $\lambda(\mathcal{M}_t)$ for $\mathcal{B}_{\mathcal{M}_t}$ with respect to t. This corresponds to the Gâteaux derivative with respect to the function h.

We remark that for a function $f : \mathcal{N} \to \mathbb{R}$ it holds that

$$\frac{d}{dt}\left(h^* f(x,t)\right) = \frac{d}{dt}\left(f(h(x,t),t)\right) = \left\langle h^*\nabla f, \frac{\partial h}{\partial t}\right\rangle + h^*\frac{\partial f}{\partial t},$$

where ∇ denotes de tangential gradient on $\mathcal{N}$. Then we denote

$$D_t = \frac{\partial}{\partial t} - \left\langle \frac{\partial h}{\partial t}, \nabla \right\rangle, \tag{3.13}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{N}$.

If $\mathcal{N} = \mathbb{R}^n$ and $\mathcal{M} = \Omega \subset \mathbb{R}^n$ this quantity can be written in coordinates as

$$D_t = \frac{\partial}{\partial t} - U(t,x) \cdot \frac{\partial}{\partial x} \quad \text{with} \quad U(t,x) = \frac{\partial h}{\partial x}^{-1} \frac{\partial h}{\partial t} \quad \text{for } x \in \Omega$$

and it is known as the anti-convective derivative D_t in the reference domain Ω.

We denote by $\mathrm{Diff}^1(\mathcal{M}) \subset \mathcal{C}^1(\mathcal{M}, \mathcal{N})$ the set of $\mathcal{C}^1$-functions $h : \mathcal{M} \mapsto \mathcal{N}$ which are embeddings. We assume that $\mathcal{N}$ has a Riemannian metric $g_{\mathcal{N}}$ and we denote by $g_h = h^* g_{\mathcal{N}}$ the metric on $\mathcal{M}$ induced by the embedding h. For instance, if $\mathcal{M} = \Omega \subset \mathbb{R}^n = \mathcal{N}$ is an open domain then the metric $h^* g_{\mathcal{N}}$ is given by $g_{ij} = \langle \partial_{x_i} h, \partial_{x_j} h \rangle$ and, if the dimension of Ω is n, the tangent spaces of $h(\Omega)$ and $\mathbb{R}^n$ agree in the interior and the volume element is $|Dh|\, dx$.

Consider the map

$$\begin{aligned} F : \mathrm{Diff}^1(\mathcal{M}) \times \mathbb{R} \times L^2(\mathcal{M}) &\mapsto L^2(\mathcal{M}) \times \mathbb{R} \\ (h, \lambda, u) &\mapsto \left(\left(h^* \mathcal{B}_{h(\mathcal{M})} h^{*-1} - \lambda \right) u, \int_{\mathcal{M}} u^2(x) dv_{g_h} \right). \end{aligned}$$

Here dv_{g_h} is the volume element of the metric on g_h. It is not difficult to see that $\mathrm{Diff}^1(\mathcal{M})$ is an open set of $\mathcal{C}^1(\mathcal{M}, \mathcal{N})$ (which denotes the space of $\mathcal{C}^1$-functions from $\mathcal{M}$ into $\mathcal{N}$ whose derivatives extend continuously to the closure $\overline{\mathcal{M}}$ with the usual supremum norm). Hence, F can be seen as a map defined between Banach spaces.

We will consider that $\mathcal{M} \subset \mathcal{N}$ (perhaps by identifying $\mathcal{M}$ with its image with an initial fixed embedding).

Notice that if $\lambda_0 \in \mathbb{R}$ is an eigenvalue for $\mathcal{B}_{\mathcal{M}}$ for some $u_0 \in L^2(\mathcal{M})$ with $\int_{\mathcal{M}} u_0^2(x)\, dv_g(x) = 1$, then $F(i_{\mathcal{M}}, \lambda_0, u_0) = (0, 1)$ where $i_{\mathcal{M}} \in \mathrm{Diff}^1(\mathcal{M})$ denotes the inclusion map of $\mathcal{M}$ into $\mathcal{N}$. On the other hand, whenever $F(h, \lambda, u) = (0, 1)$, we have from Proposition 3.1 that

$$\mathcal{B}_{\mathcal{M}_h} \tilde{u}(y) = \lambda \tilde{u}(y), \quad y \in \mathcal{M}_h, \quad \text{with} \quad \int_{\mathcal{M}_h} \tilde{u}^2(y)\, dv_g(y) = 1$$

where $\tilde{u}(y) = (u \circ h^{-1})(y)$ for $y \in \mathcal{M}_h$. In this way, we can proceed as in [3, 14] using the map F to deal with eigenvalues and eigenfunctions of $\mathcal{B}_{\mathcal{M}_h}$ and $h^* \mathcal{B}_{\mathcal{M}_h} h^{*-1}$ perturbing the eigenvalue problem to the fixed manifold $\mathcal{M}$ by diffeomorphisms h.

Now we proceed with the proof of the main theorem of this section.

Proof of Theorem 1.1 The proof of the existence of the neighbourhood $\mathcal{O} \subset \mathrm{Diff}^1(\mathcal{M})$ and the $\mathcal{C}^1$-functions u_h and λ_h satisfying (1.2) is very similar to that one performed in [3, Lemma 4.1]. As one can see, it is a consequence of the Implicit Function Theorem applied to the map F. Here, we compute the derivative of λ_h at $h = i_{\mathcal{M}}$. For this, it is enough to consider a curve of embeddings $h(t, x)$ that satisfies $h(0, x) = i_{\mathcal{M}}$ and $\frac{\partial h}{\partial t} = V(x)$ for a fixed vector field $V \in \mathcal{X}^1(\mathcal{N})$. To simplify the notation, we denote the eigenvalue and eigenfunction on $h(t, \mathcal{M})$ by λ_t and u_t respectively. It follows from

$$h(t)^* \mathcal{B}_{h(t,\mathcal{M})} h(t)^{*-1} u_t(x) = \lambda_t u_t, \quad x \in \mathcal{M},$$

that

$$\frac{\partial}{\partial t} \left(h(t)^* \mathcal{B}_{h(t,\mathcal{M})} h(t)^{*-1} u_t(x) \right) \Big|_{t=0} = \frac{\partial \lambda_t}{\partial t}\Big|_{t=0} u_0 + \lambda_0 \frac{\partial u_t}{\partial t}\Big|_{t=0} \quad \text{in } \mathcal{M}. \tag{3.14}$$

Now, we need to compute the derivative of the left-hand side of (3.14). Notice that

$$\begin{aligned} \frac{\partial}{\partial t} \left(h(t)^* \mathcal{B}_{h(t,\mathcal{M})} h(t)^{*-1} u_{h(t)} \right) \Big|_{t=0} &= \frac{\partial}{\partial t} \left(h^* a_{h(t,\mathcal{M})} u_t \right) \Big|_{t=0} \\ &- \frac{\partial}{\partial t} \left(h(t)^* \mathcal{J}_{h(t,\mathcal{M})} h(t)^{*-1} u_t \right) \Big|_{t=0} \quad \text{in } \mathcal{M}. \end{aligned}$$

Also, for any function $w : \mathcal{M} \times [0, T) \to \mathbb{R}$ it holds that $\frac{\partial}{\partial t}(h^* w) = h^* \frac{\partial}{\partial t} w + \langle h^* \nabla w, \frac{\partial h}{\partial t} \rangle$. Here ∇ denotes de tangential gradient on $\mathcal{N}$ and $\langle \cdot, \cdot \rangle$ the inner product in $\mathcal{N}$. Then we have

$$D_t \left(h(t)^* \mathcal{J}_{h(t,\mathcal{M})} h(t)^{*-1} u_t \right) = h(t)^* \frac{\partial}{\partial t} \left(\mathcal{J}_{h(t,\mathcal{M})} h(t)^{*-1} u_t \right) \quad \text{in } \mathcal{N}. \tag{3.15}$$

where D_t is defined by (3.13).

In the case of domains of $\mathbb{R}^n$ this derivative is known under the Dirichlet condition from [3, Lemma 4.1].

Hence, setting $\tilde{u}(t, y) = h(t)^{*-1} u_t(y) = u_t(h^{-1}(t, y))$, $y \in h(t, \mathcal{M})$, we get from (3.12)

$$\frac{\partial}{\partial t} \left(\mathcal{J}_{h(t,\mathcal{M})} h(t)^{*-1} u_t \right) \Big|_{t=0} = \frac{\partial}{\partial t} \left(\mathcal{J}_{h(t,\mathcal{M})} \tilde{u} \right) \Big|_{t=0}$$

$$= \frac{\partial}{\partial t}\left(\int_{h(t,\mathcal{M})} J(y,w)\tilde{u}(t,w)dv_t\right)\Big|_{t=0} \quad \text{for } y \in h(t,\mathcal{M}).$$

Here dv_t denotes the volume form related to the metric $g(\cdot,t)$. To explicitly compute this derivative we recall that $\frac{d}{dt}(dv_t)\big|_{t=0} = \text{tr}\,(g^{-1}\frac{dg}{dt}(0))\,dv_g$. Since $g_{ij}(t) = \langle \partial_{x_i}h, \partial_{x_j}h\rangle$ we have that $\frac{dg_{ij}}{dt} = \langle \partial_{x_i}h, \partial_{x_j}V\rangle + \langle \partial_{x_i}V, \partial_{x_j}h\rangle$. Now we denote $V = V^{\perp} + V^T$, where $V^{\perp}$ is normal component of V and V^T the tangential one. Then we have

$$\frac{dv_t}{dt}\Big|_{t=0} = (\,\text{div}_{\mathcal{M}}\,V^T + \langle \vec{H}, V^{\perp}\rangle)\,dv_g,$$

where $\vec{H}$ is the mean curvature vector associated to $\mathcal{M}$. To keep in mind the variable that we are using in the computation, we will add a subscript to ∇ (e.g. $\nabla_w J(w,x)$ or $\nabla_x J(w,x)$). Then

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_{h(t,\mathcal{M})} J(y,w)\tilde{u}(t,w)\,dv_t(w)\right)\Big|_{t=0}\\
&= \int_{\mathcal{M}} \text{div}_{\mathcal{M}}\,(J(y,w)\tilde{u}(0,w)V^T)\,dv_g(w)\\
&\quad + \int_{\mathcal{M}} J(y,w)\frac{d}{dt}\tilde{u}(0,w)\,dv_g(w)\\
&\quad + \int_{\mathcal{M}} J(y,w)\tilde{u}(t,w)\langle \vec{H}, V^{\perp}\rangle\,dv_g(w)\\
&\quad + \int_{\mathcal{M}} \langle \nabla_w(J(y,w)\tilde{u}(0,w)), V^{\perp}\rangle)\,dv_g(w)\\
&= \int_{\partial\mathcal{M}} J(y,w)u_0(w)\langle V^T, N\rangle\,dS\\
&\quad + \int_{\mathcal{M}} J(y,w)D_t u\,dv_g(w)\\
&\quad + \int_{\mathcal{M}} J(y,w)u_0\langle \vec{H}, V^{\perp}\rangle\,dv_g(w)\\
&\quad + \int_{\mathcal{M}} \langle \nabla_w(J(y,w)u_0(w)), V^{\perp}\rangle\,dv_g(w),
\end{aligned}$$

where $N \in T(\mathcal{M}) \cap (T(\partial\mathcal{M}))^{\perp}$ is the unitary normal vector to $\partial\mathcal{M}$. Since J is $\mathcal{C}^1$, the eigenfunctions u_t and their derivatives can be continuously extended to the border $\partial\mathcal{M}$. Hence, u_t possesses trace and the expression above is well defined. Since u_0 is a function defined on $\mathcal{M}$ we have $\nabla_{\mathcal{N}}u_0(w) = \nabla_{\mathcal{M}}u_0(w)$ and, $\nabla_{\mathcal{M}}u_0(w)$ is tangential to $\mathcal{M}$, then $\langle \nabla_{\mathcal{N}}u_0(w), V^{\perp}\rangle = 0$. We will also denote by $a_0 = a_{h(0,\mathcal{M})} = a_{\mathcal{M}}$.

Consequently, (3.13) and (3.15) imply

$$\frac{\partial}{\partial t}\left(h(t)^*\mathcal{J}_{h(t,\mathcal{M})}h(t)^{*-1}u_t\right)\Big|_{t=0} = \langle V, \nabla\,(\mathcal{J}_{\mathcal{M}}u_0)\rangle + \mathcal{J}_{\mathcal{M}}(D_t u|_{t=0})$$

$$
\begin{aligned}
&+\int_{\partial\mathcal{M}} J(y,w)u_0(w)\langle V^T, N\rangle\, dS(w)\\
&+\int_{\mathcal{M}} J(y,w)u_0(w)\,\langle \vec{H}, V^{\perp}\rangle\, dv_g(w)\\
&+\int_{\mathcal{M}} u_0(w)\,\langle \nabla_w(J(y,w)), V^{\perp}\rangle\, dv_g(w).
\end{aligned} \tag{3.16}
$$

We get from (3.14) and (3.16) that

$$
\begin{aligned}
&\frac{\partial\lambda_t}{\partial t}(0)u_0 + \lambda_0\frac{\partial u_t}{\partial t}\bigg|_{t=0} = \frac{\partial}{\partial t}\left(h^* a_{h(t,\mathcal{M})}\right)\Big|_{t=0} u_0\\
&\quad + a_0\frac{\partial u_t}{\partial t} - \langle V, \nabla\left(\mathcal{J}_{\mathcal{M}} u_0\right)\rangle - \mathcal{J}_{\mathcal{M}}(D_t u|_{t=0})\\
&\quad - \int_{\partial\mathcal{M}} J(y,w)u_0(w)\langle V^T, N\rangle\, dS(w) - \int_{\mathcal{M}} J(y,w)u_0(w)\,\langle \vec{H}, V^{\perp}\rangle\, dv_g(w)\\
&\quad - \int_{\mathcal{M}} u_0(w)\,\langle \nabla_w(J(y,w)), V^{\perp}\rangle\, dv_g(w).
\end{aligned} \tag{3.17}
$$

Thus, multiplying (3.17) by the normalized eigenfunction u_0 and integrating on $\mathcal{M}$, we obtain

$$
\begin{aligned}
&\frac{\partial\lambda_t}{\partial t}(0) + \lambda_0\int_{\mathcal{M}}\frac{\partial u_t}{\partial t}\, u_0\, dv_g(x) = \int_{\mathcal{M}}\frac{\partial}{\partial t}\left(h^* a_{h(t,\mathcal{M})}\right)\Big|_{t=0} u_0^2\, dv_g(x)\\
&\quad + \int_{\mathcal{M}} \mathcal{J}_{\mathcal{M}} u_0\,\frac{\partial u_t}{\partial t}\, dv_g(x) + \int_{\mathcal{M}} (a_0 - \mathcal{J}_{\mathcal{M}})u_0\frac{\partial u_t}{\partial t}\, dv_g(x)\\
&\quad - \int_{\mathcal{M}} \langle V, \nabla\left(\mathcal{J}_{\mathcal{M}}\, u_0\right)\rangle\, u_0(x)\, dv_g(x)\\
&\quad - \int_{\mathcal{M}} u_0\left[\mathcal{J}_{\mathcal{M}}(D_t u|_{t=0}) + \int_{\partial\mathcal{M}} J(x,z)u_0(z)\,\langle V^T, N\rangle(z)\, dS(z)\right] dv_g(x)\\
&\quad - \int_{\mathcal{M}}\int_{\mathcal{M}} J(x,w)u_0(x)u_0(w)\langle \vec{H}, V^{\perp}\rangle(w)\, dv_g(w)\, dv_g(x)\\
&\quad - \int_{\mathcal{M}}\int_{\mathcal{M}} u_0(x)\, u_0(w)\,\langle \nabla_w J(x,w), V^{\perp}\rangle(w)\, dv_g(w)\, dv_g(x),
\end{aligned}
$$

which in turn implies

$$
\begin{aligned}
&\frac{\partial\lambda_t}{\partial t}\bigg|_{t=0} = \int_{\mathcal{M}}\frac{\partial}{\partial t}\left(h^* a_{h(t,\mathcal{M})}\right)\Big|_{t=0} u_0^2\, dv_g(x)\\
&\quad + \int_{\mathcal{M}} \mathcal{J}_{\mathcal{M}} u_0\frac{\partial u_t}{\partial t}\, dv_g(x) - \int_{\mathcal{M}} u_0\,\langle V, \nabla\left(\mathcal{J}_{\mathcal{M}} u_0\right)\rangle\, dv_g(x)\\
&\quad - \int_{\mathcal{M}} u_0(x)\left[\mathcal{J}_{\mathcal{M}}(D_t u|_{t=0}) + \int_{\partial\mathcal{M}} J(x,y)u_0(z)\,\langle V^T, N\rangle(z)\, dS(z)\right] dv_g(x)
\end{aligned}
$$

$$
-\int_{\mathcal{M}} \mathcal{J}_{\mathcal{M}} u_0(w) u_0(w) \langle \vec{H}, V^{\perp} \rangle \, dv_g(w)
$$

$$
-\int_{\mathcal{M}} u_0(w) \langle \nabla_w (\mathcal{J}_{\mathcal{M}} u_0(w)), V^{\perp} \rangle \, dv_g(w), \tag{3.18}
$$

since $(a_0 - \mathcal{J}_{\mathcal{M}})u_0 = \lambda_0 u_0$ in $\mathcal{M}$. The last two integrals are obtained from the symmetry $J(x, w) = J(w, x)$, which also implies

$$
\begin{aligned}
&\int_{\mathcal{M}} u_0 \left[\mathcal{J}_{\mathcal{M}}(D_t u|_{t=0}) + \langle V, \nabla_x (\mathcal{J}_{\mathcal{M}} u_0)\rangle\right] dv_g(x) \\
&\quad = \int_{\mathcal{M}} \int_{\mathcal{M}} J(x, w) u_0(x) \left(\frac{\partial u_t}{\partial t}(w) - \langle V(w), \nabla_w u_0(w)\rangle\right) dv_g(w) dv_g(x) \\
&\qquad + \int_{\mathcal{M}} u_0 \langle V, \nabla_x (a_0 u_0 - \lambda_0 u_0)\rangle dv_0(x) \\
&\quad = \int_{\mathcal{M}} \frac{\partial u_t}{\partial t} \mathcal{J}_{\mathcal{M}} u_0 \, dv_0(x) - \int_{\mathcal{M}} (a_0 - \lambda_0) u_0 \langle V, \nabla u_0\rangle dv_g(x) \\
&\qquad + \int_{\mathcal{M}} u_0 \langle V, \nabla (a_0 u_0 - \lambda_0 u_0)\rangle dv_g(x) \\
&\quad = \int_{\mathcal{M}} \frac{\partial u_t}{\partial t} \mathcal{J}_{\mathcal{M}} u_0 \, dv_0(x) + \int_{\mathcal{M}} u_0^2 \langle V, \nabla a_0\rangle dv_0(x).
\end{aligned}
$$

Finally we observe

$$
\int_{\mathcal{M}} u_0(w) \langle \nabla_w (\mathcal{J}_{\mathcal{M}} u_0(w)), V^{\perp}\rangle dv_g(w) = \int_{\mathcal{M}} u_0^2(w) \langle \nabla_w (a_0(w)), V^{\perp}\rangle dv_g(w).
$$

Here we used that ∇u_0 is tangential to $\mathcal{M}$.

Consequently, we get from (3.18) that

$$
\begin{aligned}
\frac{\partial \lambda_t}{\partial t}(0) &= \int_{\mathcal{M}} \frac{\partial}{\partial t} \left(h^* a_{h(t,\mathcal{M})}\right)\Big|_{t=0} u_0^2 \, dv_g(x) - \int_{\mathcal{M}} u_0^2 \langle V, \nabla a_0\rangle dv_0(x) \\
&\quad - \int_{\mathcal{M}} u_0(x) \int_{\partial\mathcal{M}} J(x, y) u_0(z) \, \langle V^T, N\rangle(z) \, dS(z) \, dv_g(x) \\
&\quad - \int_{\mathcal{M}} \mathcal{J}_{\mathcal{M}} u_0(w) u_0(w) \langle \vec{H}, V^{\perp}\rangle \, dv_g(w) - \int_{\mathcal{M}} u_0^2(w) \langle \nabla_w (a_0(w)), V^{\perp}\rangle dv_g(w) \\
&= \int_{\mathcal{M}} u_0^2 \, D_t^T (h^* a_t)\big|_{t=0} \, dv_t(x) - \int_{\partial\mathcal{M}} (a_0 - \lambda_0) u_0^2 \langle V, N\rangle \, dS \\
&\quad - \int_{\mathcal{M}} (a_0 - \lambda_0) u_0^2(w) \langle \vec{H}, V^{\perp}\rangle dv_g(w) - \int_{\mathcal{M}} u_0^2(w) \langle \nabla_w a_0(w), V^{\perp}\rangle \, dv_g(w)
\end{aligned}
$$

where $D_t^T f = \frac{\partial f}{\partial t} - \langle V^T, \nabla f\rangle$. Observing that $\mathcal{J}_{\mathcal{M}} u_0 = (a_0 - \lambda_0) u_0$ we complete the proof. □

Remark 3.2 In the case that $\mathcal{M}$ is an open domain of $\mathbb{R}^{n+1}$ (with co-dimension 0) we have $V^{\perp} = 0$, and then, $V = V^T$. Hence, the Hadamard formula becomes

$$\frac{\partial \lambda}{\partial h}(i_{\mathcal{M}})V = -\int_{\partial\mathcal{M}} (a_{\mathcal{M}}(s) - \lambda_0)\, u_0^2(s)\, \langle V, N\rangle\, dS(s) + \int_{\mathcal{M}} u_0^2(x)\, D_t(h^* a_{h(t,\mathcal{M})})\big|_{t=0} dv_g(x).$$

Proof of Corollary 1.1 Let us now determine the domain derivative of the function u_h introduced by Theorem 1.1.

Due to (3.17), we have for all $V \in \mathcal{X}^1(\mathcal{N})$ that

$$\begin{aligned}\frac{\partial \lambda_t}{\partial t}u_0 + \lambda_0\frac{\partial u_t}{\partial t} = &\frac{\partial}{\partial t}\left(h^* a_{h(t,\mathcal{M})}\right)\Big|_{t=0}u_0 + \mathcal{B}_{\mathcal{M}}\left(\frac{\partial u_{i_{\mathcal{M}}}}{\partial t}\right)\\ &+ [\mathcal{J}_{\mathcal{M}}, \langle V, \nabla(\cdot)\rangle]\, u_0 - \int_{\partial\mathcal{M}} J(y,w)u_0(w)\langle V, N\rangle\, dS(w)\\ &- \int_{\mathcal{M}} J(y,w)u_0\langle \vec{H}, V^{\perp}\rangle dv_g(w)\\ &- \int_{\mathcal{M}} u_0(w)\, \langle \nabla_w(J(y,w)), V^{\perp}\rangle dv_g(w).\end{aligned}$$

where $[A, B]\, u := ABu - BAu$, and then, $[\mathcal{J}_{\mathcal{M}}, \langle V, \nabla(\cdot)\rangle]\, u_0 = \mathcal{J}_{\mathcal{M}}(\langle V, \nabla u_0\rangle) - \langle V, \nabla(\mathcal{J}_{\mathcal{M}}u_0)\rangle$.

Hence,

$$\begin{aligned}(\lambda_0 - \mathcal{B}_{\mathcal{M}})\frac{\partial u_{i_{\mathcal{M}}}}{\partial t} = &-\frac{\partial \lambda_{i_{\mathcal{M}}}}{\partial t}u_0 + \frac{\partial}{\partial t}\left(h^* a_{h(t,\mathcal{M})}\right)\Big|_{t=0}u_0 + [\mathcal{J}_{\mathcal{M}}, \langle V, \nabla(\cdot)\rangle]\, u_0\\ &- \int_{\partial\mathcal{M}} J(y,w)u_0(w)\langle V, N\rangle\, dS(w)\\ &- \int_{\mathcal{M}} J(y,w)u_0\langle \vec{H}, V^{\perp}\rangle dv_g(w)\\ &- \int_{\mathcal{M}} u_0(w)\, \langle \nabla_w(J(y,w)), V^{\perp}\rangle dv_g(w).\end{aligned} \tag{3.19}$$

Thus, we can conclude that the derivative of u_h at $h = i_{\mathcal{M}}$ in $V \in \mathcal{C}^1(\mathcal{N}, \mathcal{N})$ is the solution of

$$(\lambda_0 - \mathcal{B}_{\mathcal{M}})w = f_V$$

where $f_V \in L^2(\mathcal{M})$ is the function given by the right side of (3.19) which is well defined since u_0, λ_0, $\frac{\partial \lambda_t}{\partial t}$ and $\frac{\partial}{\partial t}\left(h^* a_{h(t,\mathcal{M})}\right)\Big|_{t=0}$ are known.

Notice that λ_0 is a simple eigenvalue of $\mathcal{B}_{\mathcal{M}}$, and then, we have

$$L^2(\mathcal{M}) = \mathrm{R}(\lambda_0 - \mathcal{B}_{\mathcal{M}}) \oplus [u_0].$$

Therefore, (3.19) possesses unique solution, if and only if, $\int_{\mathcal{M}} u_0 f_V dv_g(x) = 0$ for each $V \in \mathcal{X}^1(\mathcal{N})$.

Indeed, it follows from (1.3) and the assumption $\mathcal{B}_{\mathcal{M}} u_0 = \lambda_0 u_0$ in $\mathcal{M}$ that

$$
\begin{aligned}
&\int_{\mathcal{M}} u_0 f_V dv_g(x) \\
&= -\frac{\partial \lambda_t}{\partial t} + \int_{\mathcal{M}} D_t(h^* a_{h(t,\cdot)})\Big|_{t=0} u_0^2 \, dv_g(x) \\
&\quad + \int_{\mathcal{M}} u_0 (u_0 \langle V, \nabla(\cdot)\rangle a_{\mathcal{M}} + [\mathcal{J}_{\mathcal{M}}, \langle V, \nabla(\cdot)\rangle]\, u_0)\, dv_g(x) \\
&\quad - \int_{\mathcal{M}} \int_{\partial\mathcal{M}} J(x,z) u_0(x)\, u_0(z)\, \langle V, N\rangle \, dS(z) dv_g(x) \\
&\quad - \int_{\mathcal{M}} \int_{\mathcal{M}} J(x,z) u_0(x)\, u_0(z)\, \langle \vec{H}, V^{\perp}\rangle dv_g(z)\, dv_g(x) \\
&\quad - \int_{\mathcal{M}} \int_{\mathcal{M}} u_0(x)\, u_0(z)\, \langle (\nabla_x J), V^{\perp}\rangle dv_g(z)\, dv_g(x) \\
&= \int_{\mathcal{M}} u_0(x) \Big[\mathcal{J}_{\mathcal{M}}(\langle V^T, \nabla u_0\rangle) - a_{\mathcal{M}} \langle V^T, \nabla u_0\rangle + \lambda_0 \langle V^T, \nabla u_0\rangle \Big] dv_g(x) \\
&= \int_{\mathcal{M}} u_0(x)\, [\lambda_0 - \mathcal{B}_{\mathcal{M}}]\, (\langle V^T, \nabla u_0\rangle)\, dv_g(x).
\end{aligned}
$$

Thus, since $\mathcal{B}_{\mathcal{M}}$ is a self-adjoint operator, u_0 is a $\mathcal{C}^1$-function and $\langle V, \nabla(\cdot)\rangle u_0 \in L^2(\mathcal{M})$, one has

$$
\int_{\mathcal{M}} u_0 f_V dx = \int_{\mathcal{M}} \langle V, \nabla u_0\rangle \, (\lambda_0 - \mathcal{B}_{\mathcal{M}})\, u_0 \, dx = 0
$$

for all $V \in \mathcal{X}^1(\mathcal{N})$ which proves Corollary 1.1. □

Examples

We finish this section with some concrete examples, by setting specific nonlocal operators and manifolds.

Example 3.1 (The sphere) Consider $\mathcal{M} = \mathbb{S}^n$ and $\mathcal{N} = \mathbb{R}^{n+1}$. If we take $a \equiv 0$, then $\vec{H}(p) = \frac{n}{R} p$ and

$$
\frac{\partial \lambda_t}{\partial t}(i_{\mathbb{S}^n}) V = \frac{n\lambda_0}{R} \int_{\mathbb{S}^n} u_0^2(w) \langle w, V^{\perp}\rangle \, dv_g(w).
$$

Example 3.2 (The Dirichlet problem on the upper hemisphere) Consider $\mathcal{M} = \mathbb{S}^n_+$ (that is $p \in \mathbb{S}^n$ with $x_{n+1} \geq 0$) and $\mathcal{N} = \mathbb{R}^{n+1}$. If we take $a \equiv 1$, then $\vec{H}(p) = \frac{1}{R} p$

and $N(p) = e_{n+1}$

$$\frac{\partial \lambda_t}{\partial t}(i_{\mathbb{S}^n_+})V = -(1-\lambda_0)\int_{\partial \mathbb{S}^n_+} u_0^2 V_{n+1}\, dS - \frac{n(1-\lambda_0)}{R}\int_{\mathbb{S}^n} u_0^2(w)\langle w, V^{\perp}\rangle\, dv_g(w).$$

Example 3.3 (One parameter family of functions a) Consider $\Omega \subset \mathbb{R}^n$, $\mathcal{M} = \Omega \times [0, 1]$ and $\mathcal{N} = \mathbb{R}^{n+1}$. In this case $\vec{H} = 0$, but assume that a depends on the variable x_{n+1} then

$$\frac{\partial \lambda}{\partial h}(i_\Omega)V = -\int_{\partial\Omega} (a_\Omega(s) - \lambda_0)\, u_0^2(s)\ (V \cdot N)(s)\, dS + \int_\Omega u_0^2(x)\, D_t(h^* a_{\Omega_h})\big|_{t=0} dx$$
$$- \int_{\Omega \times [0,1]} u_0^2(w)\langle \nabla_w a_\Omega(w), V^{\perp}\rangle\, dv_g(w).$$

Some examples in Euclidean spaces

In the sequel we compute some examples assuming $\mathcal{M} = \Omega$ is a domain of $\mathbb{R}^n$, $J(x, y) = J(|x - y|)$, for some $J \in \mathcal{C}^1(\mathbb{R})$ non negative satisfying $J(0) > 0$ and $\int_{\mathbb{R}^n} J(z)dz = 1$. Here we use the Hadamard formula given by Remark 3.2. It is still worth mentioning that such examples often appear in the literature associated with nonlocal differential equations. We include below appropriate references for each considered example.

Example 3.4 (Dirichlet problem) If we take $a_\Omega(x) \equiv 1$ in (2.6), we have what is called the Dirichlet nonlocal problem. In this case, the Hadamard formula is known and it was first obtained in [9] for the first eigenvalue. In [3], we have proved that the same formula still holds for any simple eigenvalue. Since a_Ω is constant, $D_t(h^* a_{\Omega_h})\big|_{t=0} = 0$ and, from Theorem 1.1, we get

$$\frac{\partial \lambda}{\partial h}(i_\Omega)V = -\,(1-\lambda_0)\int_{\partial\Omega} u_0^2\ V \cdot N\, dS \quad \forall V \in \mathcal{C}^1(\Omega, \mathbb{R}^n)$$

with $\cdot$ denoting the scalar product in $\mathbb{R}^n$.

Example 3.5 (Neumann problem) In the literature, see for instance [1, 8, 16], the nonlocal Neumann problem is established taking

$$a_\Omega(x) = \int_\Omega J(|x - y|)dy, \quad x \in \mathbb{R}^n.$$

As expected, zero is its first eigenvalue for any measurable open set Ω which is simple and it is associated with a constant eigenfunction. Clearly, the rate of the first eigenvalue with respect to the domain must be null. Let us take its rate for any other simple eigenvalue. For this, we first compute the anti-convective derivative of a_Ω at $t = 0$ assuming $h(t, x) = x + tV(x)$ for some $V \in \mathcal{C}^1(\Omega, \mathbb{R}^n)$. We have from [14, Lemma 2.1] and [14, Theorem 1.1] that

$$D_t \left[h^*(t) a_{h(t,\Omega)}\right]\big|_{t=0} = h^*(t) \frac{\partial}{\partial t} \left[\int_{h(t,\Omega)} J(|\cdot - w|) dw\right]\Big|_{t=0}$$
$$= \int_{\partial\Omega} J(|x-s|)(V \cdot N)(s)\, dS, \quad x \in \Omega.$$

Hence, we obtain from Theorem 1.1 that

$$\begin{aligned}
\frac{\partial \lambda}{\partial h}(i_\Omega) V &= -\int_{\partial\Omega} (a_\Omega(s) - \lambda_0)\, u_0^2(s)\ (V \cdot N)(s)\, dS \\
&\quad + \int_\Omega u_0^2(x) \int_{\partial\Omega} J(|x-s|)(V \cdot N)(s)\, dS dx \\
&= -\int_{\partial\Omega} (a_\Omega(s) - \lambda_0)\, u_0^2(s)\ (V \cdot N)(s)\, dS + \int_{\partial\Omega} (\mathcal{J}_\Omega u_0^2)(s)(V \cdot N)(s)\, dS \\
&= -\int_{\partial\Omega} (\mathcal{B}_\Omega - \lambda_0)\, u_0^2\ (V \cdot N)(s)\, dS.
\end{aligned}$$

Notice in the last integral the term $\mathcal{J}_\Omega u_0^2$ which is the operator $\mathcal{J}_\Omega$ applied to the square of the normalized eigenfunction u_0.

Example 3.6 Let $D \subset \mathbb{R}^n$ be a bounded open set and take $A \subset D$, another open bounded set strictly contained in D in such way that $\partial A \cap \partial D = \emptyset$. Next, consider $\Omega = D \backslash A$ defining

$$a_\Omega(x) = \int_{\mathbb{R}^n \backslash A} J(|x-y|)\, dy, \quad x \in \mathbb{R}^n.$$

The nonlocal operator $\mathcal{B}_\Omega$ given for such function a_Ω is a kind of Dirichlet/Neumann problem. It takes Dirichlet boundary condition side out of D setting Neumann condition on the hole A. Such operator is given by

$$\mathcal{B}_\Omega(x) = \int_{\mathbb{R}^n \backslash A} J(|x-y|)(u(x) - u(y))\, dy, \quad x \in \Omega,$$

assuming $u \equiv 0$ in $\mathbb{R}^n \backslash D$ and has been studied for instance in [23]. Let us compute its Hadamard formula. Due to

$$\begin{aligned}
a_\Omega(x) &= \int_{\mathbb{R}^n} J(|x-y|) dy - \int_A J(|x-y|)\, dy \\
&= 1 - \int_D J(|x-y|)\, dy + \int_\Omega J(|x-y|)\, dy, \quad \forall x \in \mathbb{R}^n,
\end{aligned}$$

one gets again from [14, Lemma 2.1, Theorem 1.1] that

$$
\begin{aligned}
& D_t \left[h^*(t) a_{h(t,\Omega)} \right] \Big|_{t=0} \\
&= h^*(t) \frac{\partial}{\partial t} \left[1 - \int_{h(t,D)} J(|\cdot - y|)\, dy + \int_{h(t,\Omega)} J(|\cdot - y|)\, dy \right] \Big|_{t=0} \\
&= - \int_{\partial D} J(|x-s|)(V \cdot N)(s)\, dS + \int_{\partial \Omega} J(|x-s|)(V \cdot N)(s)\, dS \\
&= \int_{\partial A} J(|x-s|)(V \cdot N)(s)\, dS, \quad x \in \Omega
\end{aligned}
$$

since $\partial\Omega = \partial D \cup \partial A$ with $\partial D \cap \partial A = \emptyset$. Hence,

$$
\begin{aligned}
\frac{\partial \lambda}{\partial h}(i_\Omega)V = & - \int_{\partial \Omega} (a_\Omega(s) - \lambda_0)\, u_0^2(s)\ (V \cdot N)(s)\, dS \\
& + \int_{\partial A} (\mathcal{J}_\Omega u_0^2)(s)(V \cdot N)(s)\, dS.
\end{aligned}
$$

4 Hadamard formula for multiple eigenvalues

In this section we prove Theorem 1.2. As we will see, it is based on the implicit functional theorem and the Lyapunov–Schmidt method. Here, we use the domain of definition of the solutions as a bifurcation parameter according to the pioneering works [6, 14]. Under the additional condition J is analytic, we first ensure the existence of analytic curves of eigenvalues and eigenfunctions for the eigenvalue problem

$$
\mathcal{B}_{h(t,\mathcal{M})} \tilde{u}(y) = \lambda \tilde{u}(y), \quad y \in h(t, \mathcal{M})
$$

when $h(t,\cdot)$ is an analytic curve of diffeomorphisms in $\mathrm{Diff}^1(\mathcal{M})$. Next, we compute the Hadamard formula for an m-fold eigenvalue with $m > 1$.

It is worth noticing that the assumption of analyticity of J is not optimal. As we will see in the proof of Lemma 4.1, it is enough to hold the conditions of Puiseux's theorem, which guarantees the existence of the analytic curves. Notice that the case $m = 1$ is considered in Theorem 1.1 without the additional assumption that J is analytic. A complete discussion in this direction can be seen in [17] where a more general result, which also includes Lemma 4.1, is presented. In our context, Lemma 4.1 and its proof are important to compute the derivatives.

Lemma 4.1 *Let λ_0 be an eigenvalue of multiplicity m of the operator $\mathcal{B}_\mathcal{M}$ with $m > 1$. Assume the non-singular kernel $J : \mathcal{N} \times \mathcal{N} \mapsto \mathbb{R}$ is an analytic function satisfying condition* (H)*, $h(t,\cdot)$ is an analytic curve of diffeomorphisms of class $\mathcal{C}^1$ with $h(0,x) = x$ in $\mathcal{M}$, and $\Phi : \mathbb{R} \mapsto \mathcal{C}^1(\mathcal{M})$, given by $\Phi(t) = \left(h(t,\cdot)^* a_{\mathcal{M}_{h(t,\cdot)}}\right)$ is also an analytic map.*

Then, for some $\delta > 0$, there exist m analytic curves $\lambda_1(t),\ldots, \lambda_m(t) \in \mathbb{R}$ with $\lambda_1(0) = \cdots = \lambda_m(0) = \lambda_0$, and m analytic curves $u_1(t),\ldots, u_m(t) \in L^2(\mathcal{M})$ setting *respectively pairs of eigenvalues and eigenfunctions* $(\lambda_j(t), u_j(t))$ *for*

$$h(t,\cdot)^*\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1}u(x) = \lambda u(x), \quad x \in \mathcal{M},$$

for all $t \in (-\delta, \delta)$.

Proof Since J is an analytic function, h is a $\mathcal{C}^1$ diffeomorphism and

$$h^*\mathcal{J}_{h(\mathcal{M})}h^{*-1}u(x) = \int_{\mathcal{M}} J\left(h(x), h(z)\right) u(z) dv_{g_h}(z), \quad x \in \mathcal{M},$$

where dv_{g_h} is the volume element of the metric on g_h, we have that

$$(h, u) \in \mathrm{Diff}^1(\mathcal{M}) \times L^2(\mathcal{M}) \mapsto h^*\mathcal{J}_{h(\mathcal{M})}h^{*-1}u \in L^2(\mathcal{M})$$

is an analytic map. Thus, under the hypotheses of the lemma, we conclude that

$$\begin{aligned} G : \mathbb{R} \times \mathbb{R} \times L^2(\mathcal{M}) &\mapsto L^2(\mathcal{M}) \\ (t, \lambda, u) &\mapsto \left(h(t,\cdot)^*\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1} - \lambda\right) u \end{aligned}$$

is also an analytic map. Moreover, $(\lambda, \tilde{u}) \in \mathbb{R} \times L^2(h(t, \mathcal{M}))$ is an eigenpair for $\mathcal{B}_{h(t,\mathcal{M})}$, if and only if, $G(t, \lambda, u) = 0$ with $\tilde{u} = h(t,\cdot)^{*-1}u$.

Now, let $\{\phi_1, \ldots, \phi_m\}$ be an orthonormal basis of eigenfunctions for $\mathcal{B}_{\mathcal{M}}$ with eigenvalue $\lambda = \lambda_0$. Set $P(\cdot) = \sum_{i=1}^m \phi_i \int_{\mathcal{M}} \phi_i\,(\cdot) dv_g$ the orthogonal projection onto the span of $\{\phi_1, \ldots, \phi_m\}$. We seek λ near λ_0 and $u \neq 0$ such that $G(t, \lambda, u) = 0$ for t near 0, and then, for $h(t,\cdot)$ near $i_{\mathcal{M}}$. Notice that this is equivalent to finding $u = v + w \neq 0$ with

$$v = Pu \quad \text{and} \quad w = (I - P)u \in N(P)$$

such that

$$PG(t, \lambda, v + w) = 0 \quad \text{and} \quad (I - P)G(t, \lambda, v + w) = 0$$

where $N(P)$ denotes the kernel of the orthonormal projection P and I the identity operator. We rewrite the last equation as

$$\begin{aligned} 0 &= (I - P)\left[\left(h(t,\cdot)^*\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1} - \mathcal{B}_{\mathcal{M}}\right) + (\mathcal{B}_{\mathcal{M}} - \lambda)\right](v + w) \\ &= (I - P)\left(h(t,\cdot)^*\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1} - \mathcal{B}_{\mathcal{M}}\right) \\ &\quad \times (v + w) + (\mathcal{B}_{\mathcal{M}} - \lambda)\, w - P\,(\mathcal{B}_{\mathcal{M}} - \lambda)\, w. \end{aligned}$$

Since $w \in N(P)$ and $R(\mathcal{B}_{\mathcal{M}}) \perp N(P)$, where $R(\mathcal{B}_{\mathcal{M}})$ denotes the image of the operator $\mathcal{B}_{\mathcal{M}}$, we have $P(\mathcal{B}_{\mathcal{M}} - \lambda)w = 0$, and then,

$$0 = (I - P)\left(h(t,\cdot)^*\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1} - \mathcal{B}_{\mathcal{M}}\right)(v + w) + (\mathcal{B}_{\mathcal{M}} - \lambda)\, w.$$

Thus, $G(t,\lambda,u)=0$ with $u\neq 0$, if and only if, $u=v+w\neq 0$ with $v=Pu$ and $w=(I-P)u$ satisfies $0=PG(t,\lambda,v+w)$ and

$$0=(I-P)\left(h(t,\cdot)^{*}\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1}-\mathcal{B}_{\mathcal{M}}\right)(v+w)+(\mathcal{B}_{\mathcal{M}}-\lambda)\,w\in N(P). \tag{4.20}$$

Now, since $(\mathcal{B}_{\mathcal{M}}-\lambda):N(P)\cap L^2(\mathcal{M})\mapsto N(P)\subset L^2(\mathcal{M})$ is an isomorphism at $\lambda=\lambda_0$, by the implicit function theorem, it is also an isomorphism for λ near λ_0. Hence, for λ near λ_0, it possesses an inverse which we denote by Q_λ. In addition Q_λ is an analytic function on λ. Thus, from (4.20)

$$w=-Q_\lambda(I-P)\left(h(t,\cdot)^{*}\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1}-\mathcal{B}_{\mathcal{M}}\right)(v+w)\ \in N(P). \tag{4.21}$$

See that (4.21) is solvable for w if t is near zero. In fact, if $\Phi:L^2(\mathcal{M})\mapsto L^2(\mathcal{M})$, given by

$$\Phi_{(t,\lambda,v)}(w)=\Phi(w)=-Q_\lambda(I-P)\left(h(t,\cdot)^{*}\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1}-\mathcal{B}_{\mathcal{M}}\right)(v+w)$$

is a uniform contraction, then, by the fixed point theorem with parameter [15, Section 1.2.6], there exists a unique fixed point w for each (t,λ,v). Next, let us see that Φ is a uniform contraction. First, we have that

$$\begin{aligned}\|\Phi(w)-\Phi(z)\|_{L^2(\mathcal{M})}&\leq\|Q_\lambda\|\|(I-P)\|\\&\left\|\left(h(t,\cdot)^{*}\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1}-\mathcal{B}_{\mathcal{M}}\right)\right\|\|w-z\|_{L^2(\mathcal{M})}\end{aligned}$$

where $\|\cdot\|$ denotes the norm in the space of linear and bounded operators. Also,

$$\begin{aligned}&\left\|\left(h(t,\cdot)^{*}\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1}-\mathcal{B}_{\mathcal{M}}\right)u\right\|^2_{L^2(\mathcal{M})}\\&\leq 2\left\|h(t,\cdot)^{*}a_{h(t,\mathcal{M})}-a_{\mathcal{M}}\right\|^2_{L^\infty(\mathcal{M})}\|u\|^2_{L^2(\mathcal{M})}+4\left(\sup_{\bar{\mathcal{M}}}|\det(Dh(t,\cdot))|\right)^2\\&\times\left(\int_{\mathcal{M}}\int_{\mathcal{M}}|J(h(t,x)-h(t,y))-J(x,y)|^2dv_g(x)dv_g(y)\right)\\&\times\|u\|^2_{L^2(\mathcal{M})}4\left(\sup_{\bar{\mathcal{M}}}|1-|\det(Dh(t,\cdot))||\right)^2\\&\times\left(\int_{\mathcal{M}}\int_{\mathcal{M}}J(x,y)^2dv_g(x)dv_g(y)\right)\|u\|^2_{L^2(\mathcal{M})}.\end{aligned}$$

Hence, as $h(t,\cdot)^* a_{h(t,\mathcal{M})}$, $|\det(Dh(t,\cdot))|$ and J are continuous with $\mathcal{M}$ bounded, there exists a positive constant $C(t)$ with $C(t) \to 0$ as $t \to 0$ such that

$$\left\| \left(h(t,\cdot)^* \mathcal{B}_{h(t,\mathcal{M})} h(t,\cdot)^{*-1} - \mathcal{B}_{\mathcal{M}} \right) \right\| \leq C(t), \quad \text{for } t \approx 0.$$

Thus, as Q_λ is uniformly bounded in a neighborhood of λ_0 and P is fixed, Φ is a contraction for $\lambda \approx \lambda_0$ and $h(t,\cdot) \approx i_{\mathcal{M}}$. Moreover, Φ is analytic in t, λ and v being linear in v. Then, there exists a unique operator $S(t,\lambda)$, analytic in (t,λ), which sets the fixed points of (4.21) by $w = S(t,\lambda)v$. Also,

$$\|S(t,\lambda)v\| = O\left(\|h(t,\cdot) - i_{\mathcal{M}}\|_{\mathcal{C}^1\left(\overline{\mathcal{M}},\mathcal{N}\right)} \right).$$

In particular, $S(0,\lambda)v = 0$.

Next, we use the equation in $R(P)$ to characterize the existence of the eigenpairs. Since $v = \sum_{i=1}^m c_i \phi_i$, for some $c_i \in \mathbb{R}$, we have

$$u = v + w = \sum_{i=1}^m c_i \left(1 + S(t,\lambda)\right) \phi_i \neq 0.$$

Thus, $h(t,\cdot)^{*-1}u$ is an eigenfunction of the operator $\mathcal{B}_{h(t,\mathcal{M})}$ with eigenvalue $\lambda \approx \lambda_0$, if and only if, there exist c_i not all zero such that $0 = \sum_{k=1}^m M_{ik}(t,\lambda)c_k$ for all $1 \leq i \leq m$ where

$$M_{ik}(t,\lambda) = \int_{\mathcal{M}} \phi_i \left(h(t,\cdot)^* \mathcal{B}_{h(t,\mathcal{M})} h(t,\cdot)^{*-1} - \lambda \right) \left(1 + S(t,\lambda)\right) \phi_k dv_g. \quad (4.22)$$

Therefore, λ is an eigenvalue, if and only if, $\det\left(M(t,\lambda)\right) = 0$ where $M(t,\lambda) = (M_{ik}(t,\lambda))_{i,k=1}^m$.

Notice that $(\lambda, t) \mapsto \det\left(M(t,\lambda)\right)$ is analytic near $t = 0$ and $\lambda = \lambda_0$. Hence, since $M(t,\lambda)$ is symmetric, it follows from Puiseux's theorem (see for instance [27]) that there exist m analytic curves $\lambda_1(t)$,..., $\lambda_m(t)$, not necessarily distinct, solutions of $\det\left(M(t,\lambda)\right) = 0$. Also, for each curve $\lambda_j(t)$, there exists an analytic curve $C_j(t) = (c_1^j(t), \ldots, c_m^j(t)) \in \mathbb{R}^m$ solution of $M(t,\lambda_j(t))C_j(t) = 0$ with $C_1(t)$,..., $C_m(t)$ linearly independent. Thus,

$$u_j = \sum_{i=1}^m c_i^j(t) \left(\phi_i + S(t,\lambda_j(t))\phi_i \right)$$

is an analytic curve of associated eigenfunctions which completes the proof of the lemma. □

Proof of Theorem 1.2 We have seen in Lemma 4.1 that there is an analytic eigenpair $(\lambda(t), u(t,\cdot))$ such that

$$\Big(h(t,\cdot)^*\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1} - \lambda(t)\Big)u(t,x) = 0 \quad \text{in } \mathcal{M}$$

for all t near a neighborhood of 0. Thus

$$\frac{d}{dt}\Big(h(t,\cdot)^*\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1}u(t,x)\Big)\Big|_{t=0} = \frac{d\lambda}{dt}(0)u(t,x) + \lambda(0)\frac{du}{dt}(t,x).$$

Now, from the expression (3.16), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(h(t,\cdot)^*\mathcal{B}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1}u(t,x)\Big)\Big|_{t=0} = \frac{d}{dt}\left(h(t,\cdot)^*a_{h(t,\mathcal{M})}u(t,x)\right)\Big|_{t=0}\\
&\quad - \frac{d}{dt}\Big(h(t,\cdot)^*\mathcal{J}_{h(t,\mathcal{M})}h(t,\cdot)^{*-1}u(t,x)\Big)\Big|_{t=0}\\
&\quad = \frac{d}{dt}\left(h(t,\cdot)^*a_{h(t,\mathcal{M})}\right)\Big|_{t=0}u(t,x) + a_{\mathcal{M}}\frac{du}{dt}(t,x)\\
&\quad - \int_{\mathcal{M}} J(x,y)D_t u(t,y)dv_g(y) - \langle V^T, \nabla\left(\mathcal{J}_{\mathcal{M}}u(t,x)\right)\rangle\\
&\quad - \int_{\partial\mathcal{M}} J(x,y)u(t,y)\langle V^T, N\rangle dS(y) - \int_{\mathcal{M}} J(x,y)u(t,y)\,\langle\vec{H}, V^{\perp}\rangle\,dv_g(y)\\
&\quad - \int_{\mathcal{M}} u(t,y)\,\langle\nabla_w(J(x,y)), V^{\perp}\rangle\,dv_g(y).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{d}{dt}\left(h(t,\cdot)^*a_{h(t,\mathcal{M})}\right)\Big|_{t=0}u(t,x) + a_{\mathcal{M}}\frac{du}{dt}(t,x) - \int_{\mathcal{M}} J(x,y)D_t u(0,y)dv_g(y)\\
&-\langle V^T, \nabla\left(\mathcal{J}_{\mathcal{M}}u(t,y)\right)\rangle - \int_{\mathcal{M}} J(x,y)u(0,y)\langle\vec{H}, V^{\perp}\rangle\,dv_g(y)\\
&- \int_{\mathcal{M}} u(0)\,\langle\nabla_y(J(x,y)), V^{\perp}\rangle\,dv_g(y)\\
&- \int_{\partial\mathcal{M}} J(x,y)u(t,y)\langle V^T, N\rangle dS(y) = \frac{d\lambda}{dt}(0)u(t,x) + \lambda(0)\frac{du}{dt}(t,x). \qquad (4.23)
\end{aligned}$$

Notice that there exist $c_1,\ldots, c_m$ such that $u(0,x) = \sum_{i=1}^m c_i\phi_i(x)$. Hence, multiplying (4.23) by the eigenfunction ϕ_k and integrating on $\mathcal{M}$, we obtain

$$\begin{aligned}
\frac{d\lambda}{dt}(0)\,c_k &= \int_{\mathcal{M}}\frac{d\lambda}{dt}(0)u(0,x)\phi_k(x)dv_g(x)\\
&= \int_{\mathcal{M}}\left[\left(h(t,\cdot)^*a_{h(t,\mathcal{M})}\right)\Big|_{t=0}u(0,x) + \left(a_{\mathcal{M}} - \lambda_0\right)\frac{du}{dt}(0,x)\right]\phi_k(x)dv_g(x)\\
&\quad - \int_{\mathcal{M}}\int_{\mathcal{M}} J(x,y)\frac{du}{dt}(0,y)\phi_k(x)dv_g(y)dv_g(x)
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathcal{M}}\langle V^T, \nabla\left(\mathcal{J}_{\mathcal{M}}u(0,x)\right)\rangle\phi_k(x)dv_g(x)\\
&+\int_{\mathcal{M}}\int_{\mathcal{M}} J(x,y)\langle V^T, \nabla u(0,y)\rangle\,\phi_k(x)dv_g(y)dv_g(x)\\
&-\int_{\mathcal{M}}\int_{\partial\mathcal{M}} J(x,y)u(0,y)\langle V^T, N\rangle\phi_k(x)dS(y)dv_g(x)\\
&-\int_{\mathcal{M}}\int_{\mathcal{M}} \phi_k(x)J(x,y)u(0,y)\langle\vec{H}, V^{\perp}\rangle\,dv_g(y)\,dv_g(x)\\
&-\int_{\mathcal{M}}\int_{\mathcal{M}} \phi_k(x)\,u(0,y)\langle\nabla_y J(x,y), V^{\perp}\rangle\,dv_g(y)\,dv_g(x).
\end{aligned}
\tag{4.24}
$$

Since J is symmetric,

$$
\begin{aligned}
&\int_{\mathcal{M}}(a_{\mathcal{M}}-\lambda_0)\,\frac{du}{dt}(0,x)\phi_k(x)dv_g(x)\\
&\quad-\int_{\mathcal{M}}\int_{\mathcal{M}} J(x,y)\frac{du}{dt}(0,y)\,\phi_k(x)\,dv_g(y)dv_g(x)\\
&\int_{\mathcal{M}}\frac{du}{dt}(0,x)\left[(a_{\mathcal{M}}-\lambda_0)\,\phi_k(x)-\int_{\mathcal{M}} J(x,y)\phi_k(y)dv_g(y)\right]dv_g(x)\\
&=\int_{\mathcal{M}}\frac{du}{dt}(0,x)\,(\mathcal{B}_{\mathcal{M}}-\lambda_0)\,\phi_k(x)dv_g(x)=0.
\end{aligned}
\tag{4.25}
$$

In addition,

$$
\begin{aligned}
\int_{\mathcal{M}}\langle V, \nabla\left(\mathcal{J}_{\mathcal{M}}u(0,x)\right)\rangle\phi_k(x)dv_g(x) &= \int_{\mathcal{M}} u(0,x)\phi_k(x)\langle V, \nabla a_{\mathcal{M}}\rangle\,dv_g(x)\\
&+\int_{\mathcal{M}}(a_{\mathcal{M}}-\lambda_0)\,\langle V, \nabla u(0,x)\rangle\phi_k(x)\,dv_g(x)
\end{aligned}
$$

and then,

$$
\begin{aligned}
&-\int_{\mathcal{M}}\langle V, \nabla\left(\mathcal{J}_{\mathcal{M}}u(0,x)\right)\rangle\phi_k(x)dv_g(x)\\
&\quad+\int_{\mathcal{M}}\int_{\mathcal{M}} J(x,y)\langle V, \nabla u(0,y)\rangle\,\phi_k(x)\,dv_g(y)\,dv_g(x)\\
&=-\int_{\mathcal{M}} u(0,x)\phi_k(x)\langle V, \nabla a_{\mathcal{M}}\rangle\,dv_g(x)\\
&\quad-\int_{\mathcal{M}}\langle V, \nabla u(0,x)\rangle\left[(a_{\mathcal{M}}-\mathcal{J}_{\mathcal{M}}-\lambda_0)\,\phi_k(x)\right]\,dv_g(x)\\
&=-\int_{\mathcal{M}} u(0,x)\phi_k(x)\langle V, \nabla a_{\mathcal{M}}\rangle\,dv_g(x).
\end{aligned}
\tag{4.26}
$$

We also have

$$
\int_{\mathcal{M}}\int_{\partial\mathcal{M}} J(x,y)u(0,y)\langle V^T, N\rangle\phi_k(x)\,dS(y)dv_g(x)
$$

$$
\begin{aligned}
&= \int_{\partial\mathcal{M}} u(0,y)\langle V^T, N\rangle \left(\int_{\mathcal{M}} J(x,y)\phi_k(x)dv_g(x) \right) dS(y) \\
&= \int_{\partial\mathcal{M}} u(0,y)\langle V^T, N\rangle \,(a_{\mathcal{M}} - \lambda_0)\, \phi_k(y) dS(y). \qquad (4.27)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_{\mathcal{M}} \int_{\mathcal{M}} \phi_k(x) J(x,y) u(0,y)\langle \vec{H}, V^{\perp}\rangle \, dv_g(y)\, dv_g(x) \\
&\quad = \int_{\mathcal{M}} (\mathcal{J}_{\mathcal{M}}\phi_k) u(0,y)\langle \vec{H}, V^{\perp}\rangle \, dv_g(y) \\
&\quad = \int_{\mathcal{M}} (a_{\mathcal{M}} - \lambda_0)\, \phi_k(y)\, u(0,y)\langle \vec{H}, V^{\perp}\rangle \, dv_g(y), \qquad (4.28)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathcal{M}} \int_{\mathcal{M}} \phi_k(x)\, u(0,y)\langle \nabla_y (J(x,y)), V^{\perp}\rangle \, dv_g(y)\, dv_g(x) \\
&\quad = \int_{\mathcal{M}} u(0,y)\langle \nabla_y (\mathcal{J}_{\mathcal{M}}\phi_k), V^{\perp}\rangle \, dv_g(y) \\
&\quad = \int_{\mathcal{M}} u(0,y)\langle \nabla_y \left[(a_{\mathcal{M}} - \lambda_0)\, \phi_k \right], V^{\perp}\rangle \, dv_g(y).
\end{aligned}
$$

As in Sect. 3, since $\nabla\phi_k$ are tangential to $\mathcal{M}$, we have that

$$
\int_{\mathcal{M}} u(0,y)\langle \nabla_y \left[(a_{\mathcal{M}} - \lambda_0)\, \phi_k \right], V^{\perp}\rangle \, dv_g(y) = \int_{\mathcal{M}} u(0,y)\phi_k \langle \nabla_y a_{\mathcal{M}}, V^{\perp}\rangle \, dv_g(y). \qquad (4.29)
$$

Therefore, it follows from (4.24), (4.25), (4.26), (4.27), (4.28) and (4.29) that

$$
\begin{aligned}
\frac{d\lambda}{dt}(0)\, c_k &= \int_{\mathcal{M}} \left(h(t,\cdot)^* a_{h(t,\mathcal{M})} \right) \Big|_{t=0} u(0,x)\phi_k(x)\, dv_g(x) \\
&\quad - \int_{\mathcal{M}} u(0,x))\phi_k(x)\langle V^T, \nabla a_{\mathcal{M}}\rangle \, dv_g(x) \\
&\quad - \int_{\partial\mathcal{M}} u(0,y)\langle V^T, N\rangle \,(a_{\mathcal{M}} - \lambda_0)\, \phi_k(y) dS(y) \\
&\quad - \int_{\mathcal{M}} (a_{\mathcal{M}} - \lambda_0)\, \phi_k(y) u(0,y)\langle \vec{H}, V^{\perp}\rangle \, dv_g(y) \\
&\quad - \int_{\mathcal{M}} u(0,y)\phi_k(y)\langle \nabla_y a_{\mathcal{M}}, V^{\perp}\rangle \, dv_g(y).
\end{aligned}
$$

As $u(0, \cdot) = \sum_{i=1}^{m} c_i \phi_i(\cdot)$, we get

$$\frac{d\lambda}{dt}(0)\, c_k = \sum_{i=1}^{m} c_i \int_{\mathcal{M}} \left[\left(h(t,\cdot)^* a_{h(t,\mathcal{M})} \right) \Big|_{t=0} - \langle V^T, \nabla a_{\mathcal{M}} \rangle \right] \phi_k(x)\phi_i(x) dv_g(x)$$
$$- \sum_{i=1}^{m} c_i \int_{\mathcal{M}} (a_{\mathcal{M}} - \lambda_0)\, \phi_k(x)\phi_i(x) \langle \vec{H}, V^{\perp} \rangle \, dv_g(x)$$
$$- \sum_{i=1}^{m} c_i \int_{\partial\mathcal{M}} \langle V^T, N \rangle \, (a_{\mathcal{M}} - \lambda_0)\, \phi_k(x)\phi_i(x) dS(x)$$
$$- \sum_{i=1}^{m} c_i \int_{\mathcal{M}} \phi_i(y)\phi_k(y) \langle \nabla_y a_{\mathcal{M}}, V^{\perp} \rangle \, dv_g(y).$$

Thus, $\frac{d\lambda}{dt}(0)$ is an eigenvalue of the matrix $B = (B_{ki})_{k,i=1}^{m}$ given by

$$B_{ki} = \int_{\mathcal{M}} D_t^T \left(h(t,\cdot)^* a_{h(t,\mathcal{M})} \right) \Big|_{t=0} \phi_k \phi_i \, dv_g$$
$$- \int_{\partial\mathcal{M}} (a_{\mathcal{M}} - \lambda_0)\, \phi_i(x)\phi_k(x) \langle V^T, N \rangle dS(x)$$
$$- \int_{\mathcal{M}} (a_{\mathcal{M}} - \lambda_0)\, \phi_k(x)\phi_i(x) \langle \vec{H}, V^{\perp} \rangle \, dv_g(x)$$
$$- \int_{\mathcal{M}} \phi_i(y)\phi_k(y) \langle \nabla_y a_{\mathcal{M}}, V^{\perp} \rangle \, dv_g(y).$$

which shows the theorem. □

Remark 4.1 Notice that the examples given in Sect. 3 are easily adapted from simple to multiple eigenvalues by Theorem 1.2. In particular, if $a_{\mathcal{M}}(x) \equiv 1$ in $\mathcal{M}$, we get the Hadamard formula for the Dirichlet problem. More precisely, if $\lambda(t)$ is a curve of eigenvalues, given under the conditions of Lemma 4.1 for the family of operators $\mathcal{B}_{h(t,\mathcal{M})}$ with $a_{\mathcal{M}}(x) \equiv 1$, then, the rate $\frac{d\lambda}{dt}(0)$ is an eigenvalue of the matrix $B = (B_{ki})_{k,i=1}^{m}$ with

$$B_{ki} = - \int_{\partial\mathcal{M}} (1 - \lambda_0)\, \langle V^T, N \rangle \phi_i(x)\phi_k(x) dS(x)$$
$$- \int_{\mathcal{M}} (1 - \lambda_0)\, \phi_k(x)\phi_i(x) \langle \vec{H}, V^{\perp} \rangle \, dv_g(x).$$

Remark 4.2 It is worth mentioning that the simple eigenvalue case, given by the condition $m = 1$, is also recovered by formula (1.4).

4.1 An application

Let us finish this section giving an application of Lemma 4.1 and Theorem 1.2. Here, we assume $\mathcal{M}$ is an open bounded domain set in $\mathbb{R}^n$. Also, we suppose $\mathcal{B}_{\mathcal{M}}$ is the Dirichlet operator set by $a_{\mathcal{M}} \equiv 1$. We have the following result.

Theorem 4.1 *Let λ_0 be an m-fold eigenvalue of $\mathcal{B}_{\mathcal{M}}$ with $a_{\mathcal{M}} \equiv 1$ and under the conditions of Lemma 4.1. Also, suppose the multiplicity of λ_0 cannot be reduced by small perturbations of $\mathcal{M}$ set by $h \in \mathrm{Diff}^1(\mathcal{M})$. Then, there exists a ball $\mathcal{O} \subset \mathrm{Diff}^1(\mathcal{M})$ centered at $i_{\mathcal{M}}$ such that, for each $h \in \mathcal{O}$, there exists a unique eigenvalue $\lambda_{h(\mathcal{M})}$ of $\mathcal{B}_{h(\mathcal{M})}$, near λ_0, which also has multiplicity m and satisfies $\lambda_{h(\mathcal{M})} = \lambda_0$ for all $h \in \mathcal{O}$ (that is, $\lambda_{h(\mathcal{M})}$ is constant and equal to λ_0 in $\mathcal{O}$). Moreover, there exists a neighborhood of ∂M where any eigenfunction of $\lambda_{h(\mathcal{M})}$ must vanish.*

Proof of the Theorem 4.1 We know from Lemma 4.1 that λ is an eigenvalue of $\mathcal{B}_{h(t,\mathcal{M})}$ with $\lambda \approx \lambda_0$ and $t \approx 0$, if and only if, $\lambda = \lambda(t)$ for some $t \approx 0$, and there exists $C(t) = (c_1(t), \dots, c_m(t)) \in \mathbb{R}^n$ satisfying $M(t, \lambda(t))C(t) = 0$, where M is the matrix given by (4.22). Hence, for all $1 \le i \le m$,

$$0 = \sum_{k=1}^{m} \int_{\mathcal{M}} \phi_i \left(h^*(t,x)\mathcal{B}_{h(t,\mathcal{M})} h^*(t,x)^{-1} - \lambda(t) \right) (\phi_k(x) + S(t,\lambda(t))\phi_k(x))\, c_k(t) dv_g(x)$$
$$= \sum_{k=1}^{n} p_{ik}(t,\lambda(t))c_k(t) - \lambda(t)c_i(t)$$

where $p_{ik}(t,\lambda)$ is set by

$$p_{ik}(t,\lambda) = \sum_{k=1}^{m} \int_{\mathcal{M}} \phi_i \left(h^*(t,x)\mathcal{B}_{h(t,\mathcal{M})} h^*(t,x)^{-1} \right) (1 + S(t,\lambda(t)))\, \phi_k(x)c_k(t) dv_g(x). \tag{4.30}$$

Consequently, it follows from our hypotheses that for each $t \approx 0$, there is $\lambda(t) \approx \lambda_0$ an eigenvalue of $\mathcal{B}_{h(t,\mathcal{M})}$ having multiplicity m. Thus, from (4.30), we have

$$0 = \Delta(t,\lambda) = \det\left[(\lambda\delta_{ik} - p_{ik}(t,\lambda))_{i,k=1}^{m} \right]$$

with the map $p_{ik}(t,\lambda)$ analytic near $(0,\lambda_0)$. Moreover, we have $p_{ik}(0,\lambda) = \lambda_0\delta_{ik}$ and $\Delta(0,\lambda) = (\lambda - \lambda_0)^m$. Then, by Rouché's theorem, there are m real roots λ near λ_0 such that the matrix $(\lambda\delta_{ik} - p_{ik}(t,\lambda))_{i,k=1}^{m}$ has at least m independent null-vectors. Hence, since $p_{ik} = p_{ki}$, such matrix is symmetric, and then, we can conclude that $p_{ik}(t,\lambda) = \lambda\delta_{ik}$ for all $1 \le i, k \le m$.

See now that, by (3.16),

$$\begin{aligned}
&\frac{d}{dt}\left(p_{ik}(t,\lambda(t))\right|_{t=0} \\
&= \int_{\mathcal{M}} \phi_i \frac{d}{dt}\left[h^*(t,\cdot)\mathcal{B}_{h(t,\mathcal{M})}h^*(t,x)^{-1}\left(1+S(t,\lambda(t))\right)\phi_k(x)\right]\Big|_{t=0} dv_g(x) \\
&= \int_{\mathcal{M}} \phi_i(x)\left[\frac{d}{dt}\left(h(t,\cdot)^* a_{h(t,\mathcal{M})}\right)\Big|_{t=0}\phi_k(x) + a_{\mathcal{M}}\frac{d}{dt}\left(\phi_k(x)+S(t,\lambda(t))\phi_k(x)\right)\Big|_{t=0}\right] dv_g(x) \\
&\quad - \int_{\mathcal{M}} J(x,y)\left(D_t^T\left(\phi_k(y)+S(t,\lambda(t))\phi_k(y)\right)\right)\Big|_{t=0} dv_g(y) - V\cdot\nabla\left(\mathcal{J}_{\mathcal{M}}\phi_k\right) \\
&\quad - \int_{\partial\mathcal{M}} J(x,y)\phi_k(x) V\cdot N\, dS(y) dv_g(x) \\
&= \int_{\mathcal{M}} \phi_i(x)\mathcal{B}_{\mathcal{M}}\frac{d}{dt}\left(S(t,\lambda(t))\phi_k(x)\right)\Big|_{t=0} dv_g(x) - \int_{\mathcal{M}} J(x,y)V\cdot\nabla\phi_k(x) dv_g(y) \\
&\quad - V\cdot\nabla\left(\mathcal{J}_{\mathcal{M}}\phi_k(x)\right) - \int_{\partial\mathcal{M}} J(x,y)\phi_k(y)V\cdot N\, dS(y) dv_g(x).
\end{aligned}$$

Hence, as $\int_{\mathcal{M}} \phi_i(x) V\cdot\nabla\left(\mathcal{J}_{\mathcal{M}}\phi_k\right)\, dv_g(x) = \int_{\mathcal{M}} \phi_i(x)(1-\lambda_0)V\cdot\nabla\phi_k(x)\, dv_g(x)$, we have

$$\begin{aligned}
&\int_{\mathcal{M}} \phi_i(x)\left[\int_{\mathcal{M}} J(x,y)V\cdot\nabla\phi_k(y) dv_g(y) - V\cdot\nabla\left(\mathcal{J}_{\mathcal{M}}\phi_k\right)\right] dv_g(x) \\
&\quad = \int_{\mathcal{M}} \phi_i(x)\left(\lambda_0 - \mathcal{B}_{\mathcal{M}}\right)\left(V\cdot\nabla\phi_k\right) dv_g(x) \\
&\quad = \int_{\mathcal{M}} \left(V\cdot\nabla\phi_k\right)\left(\lambda_0 - \mathcal{B}_{\mathcal{M}}\right)\phi_i(x) dv_g(x) = 0.
\end{aligned}$$

Also, $\int_{\mathcal{M}} \phi_i \mathcal{B}_{\mathcal{M}} \frac{d}{dt}\left(S(t,\lambda(t))\phi_k\right)\Big|_{t=0} dv_g(x) = 0$, since $S(t,\lambda(t))\phi_k \in N(P)$ for all t. Therefore, as the kernel J is symmetric, we have that

$$\begin{aligned}
\frac{d}{dt}\left(p_{ik}(t,\lambda(t))\right|_{t=0} &= -\int_{\mathcal{M}}\int_{\partial\mathcal{M}} \phi_i J(x,y)\phi_k V\cdot N\, dS(y) dv_g(x) \\
&= -\int_{\partial\mathcal{M}} (1-\lambda_0)\phi_k\phi_i V\cdot N\, dS(y).
\end{aligned} \tag{4.31}$$

On the other hand, we obtain from $p_{ik}(t,\lambda(t)) = \lambda(t)\delta_{ik}$ that

$$\dot{p}_{ik} = \frac{d}{dt}\left(p_{ik}(t,\lambda(t))\right)\Big|_{t=0} = \frac{d\lambda(0)}{dt}\delta_{ik}.$$

Consequently, it follows from (4.31) that

$$\begin{aligned}
&\int_{\partial\mathcal{M}} (1-\lambda_0)\phi_k\phi_i V\cdot N\, dS(y) = 0 \quad \text{and} \\
&\quad \int_{\partial\mathcal{M}} (1-\lambda_0)(\phi_k^2 - \phi_i^2) V\cdot N\, dS(y) = 0 \quad \forall V\in\mathcal{X}^1(\mathcal{M}) \ \text{ and } \ k\neq i.
\end{aligned}$$

Then, $\phi_k\phi_i = \phi_k^2 - \phi_i^2 \equiv 0$ on $\partial\mathcal{M}$ for any $k \neq i$. Thus,

$$\phi_k \equiv 0 \quad \text{on} \quad \partial\mathcal{M}, \qquad \text{for all} \quad k = 1,\ldots,m, \tag{4.32}$$

and then, $\frac{d\lambda(0)}{dt} = 0$ by Theorem 1.2. Using once more that the multiplicity of λ_0 can not be reduced, it follows from the change of origin discussed in [14, pag. 25] that $\frac{d\lambda(0)}{dt} \equiv 0$ in a neighborhood of $t = 0$. Thus, $\lambda(t) = \lambda_0$ for all $t \approx 0$. Since we are considering an arbitrary perturbation, we conclude that $\lambda_{h(\mathcal{M})}$ is constant and equal to λ_0 in a neighborhood of $i_{\mathcal{M}}$.

Finally, let us see that there exists a family of basis $\{\psi_k(t)\}_{k=1}^m$ for the eigenfunctions of the m-fold eigenvalue λ_0 satisfying $\psi_k(t) \equiv 0$ on $\partial h(t, \mathcal{M})$ for all $t \approx 0$. We take

$$\psi_k(t) = h^*(t, \cdot)^{-1} \left(\phi_k + S(t, \lambda(t))\phi_k\right) \quad \text{for each} \quad k = 1, \ldots, m. \tag{4.33}$$

Since the multiplicity of λ_0 can not be reduced, it follows from the definition of $\phi_k + S(t, \lambda(t))\phi_k$ and the fact $h^*(t, \cdot)^{-1}$ is an isomorphism that $\{\psi_k(t)\}_{k=1}^m$ given by (4.33) is a basis for the eigenfunctions of $\lambda(t) = \lambda_0$. Also, by the change of origin [14, pag. 25], it in enough to check that $\phi_k + S(0, \lambda_0)\phi_k = \phi_k \equiv 0$ on $\partial\mathcal{M}$ for any $k \neq i$. But, we have just done that at (4.32). Hence, we can conclude that $\{\psi_k(t)\}_{k=1}^m$ is a basis for the eigenfunctions of λ_0 satisfying $\psi_k(t) \equiv 0$ on $\partial h(t, \mathcal{M})$ for all $t \approx 0$. And then, we finish the proof. □

Remark 4.3 It is worth noting that Theorem 4.1 is a kind of result which guarantees the generic simplicity of eigenvalues for operators with the unique continuation property. As we still do not know if $\mathcal{B}_{\mathcal{M}}$ satisfies such property, we are not in a position to obtain the generic simplicity of their eigenvalues. Indeed, if $\mathcal{B}_{\mathcal{M}}$ would satisfy the unique continuation property, then their eigenfunctions could not vanish in a neighborhood of $\partial\mathcal{M}$, and then, from Lemma 4.1, the multiplicity of their eigenvalues would decrease to $m = 1$ as in the case of the Laplace operator with Dirichlet boundary condition.

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