



Invariant tori via higher order averaging method: existence, regularity, convergence, stability, and dynamics

Douglas D. Novaes¹ · Pedro C. C. R. Pereira¹

Received: 20 December 2022 / Revised: 19 May 2023 / Accepted: 11 June 2023 /
Published online: 24 June 2023

© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract

Important information about the dynamical structure of a differential system can be revealed by looking into its invariant compact manifolds, such as equilibria, periodic orbits, and invariant tori. This knowledge is significantly increased if asymptotic properties of the trajectories nearby such invariant manifolds can be determined. In this paper, we present a result providing sufficient conditions for the existence of invariant tori in perturbative differential systems. The regularity, convergence, and stability of such tori as well as the dynamics defined on them are also investigated. The conditions are given in terms of their so-called higher order averaged equations. This result is an extension to a wider class of differential systems of theorems due to Krylov, Bogoliubov, Mitropolsky, and Hale.

Mathematics Subject Classification 34C23 · 34C29 · 34C45

1 Introduction and statement of the main result

The *averaging method* has been employed by Krylov, Bogoliubov, and Mitropolski [1, 16] to study the existence of invariant tori in the extended phase space of T -periodic

DDN is partially supported by São Paulo Research Foundation (FAPESP) grants 2022/09633-5, 2019/10269-3, and 2018/13481-0, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) grants 438975/2018-9 and 309110/2021-1. PCCRP is supported by São Paulo Research Foundation (FAPESP) grant 2020/14232-4.

✉ Pedro C. C. R. Pereira
pedro.pereira@ime.unicamp.br

Douglas D. Novaes
ddnovaes@unicamp.br

¹ Departamento de Matemática-Instituto de Matemática, Estatística e Computação Científica (IMECC), Universidade Estadual de Campinas (UNICAMP), Rua Sérgio Buarque de Holanda, 651, Cidade Universitária Zeferino Vaz, Campinas, SP 13083-859, Brazil

non-autonomous perturbative differential equations of the kind $\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x})$. Those results were generalized by Hale in [11] and [12] and assert that the existence of invariant tori is associated to the existence of limit cycles of the so-called averaged equation,

$$\dot{\mathbf{x}} = \frac{1}{T} \int_0^T F_1(t, \mathbf{x}) dt.$$

In this paper, we are concerned with a wider class of T -periodic non-autonomous perturbative differential equations of the following kind:

$$\dot{\mathbf{x}} = \sum_{i=1}^N \varepsilon^i F_i(t, \mathbf{x}) + \varepsilon^{N+1} \tilde{F}(t, \mathbf{x}, \varepsilon), \quad (t, \mathbf{x}, \varepsilon) \in \mathbb{R} \times D \times [0, \varepsilon_0], \quad (1)$$

where D is an open bounded subset of \mathbb{R}^n , $\varepsilon_0 > 0$, and the functions $F_i : \mathbb{R} \times D \rightarrow \mathbb{R}^n$, $i \in \{1, \dots, N\}$, and $\tilde{F} : \mathbb{R} \times D \times [0, \varepsilon_0] \rightarrow \mathbb{R}^n$ are of class C^r , $r \geq 2$, and T -periodic in the variable t . Our goal is to extend the mentioned results of Krylov, Bogoliubov, Mitropolsky, and Hale concerning the existence of invariant tori to the differential equation (1). More specifically, we aim to provide sufficient conditions for the existence of invariant tori in the extended phase space of (1) which, due to the periodicity in the variable t , can be seen as a vector field defined on a cylinder:

$$\begin{cases} \tau' = 1, \\ \mathbf{x}' = \sum_{i=1}^N \varepsilon^i F_i(\tau, \mathbf{x}) + \varepsilon^{N+1} \tilde{F}(\tau, \mathbf{x}, \varepsilon), \end{cases} \quad (\tau, \mathbf{x}) \in \mathbb{S}^1 \times D, \quad (2)$$

where $\mathbb{S}^1 = \mathbb{R}/(T\mathbb{Z})$. In addition, results concerning the regularity, convergence, and stability of such tori as well as information about the dynamics defined on them will also be presented.

1.1 Introduction to the averaging theory

Some notions from the averaging theory will appear in the statement of our main result, Theorem A. Thus, in order to state it, we must provide a brief introduction to the averaging method, with special attention to the concept of higher order averaged functions.

The *averaging method* or *averaging theory* stemmed from the works of Clairaut, Lagrange, and Laplace regarding perturbations of differential equations (see [22, Appendix A]), even though its formalization was only established much later, by Fatou, Krylov, Bogoliubov, and Mitropolsky (see [1, 2, 8, 16]). It is particularly useful in the study of nonlinear oscillating systems which are affected by small perturbations, by providing asymptotic estimates for solutions of non-autonomous differential equations given in the standard form (1).

The estimates provided by the averaging method depend on the *averaged functions*, $\mathbf{g}_i : D \rightarrow \mathbb{R}^n$ for $i \in \{1, \dots, N\}$, which appear as solutions of homological equations when transforming system (1) according to the following result.

Theorem 1 ([22, Lemma 2.9.1]) *There exists a T -periodic near-identity transformation of class C^r*

$$\mathbf{x} = U(t, \mathbf{z}, \varepsilon) = \mathbf{z} + \sum_{i=1}^N \varepsilon^i \mathbf{u}_i(t, \mathbf{z}), \tag{3}$$

satisfying $U(0, \mathbf{z}, \varepsilon) = \mathbf{z}$, such that the differential equation (1) is transformed into

$$\dot{\mathbf{z}} = \sum_{i=1}^N \varepsilon^i \mathbf{g}_i(\mathbf{z}) + \varepsilon^{N+1} r_N(t, \mathbf{z}, \varepsilon).$$

The condition $U(0, \mathbf{z}, \varepsilon) = \mathbf{z}$, called *stroboscopic condition*, ensures that the functions \mathbf{g}_i are uniquely determined. In that case, \mathbf{g}_i is named the *averaged function of order i* . One can easily verify that \mathbf{g}_1 is, indeed, the time-average of $F_1(t, \mathbf{x})$, that is,

$$\mathbf{g}_1(\mathbf{z}) = \frac{1}{T} \int_0^T F_1(s, \mathbf{z}) ds. \tag{4}$$

In general terms, the averaging theory guarantees that, for time $\mathcal{O}(1/\varepsilon)$ and ε small, any solution of (1) remains ε^N -close to the solution of the *truncated averaged equation*

$$\dot{\mathbf{z}} = \sum_{i=1}^N \varepsilon^i \mathbf{g}_i(\mathbf{z}),$$

with the same initial conditions.

In addition to the aforesaid quantitative estimates, the averaging theory has found great success when applied to investigate invariant manifolds; for instance, to guarantee the existence of invariant tori, as mentioned in the introduction of this work, [1, 4, 11, 12]. It has also been successfully applied to the study of simpler compact invariant manifolds, such as periodic solutions (see, for example, [3, 13, 17, 19, 21, 24]).

Recently, the paper [20] provided a general recursive formula for the higher order averaged functions in terms of Melnikov functions. Accordingly, define the *Melnikov function of order i* , \mathbf{f}_i , for $i \in \{0, \dots, N\}$, by

$$\mathbf{f}_0(\mathbf{z}) = 0 \quad \text{and} \quad \mathbf{f}_i(\mathbf{z}) = \frac{y_i(T, \mathbf{z})}{i!}, \tag{5}$$

where

$$y_1(t, \mathbf{z}) = \int_0^t F_1(s, \mathbf{z}) ds \quad \text{and} \tag{6}$$

$$y_i(t, \mathbf{z}) = \int_0^t \left(i! F_i(s, \mathbf{z}) + \sum_{j=1}^{i-1} \sum_{m=1}^j \frac{i!}{j!} \partial_{\mathbf{x}}^m F_{i-j}(s, \mathbf{z}) B_{j,m}(y_1, \dots, y_{j-m+1})(s, \mathbf{z}) \right) ds,$$

for $i \in \{2, \dots, N\}$. In the formulae above, for p and q positive integers, $B_{p,q}$ denotes the *partial Bell polynomials* (see, for instance, [6]). Roughly speaking, the Melnikov functions determine the N -jet in ε of the time- T -map of (1), that is, $\varphi(T, \mathbf{z}, \varepsilon) = \mathbf{z} + \sum_{i=1}^N \varepsilon^i \mathbf{f}_i(\mathbf{z}) + \mathcal{O}(\varepsilon^{N+1})$, where $\varphi(t, \mathbf{z}, \varepsilon)$ corresponds to the solution of (1) with initial condition $\varphi(0, \mathbf{z}, \varepsilon) = \mathbf{z}$ (see [17, 18]).

In particular, (4) and (6) ensure that $\mathbf{f}_1(\mathbf{z}) = T \mathbf{g}_1(\mathbf{z})$. The next result states that the same holds for higher order averaged functions provided that some conditions are satisfied.

Proposition 1 ([20, Corollary A]) *Let $\ell \in \{2, \dots, N\}$. If either $\mathbf{f}_1 = \dots = \mathbf{f}_{\ell-1} = 0$ or $\mathbf{g}_1 = \dots = \mathbf{g}_{\ell-1} = 0$, then $\mathbf{f}_i = T \mathbf{g}_i$ for $i \in \{1, \dots, \ell\}$.*

The relationship established in Proposition 1 allows us to directly calculate the first non-vanishing averaged function, thus motivating the main result of this work, i.e., an extension of the results of Krylov, Bogoliubov, Mitropolsky, and Hale to higher order averaged functions.

1.2 Statement of the main theorem

The existence of invariant tori in a differential system, as in the case of existence of equilibria and periodic orbits, reveals important information about the dynamical structure of the differential system. This knowledge is significantly increased if asymptotic properties of the trajectories nearby such invariant tori can be determined. Thus, before introducing our main result, we must set forth the following definition regarding asymptotic stability of invariant manifolds.

Definition 1 Let

$$\dot{\mathbf{x}} = F(\mathbf{x}) \tag{7}$$

be an autonomous differential system in \mathbb{R}^n and let $\mathbf{x}(t, \mathbf{x}_0)$ be the solution of (7) satisfying $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$. Let also M be an m -dimensional invariant manifold of system (7) and V be a neighborhood of the manifold M .

(a) The local stable set of M with respect to V is

$$\mathcal{S}_M^V := \left\{ \mathbf{x}_0 \in V : \mathbf{x}(t, \mathbf{x}_0) \in V \text{ for all } t > 0 \text{ and } \lim_{t \rightarrow \infty} d(\mathbf{x}(t, \mathbf{x}_0), M) = 0 \right\}.$$

(b) The local unstable set of M with respect to V is

$$\mathcal{U}_M^V := \left\{ \mathbf{x}_0 \in V : \mathbf{x}(t, \mathbf{x}_0) \in V \text{ for all } t < 0 \text{ and } \lim_{t \rightarrow -\infty} d(\mathbf{x}(t, \mathbf{x}_0), M) = 0 \right\}.$$

Now, we are ready to provide our main result.

Theorem A *Consider the C^r , $r \geq 2$, differential equation (1) and its extension (2). Suppose that, for some $\ell \in \{1, \dots, \min(N, r - 2)\}$, $\mathbf{f}_0 = \dots = \mathbf{f}_{\ell-1} = 0$, $\mathbf{f}_\ell \neq 0$.*

Assume that the guiding system $\dot{\mathbf{z}} = \mathbf{g}_\ell(\mathbf{z})$ has an ω -periodic hyperbolic limit cycle $\varphi(t)$. Then, there exists $\bar{\varepsilon} > 0$ such that, for each $\varepsilon \in (0, \bar{\varepsilon}]$, the following statements hold:

- (a) **Existence:** The differential system (2) has an invariant torus M_ε . In addition, there exists a neighborhood $V \subset D$ of $\Gamma := \{\varphi(t) : t \in \mathbb{R}\}$ such that any invariant compact manifold of (2) contained in $\mathbb{S}^1 \times V$ must be contained in M_ε . In particular, M_ε is the unique invariant torus in $\mathbb{S}^1 \times V$.
- (b) **Regularity:** The invariant torus M_ε is of class $C^{r-\ell}$. Furthermore, there is a C^0 -continuous family of $C^{r-\ell}$ functions $\mathcal{F}_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^n$, T -periodic in the first coordinate and ω -periodic in the second coordinate, such that $M_\varepsilon = \{(\tau, \mathcal{F}_\varepsilon(\tau, \theta)) : (\tau, \theta) \in \mathbb{S}^1 \times \mathbb{R}\}$.
- (c) **Convergence:** There is a continuous function $\delta : [0, \bar{\varepsilon}] \rightarrow \mathbb{R}_+$ satisfying $\delta(0) = 0$ such that $\|\mathcal{F}_\varepsilon(\tau, \theta) - U(\tau, \varphi(\theta), \varepsilon)\| < \delta(\varepsilon)$ for every $(\tau, \theta) \in \mathbb{R}^2$, where U is the transformation given by Theorem 1. In particular, M_ε converges to $\mathbb{S}^1 \times \Gamma$ in the Hausdorff distance as $\varepsilon \rightarrow 0$.
- (d) **Stability:** Let $k \leq n - 1$ be the number of characteristic multipliers of Γ with modulus less than 1. Then, there are neighborhoods $W_s, V_s, W_u,$ and V_u of M_ε such that
 - (d.1) $\mathcal{S}_{M_\varepsilon}^{V_s} \cap W_s$ is a $(k + 2)$ -dimensional manifold embedded in \mathbb{R}^{n+1} ;
 - (d.2) $\mathcal{U}_{M_\varepsilon}^{V_u} \cap W_u$ is a $(n - k + 1)$ -dimensional manifold embedded in \mathbb{R}^{n+1} .
- (e) **Dynamics:** The flow of (2) restricted to M_ε defines a first return map $p_\varepsilon : S_\varepsilon \rightarrow S_\varepsilon$ where, for $\Sigma = \{(0, \mathbf{x}) : \mathbf{x} \in D\}$ a transversal section of (2), $S_\varepsilon := \Sigma \cap M_\varepsilon$ is $C^{r-\ell}$ diffeomorphic to the circle \mathbb{S}^1 . Moreover, p_ε is of class $C^{r-\ell}$; its rotation number $\rho(\varepsilon)$ is a continuous function on $\varepsilon \in [0, \bar{\varepsilon}]$ satisfying $\rho(\varepsilon) = \varepsilon^\ell T / \omega + \mathcal{O}(\varepsilon^{\ell+1})$; and, finally, if $r - \ell \geq 4$, then ρ maps zero Lebesgue measure sets to zero Lebesgue measure sets, and there exists a positive Lebesgue measure set $E \subset [0, \bar{\varepsilon}]$ such that, for every $\varepsilon \in E$, $\rho(\varepsilon)$ is irrational and p_ε is $C^{r-\ell-3}$ conjugated to an irrational rotation.

Theorem A is proved in Sect. 3 after establishing some preliminary results in Sect. 2.

Remark 1 Since p_ε is at least of class C^2 (because $r - \ell \geq 2$), if $\rho(\varepsilon)$ is irrational, then p_ε is topologically conjugate to an irrational rotation (see, for instance, [13, Theorems 2.4 and 2.5]). In this case, the dynamics of (2) on the invariant torus M_ε corresponds to an irrational flow and, therefore, the torus M_ε is a minimal invariant compact manifold, in the sense that there is no other compact invariant manifold of (2) contained in M_ε besides itself. Therefore, since $\rho(\varepsilon) = \varepsilon^\ell T / \omega + \mathcal{O}(\varepsilon^{\ell+1})$ is continuous, we can always find $\varepsilon^* \in (0, \bar{\varepsilon}]$ such that M_{ε^*} is minimal.

Remark 2 The conclusion provided by statement e) that “if $r - \ell \geq 4$, then ρ maps zero Lebesgue measure sets to zero Lebesgue measure sets” is known as Luzin-N-property of the function ρ .

1.3 Application: invariant tori in 4D vector fields

Theorem A provides a means for investigating the existence of invariant tori also in higher dimensional vector fields. For instance, for a positive integer $N \geq 2$, consider the following 4D autonomous differential system

$$\begin{aligned} \dot{x} &= -y + \varepsilon^N f_1(x, y, u, v) + \varepsilon^{N+1} g_1(x, y, u, v) + \varepsilon^{N+2} h_1(x, y, u, v, \varepsilon), \\ \dot{y} &= x + \varepsilon^N f_2(x, y, u, v) + \varepsilon^{N+1} g_2(x, y, u, v) + \varepsilon^{N+2} h_2(x, y, u, v, \varepsilon), \\ \dot{u} &= \varepsilon^N f_3(x, y, u, v) + \varepsilon^{N+1} g_3(x, y, u, v) + \varepsilon^{N+2} h_3(x, y, u, v, \varepsilon), \\ \dot{v} &= \varepsilon^N f_4(x, y, u, v) + \varepsilon^{N+1} g_4(x, y, u, v) + \varepsilon^{N+2} h_4(x, y, u, v, \varepsilon), \end{aligned} \tag{8}$$

where ε is a small positive parameter; $\mu \in \{-1, 1\}$; f_i , for $i \in \{1, 2, 3, 4\}$, are functions of class C^r , $r \geq 4$, satisfying that

$$\begin{aligned} \theta &\mapsto \cos(\theta) f_1(r \cos(\theta), r \sin(\theta), u, v) + \sin(\theta) f_2(r \cos(\theta), r \sin(\theta), u, v), \\ \theta &\mapsto f_3(r \cos(\theta), r \sin(\theta), u, v), \text{ and} \\ \theta &\mapsto f_4(r \cos(\theta), r \sin(\theta), u, v) \end{aligned} \tag{9}$$

have vanishing average over $\theta \in [0, 2\pi]$; g_i , for $i \in \{1, 2, 3, 4\}$, are given by

$$\begin{aligned} g_1(x, y, u, v) &= \mu x(x^2 + y^2), \\ g_2(x, y, u, v) &= -\mu y(x^2 + y^2)^2, \\ g_3(x, y, u, v) &= x^2(u(-u^2 - v^2 + 1) + v), \\ g_4(x, y, u, v) &= y^2(v(-u^2 - v^2 + 1) - u); \end{aligned}$$

and h_i , for $i \in \{1, 2, 3, 4\}$, are C^r , $r \geq 4$, functions.

Proposition 2 *Assume the conditions above for the differential system (8). Then, for any integer $N \geq 2$ and $\varepsilon > 0$ sufficiently small, the differential system (8) has an invariant torus \mathbb{T}_ε converging, as ε goes to 0, to $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$. Moreover, the invariant torus is asymptotically stable provided that $\mu = 1$ and has an unstable direction provided that $\mu = -1$.*

Proposition 2 is proven in Sect. 4.

Example 1 Assuming that

$$\begin{aligned} f_1(x, y, u, v) &= yu, \quad f_2(x, y, u, v) = -xv, \\ f_3(x, y, u, v) &= x^3, \text{ and } f_4(x, y, u, v) = y^3, \end{aligned}$$

one can easily see that the functions given in (9) have vanishing average. Thus, Proposition 2 can be applied to provide the existence of an invariant torus for $\varepsilon > 0$ sufficiently

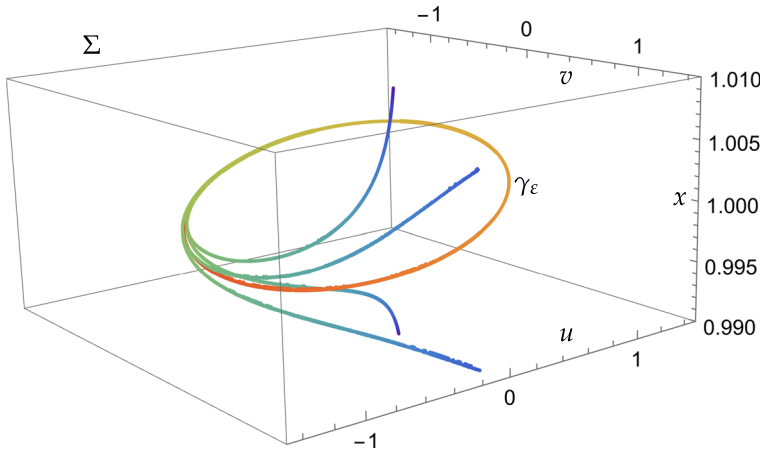


Fig. 1 Assuming $N = 2$, $\mu = 1$, $h_i = 0$, $i = 1, \dots, 4$, and $\varepsilon = 1/15$, we show 10345 iterations of the Poincaré map of (8), defined on the section $\Sigma = \{(x, 0, u, v) : x > 0\}$, for the initial values: $(1.01, 0, 2, 0)$, $(0.99, 0, 2, 0)$, $(1.01, 0, 0.5, 0)$, and $(0.99, 0, 0.5, 0)$. The orbits are attracted by the closed curve γ_ε , which corresponds to the intersection between the invariant torus \mathbb{T}_ε with the section Σ . For the web version of the paper, purple points indicate a low number of iterations, whilst red points indicate a high number of iterations

small. In Fig. 1, assuming $N = 2$, $\mu = 1$, $h_i = 0$, $i = 1, \dots, 4$, and $\varepsilon = 1/15$, we provide a numeric simulation (performed on Mathematica) of the Poincaré map defined on the section $\Sigma = \{(x, 0, u, v) : x > 0\}$ of the differential system (8). The asymptotically stable invariant tori \mathbb{T}_ε corresponds to an asymptotically stable invariant closed curve $\gamma_\varepsilon := \mathbb{T}_\varepsilon \cap \Sigma$ for the Poincaré map.

2 Fundamental Lemma

The proof of Theorem A makes use of some results concerning integral manifolds of a class of perturbed differential systems. Such results, and the methods employed for obtaining them, are similar to those established by Hale in [11, Lemmas 2.1, 2.2 and 2.3] (see also [2, Sect. 28, Lemmas 1, 2, and 3] and [12]). In this section, we state and prove those results in the form of a single Lemma, along with Propositions addressing the issue of regularity of the integral manifolds obtained.

Throughout the paper, we will adopt the notation $\text{diag}(A_1, \dots, A_n)$ to represent the direct sum $A_1 \oplus \dots \oplus A_n$ of the square matrices A_i , $i \in \{1, \dots, n\}$. We will also employ the notation $B_n(p, r)$ for the n -dimensional open ball $\{x \in \mathbb{R}^n : \|x - p\| < r\}$.

We consider a one-parameter family of differential systems of the form

$$\begin{aligned}
 \theta' &= 1 + \zeta_0(t, \theta, y, z, \varepsilon), \\
 y' &= H_1 \cdot y + \zeta_1(t, \theta, y, z, \varepsilon), \\
 z' &= H_2 \cdot z + \zeta_2(t, \theta, y, z, \varepsilon),
 \end{aligned}
 \tag{10}$$

where $\theta \in \mathbb{R}$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, ε is a real parameter, H_1 is a real $m \times m$ matrix, H_2 is a real $n \times n$ matrix, and the continuous functions $\zeta_0 : \mathbb{R} \times \mathbb{R} \times B_m(0, \rho_1) \times B_n(0, \rho_2) \times (0, \varepsilon_0] \rightarrow \mathbb{R}$, $\zeta_1 : \mathbb{R} \times \mathbb{R} \times B_m(0, \rho_1) \times B_n(0, \rho_2) \times (0, \varepsilon_0] \rightarrow \mathbb{R}^m$, and $\zeta_2 : \mathbb{R} \times \mathbb{R} \times B_m(0, \rho_1) \times B_n(0, \rho_2) \times (0, \varepsilon_0] \rightarrow \mathbb{R}^n$ have Lipschitz continuous partial derivatives with respect to (θ, y, z) up to the p -th order, where $p \geq 1$, $\rho_1, \rho_2 > 0$, and $\varepsilon_0 > 0$. For conciseness, we define, for each $(\sigma, \mu) \in (0, \rho_1) \times (0, \rho_2)$, the set

$$\Sigma_{\sigma, \mu}^{\varepsilon_0} := \mathbb{R} \times \mathbb{R} \times \bar{B}_m(0, \sigma) \times \bar{B}_n(0, \mu) \times (0, \varepsilon_0].$$

We suppose that the following hypotheses are satisfied by (10):

(i) There is $\omega > 0$ such that

$$\begin{aligned}\zeta_0(t, \theta + \omega, y, z, \varepsilon) &= \zeta_0(t, \theta, y, -z, \varepsilon), \\ \zeta_1(t, \theta + \omega, y, z, \varepsilon) &= \zeta_1(t, \theta, y, -z, \varepsilon), \\ \zeta_2(t, \theta + \omega, y, z, \varepsilon) &= -\zeta_2(t, \theta, y, -z, \varepsilon).\end{aligned}$$

(ii) There is a continuous function $M : [0, \varepsilon_0] \rightarrow \mathbb{R}_+$ such that $M(0) = 0$ and the functions ζ_i satisfy $|\zeta_0(t, \theta, 0, 0, \varepsilon)| \leq M(\varepsilon)$, $\|\zeta_1(t, \theta, 0, 0, \varepsilon)\| \leq M(\varepsilon)$, and $\|\zeta_2(t, \theta, 0, 0, \varepsilon)\| \leq M(\varepsilon)$ for all $(t, \theta, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times (0, \varepsilon_0]$.

(iii) There is a continuous function $L : (0, \varepsilon_0] \times [0, \rho_1) \times [0, \rho_2) \rightarrow \mathbb{R}_+$ such that

$$\lim_{(\varepsilon, \sigma, \mu) \rightarrow (0, 0, 0)} L(\varepsilon, \sigma, \mu) = 0,$$

and, for $(t, \theta_1, y_1, z_1, \varepsilon), (t, \theta_2, y_2, z_2, \varepsilon) \in \Sigma_{\sigma, \mu}^{\varepsilon_0}$, the following inequalities hold true:

$$\begin{aligned}|\zeta_0(t, \theta_1, y_1, z_1, \varepsilon) - \zeta_0(t, \theta_2, y_2, z_2, \varepsilon)| &\leq L(\varepsilon, \sigma, \mu) \|(\theta_1, y_1, z_1) - (\theta_2, y_2, z_2)\|, \\ \|\zeta_1(t, \theta_1, y_1, z_1, \varepsilon) - \zeta_1(t, \theta_2, y_2, z_2, \varepsilon)\| &\leq L(\varepsilon, \sigma, \mu) \|(\theta_1, y_1, z_1) - (\theta_2, y_2, z_2)\|, \\ \|\zeta_2(t, \theta_1, y_1, z_1, \varepsilon) - \zeta_2(t, \theta_2, y_2, z_2, \varepsilon)\| &\leq L(\varepsilon, \sigma, \mu) \|(\theta_1, y_1, z_1) - (\theta_2, y_2, z_2)\|.\end{aligned}$$

(iv) The eigenvalues of H_1 and H_2 have non-zero real parts.

Let $(\theta(t, t_0, \theta_0, y_0, z_0, \varepsilon), y(t, t_0, \theta_0, y_0, z_0, \varepsilon), z(t, t_0, \theta_0, y_0, z_0, \varepsilon))$ denote the solution of (10) with initial conditions $(t_0, \theta_0, y_0, z_0, \varepsilon)$. Having set forth the hypotheses above, we are now ready to state the Lemma.

Lemma 1 Consider system (10) with the hypotheses presented in this section. There are $\varepsilon_1 \in (0, \varepsilon_0)$ and families of continuous functions $f_\varepsilon : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^m$ and $g_\varepsilon : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n$ such that, for each $\varepsilon \in (0, \varepsilon_1]$, the autonomous system

$$\begin{aligned}t' &= 1, \\ \theta' &= 1 + \zeta_0(t, \theta, y, z, \varepsilon), \\ y' &= H_1 \cdot y + \zeta_1(t, \theta, y, z, \varepsilon), \\ z' &= H_2 \cdot z + \zeta_2(t, \theta, y, z, \varepsilon),\end{aligned}\tag{11}$$

has an invariant manifold given by $y = f_\varepsilon(t, \theta)$, $z = g_\varepsilon(t, \theta)$. The families f_ε and g_ε also satisfy:

(a) There is a continuous function $D : [0, \varepsilon_1] \rightarrow \mathbb{R}_+$ such that $D(0) = 0$ and

$$\|f_\varepsilon(t, \theta)\| \leq D(\varepsilon), \quad \|g_\varepsilon(t, \theta)\| \leq D(\varepsilon)$$

for all $(t, \theta, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times (0, \varepsilon_1]$. Furthermore, there is a constant $C_H > 0$, depending only on the matrices H_1 and H_2 , such that $D(\varepsilon) = C_H M(\varepsilon)$, where $M(\varepsilon)$ is the function appearing in hypothesis ii).

(b) There is a continuous function $\Delta : [0, \varepsilon_1] \rightarrow \mathbb{R}_+$ such that $\Delta(0) = 0$ and

$$\begin{aligned} \|f_\varepsilon(t, \theta_1) - f_\varepsilon(t, \theta_2)\| &\leq \Delta(\varepsilon)|\theta_1 - \theta_2|, \\ \|g_\varepsilon(t, \theta_1) - g_\varepsilon(t, \theta_2)\| &\leq \Delta(\varepsilon)|\theta_1 - \theta_2| \end{aligned}$$

for all $t \in \mathbb{R}$, all $\theta_1, \theta_2 \in \mathbb{R}$ and all $\varepsilon \in (0, \varepsilon_1]$.

(c) f_ε is ω -periodic in θ for all $\varepsilon \in (0, \varepsilon_1]$ and g_ε satisfies $g_\varepsilon(t, \theta + \omega) = -g_\varepsilon(t, \theta)$ for all $(t, \theta, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times (0, \varepsilon_1]$;

(d) If, for a given $\varepsilon \in (0, \varepsilon_1]$, the functions $\zeta_0(t, \theta, y, z, \varepsilon)$, $\zeta_1(t, \theta, y, z, \varepsilon)$, and $\zeta_2(t, \theta, y, z, \varepsilon)$ are T_ε -periodic in the variable t , then so are f_ε and g_ε ;

(e) f_ε and g_ε have bounded and uniformly continuous derivatives with respect to θ up to the p -th order for all $\varepsilon \in (0, \varepsilon_1]$;

(f) Let $\pi_1 : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\pi_2 : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the canonical projections. If $m_s \leq m$ of the eigenvalues of H_1 and $n_s \leq n$ of the eigenvalues of H_2 have negative real parts, there are positive constants r, λ, C, σ_0 , and σ_1 such that $r \leq \sigma_0$, $D(\varepsilon) < \sigma_0 < \sigma_1$, and, for each $(t_0, \theta_0, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times (0, \varepsilon_1]$, there is in $\bar{B}_m(0, \sigma_0) \times \bar{B}_n(0, \sigma_0)$ a local $(m_s + n_s)$ -dimensional embedded submanifold $S(t_0, \theta_0, \varepsilon)$ of $\mathbb{R}^m \times \mathbb{R}^n$, containing the point $(f_\varepsilon(t_0, \theta_0), g_\varepsilon(t_0, \theta_0))$, and having the following properties:

(f.1) If $(y_0, z_0) \in \bar{B}_m(0, \sigma_0) \times \bar{B}_n(0, \sigma_0) \setminus S(t_0, \theta_0, \varepsilon)$, there is $t_* > t_0$ for which

$$(y(t_*, t_0, \theta_0, y_0, z_0, \varepsilon), z(t_*, t_0, \theta_0, y_0, z_0, \varepsilon)) \notin \bar{B}_m(0, \sigma_1) \times \bar{B}_n(0, \sigma_1).$$

(f.2) Reciprocally, if $(y_0, z_0) \in S(t_0, \theta_0, \varepsilon)$, then, for all $t \geq t_0$,

$$(y(t, t_0, \theta_0, y_0, z_0, \varepsilon), z(t, t_0, \theta_0, y_0, z_0, \varepsilon)) \in \bar{B}_m(0, \sigma_1) \times \bar{B}_n(0, \sigma_1)$$

and the following inequalities hold:

$$\begin{aligned} \|y(t, t_0, \theta_0, y_0, z_0, \varepsilon) - f_\varepsilon(t, \theta(t, t_0, \theta_0, f_\varepsilon(t_0, \theta_0), g_\varepsilon(t_0, \theta_0), \varepsilon))\| \\ \leq C e^{-\lambda(t-t_0)} \|y_0 - f_\varepsilon(t_0, \theta_0)\|, \\ \|z(t, t_0, \theta_0, y_0, z_0, \varepsilon) - g_\varepsilon(t, \theta(t, t_0, \theta_0, f_\varepsilon(t_0, \theta_0), g_\varepsilon(t_0, \theta_0), \varepsilon))\| \\ \leq C e^{-\lambda(t-t_0)} \|z_0 - g_\varepsilon(t_0, \theta_0)\|. \end{aligned}$$

(f.3) *There is a continuous function $\phi_1^\varepsilon : \mathbb{R} \times \mathbb{R} \times \bar{B}_{m_s}(0, r) \times \bar{B}_{n_s}(0, r) \rightarrow \bar{B}_{m-m_s}(0, \sigma_0)$ such that*

$$\pi_1(S(t_0, \theta_0, \varepsilon)) = \{(\phi_1^\varepsilon(t_0, \theta_0, \xi_1, \xi_2), \xi_1) : (\xi_1, \xi_2) \in \bar{B}_{m_s}(0, r) \times \bar{B}_{n_s}(0, r)\}.$$

Similarly, there is a continuous function $\phi_2^\varepsilon : \mathbb{R} \times \mathbb{R} \times \bar{B}_{m_s}(0, r) \times \bar{B}_{n_s}(0, r) \rightarrow \bar{B}_{n-n_s}(0, \sigma_0)$ such that

$$\pi_2(S(t_0, \theta_0, \varepsilon)) = \{(\phi_2^\varepsilon(t_0, \theta_0, \xi_1, \xi_2), \xi_2) : (\xi_1, \xi_2) \in \bar{B}_{m_s}(0, r) \times \bar{B}_{n_s}(0, r)\}.$$

(f.4) *The functions ϕ_1^ε and ϕ_2^ε satisfy*

$$\phi_1^\varepsilon(t_0, \theta_0 + \omega, \xi_1, \xi_2) = \phi_1^\varepsilon(t_0, \theta_0, \xi_1, -\xi_2)$$

and

$$\phi_2^\varepsilon(t_0, \theta_0 + \omega, \xi_1, \xi_2) = -\phi_2^\varepsilon(t_0, \theta_0, \xi_1, -\xi_2)$$

for all $(t_0, \theta_0, \xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R} \times \bar{B}_{m_s}(0, r) \times \bar{B}_{n_s}(0, r)$.

Proof The argument is very similar to the one found in [11, Lemmas 2.1, 2.2, and 2.3] (see also [2, Sect. 28, Lemmas 1, 2, and 3]). We will omit computations when analogous ones can be found in those references, simply referring the reader to them.

Without loss of generality, suppose that $H_1 = \text{diag}(H_1^+, H_1^-)$ and $H_2 = \text{diag}(H_2^+, H_2^-)$, with the eigenvalues of H_i^+ and H_i^- , $i \in \{1, 2\}$, having respectively positive and negative real parts. For each $i \in \{1, 2\}$, define

$$J_i(t) = \begin{cases} - \begin{bmatrix} e^{-tH_i^+} & 0 \\ 0 & 0 \end{bmatrix}, & t > 0; \\ \begin{bmatrix} 0 & 0 \\ 0 & e^{-tH_i^-} \end{bmatrix}, & t < 0. \end{cases}$$

Consider the complete metric space $\mathcal{P}_\omega(D, \Delta)$ of continuous functions $F \in C(\mathbb{R}^2; \mathbb{R}^m)$ satisfying:

- $F(t, \theta + \omega) = F(t, \theta)$ for all $(t, \theta) \in \mathbb{R} \times \mathbb{R}$;
- $\|F(t, \theta)\| \leq D$ for all $(t, \theta) \in \mathbb{R} \times \mathbb{R}$;
- $\|F(t, \theta_1) - F(t, \theta_2)\| \leq \Delta|\theta_2 - \theta_1|$ for all $(t, \theta_1), (t, \theta_2) \in \mathbb{R} \times \mathbb{R}$,

where the metric is given by the uniform norm. Similarly, consider the complete metric space $\mathcal{A}_\omega(D, \Delta)$ of continuous functions $G \in C(\mathbb{R}^2; \mathbb{R}^n)$ satisfying:

- $G(t, \theta + \omega) = -G(t, \theta)$ for all $(t, \theta) \in \mathbb{R} \times \mathbb{R}$;
- $\|G(t, \theta)\| \leq D$ for all $(t, \theta) \in \mathbb{R} \times \mathbb{R}$;
- $\|G(t, \theta_1) - G(t, \theta_2)\| \leq \Delta|\theta_2 - \theta_1|$ for all $(t, \theta_1), (t, \theta_2) \in \mathbb{R} \times \mathbb{R}$.

For each $(F, G) \in \mathcal{P}_\omega(D, \Delta) \times \mathcal{A}_\omega(D, \Delta)$, let $T_{F,G}(t, t_0, \theta_0, \varepsilon)$ denote the solution of

$$\theta' = 1 + \zeta_0(t, \theta, F(t, \theta), G(t, \theta), \varepsilon)$$

satisfying $T_{F,G}(t_0, t_0, \theta_0, \varepsilon) = \theta_0$. Since $T_{F,G}(t, t_0, \theta_0 + \omega, \varepsilon)$ and $T_{F,G}(t, t_0, \theta_0, \varepsilon) + \omega$ are both solutions of the same initial problem, it follows that

$$T_{F,G}(t, t_0, \theta_0 + \omega, \varepsilon) = T_{F,G}(t, t_0, \theta_0, \varepsilon) + \omega. \tag{12}$$

For each $\varepsilon \in (0, \varepsilon_0]$, define the function $S^\varepsilon(F, G) = (S_1^\varepsilon(F, G), S_2^\varepsilon(F, G)) \in C(\mathbb{R}^2; \mathbb{R}^m) \times C(\mathbb{R}^2; \mathbb{R}^n)$, acting on the metric space $\mathcal{P}_\omega(D, \Delta) \times \mathcal{A}_\omega(D, \Delta)$ and given by

$$\begin{aligned} S_1^\varepsilon(F, G)(t, \theta) &= \int_{-\infty}^\infty J_1(x)\zeta_1(t+x, T_{F,G}(t+x, t, \theta, \varepsilon), F(t+x, T_{F,G}(t+x, t, \theta, \varepsilon)), G(t+x, T_{F,G}(t+x, t, \theta, \varepsilon)), \varepsilon)dx, \\ S_2^\varepsilon(F, G)(t, \theta) &= \int_{-\infty}^\infty J_2(x)\zeta_2(t+x, T_{F,G}(t+x, t, \theta, \varepsilon), F(t+x, T_{F,G}(t+x, t, \theta, \varepsilon)), G(t+x, T_{F,G}(t+x, t, \theta, \varepsilon)), \varepsilon)dx. \end{aligned}$$

By performing the change of variable of integration $\tau = x + t$ and differentiating the compositions $F(t, T_{F,G}(t, t_0, \theta_0, \varepsilon))$ and $G(t, T_{F,G}(t, t_0, \theta_0, \varepsilon))$ with respect to t , it is easy to see that, if $(f_\varepsilon, g_\varepsilon)$ is a fixed-point of S^ε , then the manifold given by $y = f_\varepsilon(t, \theta)$, $z = g_\varepsilon(t, \theta)$ is invariant under (11). Thus, the problem is reduced to proving that S^ε admits a fixed-point.

Following the arguments found in [11, Lemma 2.1], we conclude that it is possible to find $\varepsilon_1 \in (0, \varepsilon_0]$ and define $D(\varepsilon)$ and $\Delta(\varepsilon)$ such that, for $\varepsilon \in (0, \varepsilon_1]$, the function S^ε is a contraction of $\mathcal{P}_\omega(D(\varepsilon), \Delta(\varepsilon)) \times \mathcal{A}_\omega(D(\varepsilon), \Delta(\varepsilon))$ into itself. The only change compared to the arguments found in the reference is that, in order to guarantee that $S_2^\varepsilon(F, G) \in \mathcal{A}(D(\varepsilon), \Delta(\varepsilon))$, we need to show that $S_2^\varepsilon(F, G)$ satisfies $S_2^\varepsilon(F, G)(t, \theta + \omega) = -S_2^\varepsilon(F, G)(t, \theta)$. However, this is easily seen by considering (12) and the properties of functions ζ_1 and ζ_2 . Furthermore, since in [11] it is proved that $\Delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and that there is $C_H > 0$ such that $D(\varepsilon) = C_H M(\varepsilon)$, properties (a) and (b) are ensured to hold.

It remains to show that properties (c) to (f) are valid. Observe that property (c) follows directly from the fact that $(f_\varepsilon, g_\varepsilon) \in \mathcal{P}_\omega(D(\varepsilon), \Delta(\varepsilon)) \times \mathcal{A}_\omega(D(\varepsilon), \Delta(\varepsilon))$. Property (d) follows from the same argument found in [11, Lemma 2.2]. Property (e) is not directly discussed in [11], but it is stated and proved in [2, Sect. 28, Lemma 1]. The proof in our case is essentially the same.

Finally, we proceed to discussing property (f). For $t_0, \sigma_0, \nu \in \mathbb{R}$ and $k \in \mathbb{N}$, let $C_k^{t_0}(\sigma_0, \nu)$ be the complete metric space of continuous functions $W : [t_0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^k$ satisfying:

- $\|W(t, \theta)\| \leq \sigma_0$ for all $(t, \theta) \in [t_0, \infty) \times \mathbb{R}$;

- $\|W(t, \theta_1) - W(t, \theta_2)\| \leq \nu|\theta_1 - \theta_2|$ for all $(t, \theta_1), (t, \theta_2) \in [t_0, \infty) \times \mathbb{R}$.

The metric of $C_k^{t_0}(\sigma_0, \nu)$ is given by the uniform norm.

Let $t_0, \sigma_0, \nu \in \mathbb{R}$ be given. For each $b_1 \in \mathbb{R}^m$ and each $b_2 \in \mathbb{R}^n$, define the following functions acting on $C_m^{t_0}(\sigma_0, \nu)$ and $C_n^{t_0}(\sigma_0, \nu)$, respectively:

$$\begin{aligned}
 S_1^{\varepsilon, b_1}(W_1, W_2)(t, \theta) &= J_1(t_0 - t)b_1 \\
 &+ \int_{t_0}^{\infty} J_1(\tau - t)\xi_1(\tau, T_{W_1, W_2}(\tau, t, \theta, \varepsilon), W_1(\tau, T_{W_1, W_2}(\tau, t, \theta, \varepsilon)), \\
 &W_2(\tau, T_{W_1, W_2}(\tau, t, \theta, \varepsilon)), \varepsilon)d\tau, \\
 S_2^{\varepsilon, b_2}(W_1, W_2)(t, \theta) &= J_2(t_0 - t)b_2 \\
 &+ \int_{t_0}^{\infty} J_2(\tau - t)\xi_2(\tau, T_{W_1, W_2}(\tau, t, \theta, \varepsilon), W_1(\tau, T_{W_1, W_2}(\tau, t, \theta, \varepsilon)), \\
 &W_2(\tau, T_{W_1, W_2}(\tau, t, \theta, \varepsilon)), \varepsilon)d\tau.
 \end{aligned}$$

Let $S^{\varepsilon, b_1, b_2}$ act on $C_m^{t_0}(\sigma_0, \nu) \times C_n^{t_0}(\sigma_0, \nu)$ by $S^{\varepsilon, b_1, b_2}(W_1, W_2) = (S_1^{\varepsilon, b_1}(W_1, W_2), S_2^{\varepsilon, b_2}(W_1, W_2))$. Then, following the same procedure as before, we can ensure that, by taking $\varepsilon_1, \sigma_0, \nu$, and $r < \sigma_0$ sufficiently small, $S^{\varepsilon, b_1, b_2}$ becomes a contraction of $C_m^{t_0}(\sigma_0, \nu) \times C_n^{t_0}(\sigma_0, \nu)$ into itself if $\|b_1\|, \|b_2\| \leq r$.

Define $\Psi_1^{\varepsilon, t_0}$ and $\Psi_2^{\varepsilon, t_0}$ to be such that $(t, \theta) \mapsto (\Psi_1^{\varepsilon, t_0}(t, \theta, b_1, b_2), \Psi_2^{\varepsilon, t_0}(t, \theta, b_1, b_2))$ is the fixed point of the operator $S^{\varepsilon, b_1, b_2}$. Then, it is easy to see that there is $C_0 > 0$ such that

$$\begin{aligned}
 &\|\Psi_i^{\varepsilon, t_0}(t, \theta, b_1, b_2) - \Psi_i^{\varepsilon, t_0}(t, \tilde{\theta}, \tilde{b}_1, \tilde{b}_2)\| \\
 &\leq C_0 e^{-\frac{\alpha}{2}(t-t_0)} \left(\|b_1 - \tilde{b}_1\| + \|b_2 - \tilde{b}_2\| \right) + \nu|\theta - \tilde{\theta}| \tag{13}
 \end{aligned}$$

for $i \in \{1, 2\}, t \in (t_0, +\infty), \theta, \tilde{\theta} \in \mathbb{R}, b_1, \tilde{b}_1 \in \bar{B}_m(0, r)$, and $b_2, \tilde{b}_2 \in \bar{B}_n(0, r)$. This ensures, in particular, that, for $i \in \{1, 2\}, \Psi_i^{\varepsilon, t_0}$ is continuous if seen as a function on $[t_0, +\infty) \times \mathbb{R} \times \bar{B}_m(0, r) \times \bar{B}_n(0, r)$.

Following the argument in [2, Sect. 28, Lemma 3], we can prove that, if $\sigma_1 \geq \sigma_0$, every solution of (11) satisfying

- $y_0 \in \bar{B}_m(0, \sigma_0)$ and $y(t, t_0, \theta_0, y_0, z_0, \varepsilon) \in \bar{B}_m(0, \sigma_1)$;
- $z_0 \in \bar{B}_n(0, \sigma_0)$ and $z(t, t_0, \theta_0, y_0, z_0, \varepsilon) \in \bar{B}_n(0, \sigma_1)$

must be of the form $(t, \theta(t), \Psi_1^\varepsilon(t, \theta(t), b_1, b_2), \Psi_2^\varepsilon(t, \theta(t), b_1, b_2))$ for some $(b_1, b_2) \in B_m(0, r) \times B_n(0, r)$, where $\theta(t)$ denotes $T_{\Psi_1^\varepsilon, \Psi_2^\varepsilon}(t, t_0, \theta_0, \varepsilon)$. Conversely, every solution of the form given above clearly satisfies the two conditions set forth. Therefore, define

$$S(t_0, \theta_0, \varepsilon) := \left\{ \lim_{t \rightarrow t_0^+} (\Psi_1^{\varepsilon, t_0}(t, \theta_0, b_1, b_2), \Psi_2^{\varepsilon, t_0}(t, \theta_0, b_1, b_2)) : (b_1, b_2) \in \bar{B}_m(0, r) \times \bar{B}_n(0, r) \right\}.$$

Then, considering also (13), properties (f.1) and (f.2) follow immediately.

Observe that, from the definition of the operators S_1^{ε, b_1} and S_2^{ε, b_2} , it follows that the last m_s and n_s elements of the vectors $\Psi_1^{\varepsilon, t_0}(t_0^+, \theta, b_1, b_2)$ and $\Psi_2^{\varepsilon, t_0}(t_0^+, \theta, b_1, b_2)$ coincide with, respectively, the last m_s and n_s elements of the vectors b_1 and b_2 . Thus, for each $i \in \{1, 2\}$, define ϕ_i^ε by

$$(\phi_i^\varepsilon(t_0, \theta_0, \xi_1, \xi_2), \xi_i) = \lim_{t \rightarrow t_0^+} \Psi_i^{\varepsilon, t_0}(t, \theta_0, (0, \xi_1), (0, \xi_2)).$$

Then, it is clear that ϕ_i^ε is continuous. It is also clear from this definition that property (f.3) holds.

Finally, since the change of variables $(t, \theta, y, z) \rightarrow (\tilde{t}, \tilde{\theta}, \tilde{y}, \tilde{z}) = (t, \theta - \omega, y, -z)$ carries system (11) into an identical system, it follows from the already proved properties (f.1) and (f.2) that: a point $(y_*, z_*) \in \mathbb{R}^m \times \mathbb{R}^n$ is in $S(t_0, \theta_0 + \omega, \varepsilon)$ if, and only if, $(y_*, -z_*) \in S(t_0, \theta_0, \varepsilon)$. Hence, it follows that, for each $(\xi_1, \xi_2) \in \bar{B}_{m_s}(0, r) \times \bar{B}_{n_s}(0, r)$, there is $(\tilde{\xi}_1, \tilde{\xi}_2) \in \bar{B}_{m_s}(0, r) \times \bar{B}_{n_s}(0, r)$ such that

$$(\phi_1^\varepsilon(t_0, \theta_0 + \omega, \xi_1, \xi_2), \xi_1) = (\phi_1^\varepsilon(t_0, \theta_0, \tilde{\xi}_1, \tilde{\xi}_2), \tilde{\xi}_1),$$

and

$$(\phi_2^\varepsilon(t_0, \theta_0 + \omega, \xi_1, \xi_2), \xi_2) = -(\phi_2^\varepsilon(t_0, \theta_0, \tilde{\xi}_1, \tilde{\xi}_2), \tilde{\xi}_2).$$

Therefore, $\xi_1 = \tilde{\xi}_1$ and $\xi_2 = -\tilde{\xi}_2$, and property (f.4) follows. This concludes the proof of the Lemma. □

The following corollary addresses the issue of uniqueness of the invariant manifold found in the previous Lemma. Its proof will not be presented here, but it follows essentially from the stability property provided by statement (f) (see, for instance, [11, Remark 2.2] and, for more details, [2, Remark of page 494]).

Corollary 1 *For each $\varepsilon \in (0, \varepsilon_1]$, the invariant manifold given by $y = f_\varepsilon(t, \theta)$, $z = g_\varepsilon(t, \theta)$ is unique in $\mathbb{R} \times \mathbb{R} \times B_m(0, \sigma_0) \times B_n(0, \sigma_0)$, that is, every invariant manifold contained in $\mathbb{R} \times \mathbb{R} \times B_m(0, \sigma_0) \times B_n(0, \sigma_0)$ must be contained in the set given by $y = f_\varepsilon(t, \theta)$, $z = g_\varepsilon(t, \theta)$.*

The remainder of this section is devoted to present technical propositions to address issues of regularity. Their proofs are provided in the Appendix. The first proposition is concerned with the regularity of the invariant manifold whose existence was established in the previous Lemma.

Proposition 3 *Consider system (10) with the hypotheses presented in this section. Suppose that, for each $\varepsilon \in (0, \varepsilon_1]$ and each $i \in \{0, 1, 2\}$ fixed, the functions $(t, \theta, y, z) \mapsto \zeta_i(t, \theta, y, z, \varepsilon)$ are of class C^p . Then, the invariant manifold found in Lemma 1 above, that is, the manifold $M_\varepsilon = \{(t, \theta, f_\varepsilon(t, \theta), g_\varepsilon(t, \theta)) \in \mathbb{R} \times \mathbb{R} \times B_m(0, \sigma_0) \times B_n(0, \sigma_0) : (t, \theta) \in \mathbb{R} \times \mathbb{R}\}$, is of class C^p . Moreover, for each $\varepsilon \in (0, \varepsilon_1]$, the functions $(t, \theta) \mapsto f_\varepsilon(t, \theta)$ and $(t, \theta) \mapsto g_\varepsilon(t, \theta)$ are also of class C^p .*

The next three propositions consider the issue of regularity of the family $(f_\varepsilon, g_\varepsilon)$ with respect to the parameter ε . They will be crucial when we discuss the statement concerning dynamics of Theorem A.

Proposition 4 Consider system (10) with the hypotheses presented in this section. Suppose that $\zeta_0, \zeta_1,$ and ζ_2 are of class C^2 . Let $c : (0, \varepsilon_1) \rightarrow C(\mathbb{R}^2; \mathbb{R}^m) \times C(\mathbb{R}^2; \mathbb{R}^n)$ be defined by $c(\varepsilon) = (f_\varepsilon, g_\varepsilon)$, where $C(\mathbb{R}^2; \mathbb{R}^m)$ and $C(\mathbb{R}^2; \mathbb{R}^n)$ are equipped with the uniform norm. If $\varepsilon_1 > 0$ is sufficiently small, then c is of class C^1 .

Proposition 5 Consider system (10) with the hypotheses presented in this section. Suppose that $\zeta_0, \zeta_1,$ and ζ_2 are of class C^{p+1} . Let $T_{F,G}$ be defined as in the proof of Lemma 1. Then, if $\varepsilon_1 > 0$ is sufficiently small, then the following holds: there is $N_T \in \mathbb{N}$ and, for each compact interval $[a, b] \subset (0, \varepsilon_1]$, there are $C_{[a,b]} > 0$ and $M_{[a,b]} > 0$ such that

$$\left\| \frac{\partial^q f_\varepsilon}{\partial \theta^q} \right\| \leq C_{[a,b]}, \quad \left\| \frac{\partial^q g_\varepsilon}{\partial \theta^q} \right\| \leq C_{[a,b]},$$

and

$$\left| \frac{\partial^q T_{f_\varepsilon, g_\varepsilon}}{\partial \theta_0^q}(t + x, t, \theta, \varepsilon) \right| \leq M_{[a,b]} e^{N_T L(\varepsilon)(1+2\Delta(\varepsilon))|x|}$$

for all $q \in \{1, \dots, p + 1\}$ and all $\varepsilon \in [a, b]$.

Proposition 5 admits the following corollary which is a straightforward application of the mean value inequality.

Corollary 2 Consider system (10) with the hypotheses presented in this section. Suppose that $\zeta_0, \zeta_1,$ and ζ_2 are of class C^{p+1} . Then, if $\varepsilon_1 > 0$ is sufficiently small, then the following holds: for each compact interval $[a, b] \subset (0, \varepsilon_1]$, there is $C_{[a,b]} > 0$ such that

$$\left\| \frac{\partial^q f_\varepsilon}{\partial \theta^q}(t, \theta_2) - \frac{\partial^q f_\varepsilon}{\partial \theta^q}(t, \theta_1) \right\| \leq C_{[a,b]} |\theta_2 - \theta_1|,$$

$$\left\| \frac{\partial^q g_\varepsilon}{\partial \theta^q}(t, \theta_2) - \frac{\partial^q g_\varepsilon}{\partial \theta^q}(t, \theta_1) \right\| \leq C_{[a,b]} |\theta_2 - \theta_1|$$

for all $q \in \{1, \dots, p\}$, all $t, \theta_1, \theta_2 \in \mathbb{R}$, and all $\varepsilon \in [a, b]$.

Proposition 6 Consider system (10) with the hypotheses presented in this section. Suppose that $\zeta_0, \zeta_1,$ and ζ_2 are of class C^{p+1} . Let $q \leq p$ be a non-negative integer. Then, if $\varepsilon_1 > 0$ is sufficiently small, then the functions

$$\varepsilon \mapsto \frac{\partial^q f_\varepsilon}{\partial \theta^q} \quad \text{and} \quad \varepsilon \mapsto \frac{\partial^q g_\varepsilon}{\partial \theta^q}$$

are locally Lipschitz continuous in the uniform norm for $\varepsilon \in (0, \varepsilon_1]$.

3 Proof of Theorem A

This section is devoted to the proof of Theorem A. In Sect. 3.1, we perform the change of variables that transforms system (1) into a system to which Lemma 1 can be applied. In Sect. 3.2, we apply this Lemma in order to prove the statements regarding existence, regularity, and convergence of Theorem A. In Sect. 3.3, we prove the statement regarding stability, and finally, in Sect. 3.4, the statement regarding the dynamics on the object M_ε of the same Theorem.

3.1 Change of variables

Consider the differential equation (1). We shall first find a change of coordinates transforming this system into one to which we can apply Lemma 1. Thus, let $\ell \in \{1, \dots, \min(N, r-2)\}$ be such that $\mathbf{f}_1 = \dots = \mathbf{f}_{\ell-1} = 0$ and $\mathbf{f}_\ell \neq 0$. By Theorem 1, there exists a T -periodic near-identity transformation (3) that transforms the differential equation (1) into

$$\dot{\mathbf{z}} = \varepsilon^\ell \mathbf{g}_\ell(\mathbf{z}) + \varepsilon^{\ell+1} r_\ell(t, \mathbf{z}, \varepsilon). \tag{14}$$

Observe that the formulas given in (6) ensure that \mathbf{g}_ℓ is of class $C^{r-\ell+1}$. Moreover, r_ℓ is of class $C^{r-\ell}$.

Consider the ω -periodic hyperbolic limit cycle $\varphi(s)$ of the guiding system $\dot{\mathbf{z}} = \mathbf{g}_\ell(\mathbf{z})$. Also, consider the linear variational equation

$$\frac{dy}{dt} = D\mathbf{g}_\ell(\varphi(t)) \cdot y. \tag{15}$$

Observe that $\varphi'(t)$ is a solution to the linear periodic system (15). Let $\Phi(t)$ denote a fundamental matrix solution of this system. We will use Floquet theory to obtain a useful change of variables in a neighborhood of the limit cycle Γ .

We remind the reader that the characteristic multipliers of (15) are, for any choice of Φ , the eigenvalues of the monodromy matrix $\Phi^{-1}(0)\Phi(\omega)$ (see, for instance, [5]). Since Γ is hyperbolic, we know that 1 is an eigenvalue of multiplicity exactly 1 of this matrix, all its other eigenvalues being outside the unit circle. By taking into account the real Jordan canonical form of the monodromy matrix, we see that $\Phi(t)$ can be chosen satisfying

$$\Phi^{-1}(0)\Phi(\omega) = \text{diag}(1, \mathcal{J}_1, \mathcal{J}_2), \tag{16}$$

where $\mathcal{J}_1 \in \mathbb{R}^{(n-d-1) \times (n-d-1)}$ and $\mathcal{J}_2 \in \mathbb{R}^{d \times d}$ are matrices in the real Jordan canonical form satisfying the following condition: each Jordan block of \mathcal{J}_1 associated to a real negative eigenvalue appears an even number of times, and every Jordan block of \mathcal{J}_2 is associated to a real negative eigenvalue and appears only once in this matrix. We remark that, with this choice, the first of column of Φ must be the only ω -periodic solution of (15), which is given by $\varphi'(t)$.

Let I_d denote the $d \times d$ identity matrix. Under the above-mentioned conditions, by considering the logarithm of the matrices R_1 and R_2 (see, [7] and [9, page 100]), we know that there exist real matrices R_1 and R_2 such that $e^{\omega R_1} = \mathcal{J}_1$ and $e^{\omega R_2 + i\pi I_d} = \mathcal{J}_2$. In particular, we remark that the eigenvalues of R_1 and R_2 all have non-zero real parts. For the same reason, the number of eigenvalues (counting multiplicity) of the Poincaré map defined in a transversal section of Γ with modulus less than 1 is equal to the number of eigenvalues (counting multiplicity) of $R := \text{diag}(R_1, R_2)$ with strictly negative real part.

Define the matrices

$$\tilde{B} := \text{diag} \left(0, R_1, R_2 + i \frac{\pi}{\omega} I_d \right)$$

and

$$B := \text{diag} (0, R_1, R_2) = \text{diag}(0, R).$$

It is easy to see that $e^{\omega \tilde{B}} = \Phi^{-1}(0)\Phi(\omega)$ and $e^{2\omega B} = e^{2\omega \tilde{B}} = (\Phi^{-1}(0)\Phi(\omega))^2$.

Since $D\mathbf{g}_\ell(\varphi(t))$ is ω -periodic, Floquet’s theorem ensures that there are a ω -periodic matrix function $t \mapsto \tilde{P}(t) \in \mathbb{C}^{n \times n}$ and a 2ω -periodic matrix function $t \mapsto P(t) \in \mathbb{R}^{n \times n}$, both of class $C^{r-\ell+1}$, such that

$$\Phi(t) = \tilde{P}(t)e^{t\tilde{B}} = P(t)e^{tB}.$$

In particular, since the first column of $\Phi(t)$ is $\varphi'(t)$, it follows that $P(t)$ is of the form

$$P(t) = [\varphi'(t) \mid Q(t)],$$

where $t \mapsto Q(t) \in \mathbb{R}^{n \times (n-1)}$ is 2ω -periodic. Also, considering that B and \tilde{B} clearly commute, it follows that

$$P(t + \omega) = P(t)e^{\omega(\tilde{B}-B)} = [\varphi'(t) \mid Q(t)A],$$

where

$$A := \text{diag}(I_{n-d-1}, -I_d). \tag{17}$$

Thus, it is clear that Q satisfies

$$Q(t + \omega) = Q(t)A \tag{18}$$

for all $t \in \mathbb{R}$.

Since $\Phi(t)$ solves (15), it follows that

$$P'(t) + P(t) \cdot B = D\mathbf{g}_\ell(\varphi(t)) \cdot P(t).$$

Thus, by restricting the equality above to the last $n - 1$ columns, we obtain

$$Q'(t) + Q(t) \cdot R = Dg_\ell(\varphi(t)) \cdot Q(t), \tag{19}$$

for all $t \in \mathbb{R}$.

We apply the transformation $\mathbf{z} \mapsto (s, \mathbf{h}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ given by

$$\mathbf{z} = \varphi(s) + Q(s) \cdot \mathbf{h}. \tag{20}$$

Observe that, by taking \mathbf{h} to be sufficiently small and $s \in [0, \omega)$, we can ensure that the transformation $(s, \mathbf{h}) \mapsto \mathbf{z}$ is injective. Accordingly, we will assume henceforth that $\|\mathbf{h}\| \leq 4\rho$, ensuring that our transformation is bijective. Let us find the differential equation in (s, \mathbf{h}) that is equivalent to (14). In order to do so, we differentiate (20) with respect to t and obtain

$$\dot{\mathbf{z}} = (\varphi'(s) + Q'(s) \cdot \mathbf{h}) \dot{s} + Q(s) \dot{\mathbf{h}}.$$

Thus, by (14), it follows that

$$\begin{aligned} (\varphi'(s) + Q'(s) \cdot \mathbf{h}) \dot{s} + Q(s) \cdot \dot{\mathbf{h}} &= \varepsilon^\ell g_\ell(\varphi(s) + Q(s) \cdot \mathbf{h}) \\ &\quad + \varepsilon^{\ell+1} r(t, \varphi(s) + Q(s) \cdot \mathbf{h}, \varepsilon). \end{aligned} \tag{21}$$

Observe that (19) ensures that

$$\varepsilon^\ell \varphi'(s) + \varepsilon^\ell Q'(s) \cdot \mathbf{h} + \varepsilon^\ell Q(s) R \cdot \mathbf{h} = \varepsilon^\ell g_\ell(\varphi(s)) + \varepsilon^\ell Dg_\ell(\varphi(s)) \cdot Q(s) \cdot \mathbf{h}. \tag{22}$$

Let us define the functions

$$\begin{aligned} Y(s, \mathbf{h}) &:= g_\ell(\varphi(s) + Q(s) \cdot \mathbf{h}) - g_\ell(\varphi(s)) - Dg_\ell(\varphi(s)) \cdot Q(s) \cdot \mathbf{h}, \\ Z(t, s, \mathbf{h}, \varepsilon) &:= r_\ell(t, \varphi(s) + Q(s) \cdot \mathbf{h}, \varepsilon). \end{aligned}$$

By subtracting (22) from (21), we obtain

$$(\varphi'(s) + Q'(s) \cdot \mathbf{h})(\dot{s} - \varepsilon^\ell) + Q(s)(\dot{\mathbf{h}} - \varepsilon^\ell R \cdot \mathbf{h}) = \varepsilon^\ell Y(s, \mathbf{h}) + \varepsilon^{\ell+1} Z(t, s, \mathbf{h}, \varepsilon). \tag{23}$$

Observe that (23) can be rewritten as:

$$\left[\begin{array}{c|c} \varphi'(s) + Q'(s) \cdot \mathbf{h} & \\ \hline & Q(s) \end{array} \right] \cdot \left[\begin{array}{c} \dot{s} - \varepsilon^\ell \\ \dot{\mathbf{h}} - \varepsilon^\ell R \cdot \mathbf{h} \end{array} \right] = \varepsilon^\ell Y(s, \mathbf{h}) + \varepsilon^{\ell+1} Z(t, s, \mathbf{h}, \varepsilon). \tag{24}$$

Observe that the $C^{r-\ell}$ matrix function

$$C(s, \mathbf{h}) = \begin{bmatrix} \varphi'(s) + Q'(s) \cdot \mathbf{h} \\ Q(s) \end{bmatrix}$$

satisfies $C(s, 0) = P(s)$ for all $s \in \mathbb{R}$. Since $P(s)$ is non-singular, for sufficiently small values of \mathbf{h} , the matrix $C(s, \mathbf{h})$ can be inverted. Therefore, assuming that $\rho > 0$ is sufficiently small, if $\|\mathbf{h}\| \leq 4\rho$, then (24) can be transformed into

$$\begin{bmatrix} \dot{s} \\ \dot{\mathbf{h}} \end{bmatrix} = \varepsilon^\ell \begin{bmatrix} 1 \\ R\mathbf{h} \end{bmatrix} + \varepsilon^\ell (C(s, \mathbf{h}))^{-1} \cdot Y(s, \mathbf{h}) + \varepsilon^{\ell+1} (C(s, \mathbf{h}))^{-1} \cdot Z(t, s, \mathbf{h}, \varepsilon). \tag{25}$$

Set $\mathbf{h} = (\mathbf{v}, \mathbf{w}) \in \mathbb{R}^{n-d-1} \times \mathbb{R}^d$. Define $\Lambda_0(s, \mathbf{v}, \mathbf{w})$ and $\tilde{\Lambda}_0(t, s, \mathbf{v}, \mathbf{w}, \varepsilon)$ to be the first line of the products $(C(s, \mathbf{h}))^{-1} \cdot Y(s, \mathbf{h})$ and $(C(s, \mathbf{h}))^{-1} \cdot Z(t, s, \mathbf{h}, \varepsilon)$, respectively. Similarly, define $\Lambda_1(s, \mathbf{v}, \mathbf{w})$ and $\tilde{\Lambda}_1(t, s, \mathbf{v}, \mathbf{w}, \varepsilon)$ to be the next $n - d - 1$ lines and $\Lambda_2(s, \mathbf{v}, \mathbf{w})$ and $\tilde{\Lambda}_2(t, s, \mathbf{v}, \mathbf{w}, \varepsilon)$ to be the last d lines of those products. Then, (25) becomes

$$\begin{aligned} \dot{s} &= \varepsilon^\ell + \varepsilon^\ell \Lambda_0(s, \mathbf{v}, \mathbf{w}) + \varepsilon^{\ell+1} \tilde{\Lambda}_0(t, s, \mathbf{v}, \mathbf{w}, \varepsilon), \\ \dot{\mathbf{v}} &= \varepsilon^\ell R_1 \cdot \mathbf{v} + \varepsilon^\ell \Lambda_1(s, \mathbf{v}, \mathbf{w}) + \varepsilon^{\ell+1} \tilde{\Lambda}_1(t, s, \mathbf{v}, \mathbf{w}, \varepsilon), \\ \dot{\mathbf{w}} &= \varepsilon^\ell R_2 \cdot \mathbf{w} + \varepsilon^\ell \Lambda_2(s, \mathbf{v}, \mathbf{w}) + \varepsilon^{\ell+1} \tilde{\Lambda}_2(t, s, \mathbf{v}, \mathbf{w}, \varepsilon). \end{aligned} \tag{26}$$

We apply the time rescaling $\varepsilon^\ell t = \tilde{t}$ to (26) and finally obtain

$$\begin{aligned} s' &= 1 + \Lambda_0(s, \mathbf{v}, \mathbf{w}) + \varepsilon \tilde{\Lambda}_0(\tilde{t}/\varepsilon^\ell, s, \mathbf{v}, \mathbf{w}, \varepsilon), \\ \mathbf{v}' &= R_1 \cdot \mathbf{v} + \Lambda_1(s, \mathbf{v}, \mathbf{w}) + \varepsilon \tilde{\Lambda}_1(\tilde{t}/\varepsilon^\ell, s, \mathbf{v}, \mathbf{w}, \varepsilon), \\ \mathbf{w}' &= R_2 \cdot \mathbf{w} + \Lambda_2(s, \mathbf{v}, \mathbf{w}) + \varepsilon \tilde{\Lambda}_2(\tilde{t}/\varepsilon^\ell, s, \mathbf{v}, \mathbf{w}, \varepsilon). \end{aligned} \tag{27}$$

where $'$ denotes a derivative with respect to \tilde{t} . Such differential system is well defined on $(\tilde{t}, s, \mathbf{v}, \mathbf{w}, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times B_{n-d-1}(0, 2\rho) \times B_d(0, 2\rho) \times (0, \varepsilon_0]$.

3.2 Existence, regularity, and convergence

Henceforth, we consider that (27) is defined over $\mathbb{R} \times \mathbb{R} \times B_{n-d-1}(0, \rho) \times B_d(0, \rho) \times (0, \varepsilon_0]$. Observe that (27) is of the form considered in Lemma 1. We must now show that the hypotheses required for the application of that Lemma hold in our case. Observe that the fact that the parameter ε appears in the denominator of the first argument of $\tilde{\Lambda}_0$, $\tilde{\Lambda}_1$, and $\tilde{\Lambda}_2$ in (27) will not be an impediment to the application of the Lemma, since it is not required in its hypotheses that the functions appearing in the system be defined at $\varepsilon = 0$. In fact, the conditions concerning boundedness, be it of the functions themselves or of their Lipschitz constants, can still be proved by resorting to the periodicity of Z .

For convenience, we will henceforth adopt the following notation

$$\begin{aligned} (C^{-1} \cdot Y)(s, \mathbf{h}) &:= (C(s, \mathbf{h}))^{-1} \cdot Y(s, \mathbf{h}), \\ (C^{-1} \cdot Z)(t, s, \mathbf{h}, \varepsilon) &:= (C(s, \mathbf{h}))^{-1} \cdot Z(t, s, \mathbf{h}, \varepsilon), \\ (C^{-1} \cdot Y + \varepsilon C^{-1} \cdot Z)(t, s, \mathbf{h}, \varepsilon) &:= (C(s, \mathbf{h}))^{-1} \cdot Y(s, \mathbf{h}) + \varepsilon(C(s, \mathbf{h}))^{-1} \cdot Z(t, s, \mathbf{h}, \varepsilon). \end{aligned}$$

With that in mind, we proceed to proving that Lemma 1 can be applied to (27).

As remarked before, the eigenvalues of the matrix R have non-zero real parts, so that it is immediate that hypothesis (iv) holds. Regarding hypothesis (i), observe that, by their definitions, we know that $Y(s + \omega, \mathbf{h}) = Y(s, A \cdot \mathbf{h})$ and $Z(t, s + \omega, \mathbf{h}, \varepsilon) = Z(t, s, A \cdot \mathbf{h}, \varepsilon)$. Furthermore,

$$C(s + \omega, \mathbf{h}) = \left[\begin{array}{c|c} \varphi'(s) + Q'(s)A \cdot \mathbf{h} & \\ \hline & Q(s)A \end{array} \right] = C(s, A \cdot \mathbf{h}) \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array} \right].$$

Thus, since $A^2 = I_{n-1}$, it follows that

$$(C(s + \omega, \mathbf{h}))^{-1} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array} \right] (C(s, A \cdot \mathbf{h}))^{-1}.$$

Hence, it is easily verified that the following conditions hold:

- $\Lambda_0(s + \omega, \mathbf{v}, \mathbf{w}) = \Lambda_0(s, \mathbf{v}, -\mathbf{w})$;
- $\tilde{\Lambda}_0(t, s + \omega, \mathbf{v}, \mathbf{w}, \varepsilon) = \tilde{\Lambda}_0(t, s + \omega, \mathbf{v}, -\mathbf{w}, \varepsilon)$;
- $\Lambda_1(s + \omega, \mathbf{v}, \mathbf{w}) = \Lambda_1(s, \mathbf{v}, -\mathbf{w})$;
- $\tilde{\Lambda}_1(t, s + \omega, \mathbf{v}, \mathbf{w}, \varepsilon) = \tilde{\Lambda}_1(t, s + \omega, \mathbf{v}, -\mathbf{w}, \varepsilon)$;
- $\Lambda_2(s + \omega, \mathbf{v}, \mathbf{w}) = -\Lambda_2(s, \mathbf{v}, -\mathbf{w})$;
- $\tilde{\Lambda}_2(t, s + \omega, \mathbf{v}, \mathbf{w}, \varepsilon) = -\tilde{\Lambda}_2(t, s + \omega, \mathbf{v}, -\mathbf{w}, \varepsilon)$.

This ensures that (i) is valid. It remains to show that hypotheses (ii) and (iii) hold.

In order to do so, let $B_n(p, r)$ denote the open ball $\{x \in \mathbb{R}^n : \|x - p\| < r\}$. Also, let us define the following functions:

$$\begin{aligned} \alpha_Y(s, \mathbf{h}) &:= \left\| \frac{\partial(C^{-1} \cdot Y)}{\partial(s, \mathbf{h})}(s, \mathbf{h}) \right\|, \\ \alpha_Z(s, \mathbf{h}, t, \varepsilon) &:= \left\| \frac{\partial(C^{-1} \cdot \tilde{Z})}{\partial(s, \mathbf{h})}(t, s, \mathbf{h}, \varepsilon) \right\|, \end{aligned}$$

where $\|\cdot\|$ denotes the operator norm.

Let $\varepsilon_0 > 0$ be fixed. For $\sigma \in (0, \rho)$, define

$$L(\sigma) := \sup \{ \alpha_Y(s, \mathbf{h}) : (s, \mathbf{h}) \in \mathbb{R} \times \bar{B}_{n-1}(0, \sigma) \}.$$

Observe that $\lim_{\sigma \rightarrow 0} L(\sigma) = 0$, because

$$\alpha_Y(s, \mathbf{h}) \leq \left\| \frac{\partial(C^{-1} \cdot Y)}{\partial s}(s, \mathbf{h}) \right\| + \left\| \frac{\partial(C^{-1} \cdot Y)}{\partial \mathbf{h}}(s, \mathbf{h}) \right\|$$

and $Y(s, 0) = \frac{\partial Y}{\partial s}(s, 0) = \frac{\partial Y}{\partial \mathbf{h}}(s, 0) = 0$ for all $s \in \mathbb{R}$. Thus, we can extend L continuously to $[0, \rho)$ by setting $L(0) = 0$. Moreover, since $(C^{-1} \cdot Y)$ is also of class $C^{r-\ell}$, with $r - \ell \geq 2$, it follows by the mean value inequality that there is $\mathcal{M}_\rho > 0$ such that

$$\alpha_Y(s, \mathbf{h}) \leq \mathcal{M}_\rho \|\mathbf{h}\| \leq \mathcal{M}_\rho \sigma$$

for all $(s, \mathbf{h}) \in \mathbb{R} \times \bar{B}_{n-1}(0, \sigma)$, where $\sigma \in (0, \rho)$. Then, again by the mean value inequality, we conclude that

$$\begin{aligned} \|(C^{-1} \cdot Y)(s_1, \mathbf{h}_1) - (C^{-1} \cdot Y)(s_2, \mathbf{h}_2)\| &\leq L(\sigma) \|(s_1, \mathbf{h}_1) - (s_2, \mathbf{h}_2)\| \\ &\leq \mathcal{M}_\rho \sigma \|(s_1, \mathbf{h}_1) - (s_2, \mathbf{h}_2)\|, \end{aligned} \tag{28}$$

for all $(s_1, \mathbf{h}_1), (s_2, \mathbf{h}_2) \in \mathbb{R} \times \bar{B}_{n-1}(0, \sigma)$.

Since $r - \ell \geq 2$, and since the $C^{r-\ell}$ function $\|C^{-1} \cdot Z\|$ is T -periodic in its first entry and 2ω -periodic in its second entry, it follows that there is $M > 0$ such that

$$\sup \left\{ \|(C^{-1} \cdot Z)(\tilde{t}/\varepsilon^\ell, s, \mathbf{h}, \varepsilon)\| : (\tilde{t}, s, \mathbf{h}, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times \{0\} \times (0, \varepsilon_0] \right\} \leq M,$$

and

$$\sup \left\{ \alpha_Z(\tilde{t}/\varepsilon^\ell, s, \mathbf{h}, \varepsilon) : (\tilde{t}, s, \mathbf{h}, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times \bar{B}_{n-1}(0, \rho) \times (0, \varepsilon_0] \right\} \leq M.$$

Thus, it follows on the one hand that

$$\begin{aligned} \|\varepsilon(C^{-1} \cdot Z)(\tilde{t}/\varepsilon^\ell, s, 0, \varepsilon)\| &= \|(C^{-1} \cdot Y)(s, 0) \\ &+ \varepsilon(C^{-1} \cdot Z)(\tilde{t}/\varepsilon^\ell, s, 0, \varepsilon)\| \leq \varepsilon M, \end{aligned} \tag{29}$$

for all $(\tilde{t}, s, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times (0, \varepsilon_0]$, proving that (ii) is valid with $M(\varepsilon) = \varepsilon M$. On the other hand, the mean value inequality ensures that

$$\begin{aligned} \|(C^{-1} \cdot Z)(\tilde{t}/\varepsilon^\ell, s_1, \mathbf{h}_1, \varepsilon) - (C^{-1} \cdot Z)(\tilde{t}/\varepsilon^\ell, s_2, \mathbf{h}_2, \varepsilon)\| \\ \leq M \|(s_1, \mathbf{h}_1) - (s_2, \mathbf{h}_2)\|, \end{aligned} \tag{30}$$

for all $(s_1, \mathbf{h}_1), (s_2, \mathbf{h}_2) \in \mathbb{R} \times \bar{B}_{n-1}(0, \rho)$ and all $(\tilde{t}, \varepsilon) \in \mathbb{R} \times (0, \varepsilon_0]$. Hence, combining (28) and (30), we conclude that

$$\begin{aligned} & \| (C^{-1} \cdot Y + \varepsilon C^{-1} \cdot Z)(\tilde{t}/\varepsilon^\ell, s_1, \mathbf{h}_1, \varepsilon) - (C^{-1} \cdot Y + \varepsilon C^{-1} \cdot Z)(\tilde{t}/\varepsilon^\ell, s_2, \mathbf{h}_2, \varepsilon) \| \\ & \leq (\mathcal{M}_L \sigma + \varepsilon M) \| (s_1, \mathbf{h}_1) - (s_2, \mathbf{h}_2) \|, \end{aligned} \tag{31}$$

for $(s_1, \mathbf{h}_1), (s_2, \mathbf{h}_2) \in \mathbb{R} \times \bar{B}_{n-1}(0, \sigma)$ and $(\tilde{t}, \varepsilon) \in \mathbb{R} \times (0, \varepsilon_0]$, ensuring that (iii) also holds.

Hence, all the hypotheses required for the application of Lemma 1 are valid for system (27). Since $(C^{-1} \cdot Y)$ and $(t, s, \mathbf{h}) \mapsto (C^{-1} \cdot Z)(t, s, \mathbf{h}, \varepsilon)$ are of class $C^{r-\ell}$, Proposition 3 may also be applied with $p = r - \ell$. Applying the above-mentioned results, we obtain $\varepsilon_1 > 0$ and families of functions $\{f_\varepsilon \in C^{r-\ell}(\mathbb{R}^2; \mathbb{R}^{n-d-1}) : \varepsilon \in (0, \varepsilon_1]\}$ and $\{g_\varepsilon \in C^{r-\ell}(\mathbb{R}^2; \mathbb{R}^d) : \varepsilon \in (0, \varepsilon_1]\}$ such that

- (I) For each $\varepsilon \in (0, \varepsilon_1]$, the set defined by the relation $\mathbf{h} = (f_\varepsilon(\tilde{t}, s), g_\varepsilon(\tilde{t}, s))$ is an invariant manifold for system

$$\begin{bmatrix} s' \\ \mathbf{h}' \end{bmatrix} = \begin{bmatrix} 1 \\ H\mathbf{h} \end{bmatrix} + (C^{-1} \cdot Y)(s, \mathbf{h}) + \varepsilon(C^{-1} \cdot Z)(\tilde{t}/\varepsilon^\ell, s, \mathbf{h}, \varepsilon), \quad \tilde{t}' = 1. \tag{32}$$

- (II) There is $D(\varepsilon) > 0$ such that $\|f_\varepsilon\|_{C^0} \leq D(\varepsilon)$, $\|g_\varepsilon\|_{C^0} \leq D(\varepsilon)$ and $\lim_{\varepsilon \rightarrow 0} D(\varepsilon) = 0$. Furthermore, there is a constant $C_R > 0$, depending only on the matrix R , such that $D(\varepsilon) = C_R M(\varepsilon) = C_R \varepsilon M$.
- (III) There is $\Delta(\varepsilon) > 0$ such that f_ε and g_ε are Lipschitz continuous in s with Lipschitz constant $\Delta(\varepsilon)$ and $\lim_{\varepsilon \rightarrow 0} \Delta(\varepsilon) = 0$.
- (IV) f_ε is ω -periodic in s and g_ε satisfies $g_\varepsilon(\tilde{t}, s + \omega) = -g_\varepsilon(\tilde{t}, s)$.
- (V) f_ε and g_ε are $\varepsilon^\ell T$ -periodic in \tilde{t} .
- (VI) Let $\pi_1 : \mathbb{R}^{n-d-1} \times \mathbb{R}^d \rightarrow \mathbb{R}^{n-d-1}$ and $\pi_2 : \mathbb{R}^{n-d-1} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the canonical projections. Also, let $k_1 \leq n - d - 1$ and $k_2 \leq d$ of the eigenvalues of R_1 and R_2 , respectively, have negative real parts. There are positive constants r, λ, C, σ_0 , and σ_1 such that $r < \sigma_0, D(\varepsilon) < \sigma_0 < \sigma_1 < \rho$, and, for each $(\tilde{t}_0, s_0, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times (0, \varepsilon_1]$, there is in $\bar{B}_{n-d-1}(0, \sigma_0) \times \bar{B}_d(0, \sigma_0)$ a local $(k_1 + k_2)$ -dimensional embedded submanifold $S(\tilde{t}_0, s_0, \varepsilon)$ of $\mathbb{R}^{n-d-1} \times \mathbb{R}^d$, containing the point $(f_\varepsilon(\tilde{t}_0, s_0), g_\varepsilon(\tilde{t}_0, s_0))$, and having the following properties:

- (VI.1) If $\mathbf{h}_0 \in \bar{B}_{n-d-1}(0, \sigma_0) \times \bar{B}_d(0, \sigma_0) \setminus S(\tilde{t}_0, s_0, \varepsilon)$, there is $\tilde{t}_* > \tilde{t}_0$ for which

$$\mathbf{h}(\tilde{t}_*, \tilde{t}_0, s_0, \mathbf{h}_0, \varepsilon) \notin \bar{B}_{n-d-1}(0, \sigma_1) \times \bar{B}_d(0, \sigma_1).$$

- (VI.2) Reciprocally, if $\mathbf{h}_0 \in S(\tilde{t}_0, s_0, \varepsilon)$, then, for all $\tilde{t} \geq \tilde{t}_0$,

$$\mathbf{h}(\tilde{t}, \tilde{t}_0, s_0, \mathbf{h}_0, \varepsilon) \in \bar{B}_{n-d-1}(0, \sigma_1) \times \bar{B}_d(0, \sigma_1)$$

and the following inequality holds:

$$\begin{aligned} & \| \mathbf{h}(\tilde{t}, \tilde{t}_0, s_0, \mathbf{h}_0, \varepsilon) - \mathbf{h}(\tilde{t}, \tilde{t}_0, s_0, (f_\varepsilon(\tilde{t}_0, s_0), g_\varepsilon(\tilde{t}_0, s_0)), \varepsilon) \| \\ & \leq C e^{-\lambda(\tilde{t}-\tilde{t}_0)} \| \mathbf{h}_0 - (f_\varepsilon(\tilde{t}_0, s_0), g_\varepsilon(\tilde{t}_0, s_0)) \|. \end{aligned}$$

(VI.3) There is a continuous function $\phi_1^\varepsilon : \mathbb{R} \times \mathbb{R} \times \bar{B}_{k_1}(0, r) \times \bar{B}_{k_2}(0, r) \rightarrow \bar{B}_{n-d-k_1-1}(0, \sigma_0)$ such that

$$\pi_1(S(\tilde{t}_0, s_0, \varepsilon)) = \{(\phi_1^\varepsilon(\tilde{t}_0, s_0, \xi_1, \xi_2), \xi_1) : (\xi_1, \xi_2) \in \bar{B}_{k_1}(0, r) \times \bar{B}_{k_2}(0, r)\}.$$

Similarly, there is a continuous function $\phi_2^\varepsilon : \mathbb{R} \times \mathbb{R} \times \bar{B}_{k_1}(0, r) \times \bar{B}_{k_2}(0, r) \rightarrow \bar{B}_{d-k_2}(0, \sigma_0)$ such that

$$\pi_2(S(\tilde{t}_0, s_0, \varepsilon)) = \{(\phi_2^\varepsilon(\tilde{t}_0, s_0, \xi_1, \xi_2), \xi_2) : (\xi_1, \xi_2) \in \bar{B}_{k_1}(0, r) \times \bar{B}_{k_2}(0, r)\}.$$

(VI.4) The functions ϕ_1^ε and ϕ_2^ε satisfy

$$\phi_1^\varepsilon(\tilde{t}_0, s_0 + \omega, \xi_1, \xi_2) = \phi_1^\varepsilon(\tilde{t}_0, s_0, \xi_1, -\xi_2)$$

and

$$\phi_2^\varepsilon(\tilde{t}_0, s_0 + \omega, \xi_1, \xi_2) = -\phi_2^\varepsilon(\tilde{t}_0, s_0, \xi_1, -\xi_2)$$

for all $(\tilde{t}_0, s_0, \xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R} \times \bar{B}_{k_1}(0, r) \times \bar{B}_{k_2}(0, r)$.

Let \mathcal{X} be the function associated to the change of coordinates we have performed, that is,

$$\mathcal{X}(s, \mathbf{h}) = \varphi(s) + Q(s) \cdot \mathbf{h}.$$

Define $w_\varepsilon : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}^n$ by

$$w_\varepsilon(\tau, s) := \left(\tau, \mathcal{X} \left(s, (f_\varepsilon(\varepsilon^\ell \tau, s), g_\varepsilon(\varepsilon^\ell \tau, s)) \right) \right).$$

Since \mathcal{X} is injective for $s \in [0, \omega)$ and $\|\mathbf{h}\| \leq \rho$, the function w_ε restricted to $\mathbb{R} \times [0, \omega)$ is injective. It is also clear that ε_1 can be taken sufficiently small as to ensure that w_ε is an immersion for all $\varepsilon \in (0, \varepsilon_1]$.

It is easy to see that property (IV) above guarantees that w_ε is ω -periodic in s . In fact, considering (18) and the definition of A given in (17), we have that

$$w_\varepsilon(\tau, s + \omega) = \left(\tau, \varphi(s) + Q(s) A \cdot (f_\varepsilon(\varepsilon^\ell \tau, s), -g_\varepsilon(\varepsilon^\ell \tau, s)) \right) = w_\varepsilon(\tau, s).$$

Thus, $W_\varepsilon := \{w_\varepsilon(\tau, s) : (\tau, s) \in \mathbb{R} \times \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}^n$ is an embedded cylinder of class $C^{r-\ell}$ that is invariant under the flow of

$$\begin{cases} \mathbf{z}' = \varepsilon^\ell \mathbf{g}_\ell(\mathbf{z}) + \varepsilon^{\ell+1} r_\ell(\mathbf{z}, \tau, \varepsilon), \\ \tau' = 1. \end{cases} \tag{33}$$

Property (V) ensures that w_ε satisfies $w_\varepsilon(\tau + T, s) = (T, 0) + w_\varepsilon(\tau, s)$. We can thus consider τ an angular variable modulo T in (33), and W_ε becomes an invariant torus in $\mathbb{S}^1 \times \mathbb{R}^n$. Finally, the torus M_ε , invariant under (2), is obtained from W_ε by reverting the near-identity periodic transformation $\mathbf{x} = U(\tau, \mathbf{z}, \varepsilon)$ that we employed in the beginning of the proof. This proves the existence of M_ε stated in Theorem A. The fact that there is a neighborhood V of Γ such that any compact manifold that is invariant under (2) and contained in $\mathbb{S}^1 \times V$ must also be contained in M_ε follows from Corollary 1.

We proceed to proving the statement regarding regularity of M_ε in Theorem A. Define \mathcal{F}_ε by

$$\mathcal{F}_\varepsilon(\tau, s) = U\left(\tau, \mathcal{X}\left(s, (f_\varepsilon(\varepsilon^\ell \tau, s), g_\varepsilon(\varepsilon^\ell \tau, s))\right), \varepsilon\right).$$

Observe that $\{\mathcal{F}_\varepsilon\}_\varepsilon$ is a family of $C^{r-\ell}$ functions that are also ω -periodic in s and T -periodic in τ , and that M_ε is given by the relation $\mathbf{x} = \mathcal{F}_\varepsilon(\tau, s)$, i.e.,

$$M_\varepsilon = \{(\tau, \mathcal{F}_\varepsilon(\tau, s)) \in \mathbb{S}^1 \times \mathbb{R}^n : (\tau, s) \in \mathbb{R} \times \mathbb{R}\}.$$

Moreover, by Proposition 4, it follows that the family $\{\mathcal{F}_\varepsilon\}_\varepsilon$ is C^0 -continuous, that is, continuous in the C^0 -norm, provided that ε_1 is chosen sufficiently small. In fact, this Proposition guarantees that this family is C^1 in the C^0 -norm.

Regarding the statement about convergence, observe that it follows from property (II) that there is $D^*(\varepsilon)$ such that

$$\left\| \mathcal{X}\left(s, (f_\varepsilon(\varepsilon^\ell \tau, s), g_\varepsilon(\varepsilon^\ell \tau, s))\right) - \varphi(s) \right\| < D^*(\varepsilon)$$

and $\lim_{\varepsilon \rightarrow 0} D^*(\varepsilon) = 0$. Then, considering that U is locally Lipschitz in its second argument and that both functions appearing inside the norm of the inequality above are periodic, it follows that there is $\delta(\varepsilon) \geq 0$ such that $\delta(0) = 0$ and $\|\mathcal{F}_\varepsilon(\tau, s) - U(\tau, \varphi(s), \varepsilon)\| < \delta(\varepsilon)$.

3.3 Stability

Let the non-negative integers $k_1 \leq n - d - 1$ and $k_2 \leq d$ denote the number of eigenvalues with negative real parts of the matrices R_1 and R_2 respectively. Define the function $q_\varepsilon : \mathbb{R} \times \mathbb{R} \times B_{k_1}(0, r) \times B_{k_2}(0, r) \rightarrow \mathbb{R} \times \mathbb{R}^n$ by

$$q_\varepsilon(\tau, s, \xi_1, \xi_2) = \left(\tau, U\left(\tau, \mathcal{X}\left(s, (\phi_1^\varepsilon(\varepsilon^\ell \tau, s, \xi_1, \xi_2), \xi_1, \phi_2^\varepsilon(\varepsilon^\ell \tau, s, \xi_1, \xi_2), \xi_2)\right), \varepsilon\right)\right).$$

Let S_{M_ε} be the image of q_ε . We will show that S_{M_ε} is an embedded submanifold in $\mathbb{R} \times \mathbb{R}^n$.

For convenience, we denote by $q_\varepsilon|_I$ the restriction of q_ε to the set $\mathbb{R} \times I \times B_{k_1}(0, r) \times B_{k_2}(0, r)$, where $I \subset \mathbb{R}$. Observe that the properties of ϕ_1^ε and ϕ_2^ε given in (VI.4), along with (18), ensure that S_{M_ε} is contained in the image of $q_\varepsilon|_{(0, \omega)}$. Hence, S_{M_ε} is contained in the union of the images of $q_\varepsilon|_{(0, \omega)}$ and $q_\varepsilon|_{(-\frac{\omega}{2}, \frac{\omega}{2})}$.

Now, since $r \leq \sigma_0 \leq \rho$, $\mathcal{X}(s, \mathbf{h})$ is injective for $(s, \mathbf{h}) \in [0, \omega) \times \bar{B}_{n-1}(0, \sigma_0)$, ensuring that $q_\varepsilon|_{(0, \omega)}$ and $q_\varepsilon|_{(-\frac{\omega}{2}, \frac{\omega}{2})}$ are injective. It is then easy to see that $q_\varepsilon|_{(0, \omega)}$ and $q_\varepsilon|_{(-\frac{\omega}{2}, \frac{\omega}{2})}$ are homeomorphisms onto their images, proving that S_{M_ε} is a $(k_1 + k_2 + 2)$ -dimensional embedded submanifold of $\mathbb{R} \times \mathbb{R}^n$. As remarked before, in Sect. 3.1, if k is the number of characteristic multipliers of the limit cycle Γ whose absolute values are less than 1, then $k = k_1 + k_2$. Thus, S_{M_ε} is k -dimensional.

We will prove that S_{M_ε} is locally the stable set of M_ε . Suppose that $\mathbf{h}_0 \in S(\tilde{t}_0, s_0, \varepsilon)$. For convenience, let us define $s_*(\tilde{t}) := s(\tilde{t}, \tilde{t}_0, s_0, \mathbf{h}_0, \varepsilon)$, $s_{M_\varepsilon}(\tilde{t}) := s(\tilde{t}, \tilde{t}_0, s_0, (f_\varepsilon(\tilde{t}_0, s_0), g_\varepsilon(\tilde{t}_0, s_0)), \varepsilon)$, $\mathbf{h}_*(\tilde{t}) := \mathbf{h}(\tilde{t}, \tilde{t}_0, s_0, \mathbf{h}_0, \varepsilon)$, and finally $\mathbf{h}_{M_\varepsilon}(\tilde{t}) := \mathbf{h}(\tilde{t}, \tilde{t}_0, s_0, (f_\varepsilon(\tilde{t}_0, s_0), g_\varepsilon(\tilde{t}_0, s_0)), \varepsilon)$. Also, let

$$u(\tilde{t}) := \|s_*(\tilde{t}) - s_b(\tilde{t})\| + \|\mathbf{h}_*(\tilde{t}) - \mathbf{h}_{M_\varepsilon}(\tilde{t})\|.$$

Observe that, considering (27), along with the boundedness and Lipschitz continuity properties that Λ_0 and $\tilde{\Lambda}_0$ are proved to satisfy, we have that

$$u(\tilde{t}) \leq \|\mathbf{h}_*(\tilde{t}) + \mathbf{h}_{M_\varepsilon}(\tilde{t})\| + \int_{\tilde{t}_0}^{\tilde{t}} (\mathcal{M}_L \sigma_1 + \varepsilon M) u(x) dx.$$

Thus, considering property (VI.2) and applying Grönwall’s inequality, it follows that

$$u(\tilde{t}) \leq C e^{(-\lambda + \mathcal{M}_L \sigma_1 + \varepsilon M)(\tilde{t} - \tilde{t}_0)} \|\mathbf{h}_0 - (f_\varepsilon(\tilde{t}_0, s_0), g_\varepsilon(\tilde{t}_0, s_0))\|.$$

Hence, if ρ and ε_1 are chosen sufficiently small, we ensure that $u(\tilde{t}) \rightarrow 0$ as $\tilde{t} \rightarrow \infty$. Thus, it follows that, if $\mathbf{h}_0 \in S(\tilde{t}_0, s_0, \varepsilon)$, then

$$\lim_{\tilde{t} \rightarrow \infty} \|\mathcal{X}(s_*(\tilde{t}), \mathbf{h}_*(\tilde{t})) - \mathcal{X}(s_{M_\varepsilon}(\tilde{t}), \mathbf{h}_{M_\varepsilon}(\tilde{t}))\| = 0. \tag{34}$$

Since $\sigma_1 < \rho$, we know that $\mathcal{X}(s, \mathbf{h})$ is injective for $(s, \mathbf{h}) \in [0, \omega) \times \bar{B}_{n-1}(0, \sigma_1)$. Consider the following neighborhoods of M_ε :

$$\begin{aligned} V_s &:= \{(\tau, U(\tau, \mathcal{X}(s, \mathbf{h}), \varepsilon)) : (\tau, s, \mathbf{h}) \in \mathbb{R} \times \mathbb{R} \times B_{n-1}(0, \sigma_1)\}, \\ W_s &:= \{(\tau, U(\tau, \mathcal{X}(s, \mathbf{h}), \varepsilon)) : (\tau, s, \mathbf{h}) \in \mathbb{R} \times \mathbb{R} \times B_{n-1}(0, \sigma_0)\}. \end{aligned}$$

It is then clear, considering (34) and the fact that M_ε is an invariant manifold, that the local stable set of M_ε with respect to V_s satisfies $S_{M_\varepsilon}^{V_s} \cap W_s = S_{M_\varepsilon}$.

The same argument with time reversed proves the analogous statement for the local unstable set $\mathcal{U}_{M_\varepsilon}^{W_u} \cap W_u$. In this case, the dimension of the manifold obtained is

$2 + (n - 1 - k) = n - k + 1$, because the number of eigenvalues of R with positive real part is $n - 1 - k$.

3.4 Dynamics

Let $S_\varepsilon \subset \mathbb{R}^{n+1}$ be defined as the section $\tau = 0$ of the torus M_ε , that is, the image of the real 1-periodic function $\Pi_\varepsilon : \theta \mapsto (0, \mathcal{F}_\varepsilon(0, \omega\theta))$. It is clear that S_ε is $C^{r-\ell}$ -diffeomorphic to the circle S^1 . Once more, let $t \mapsto (s(t, t_0, s_0, \mathbf{h}_0, \varepsilon), \mathbf{h}(t, t_0, s_0, \mathbf{h}_0, \varepsilon))$ be the solution of (25) satisfying $(s(t_0, t_0, s_0, \mathbf{h}_0, \varepsilon), \mathbf{h}(t_0, t_0, s_0, \mathbf{h}_0, \varepsilon)) = (s_0, \mathbf{h}_0)$. Define, for $(\nu, \theta) \in \mathbb{R} \times \mathbb{R}$,

$$s_\varepsilon(\nu, \theta) := s(\nu T, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon).$$

Since $\tau' = 1$ in (2), it follows that the first-return map p_ε defined on S_ε under the action of this differential system is well defined. Moreover, it is clear that

$$p_\varepsilon(\Pi_\varepsilon(\theta)) = \Pi_\varepsilon\left(\frac{s_\varepsilon(1, \omega\theta)}{\omega}\right).$$

Thus, the real function

$$\tilde{p}_\varepsilon : \theta \mapsto \frac{s_\varepsilon(1, \omega\theta)}{\omega}$$

is a lift of p_ε with respect to the covering map $\Pi_\varepsilon : \mathbb{R} \rightarrow S_\varepsilon$. Moreover, this ensures that p_ε is at least of class $C^{r-\ell}$.

Observe that

$$\tilde{p}_\varepsilon^n(\theta) = \frac{s_\varepsilon(n, \omega\theta)}{\omega}$$

for all $n \in \mathbb{N}$. Then, it is clear that the rotation number of p_ε is given by

$$\rho(\varepsilon) := \lim_{n \rightarrow \infty} \frac{\tilde{p}_\varepsilon^n(\theta) - \theta}{n} = \lim_{n \rightarrow \infty} \frac{s_\varepsilon(n, \omega\theta) - \omega\theta}{n\omega}.$$

We will rewrite this limit so as to be able to calculate it up to ℓ -th order of ε .

Integrating the first equation of (26) from $t = 0$ to $t = nT$, we obtain

$$\begin{aligned} s_\varepsilon(n, \theta) &= \theta + \varepsilon^\ell nT + \varepsilon^\ell \int_0^{nT} \Lambda_0(s(\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon), \\ &\quad \mathbf{h}(\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon)) d\tau \\ &\quad + \varepsilon^{\ell+1} \int_0^{nT} \tilde{\Lambda}_0(\tau, s(\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon), \\ &\quad \mathbf{h}(\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon)) d\tau. \end{aligned} \tag{35}$$

Thus, we can define the sequence of functions

$$G_n(\theta, \varepsilon) := \frac{1}{n} \int_0^{nT} \frac{\Lambda_0}{\varepsilon}(s(\tau, 0, \theta(f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon), \mathbf{h}(\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon))d\tau + \frac{1}{n} \int_0^{nT} \tilde{\Lambda}_0(\tau, s(\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon), \mathbf{h}(\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon))d\tau,$$

so that (35) becomes

$$s_\varepsilon(n, \theta) = \theta + \varepsilon^\ell nT + \varepsilon^{\ell+1} n G_n(\theta, \varepsilon).$$

Since M_ε is an invariant manifold, it is clear that

$$\mathbf{h}(\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon) = (f_\varepsilon(\tau, s(\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon)), g_\varepsilon(\tau, s(\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon))).$$

Then, by changing the variables in the integral, it follows that

$$G_n(\theta, \varepsilon) := \int_0^T \frac{\Lambda_0}{\varepsilon}(s(n\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon), \mathbf{h}(n\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon))d\tau + \int_0^T \tilde{\Lambda}_0(n\tau, s(n\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon), \mathbf{h}(n\tau, 0, \theta, (f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)), \varepsilon))d\tau.$$

Observe that

$$\|\Lambda_0(s, \mathbf{h}) + \varepsilon \tilde{\Lambda}_0(t, s, \mathbf{h}, \varepsilon)\| \leq \|\Lambda_0(s, \mathbf{h}) + \varepsilon \tilde{\Lambda}_0(t, s, \mathbf{h}, \varepsilon) - \varepsilon \tilde{\Lambda}_0(t, s, 0, \varepsilon)\| + \|\varepsilon \tilde{\Lambda}_0(t, s, 0, \varepsilon)\|$$

for all $(t, s, \mathbf{h}, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times B_{n-1}(0, \rho) \times (0, \varepsilon_0]$. Then, considering that $\Lambda(s, 0) = 0$, it follows from (29) and (31) that

$$\|\Lambda_0(s, (f_\varepsilon(n\tau, s), g_\varepsilon(n\tau, s))) + \tilde{\Lambda}_0(n\tau, s, (f_\varepsilon(n\tau, s), g_\varepsilon(n\tau, s)), \varepsilon)\| \leq \mathcal{M}_L(\|f_\varepsilon\| + \|g_\varepsilon\|) + \varepsilon M$$

for all $(s, \tau, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times (0, \varepsilon_1]$. Then, from property (II), it follows that

$$\|\Lambda_0(s, (f_\varepsilon(n\tau, s), g_\varepsilon(n\tau, s))) + \tilde{\Lambda}_0(n\tau, s, (f_\varepsilon(n\tau, s), g_\varepsilon(n\tau, s)), \varepsilon)\| \leq C_G \varepsilon,$$

where $C_G := 2\mathcal{M}_L C_R M + M$. Hence, it is easy to see that

$$|G_n(\theta, \varepsilon)| \leq C_G T \tag{36}$$

for all $\theta \in \mathbb{R}$ and all $\varepsilon \in (0, \varepsilon_1]$.

Considering that

$$\frac{s_\varepsilon(n, \omega\theta) - \omega\theta}{n\omega} = \varepsilon^\ell \frac{T}{\omega} + \varepsilon^{\ell+1} \frac{G_n(\omega\theta, \varepsilon)}{\omega},$$

and since the limit

$$\lim_{n \rightarrow \infty} \frac{s_\varepsilon(n, \omega\theta) - \omega\theta}{n\omega}$$

corresponding to the rotation number exists and does not depend on θ , it is ensured that

$$G(\varepsilon) := \lim_{n \rightarrow \infty} G_n(\omega\theta, \varepsilon)$$

is well defined. Moreover, from (36), it is clear that $|G(\varepsilon)| \leq C_G T$. Hence, it follows at once that

$$\rho(\varepsilon) = \lim_{n \rightarrow \infty} \varepsilon^\ell \frac{T}{\omega} + \varepsilon^{\ell+1} \frac{G_n(\omega\theta, \varepsilon)}{\omega} = \varepsilon^\ell \frac{T}{\omega} + \varepsilon^{\ell+1} \frac{G(\varepsilon)}{\omega} = \varepsilon^\ell \frac{T}{\omega} + \mathcal{O}(\varepsilon^{\ell+1}).$$

By Proposition 6 combined with the definition of \tilde{p}_ε , it follows that the family $\{\tilde{p}_\varepsilon\}_\varepsilon$ is continuous in the space of homeomorphisms of \mathbb{S}^1 with the C^0 topology. Hence, ρ is continuous in $(0, \varepsilon_1]$. Since we also know that system (2) becomes $\tau' = 1, \mathbf{x}' = 0$, when $\varepsilon = 0$, it follows that $\rho(0) = 0$, so that ρ is actually continuous in $[0, \varepsilon_1]$. In particular, the relation $\omega\rho(\varepsilon) = \varepsilon^\ell T + \varepsilon^{\ell+1} G(\varepsilon)$ ensures that G is also continuous in $(0, \varepsilon_1]$.

In order to prove the rest of the statement concerning Dynamics of Theorem A, we will make use of the following result, which can be found in [15, Theorem 6.1].

Theorem 2 *Let $\gamma \geq 3$ and $D^\gamma(\mathbb{S}^1)$ be the class of C^γ -diffeomorphisms of the circle \mathbb{S}^1 endowed with the norm C^γ . Let $c : [a, b] \rightarrow D^\gamma(\mathbb{S}^1)$ be a continuous path satisfying: c is of class C^1 if considered as a function on $D^0(\mathbb{S}^1)$. Let $\rho(\lambda)$ denote the rotation number of $c(\lambda)$, $\lambda \in [a, b]$. If $\rho(a) \neq \rho(b)$, then the Lebesgue measure λ of the set*

$$\{x \in [a, b] : c(x) \text{ is } C^{\gamma-2}\text{-conjugated to an irrational rotation}\}$$

is strictly positive. Also, ρ maps zero Lebesgue measure sets to zero Lebesgue measure sets.

Suppose that $r - \ell \geq 4$. Then, Propositions 4 and 6 ensure that $\varepsilon \mapsto \tilde{p}_\varepsilon$ satisfies the regularity conditions stated in Theorem 2 with $\gamma = r - \ell - 1$. Moreover, from the fact that $\omega\rho(\varepsilon) = \varepsilon^\ell T + \mathcal{O}(\varepsilon^{\ell+1})$, it is clear that there is an interval $[a, b] \subset (0, \varepsilon_1]$ such that $\rho(a) \neq \rho(b)$. Hence, there is a subset of $I \subset [a, b]$ of positive Lebesgue measure such that \tilde{p}_ε is $C^{r-\ell-3}$ -conjugated to an irrational rotation for all $\varepsilon \in I$.

4 Invariant torus in 4D vector fields

Consider the differential system (8) under the assumptions established in Sect. 1.3. By applying the cylindrical change of coordinates $(x, y, u, v) = (r \cos \theta, r \sin \theta, u, v)$, $r > 0$, system (8) becomes

$$\begin{aligned}
 \dot{r} &= \varepsilon^N (\cos(\theta)f_1(r \cos(\theta), r \sin(\theta), u, v) + \sin(\theta)f_2(r \cos(\theta), r \sin(\theta), u, v)) \\
 &\quad - \frac{\varepsilon^{N+1}}{2} r^3 \mu (r^2 - (r^2 + 1) \cos(2\theta) - 1) + \mathcal{O}(\varepsilon^{N+2}), \\
 \dot{\theta} &= 1 + \varepsilon^N \left(\frac{\cos(\theta)f_2(r \cos(\theta), r \sin(\theta), u, v) - \sin(\theta)f_1(r \cos(\theta), r \sin(\theta), u, v)}{r} \right) \\
 &\quad - \varepsilon^{N+1} \mu (r^2 \sin(\theta) \cos(\theta) + r^4 \sin(\theta) \cos(\theta)) + \mathcal{O}(\varepsilon^{N+2}), \\
 \dot{u} &= \varepsilon^N f_3(r \cos(\theta), r \sin(\theta), u, v) + \varepsilon^{N+1} r^2 \cos^2(\theta)(u - u^3 + v - uv^2) + \mathcal{O}(\varepsilon^{N+2}), \\
 \dot{v} &= \varepsilon^N f_4(r \cos(\theta), r \sin(\theta), u, v) + \varepsilon^{N+1} r^2 \sin^2(\theta)(v - u - u^2v - v^3) + \mathcal{O}(\varepsilon^{N+2}).
 \end{aligned}
 \tag{37}$$

Since $\dot{\theta} = 1 + \mathcal{O}(\varepsilon^2) > 0$, it follows that $\dot{\theta} > 0$ for ε sufficiently small. Thus, we can take θ to be the independent variable, and system (37) becomes

$$\begin{aligned}
 r' &= \varepsilon^N R_N(\theta, r, u, v) + \varepsilon^{N+1} R_{N+1}(\theta, r, u, v) + \mathcal{O}(\varepsilon^{N+2}), \\
 u' &= \varepsilon^N U_N(\theta, r, u, v) + \varepsilon^{N+1} U_{N+1}(\theta, r, u, v) + \mathcal{O}(\varepsilon^{N+2}), \\
 v' &= \varepsilon^N V_N(\theta, r, u, v) + \varepsilon^{N+1} V_{N+1}(\theta, r, u, v) + \mathcal{O}(\varepsilon^{N+2}),
 \end{aligned}
 \tag{38}$$

where ' indicates derivative with respect to the variable θ , and the functions R_i, U_i , and $V_i, i \in \{N, N + 1\}$, are given by

$$\begin{aligned}
 R_N(\theta, r, u, v) &= \cos(\theta) f_1(r \cos(\theta), r \sin(\theta), u, v) \\
 &\quad + \sin(\theta) f_2(r \cos(\theta), r \sin(\theta), u, v); \\
 R_{N+1}(\theta, r, u, v) &= \frac{1}{2} r^3 \mu \left((r^2 + 1) \cos(2\theta) - r^2 + 1 \right); \\
 U_N(\theta, r, u, v) &= f_3(r \cos(\theta), r \sin(\theta), u, v); \\
 U_{N+1}(\theta, r, u, v) &= r^2 \cos^2(\theta) \left(-u^3 - uv^2 + u + v \right); \\
 V_N(\theta, r, u, v) &= f_4(r \cos(\theta), r \sin(\theta), u, v); \\
 V_{N+1}(\theta, r, u, v) &= -r^2 \sin^2(\theta) \left(u^2v + u + v^3 - v \right).
 \end{aligned}
 \tag{39}$$

We remark that each of the functions defined above is 2π -periodic in θ . By defining $\mathbf{x} = (r, u, v)$, system (38) can be written as

$$\mathbf{x}' = \varepsilon^N F_N(\theta, \mathbf{x}) + \varepsilon^{N+1} F_{N+1}(\theta, \mathbf{x}) + \varepsilon^{N+2} \tilde{F}(\theta, \mathbf{x}, \varepsilon),
 \tag{40}$$

where

$$F_i(\theta, \mathbf{x}) = (R_i(\theta, \mathbf{x}), U_i(\theta, \mathbf{x}), V_i(\theta, \mathbf{x})). \tag{41}$$

Using formulas (5) and (6), we can calculate the Melnikov function of order N for this system as

$$\mathbf{f}_N(\mathbf{x}) = \int_0^{2\pi} F_N(s, \mathbf{x}) ds. \tag{42}$$

Since, by hypothesis, the average of functions $R_N, U_N,$ and V_N over $\theta \in [0, 2\pi]$ vanish identically, then it follows that $\mathbf{f}_N = 0$, so that formulas (5) and (6) provide

$$\begin{aligned} \mathbf{f}_{N+1}(\mathbf{x}) &= \int_0^{2\pi} F_{N+1}(s, \mathbf{x}) ds \\ &= \left(\mu \frac{r^3}{2} (1 - r^2), \frac{r^2}{2} (-u^3 - uv^2 + u + v), -\frac{r^2}{2} (u^2v + u + v^3 - v) \right), \end{aligned} \tag{43}$$

because $F_i = 0$ for all $i \in \{1, 2, \dots, N - 1\}$ and $y_1 = 0$ in this case. Thus, since it is clear that $\mathbf{f}_i = 0$ for all $i \in \{1, 2, \dots, N - 1\}$, it follows from Proposition 1 that

$$\mathbf{g}_{N+1}(\mathbf{x}) = \frac{1}{2\pi} \mathbf{f}_{N+1}(\mathbf{x}).$$

Let us prove that the guiding system $\mathbf{x}' = \mathbf{g}_{N+1}(\mathbf{x})$ has a hyperbolic limit cycle. First, observe that the curve

$$\gamma(t) = \left(1, \cos\left(\frac{t}{4\pi}\right), -\sin\left(\frac{t}{4\pi}\right) \right)$$

satisfies

$$\gamma'(t) = \left(0, -\frac{1}{4\pi} \sin\left(\frac{t}{4\pi}\right), -\frac{1}{4\pi} \cos\left(\frac{t}{4\pi}\right) \right) = \mathbf{g}_{N+1}(\gamma(t)),$$

and is therefore a $8\pi^2$ -periodic orbit of $\mathbf{x}' = \mathbf{g}_{N+1}(\mathbf{x})$. Define Γ as the image of $\gamma(t)$. Notice that $\Gamma = \{1\} \times \mathbb{S}^1$.

In order to show that Γ is indeed a hyperbolic limit cycle, we shall find the eigenvalues of the Poincaré map P associated to it. Observe that

$$\operatorname{div} \mathbf{g}_{N+1}(\mathbf{x}) = -\mu \frac{5r^4}{4\pi} + \frac{r^2}{4\pi} (2 + 3\mu - 4u^2 - 4v^2).$$

By [23, Corollary 12.5], we know that the determinant of the derivative of P at a point \mathbf{x}_0 in the periodic orbit Γ is equal to the determinant of the monodromy matrix

associated to Γ . Thus, by Liouville’s formula, we have

$$\det(DP(\mathbf{x}_0)) = \exp \int_0^{8\pi^2} \operatorname{div} \mathbf{g}_{N+1}(\gamma(s)) ds = e^{-4\pi(1+\mu)}. \tag{44}$$

Since the surface given by $r = 1$ is an invariant manifold for this system, we can also study Γ as a periodic orbit of the system $\mathbf{x}' = \mathbf{g}_{N+1}(\mathbf{x})$ restricted to such surface, which is the planar system $(u', v') = \bar{\mathbf{g}}_{n+1}(u, v)$, given by

$$\begin{aligned} u' &= \frac{1}{4\pi}(-u^3 - uv^2 + u + v), \\ v' &= -\frac{1}{4\pi}(u^2v + u + v^3 - v). \end{aligned} \tag{45}$$

Let L be the intersection of the surface $r = 1$ with the transversal section corresponding to the Poincaré map P . Then, once again by [23, Corollary 12.5], the determinant of derivative of the restriction $P|_L$ at \mathbf{x}_0 is given by

$$\det(D(P|_L)(\mathbf{x}_0)) = \exp \int_0^{8\pi^2} \operatorname{div} \bar{\mathbf{g}}_{N+1} \left(\cos \left(\frac{s}{4\pi} \right), -\sin \left(\frac{s}{4\pi} \right) \right) ds = e^{-4\pi}.$$

Since $D(P|_L)(\mathbf{x}_0)$ acts on a one-dimensional space, it follows that its eigenvalue is equal to $e^{-4\pi}$.

We have thus found one of the eigenvalues of $DP(\mathbf{x}_0)$, to wit, $e^{-4\pi} < 1$. In order to find the other, it suffices to notice that the determinant of $DP(\mathbf{x}_0)$ must be equal to the product of its two eigenvalues. Therefore, it follows from (44) that the other eigenvalue is $e^{-4\pi\mu} \neq 1$. Hence, it follows that Γ is a hyperbolic limit cycle and that the eigenvalues of the derivative of the Poincaré map associated to it are $\lambda_1 = e^{-4\pi}$ and $\lambda_2 = e^{-4\pi\mu}$.

Thus, Theorem A ensures that there is $\varepsilon_0 > 0$ such that, for each $\varepsilon \in [0, \varepsilon_0]$, system

$$\theta' = 1, \quad \mathbf{x}' = \varepsilon^N F_N(\theta, \mathbf{x}) + \varepsilon^{N+1} F_{N+1}(\theta, \mathbf{x}) + \varepsilon^{N+2} \tilde{F}(\theta, \mathbf{x}, \varepsilon)$$

admits an invariant torus M_ε of class C^{r-3} . Moreover, M_ε converges to $\mathbb{S}^1 \times \Gamma$ as $\varepsilon \rightarrow 0$. The stability of M_ε is controlled by the parameter μ . If $\mu = 1$, then M_ε is asymptotically stable, since $\mathcal{S}_{M_\varepsilon}^{V_s}$ locally becomes a neighborhood of M_ε . If, on the other hand, $\mu = -1$, then $\mathcal{S}_{M_\varepsilon}^{V_s}$ is locally a 3-dimensional manifold embedded in \mathbb{R}^4 .

Transforming back to the original coordinates, we obtain, for each $\varepsilon \in [0, \varepsilon_0]$, an invariant torus \mathbb{T}_ε converging as $\varepsilon \rightarrow 0$ to the torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ parameterized by $(\theta, t) \in [0, 2\pi] \times [0, 2\pi] \mapsto (\cos \theta, \sin \theta, \cos t, -\sin t)$.

Data availability All data generated or analysed during this study are included in this published article

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix

Proof of Proposition 3

Let $\varepsilon \in (0, \varepsilon_1]$ be fixed throughout all the proof. We shall prove that M_ε can be parameterized by a C^p function $\alpha_\varepsilon(t, \theta)$. In fact, let $\tau \mapsto \varphi_\varepsilon(\tau, t_0, \theta_0, y_0, z_0)$ be the flow of system (11) satisfying $\varphi_\varepsilon(0, t_0, \theta_0, y_0, z_0) = (t_0, \theta_0, y_0, z_0)$. Then, results about smooth dependence on initial conditions (see, for instance, [14, Corollary 4.1 of Chapter V]) ensure that φ_ε is of class C^p . Define $\alpha_\varepsilon : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \times B_m(0, \sigma_0) \times B_n(0, \sigma_0)$ by

$$\alpha_\varepsilon(t, \theta) := \varphi_\varepsilon(t, 0, \theta, f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)).$$

Observe that statement (e) of Lemma 1 guarantees that α_ε is of class C^p . Let us prove that α_ε is injective and that its image is M_ε .

In order to prove that α_ε is injective, let $(t_1, \theta_1), (t_2, \theta_2) \in \mathbb{R} \times \mathbb{R}$ be such that $\alpha_\varepsilon(t_1, \theta_1) = \alpha_\varepsilon(t_2, \theta_2)$. Define the functions $t_\varepsilon(\tau, t_0, \theta_0, y_0, z_0), \theta_\varepsilon(\tau, t_0, \theta_0, y_0, z_0), y_\varepsilon(\tau, t_0, \theta_0, y_0, z_0)$, and $z_\varepsilon(\tau, t_0, \theta_0, y_0, z_0)$ as being the components of the flow $\varphi_\varepsilon(\tau, t_0, \theta_0, y_0, z_0)$. Then, it is clear by (11) that $t_\varepsilon(\tau, t_0, \theta_0, y_0, z_0) = t_0 + \tau$. Hence, $\alpha(t_1, \theta_1) = \alpha(t_2, \theta_2)$ implies at once that $t_1 + t_0 = t_2 + t_0$, that is, $t_1 = t_2$. Therefore, the uniqueness of the flow φ_ε ensures that the points $(0, \theta_1, f_\varepsilon(0, \theta_1), g_\varepsilon(0, \theta_1))$ and $(0, \theta_2, f_\varepsilon(0, \theta_2), g_\varepsilon(0, \theta_2))$ must be the same. Thus, $\theta_1 = \theta_2$, and α_ε is indeed injective.

To show that the image of α_ε is M_ε , we first observe that, since $(0, \theta, f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)) \in M_\varepsilon$ for all $\theta \in \mathbb{R}$ and M_ε is invariant, it follows that $\alpha_\varepsilon(t, \theta) = \varphi_\varepsilon(t, 0, \theta, f_\varepsilon(0, \theta), g_\varepsilon(0, \theta)) \in M_\varepsilon$ for all $(t, \theta) \in \mathbb{R} \times \mathbb{R}$, i.e., the image of α_ε is contained in M_ε . On the other hand, every point in M_ε is, by definition, of the form $(t, \theta, f_\varepsilon(t, \theta), g_\varepsilon(t, \theta))$ for some $(t, \theta) \in \mathbb{R} \times \mathbb{R}$. By properties of the flow, defining $\tilde{\theta}_\varepsilon = \theta_\varepsilon(-t, t, \theta, f_\varepsilon(t, \theta), g_\varepsilon(t, \theta))$, we have

$$(t, \theta, f_\varepsilon(t, \theta), g_\varepsilon(t, \theta)) = \varphi_\varepsilon(t, 0, \tilde{\theta}_\varepsilon, f_\varepsilon(0, \tilde{\theta}_\varepsilon), g_\varepsilon(0, \tilde{\theta}_\varepsilon)) = \alpha_\varepsilon(t, \tilde{\theta}_\varepsilon),$$

which implies that M_ε is contained in the image of α_ε . Thus, we have proved that α_ε is an injective function of class C^p whose image is M_ε and, therefore, is a C^p parametrization of M_ε . This ensures that M_ε is of class C^p .

We shall now prove that f_ε and g_ε are of class C^p . In order to do so, we remark that $(t, \theta, f_\varepsilon(t, \theta), g_\varepsilon(t, \theta)) \in M_\varepsilon$ for every $(t, \theta) \in \mathbb{R} \times \mathbb{R}$. Then, for each $(t, \theta) \in \mathbb{R} \times \mathbb{R}$, there is $(\tilde{t}, \tilde{\theta}) \in \mathbb{R} \times \mathbb{R}$ such that

$$(t, \theta, f_\varepsilon(t, \theta), g_\varepsilon(t, \theta)) = \alpha_\varepsilon(\tilde{t}, \tilde{\theta}) = (\tilde{t}, \theta_\varepsilon(\tilde{t}, 0, \tilde{\theta}), f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta})), y_\varepsilon(\tilde{t}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta})), z_\varepsilon(\tilde{t}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta})). \tag{46}$$

Define the function $h(\tilde{t}, \tilde{\theta}) = (\tilde{t}, \theta_\varepsilon(\tilde{t}, 0, \tilde{\theta}), f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta}))$. It is then clear that the inverse of h exists and is given by

$$h^{-1}(t, \theta) = (t, \theta_\varepsilon(-t, t, \theta, f_\varepsilon(t, \theta), g_\varepsilon(t, \theta))).$$

Now, by taking $(\tilde{t}(t, \theta), \tilde{\theta}(t, \theta)) = h^{-1}(t, \theta)$, we get from (46) that

$$\begin{aligned} f_\varepsilon(t, \theta) &= y_\varepsilon(\tilde{t}(t, \theta), 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}(t, \theta)), g_\varepsilon(0, \tilde{\theta}(t, \theta))), \\ g_\varepsilon(t, \theta) &= z_\varepsilon(\tilde{t}(t, \theta), 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}(t, \theta)), g_\varepsilon(0, \tilde{\theta}(t, \theta))). \end{aligned}$$

Thus, since $y_\varepsilon, z_\varepsilon,$ and $\theta \mapsto g_\varepsilon(0, \theta)$ are of class C^p , in order to prove that f_ε and g_ε are of class C^p it only remains to show that h^{-1} is of class C^p . First, observe that h is clearly of class C^p , because $\tilde{t} \mapsto f_\varepsilon(0, \tilde{t})$ and $\tilde{t} \mapsto g_\varepsilon(0, \tilde{t})$ are of class C^p by statement (e) of Lemma 1. From the Inverse Function Theorem, it suffices then to prove that the derivative of h is non-singular at every point $(t, \theta) \in \mathbb{R} \times \mathbb{R}$. Observe that

$$Dh(\tilde{t}, \tilde{\theta}) = \begin{bmatrix} 1 & 0 \\ 1 + \zeta_0(\tilde{t}, \theta_\varepsilon, f_\varepsilon(\tilde{t}, \theta_\varepsilon), g_\varepsilon(\tilde{t}, \theta_\varepsilon), \varepsilon) \frac{\partial \theta_\varepsilon}{\partial \tilde{t}_0} + \frac{\partial \theta_\varepsilon}{\partial y_0} \cdot \frac{\partial f_\varepsilon}{\partial \theta}(0, \tilde{\theta}) + \frac{\partial \theta_\varepsilon}{\partial z_0} \cdot \frac{\partial g_\varepsilon}{\partial \theta}(0, \tilde{\theta}) \end{bmatrix},$$

where the argument of θ_ε and its partial derivatives is $(\tilde{t}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta}))$ and has been omitted for conciseness. Thus, $Dh(\tilde{t}, \tilde{\theta})$ is non-singular if, and only if,

$$\begin{aligned} &\frac{\partial \theta_\varepsilon}{\partial \theta_0}(\tilde{t}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta})) \\ &+ \frac{\partial \theta_\varepsilon}{\partial y_0}(\tilde{t}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta})) \cdot \frac{\partial f_\varepsilon}{\partial \theta}(0, \tilde{\theta}) \\ &+ \frac{\partial \theta_\varepsilon}{\partial z_0}(\tilde{t}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta})) \cdot \frac{\partial g_\varepsilon}{\partial \theta}(0, \tilde{\theta}) \neq 0. \end{aligned} \tag{47}$$

The matrix

$$\mathcal{M}(\tau, t_0, \theta_0, y_0, z_0) := \begin{bmatrix} \frac{\partial t_\varepsilon}{\partial t_0} & \frac{\partial t_\varepsilon}{\partial \theta_0} & \frac{\partial t_\varepsilon}{\partial y_0} & \frac{\partial t_\varepsilon}{\partial z_0} \\ \frac{\partial \theta_\varepsilon}{\partial t_0} & \frac{\partial \theta_\varepsilon}{\partial \theta_0} & \frac{\partial \theta_\varepsilon}{\partial y_0} & \frac{\partial \theta_\varepsilon}{\partial z_0} \\ \frac{\partial y_\varepsilon}{\partial t_0} & \frac{\partial y_\varepsilon}{\partial \theta_0} & \frac{\partial y_\varepsilon}{\partial y_0} & \frac{\partial y_\varepsilon}{\partial z_0} \\ \frac{\partial z_\varepsilon}{\partial t_0} & \frac{\partial z_\varepsilon}{\partial \theta_0} & \frac{\partial z_\varepsilon}{\partial y_0} & \frac{\partial z_\varepsilon}{\partial z_0} \end{bmatrix},$$

where the argument of each entry is given by $(\tau, t_0, \theta_0, y_0, z_0)$ is a fundamental solution of the first variational equation associated to (11). Thus, $\mathcal{M}(\tau, t_0, \theta_0, y_0, z_0)$ is

invertible. Moreover, since $t_\varepsilon(\tau, t_0, \theta_0, y_0) = \tau + t_0$, it follows that

$$\mathcal{M}(\tau, t_0, \theta_0, y_0, z_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{\partial \theta_\varepsilon}{\partial t_0} & \frac{\partial \theta_\varepsilon}{\partial \theta_0} & \frac{\partial \theta_\varepsilon}{\partial y_0} & \frac{\partial \theta_\varepsilon}{\partial z_0} \\ \frac{\partial y_\varepsilon}{\partial t_0} & \frac{\partial y_\varepsilon}{\partial \theta_0} & \frac{\partial y_\varepsilon}{\partial y_0} & \frac{\partial y_\varepsilon}{\partial z_0} \\ \frac{\partial z_\varepsilon}{\partial t_0} & \frac{\partial z_\varepsilon}{\partial \theta_0} & \frac{\partial z_\varepsilon}{\partial y_0} & \frac{\partial z_\varepsilon}{\partial z_0} \end{bmatrix},$$

Hence, we conclude that

$$\mathcal{N}(\tau, t_0, \theta_0, y_0, z_0) := \begin{bmatrix} \frac{\partial \theta_\varepsilon}{\partial \theta_0}(\tau, t_0, \theta_0, y_0, z_0) & \frac{\partial \theta_\varepsilon}{\partial y_0}(\tau, t_0, \theta_0, y_0, z_0) & \frac{\partial \theta_\varepsilon}{\partial z_0}(\tau, t_0, \theta_0, y_0, z_0) \\ \frac{\partial y_\varepsilon}{\partial \theta_0}(\tau, t_0, \theta_0, y_0, z_0) & \frac{\partial y_\varepsilon}{\partial y_0}(\tau, t_0, \theta_0, y_0, z_0) & \frac{\partial y_\varepsilon}{\partial z_0}(\tau, t_0, \theta_0, y_0, z_0) \\ \frac{\partial z_\varepsilon}{\partial \theta_0}(\tau, t_0, \theta_0, y_0, z_0) & \frac{\partial z_\varepsilon}{\partial y_0}(\tau, t_0, \theta_0, y_0, z_0) & \frac{\partial z_\varepsilon}{\partial z_0}(\tau, t_0, \theta_0, y_0, z_0) \end{bmatrix}$$

is invertible for all $(\tau, t_0, \theta_0, y_0, z_0) \in [-\Omega, \Omega] \times \mathbb{R} \times \mathbb{R} \times B_m(0, \rho) \times B_m(0, \rho)$, where $[-\Omega, \Omega]$ is the maximal interval where the flow is defined. In particular, if $t_0 = 0$, $\theta_0 = \tilde{\theta}$, $y_0 = f_\varepsilon(0, \tilde{\theta})$, and $z_0 = g_\varepsilon(0, \tilde{\theta})$, then the flow is defined for all $\tau \in \mathbb{R}$, and it follows that $\mathcal{N}(\tilde{\tau}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta}))$ is invertible for all $(\tilde{\tau}, \tilde{\theta}) \in \mathbb{R} \times \mathbb{R}$. Thus, the product

$$\mathcal{N}(\tilde{\tau}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta})) \cdot \begin{bmatrix} 1 \\ \frac{\partial f_\varepsilon}{\partial \theta}(0, \tilde{\theta}) \\ \frac{\partial g_\varepsilon}{\partial \theta}(0, \tilde{\theta}) \end{bmatrix} = \begin{bmatrix} \frac{\partial \theta_\varepsilon}{\partial \theta_0} + \frac{\partial \theta_\varepsilon}{\partial y_0} \cdot \frac{\partial f_\varepsilon}{\partial \theta} + \frac{\partial \theta_\varepsilon}{\partial z_0} \cdot \frac{\partial g_\varepsilon}{\partial \theta} \\ \frac{\partial y_\varepsilon}{\partial \theta_0} + \frac{\partial y_\varepsilon}{\partial y_0} \cdot \frac{\partial f_\varepsilon}{\partial \theta} + \frac{\partial y_\varepsilon}{\partial z_0} \cdot \frac{\partial g_\varepsilon}{\partial \theta} \\ \frac{\partial z_\varepsilon}{\partial \theta_0} + \frac{\partial z_\varepsilon}{\partial y_0} \cdot \frac{\partial f_\varepsilon}{\partial \theta} + \frac{\partial z_\varepsilon}{\partial z_0} \cdot \frac{\partial g_\varepsilon}{\partial \theta} \end{bmatrix}$$

cannot vanish, where the arguments of the derivatives of θ_ε , y_ε and z_ε , as well as the arguments of the derivatives of f_ε and g_ε , have been omitted, but should be read, respectively, as $(\tilde{\tau}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta}))$ and $(0, \tilde{\theta})$. Suppose, by contradiction, that (47) does not hold at $(\tilde{\tau}_*, \tilde{\theta}_*) \in \mathbb{R}^2$, so that the first line of product above vanishes. Let us show that this implies that the product vanishes altogether.

In fact, observe that the invariance of M_ε ensures that, for all $(\tilde{\tau}, \tilde{\theta}) \in \mathbb{R}^2$,

$$\begin{aligned} y_\varepsilon(\tilde{\tau}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta})) &= f_\varepsilon(\tilde{\tau}, \theta_\varepsilon(\tilde{\tau}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta}))), \\ z_\varepsilon(\tilde{\tau}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta})) &= g_\varepsilon(\tilde{\tau}, \theta_\varepsilon(\tilde{\tau}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta}))). \end{aligned}$$

Thus, differentiating with respect to $\tilde{\theta}$, it follows that

$$\frac{\partial y_\varepsilon}{\partial \theta_0} + \frac{\partial y_\varepsilon}{\partial y_0} \cdot \frac{\partial f_\varepsilon}{\partial \theta} + \frac{\partial y_\varepsilon}{\partial z_0} \cdot \frac{\partial g_\varepsilon}{\partial \theta} = \frac{\partial f_\varepsilon}{\partial \theta} \left[\frac{\partial \theta_\varepsilon}{\partial \theta_0} + \frac{\partial \theta_\varepsilon}{\partial y_0} \cdot \frac{\partial f_\varepsilon}{\partial \theta} + \frac{\partial \theta_\varepsilon}{\partial z_0} \cdot \frac{\partial g_\varepsilon}{\partial \theta} \right]$$

and

$$\frac{\partial z_\varepsilon}{\partial \theta_0} + \frac{\partial z_\varepsilon}{\partial y_0} \cdot \frac{\partial f_\varepsilon}{\partial \theta} + \frac{\partial z_\varepsilon}{\partial z_0} \cdot \frac{\partial g_\varepsilon}{\partial \theta} = \frac{\partial g_\varepsilon}{\partial \theta} \left[\frac{\partial \theta_\varepsilon}{\partial \theta_0} + \frac{\partial \theta_\varepsilon}{\partial y_0} \cdot \frac{\partial f_\varepsilon}{\partial \theta} + \frac{\partial \theta_\varepsilon}{\partial z_0} \cdot \frac{\partial g_\varepsilon}{\partial \theta} \right]$$

where, once again, the arguments of the derivatives of θ_ε , y_ε and z_ε , as well as the arguments of the derivatives of f_ε and g_ε , have been omitted, but should be read, respectively, as $(\tilde{t}, 0, \tilde{\theta}, f_\varepsilon(0, \tilde{\theta}), g_\varepsilon(0, \tilde{\theta}))$ and $(0, \tilde{\theta})$.

Since we assumed that (47) does not hold at $(\tilde{t}_*, \tilde{\theta}_*)$, it follows at once that

$$\begin{bmatrix} \frac{\partial \theta_\varepsilon}{\partial \theta_0} + \frac{\partial \theta_\varepsilon}{\partial y_0} \cdot \frac{\partial f_\varepsilon}{\partial \theta} + \frac{\partial \theta_\varepsilon}{\partial z_0} \cdot \frac{\partial g_\varepsilon}{\partial \theta} \\ \frac{\partial y_\varepsilon}{\partial \theta_0} + \frac{\partial y_\varepsilon}{\partial y_0} \cdot \frac{\partial f_\varepsilon}{\partial \theta} + \frac{\partial y_\varepsilon}{\partial z_0} \cdot \frac{\partial g_\varepsilon}{\partial \theta} \\ \frac{\partial z_\varepsilon}{\partial \theta_0} + \frac{\partial z_\varepsilon}{\partial y_0} \cdot \frac{\partial f_\varepsilon}{\partial \theta} + \frac{\partial z_\varepsilon}{\partial z_0} \cdot \frac{\partial g_\varepsilon}{\partial \theta} \end{bmatrix} = 0$$

when the arguments of the derivatives of θ_ε , y_ε and z_ε are given by $(\tilde{t}_*, 0, \tilde{\theta}_*, f_\varepsilon(0, \tilde{\theta}_*), g_\varepsilon(0, \tilde{\theta}_*))$ and the arguments of the derivatives of f_ε and g_ε are given by $(0, \tilde{\theta}_*)$. Therefore,

$$\mathcal{N}(\tilde{t}_*, 0, \tilde{\theta}_*, f_\varepsilon(0, \tilde{\theta}_*), g_\varepsilon(0, \tilde{\theta}_*)) \cdot \begin{bmatrix} 1 \\ \frac{\partial f_\varepsilon}{\partial \theta}(0, \tilde{\theta}_*) \\ \frac{\partial g_\varepsilon}{\partial \theta}(0, \tilde{\theta}_*) \end{bmatrix} = 0.$$

As remarked above, this would imply that $\mathcal{N}(\tilde{t}_*, 0, \tilde{\theta}_*, f_\varepsilon(0, \tilde{\theta}_*), g_\varepsilon(0, \tilde{\theta}_*))$, which we have proved to be invertible, is not invertible. Since we have reached a contradiction, it is proved that h^{-1} is indeed of class C^p , which concludes the proof of the proposition.

Proof of Proposition 4

It is not difficult to see that we can assume without loss of generality that the function L appearing in hypothesis (ii) of the Lemma satisfies: $L(\varepsilon, \sigma, \mu) \geq \varepsilon$ for all $(\varepsilon, \sigma, \mu) \in (0, \varepsilon_0) \times [0, \rho_1) \times [0, \rho_2)$.

Let $D(\varepsilon)$, $\Delta(\varepsilon)$, J_1 , J_2 be given as in the proof of Lemma 1. It is clear that there are $K > 0$ and $\alpha > 0$ such that $\|J_i(t)\| \leq Ke^{-\alpha|t|}$ for all $i \in \{1, 2\}$. By choosing ε_1 to be sufficiently small, we can then ensure that the following inequalities hold for all $\varepsilon \in (0, \varepsilon_1]$:

- $\Delta(\varepsilon) < \frac{1}{2}$ and $D(\varepsilon) < \rho := \min(\rho_1, \rho_2)$;
- $32L(\varepsilon, D(\varepsilon), D(\varepsilon)) < \alpha$;
- $64KL(\varepsilon, D(\varepsilon), D(\varepsilon)) < \alpha$.

Let $\mathcal{P}_\omega(D, \Delta)$, $\mathcal{A}_\omega(D, \Delta)$, and the operator S^ε be given as in the proof of Lemma 1. For each $\varepsilon \in (0, \varepsilon_1]$, define the sequence $(P_k, A_k)_{k \in \mathbb{N}}$, where $P_k : (0, \varepsilon_1) \rightarrow \mathcal{P}_\omega(D, \Delta)$ and $A_k : (0, \varepsilon_1) \rightarrow \mathcal{A}_\omega(D, \Delta)$ are functions of class C^1 given by:

- $(P_0(\varepsilon), A_0(\varepsilon)) = (0, 0)$ for all $\varepsilon \in (0, \varepsilon_1)$;
- $(P_{k+1}(\varepsilon), A_{k+1}(\varepsilon)) = S^\varepsilon(P_k(\varepsilon), A_k(\varepsilon))$ for all $k \in \mathbb{N}$ and all $\varepsilon \in (0, \varepsilon_1)$.

From Lemma 1, it is clear that this sequence satisfies

$$\lim_{k \rightarrow \infty} (P_k(\varepsilon), A_k(\varepsilon)) = (f_\varepsilon, g_\varepsilon). \tag{48}$$

Let $a, b \in (0, \varepsilon_1)$ be fixed. Effecting cumbersome calculations, which are very similar to those presented in the proofs of Propositions 5 and 6, and for this reason are omitted, we can show that the following hold for all $\varepsilon \in [a, b]$ and all $k \in \mathbb{N}$:

- (1) $\|P_{k+2}(\varepsilon) - P_{k+1}(\varepsilon)\| + \|A_{k+2}(\varepsilon) - A_{k+1}(\varepsilon)\| \leq \frac{1}{4}[\|P_{k+1}(\varepsilon) - P_k(\varepsilon)\| + \|A_{k+1}(\varepsilon) - A_k(\varepsilon)\|]$.
- (2) There is $C_1 > 0$ such that

$$\begin{aligned} & \left\| \frac{\partial P_{k+2}(\varepsilon)}{\partial \theta} - \frac{\partial P_{k+1}(\varepsilon)}{\partial \theta} \right\| + \left\| \frac{\partial A_{k+2}(\varepsilon)}{\partial \theta} - \frac{\partial A_{k+1}(\varepsilon)}{\partial \theta} \right\| \\ & \leq C_1 [\|P_{k+1}(\varepsilon) - P_k(\varepsilon)\| + \|A_{k+1}(\varepsilon) - A_k(\varepsilon)\|] \\ & \quad + \frac{1}{4} \left[\left\| \frac{\partial P_{k+1}(\varepsilon)}{\partial \theta} - \frac{\partial P_k(\varepsilon)}{\partial \theta} \right\| + \left\| \frac{\partial A_{k+1}(\varepsilon)}{\partial \theta} - \frac{\partial A_k(\varepsilon)}{\partial \theta} \right\| \right]. \end{aligned}$$

- (3) There is $C_2 > 0$ such that

$$\begin{aligned} & \|P'_{k+2}(\varepsilon) - P'_{k+1}(\varepsilon)\| + \|A'_{k+2}(\varepsilon) - A'_{k+1}(\varepsilon)\| \\ & \leq C_2 \left[\|P_{k+1}(\varepsilon) - P_k(\varepsilon)\| + \|A_{k+1}(\varepsilon) - A_k(\varepsilon)\| \right. \\ & \quad \left. + \left\| \frac{\partial P_{k+1}(\varepsilon)}{\partial \theta} - \frac{\partial P_k(\varepsilon)}{\partial \theta} \right\| + \left\| \frac{\partial A_{k+1}(\varepsilon)}{\partial \theta} - \frac{\partial A_k(\varepsilon)}{\partial \theta} \right\| \right] \\ & \quad + \frac{1}{4} [\|P'_{k+1}(\varepsilon) - P'_k(\varepsilon)\| + \|A'_{k+1}(\varepsilon) - A'_k(\varepsilon)\|]. \end{aligned}$$

From those inequalities, it follows easily that there is $C > 0$ such that

$$\sup_{\varepsilon \in [a, b]} \|P'_{k+1}(\varepsilon) - P'_k(\varepsilon)\| + \|A'_{k+1}(\varepsilon) - A'_k(\varepsilon)\| \leq \frac{C}{2^k}$$

for all $k \in \mathbb{N}$. Hence, the sequence $(P'_k, A'_k)_{k \in \mathbb{N}}$ converges uniformly on $[a, b]$. Since a and b were arbitrary, this implies that $(P'_k, A'_k)_{k \in \mathbb{N}}$ converges uniformly on compact subsets of $(0, \varepsilon_1)$. Therefore, considering (48), it follows that the function c given in the statement of this Lemma is of class C^1 (see, for instance, [10, Theorem 85, Chapter 1]), concluding the proof.

Proof of Proposition 5

Let $D(\varepsilon)$, $\Delta(\varepsilon)$, J_1 , J_2 , and $T_{F,G}$ be given as in the proof of Lemma 1. It is clear that there are $K > 0$ and $\alpha > 0$ such that $\|J_i(t)\| \leq Ke^{-\alpha|t|}$ for all $i \in \{1, 2\}$. For convenience, we will denote $L(\varepsilon, D(\varepsilon), D(\varepsilon))$ by $L(\varepsilon)$ throughout the proof. As in Proposition 4, we assume that the function L appearing in hypothesis (ii) of the Lemma satisfies: $L(\varepsilon, \sigma, \mu) \geq \varepsilon$ for all $(\varepsilon, \sigma, \mu) \in (0, \varepsilon_0] \times [0, \rho_1] \times [0, \rho_2]$.

Let $(P_k, A_k)_{k \in \mathbb{N}}$ be the sequence defined in Proposition 4. For convenience, define

- $T_k^{x,t,\varepsilon}(\theta) := T_{P_k(\varepsilon), A_k(\varepsilon)}(t + x, t, \theta, \varepsilon)$;
- $\Lambda_k^{x,t,\varepsilon}(\theta) = (t + x, \theta, P_k(\varepsilon)(t + x, \theta), A_k(\varepsilon)(t + x, \theta), \varepsilon)$;
- $\zeta_{i,k}^{x,t,\varepsilon}(\theta) := \zeta_i(t + x, \theta, P_k(\varepsilon)(t + x, \theta), A_k(\varepsilon)(t + x, \theta), \varepsilon) = \zeta_i \circ \Lambda_k^{x,t,\varepsilon}(\theta)$.

We will prove by induction on q that, if ε_1 is sufficiently small, the following hold for each $q \in \{1, \dots, p + 1\}$:

(P1.) There is $N_q \in \mathbb{N}$ and, for each $[a, b] \subset (0, \varepsilon_1]$, there is $C_{0,q} > 0$ such that

$$\left| (T_k^{x,t,\varepsilon})^{(q)}(\theta) \right| \leq C_{0,q} e^{N_q L(\varepsilon)(1+2\Delta(\varepsilon))|x|},$$

for all $k \in \mathbb{N}$ and all $(x, t, \theta, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [a, b]$.

(P2.) For each $[a, b] \subset (0, \varepsilon_1]$, there is $C_{1,q} > 0$ such that

$$\left\| \frac{\partial^q P_k(\varepsilon)}{\partial \theta^q} \right\| + \left\| \frac{\partial^q A_k(\varepsilon)}{\partial \theta^q} \right\| \leq C_{1,q},$$

for all $k \in \mathbb{N}$ and all $\varepsilon \in [a, b]$.

Let us then consider the case $q = 1$. Observe that, from the definition of $T_k^{x,t,\varepsilon}$, it follows that

$$\begin{aligned} & \frac{\partial}{\partial \theta} \left(\frac{\partial T_{P_k(\varepsilon), A_k(\varepsilon)}}{\partial x} \right) (t + x, t, \theta, \varepsilon) \\ &= D\zeta_0 \left(\Lambda_k^{x,t,\varepsilon} (T_k^{x,t,\varepsilon}(\theta)) \right) \cdot (\Lambda_k^{x,t,\varepsilon})' (T_k^{x,t,\varepsilon}(\theta)) \cdot (T_k^{x,t,\varepsilon})'(\theta). \end{aligned} \tag{49}$$

Hence, we obtain by changing order of derivatives and integrating

$$\left| (T_k^{x,t,\varepsilon})'(\theta) - (T_k^{0,t,\varepsilon})'(\theta) \right| \leq \int_0^x L(\varepsilon)(1 + 2\Delta(\varepsilon)) \left| (T_k^{x,t,\varepsilon})'(\theta) \right| dx.$$

Since

$$(T_k^{0,t,\varepsilon})'(\theta) = 1,$$

it follows by an application of Grönwall’s inequality that

$$\left| (T_k^{x,t,\varepsilon})'(\theta) \right| \leq e^{L(\varepsilon)(1+2\Delta(\varepsilon))|x|}. \tag{50}$$

This proves property (P1.). Property (P2.) follows directly, with $C_{1,1} = \sup_{\varepsilon \in [a,b]} \Delta(\varepsilon)$, from the fact that $(P_k, A_k) \in \mathcal{P}_\omega(D(\varepsilon), \Delta(\varepsilon)) \times \mathcal{A}_\omega(D(\varepsilon), \Delta(\varepsilon))$ for all $k \in \mathbb{N}$.

Let $N \in \{2, \dots, p + 1\}$ be given and suppose that the Lemma is true for every $q \in \mathbb{N}$ such that $1 \leq q \leq N - 1$. We will show that the Lemma also holds for $q = N$.

Henceforth, we will employ the index i to denote any element of the set $\{0, 1, 2\}$, since the considerations done below are the same. By Faà di Bruno’s formula, since $\zeta_i^{x,t,\varepsilon} = \zeta_i \circ \Lambda_k^{x,t,\varepsilon}$, it follows that

$$\left(\zeta_{i,k}^{x,t,\varepsilon}\right)^{(q)}(\theta) = \sum_{j=1}^q D^{(j)}\zeta_i(\Lambda_k^{x,t,\varepsilon}(\theta)) \cdot B_{q,j}\left(\left(\Lambda_k^{x,t,\varepsilon}\right)'(\theta), \dots, \left(\Lambda_k^{x,t,\varepsilon}\right)^{(q-j+1)}(\theta)\right),$$

for each $q \in \{1, \dots, p + 1\}$, where $B_{q,j}$ is a Bell polynomial. Observe that $D^{(j)}\zeta_i(\Lambda_k^{x,t,\varepsilon}(\theta))$ is a symmetric multilinear map that can be thought of as being applied to a “product” of vectors. Its application to a polynomial is simply a linear combination of different applications to such “products”. In particular, for $q = N$, we can write

$$\begin{aligned} \left(\zeta_{i,k}^{x,t,\varepsilon}\right)^{(N)}(\theta) &= D^{(N)}\zeta_i(\Lambda_k^{x,t,\varepsilon}(\theta)) \cdot \left(\left(\Lambda_k^{x,t,\varepsilon}\right)'(\theta)\right)^N \\ &\quad + D\zeta_i(\Lambda_k^{x,t,\varepsilon}(\theta)) \cdot \left(\left(\Lambda_k^{x,t,\varepsilon}\right)^{(N)}(\theta)\right) \\ &\quad + \sum_{j=2}^{N-1} D^{(j)}\zeta_i(\Lambda_k^{x,t,\varepsilon}(\theta)) \cdot B_{N,j}\left(\left(\Lambda_k^{x,t,\varepsilon}\right)'(\theta), \dots, \left(\Lambda_k^{x,t,\varepsilon}\right)^{(N-j+1)}(\theta)\right). \end{aligned}$$

By the same formula, we also have:

$$\begin{aligned} \left(\zeta_{i,k}^{x,t,\varepsilon}\left(T_k^{x,t,\varepsilon}(\theta)\right)\right)^{(N)} &= \left(\zeta_{i,k}^{x,t,\varepsilon}\right)^{(N)}\left(T_k^{x,t,\varepsilon}(\theta)\right) \cdot \left(\left(T_k^{x,t,\varepsilon}\right)'(\theta)\right)^N \\ &\quad + \left(\zeta_{i,k}^{x,t,\varepsilon}\right)'\left(T_k^{x,t,\varepsilon}(\theta)\right) \cdot \left(\left(T_k^{x,t,\varepsilon}\right)^{(N)}(\theta)\right) \\ &\quad + \sum_{j=2}^{N-1} \left(\zeta_{i,k}^{x,t,\varepsilon}\right)^{(j)}\left(T_k^{x,t,\varepsilon}(\theta)\right) \cdot \\ &\quad B_{N,j}\left(\left(T_k^{x,t,\varepsilon}\right)'(\theta), \dots, \left(T_k^{x,t,\varepsilon}\right)^{(N-j+1)}(\theta)\right). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} &\left(\zeta_{i,k}^{x,t,\varepsilon}\left(T_k^{x,t,\varepsilon}(\theta)\right)\right)^{(N)} \\ &= D^{(N)}\zeta_i\left(\Lambda_k^{x,t,\varepsilon}\left(T_k^{x,t,\varepsilon}(\theta)\right)\right) \cdot \left(\left(\Lambda_k^{x,t,\varepsilon}\right)'\left(T_k^{x,t,\varepsilon}(\theta)\right)\right)^N \cdot \left(\left(T_k^{x,t,\varepsilon}\right)'(\theta)\right)^N \\ &\quad + D\zeta_i\left(\Lambda_k^{x,t,\varepsilon}\left(T_k^{x,t,\varepsilon}(\theta)\right)\right) \cdot \left(\left(\Lambda_k^{x,t,\varepsilon}\right)^{(N)}\left(T_k^{x,t,\varepsilon}(\theta)\right)\right) \cdot \left(\left(T_k^{x,t,\varepsilon}\right)'(\theta)\right)^N \\ &\quad + \sum_{j=2}^{N-1} D^{(j)}\zeta_i\left(\Lambda_k^{x,t,\varepsilon}\left(T_k^{x,t,\varepsilon}(\theta)\right)\right) \\ &\quad \cdot B_{N,j}\left(\left(\Lambda_k^{x,t,\varepsilon}\right)'\left(T_k^{x,t,\varepsilon}(\theta)\right), \dots, \left(\Lambda_k^{x,t,\varepsilon}\right)^{(N-j+1)}\left(T_k^{x,t,\varepsilon}(\theta)\right)\right) \cdot \left(\left(T_k^{x,t,\varepsilon}\right)'(\theta)\right)^N \\ &\quad + D\zeta_i\left(\Lambda_k^{x,t,\varepsilon}\left(T_k^{x,t,\varepsilon}(\theta)\right)\right) \cdot \left(\left(\Lambda_k^{x,t,\varepsilon}\right)'\left(T_k^{x,t,\varepsilon}(\theta)\right)\right) \cdot \left(\left(T_k^{x,t,\varepsilon}\right)^{(N)}(\theta)\right) \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{j=2}^{N-1} \sum_{l=1}^j \left[D^{(l)} \zeta_i \left(\Lambda_k^{x,t,\varepsilon} \left(T_k^{x,t,\varepsilon}(\theta) \right) \right) \right. \\
 &\quad \cdot B_{j,l} \left(\left(\Lambda_k^{x,t,\varepsilon} \right)' \left(T_k^{x,t,\varepsilon}(\theta) \right), \dots, \left(\Lambda_k^{x,t,\varepsilon} \right)^{(j-l+1)} \left(T_k^{x,t,\varepsilon}(\theta) \right) \right) \Big] \\
 &\quad \times B_{N,j} \left(\left(T_k^{x,t,\varepsilon} \right)'(\theta), \dots, \left(T_k^{x,t,\varepsilon} \right)^{(N-j+1)}(\theta) \right).
 \end{aligned} \tag{51}$$

For simplicity, we will denote the summands on the right-hand side of this equation by *I*, *II*, *III*, *IV*, and *V*, respectively.

By definition of $T_k^{x,t,\varepsilon}(\theta)$, it follows that

$$\frac{\partial^N}{\partial \theta^N} \left(\frac{\partial T_{P_k(\varepsilon), A_k(\varepsilon)}}{\partial x} \right) (t+x, t, \theta, \varepsilon) = \left(\zeta_{0,k}^{x,t,\varepsilon} \left(T_k^{x,t,\varepsilon}(\theta) \right) \right)^{(N)}. \tag{52}$$

Observe that, for $q \in \{1, \dots, p+1\}$,

$$\left(\Lambda_k^{x,t,\varepsilon} \right)^{(q)}(\theta) = \left(0, \delta_{1q}, \frac{\partial^q P_k(\varepsilon)}{\partial \theta^q}(t+x, \theta), \frac{\partial^q A_k(\varepsilon)}{\partial \theta^q}(t+x, \theta), 0 \right),$$

where δ_{ij} is the Kronecker delta. Thus, since $N \geq 2$, it follows that

$$\begin{aligned}
 &\left\| D \zeta_i \left(\Lambda_k^{x,t,\varepsilon} \left(T_k^{x,t,\varepsilon}(\theta) \right) \right) \cdot \left(\Lambda_k^{x,t,\varepsilon} \right)^{(N)} \left(T_k^{x,t,\varepsilon}(\theta) \right) \right\| \\
 &\leq L(\varepsilon) \left[\left\| \frac{\partial^N P_k(\varepsilon)}{\partial \theta^N} \right\| + \left\| \frac{\partial^N A_k(\varepsilon)}{\partial \theta^N} \right\| \right],
 \end{aligned}$$

which, combined with (50), ensures that

$$\|II\| \leq L(\varepsilon) \left[\left\| \frac{\partial^N P_k(\varepsilon)}{\partial \theta^N} \right\| + \left\| \frac{\partial^N A_k(\varepsilon)}{\partial \theta^N} \right\| \right] e^{NL(\varepsilon)(1+2\Delta(\varepsilon))|x|}. \tag{53}$$

Moreover, we also have

$$\begin{aligned}
 &\left\| D \zeta_i \left(\Lambda_k^{x,t,\varepsilon} \left(T_k^{x,t,\varepsilon}(\theta) \right) \right) \cdot \left(\Lambda_k^{x,t,\varepsilon} \right)' \left(T_k^{x,t,\varepsilon}(\theta) \right) \right\| \\
 &\leq L(\varepsilon) \left[1 + \left\| \frac{\partial P_k(\varepsilon)}{\partial \theta} \right\| + \left\| \frac{\partial A_k(\varepsilon)}{\partial \theta} \right\| \right],
 \end{aligned}$$

so that

$$\|IV\| \leq L(\varepsilon)(1+2\Delta(\varepsilon)) \left(T_k^{x,t,\varepsilon} \right)^{(N)}(\theta). \tag{54}$$

Observe that, since ζ_i is of class C^{p+1} and periodic in its first two entries, there is $C_\zeta > 0$ such that

$$\left\| D^{(q)} \zeta_i \left(\Lambda_k^{x,t,\varepsilon} \left(T_k^{x,t,\varepsilon}(\theta) \right) \right) \right\| \leq C_\zeta$$

for all $q \in \{1, \dots, p + 1\}$, all $k \in \mathbb{N}$, and all $(x, t, \theta, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [a, b]$. Thus, considering the hypothesis of induction, it follows that there are $\tilde{C} > 0$ and $\tilde{N} \in \mathbb{N}$, where \tilde{C} depends on the choice of the interval $[a, b]$ but \tilde{N} does not, such that

$$\|I\| + \|III\| + \|V\| \leq \tilde{C} e^{\tilde{N}L(\varepsilon)(1+2\Delta(\varepsilon))|x|}. \tag{55}$$

Therefore, considering (53), (54), and (55), it follows by changing the order of derivatives of (52) and integrating with respect to x that

$$\begin{aligned} \left| (T_k^{x,t,\varepsilon})^{(N)}(\theta) \right| &\leq \int_0^x L(\varepsilon)(1 + 2\Delta(\varepsilon)) \left| (T_k^{\tau,t,\varepsilon})^{(N)}(\theta) \right| d\tau + \frac{\tilde{C} e^{\tilde{N}L(\varepsilon)(1+2\Delta(\varepsilon))|x|}}{\tilde{N}L(\varepsilon)(1 + 2\Delta(\varepsilon))} \\ &\quad + \frac{1}{N(1 + 2\Delta(\varepsilon))} \left[\left\| \frac{\partial^N P_k(\varepsilon)}{\partial \theta^N} \right\| + \left\| \frac{\partial^N A_k(\varepsilon)}{\partial \theta^N} \right\| \right] e^{N L(\varepsilon)(1+2\Delta(\varepsilon))|x|}. \end{aligned}$$

Thus, by taking $N_q := \max(N + 1, \tilde{N} + 1)$, an application of Grönwall’s inequality ensures that

$$\left| (T_k^{x,t,\varepsilon})^{(N)}(\theta) \right| \leq \left(\frac{\tilde{C}}{\tilde{N}L(\varepsilon)} + \frac{1}{1 + 2\Delta(\varepsilon)} \left[\left\| \frac{\partial^N P_k(\varepsilon)}{\partial \theta^N} \right\| + \left\| \frac{\partial^N A_k(\varepsilon)}{\partial \theta^N} \right\| \right] \right) e^{N_q L(\varepsilon)(1+2\Delta(\varepsilon))|x|}. \tag{56}$$

Having proved (56), we proceed to showing that (P2.) holds for $q = N$. This will be done by induction on $k \in \mathbb{N}$. Define

$$C_{1,N} := \frac{48K\tilde{C}}{\alpha}.$$

Since $(P_0, A_0) = (0, 0)$, property (P2.) is trivially true with this constant for $q = N$ and $k = 0$. Suppose it holds for all non-negative integers up to a given $k \in \mathbb{N}$. Let us show that it must also hold for $k + 1$. Observe that

$$\frac{\partial^N P_{k+1}(\varepsilon)}{\partial \theta^N}(t, \theta) = \int_{-\infty}^{\infty} J_1(x) \left(\zeta_{1,k}^{x,t,\varepsilon} \left(T_k^{x,t,\varepsilon}(\theta) \right) \right)^{(N)} dx.$$

Hence, considering (53), (54), (55), and (56), it follows that

$$\begin{aligned} \left\| \frac{\partial^N P_{k+1}(\varepsilon)}{\partial \theta^N} \right\| &\leq \int_{-\infty}^{\infty} K e^{-\alpha|x|} \left(\frac{\tilde{C}(1+2\Delta(\varepsilon))}{\tilde{N}} + \tilde{C} \right) e^{N_q L(\varepsilon)(1+2\Delta(\varepsilon))|x|} dx \\ &\quad + \int_{-\infty}^{\infty} K e^{-\alpha|x|} 2L(\varepsilon) \left[\left\| \frac{\partial^N P_k(\varepsilon)}{\partial \theta^N} \right\| + \left\| \frac{\partial^N A_k(\varepsilon)}{\partial \theta^N} \right\| \right] \\ &\quad e^{N_q L(\varepsilon)(1+2\Delta(\varepsilon))|x|} dx. \end{aligned}$$

If ε_1 is sufficiently small as to ensure that $2N_q L(\varepsilon)(1+2\Delta(\varepsilon)) \leq \alpha$ for all $\varepsilon \in (0, \varepsilon_1]$, and considering the hypothesis of induction, it follows that

$$\left\| \frac{\partial^N P_{k+1}(\varepsilon)}{\partial \theta^N} \right\| \leq \frac{4K}{\alpha} \left(\frac{\tilde{C}(1+2\Delta(\varepsilon))}{\tilde{N}} + \tilde{C} \right) + \frac{8KL(\varepsilon)}{\alpha} C_{1,N}. \tag{57}$$

We proceed identically for A_{k+1} and obtain

$$\left\| \frac{\partial^N A_{k+1}(\varepsilon)}{\partial \theta^N} \right\| \leq \frac{4K}{\alpha} \left(\frac{\tilde{C}(1+2\Delta(\varepsilon))}{\tilde{N}} + \tilde{C} \right) + \frac{8KL(\varepsilon)}{\alpha} C_{1,N}. \tag{58}$$

If ε_1 is also chosen sufficiently small as to ensure that $32KL(\varepsilon) < \alpha$ and $2\Delta(\varepsilon) < 1$ for all $\varepsilon \in (0, \varepsilon_1]$, then

$$\left\| \frac{\partial^N P_{k+1}(\varepsilon)}{\partial \theta^N} \right\| + \left\| \frac{\partial^N A_{k+1}(\varepsilon)}{\partial \theta^N} \right\| \leq \frac{24K\tilde{C}}{\alpha} + \frac{C_{1,N}}{2} \leq C_{1,N},$$

proving property (P2.).

Observe that the validity of property (P1.) for $q = N$ follows immediately from (56) and the fact that (P2.) holds for $q = N$. Therefore, by induction on q , it is proved that both properties hold for all $q \in \{1, \dots, p+1\}$. The Lemma then follows by defining

$$N_T := \max_{q \in \{1, \dots, p+1\}} N_q,$$

and, for each interval $[a, b] \subset (0, \varepsilon_1]$, the positive constants

$$C_{[a,b]} := \max_{q \in \{1, \dots, p+1\}} C_{1,q}, \quad M_{[a,b]} := \max_{q \in \{1, \dots, p+1\}} C_{0,q},$$

and observing that $(f_\varepsilon, g_\varepsilon)$ is the limit of the sequence $(P_k(\varepsilon), A_k(\varepsilon))_{k \in \mathbb{N}}$.

Proof of Proposition 6

Let $D(\varepsilon)$, $\Delta(\varepsilon)$, J_1 , J_2 , and $T_{F,G}$ be given as in the proof of Lemma 1. It is clear that there are $K > 0$ and $\alpha > 0$ such that $\|J_i(t)\| \leq K e^{-\alpha|t|}$ for all $i \in \{1, 2\}$. Once again,

we will denote $L(\varepsilon, D(\varepsilon), D(\varepsilon))$ by $L(\varepsilon)$ throughout the proof. As in Proposition 4, we assume that the function L appearing in hypothesis (ii) of the Lemma satisfies: $L(\varepsilon, \sigma, \mu) \geq \varepsilon$ for all $(\varepsilon, \sigma, \mu) \in (0, \varepsilon_0] \times [0, \rho_1] \times [0, \rho_2]$.

For convenience, define

- $T^{x,t,\varepsilon}(\theta) := T_{f_\varepsilon, g_\varepsilon}(t+x, t, \theta)$;
- $\Lambda^{x,t,\varepsilon}(\theta) = (t+x, \theta, f_\varepsilon(t+x, \theta), g_\varepsilon(t+x, \theta), \varepsilon)$;
- $\zeta_i^{x,t,\varepsilon}(\theta) := \zeta_i(t+x, \theta, f_\varepsilon(t+x, \theta), g_\varepsilon(t+x, \theta), \varepsilon) = \zeta_i \circ \Lambda^{x,t,\varepsilon}(\theta)$.

First, let us consider $k = 0$. Let us restrict the possible values of the parameter ε to a compact interval $[a, b] \subset (0, \varepsilon_1]$, and let $\rho > 0$ be such that $D(\varepsilon) < \rho$ for all $\varepsilon \in (0, \varepsilon_1]$. In this case, the functions $\zeta_0, \zeta_1, \zeta_2$ are Lipschitz continuous with Lipschitz constant R over $\mathbb{R} \times \mathbb{R} \times \bar{B}_m(0, \rho) \times \bar{B}_n(0, \rho) \times [a, b]$. Thus, it is clear that, if $\varepsilon, \tilde{\varepsilon} \in [a, b] \subset (0, \varepsilon_1]$, then

$$\begin{aligned} |T^{x,t,\varepsilon}(\theta) - T^{x,t,\tilde{\varepsilon}}(\theta)| &\leq \int_0^x L(\varepsilon)(1 + 2\Delta(\varepsilon))|T^{\tau,t,\varepsilon}(\theta) - T^{\tau,t,\tilde{\varepsilon}}(\theta)|d\tau \\ &\quad + \int_0^x L(\varepsilon) [\|f_\varepsilon - f_{\tilde{\varepsilon}}\| + \|g_\varepsilon - g_{\tilde{\varepsilon}}\|] dx \\ &\quad + \int_0^x R|\varepsilon - \tilde{\varepsilon}|dx. \end{aligned}$$

Hence, from Grönwall’s inequality, it follows that

$$\begin{aligned} |T^{x,t,\varepsilon}(\theta) - T^{x,t,\tilde{\varepsilon}}(\theta)| &\leq \frac{e^{L(\varepsilon)(1+2\Delta(\varepsilon))|x|} - 1}{1 + 2\Delta(\varepsilon)} [\|f_\varepsilon - f_{\tilde{\varepsilon}}\| + \|g_\varepsilon - g_{\tilde{\varepsilon}}\|] \\ &\quad + \frac{R(e^{L(\varepsilon)(1+2\Delta(\varepsilon))|x|} - 1)}{L(\varepsilon)(1 + 2\Delta(\varepsilon))} |\varepsilon - \tilde{\varepsilon}|. \end{aligned} \tag{59}$$

Now, since $(f_\varepsilon, g_\varepsilon)$ is a fixed point of the operator S^ε given in the proof of Lemma 1, it follows by subtracting $S_1^{\tilde{\varepsilon}}(f_{\tilde{\varepsilon}}, g_{\tilde{\varepsilon}})$ from $S_1^\varepsilon(f_\varepsilon, g_\varepsilon)$ that

$$\begin{aligned} \|f_\varepsilon(t, \theta) - f_{\tilde{\varepsilon}}(t, \theta)\| &\leq \int_{-\infty}^\infty K e^{-\alpha|x|} L(\varepsilon)(1 + 2\Delta(\varepsilon))|T^{x,t,\varepsilon}(\theta) - T^{x,t,\tilde{\varepsilon}}(\theta)|dx \\ &\quad + \int_{-\infty}^\infty K e^{-\alpha|x|} L(\varepsilon) [\|f_\varepsilon - f_{\tilde{\varepsilon}}\| + \|g_\varepsilon - g_{\tilde{\varepsilon}}\|] dx \\ &\quad + \int_{-\infty}^\infty K e^{-\alpha|x|} R|\varepsilon - \tilde{\varepsilon}|dx. \end{aligned}$$

Thus, considering (59), if ε_1 is chosen sufficiently small so that $2L(\varepsilon)(1 + 2\Delta(\varepsilon)) < \alpha$ for all $\varepsilon \in (0, \varepsilon_1]$, it follows that

$$\|f_\varepsilon(t, \theta) - f_{\tilde{\varepsilon}}(t, \theta)\| \leq \frac{4KL(\varepsilon)}{\alpha} [\|f_\varepsilon - f_{\tilde{\varepsilon}}\| + \|g_\varepsilon - g_{\tilde{\varepsilon}}\|] + R|\varepsilon - \tilde{\varepsilon}|.$$

A similar argument ensures that

$$\|g_\varepsilon(t, \theta) - g_{\tilde{\varepsilon}}(t, \theta)\| \leq \frac{4KL(\varepsilon)}{\alpha} [\|f_\varepsilon - f_{\tilde{\varepsilon}}\| + \|g_\varepsilon - g_{\tilde{\varepsilon}}\|] + R|\varepsilon - \tilde{\varepsilon}|.$$

Therefore, if ε_1 is also small enough to ensure that $16KL(\varepsilon) < \alpha$ for all $\varepsilon \in (0, \varepsilon_1]$, it follows that

$$\|f_\varepsilon - f_{\tilde{\varepsilon}}\| + \|g_\varepsilon - g_{\tilde{\varepsilon}}\| \leq 4R|\varepsilon - \tilde{\varepsilon}| \tag{60}$$

if $\varepsilon, \tilde{\varepsilon} \in [a, b]$. The procedure can be repeated for any choice of interval $[a, b]$ with the exact same conditions required for the choice of ε_1 , yielding generally different constants R , but ensuring local Lipschitz continuity nonetheless.

Consider the following properties, where $q \in \{0, \dots, p\}$:

(Q.1) There is $N_q \in \mathbb{N}$ and, for each $[a, b] \subset (0, \varepsilon_1]$, there is $C_{0,q} > 0$ such that

$$\left| (T^{x,t,\varepsilon})^{(q)}(\theta) - (T^{x,t,\tilde{\varepsilon}})^{(q)}(\theta) \right| \leq C_{0,q}|\varepsilon - \tilde{\varepsilon}|e^{N_q L(\varepsilon)(1+2\Delta(\varepsilon))|x|}$$

for all $(x, t, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and all $\varepsilon, \tilde{\varepsilon} \in [a, b]$.

(Q.2) For each $[a, b] \subset (0, \varepsilon_1]$, there is $C_{1,q} > 0$ such that

$$\left\| \frac{\partial^q f_\varepsilon}{\partial \theta^q} - \frac{\partial^q f_{\tilde{\varepsilon}}}{\partial \theta^q} \right\| + \left\| \frac{\partial^q g_\varepsilon}{\partial \theta^q} - \frac{\partial^q g_{\tilde{\varepsilon}}}{\partial \theta^q} \right\| \leq C_{1,q}|\varepsilon - \tilde{\varepsilon}|$$

for all $\varepsilon, \tilde{\varepsilon} \in [a, b]$.

We will prove by induction that those properties hold for all $q \in \{0, \dots, p\}$.

Before we proceed to the proof itself, we make some considerations. Once again, the index i will be used to denote any element of the set $\{0, 1, 2\}$, since the arguments are the same. Let the interval $[a, b] \subset (0, \varepsilon_1]$ be fixed. First, since ζ_i is of class C^{p+1} , periodic in its first two entries, and since f_ε and g_ε are bounded for $\varepsilon \in [a, b]$, it follows that there are constants $C_\zeta > 0$ and $L_\zeta > 0$ such that, for all $j \in \{0, 1, \dots, p\}$, the function $D^{(j)}\zeta_i$ satisfies

$$\|D^{(j)}\zeta_i(\Lambda^{x,t,\varepsilon}(\theta))\| \leq C_\zeta \tag{61}$$

and

$$\|D^{(j)}\zeta_i(\Lambda^{x,t,\varepsilon}(\theta)) - D^{(j)}\zeta_i(\Lambda^{x,t,\tilde{\varepsilon}}(\tilde{\theta}))\| \leq L_\zeta \|\Lambda^{x,t,\varepsilon}(\theta) - \Lambda^{x,t,\tilde{\varepsilon}}(\tilde{\theta})\| \tag{62}$$

for all $j \in \{0, \dots, p\}$, all $(x, t) \in \mathbb{R} \times \mathbb{R}$ and all $(\theta, \varepsilon), (\tilde{\theta}, \tilde{\varepsilon}) \in \mathbb{R} \times [a, b]$.

Furthermore, considering the definition of $\Lambda^{x,t,\varepsilon}$ and (60), it follows that, for each $[a, b] \subset (0, \varepsilon_1]$, there is $R > 0$ such that

$$\|\Lambda^{x,t,\varepsilon}(\theta) - \Lambda^{x,t,\tilde{\varepsilon}}(\tilde{\theta})\| \leq 4R|\varepsilon - \tilde{\varepsilon}| + (1 + 2\Delta(\varepsilon))|\theta - \tilde{\theta}| \tag{63}$$

for all $(x, t) \in \mathbb{R} \times \mathbb{R}$ and all $(\theta, \varepsilon), (\tilde{\theta}, \tilde{\varepsilon}) \in \mathbb{R} \times [a, b]$. Also, observe that for any $j \in \{1, \dots, p + 1\}$,

$$(\Lambda^{x,t,\varepsilon})^{(j)}(\theta) = \left(0, \delta_{1j}, \frac{\partial^j f_\varepsilon}{\partial \theta^j}(t + x, \theta), \frac{\partial^j g_\varepsilon}{\partial \theta^j}(t + x, \theta), 0 \right), \tag{64}$$

where δ_{ij} is the Kronecker delta. Thus, it follows that

$$\left\| (\Lambda^{x,t,\varepsilon})'(\theta) \right\| \leq 1 + 2\Delta(\varepsilon), \tag{65}$$

for all $(x, t, \theta, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (0, \varepsilon_1]$. Moreover, from Proposition 5, there is, for each $[a, b] \subset (0, e_1]$, a constant $C_\Lambda > 0$ such that

$$\left\| (\Lambda^{x,t,\varepsilon})^{(j)}(\theta) \right\| \leq C_\Lambda \tag{66}$$

for all $j \in \{2, \dots, p + 1\}$ and all $(x, t, \theta, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [a, b]$. Also, from Corollary 2, it follows that

$$\begin{aligned} \left\| (\Lambda^{x,t,\varepsilon})^{(j)}(\theta) - (\Lambda^{x,t,\tilde{\varepsilon}})^{(j)}(\tilde{\theta}) \right\| &\leq \left\| \frac{\partial^j f_\varepsilon}{\partial \theta^j} - \frac{\partial^j f_{\tilde{\varepsilon}}}{\partial \theta^j} \right\| + \left\| \frac{\partial^j g_\varepsilon}{\partial \theta^j} - \frac{\partial^j g_{\tilde{\varepsilon}}}{\partial \theta^j} \right\| \\ &\quad + C_\Lambda |\theta - \tilde{\theta}| \end{aligned} \tag{67}$$

for all $j \in \{1, \dots, p\}$, all $(x, t) \in \mathbb{R} \times \mathbb{R}$ and all $(\theta, \varepsilon), (\tilde{\theta}, \tilde{\varepsilon}) \in \mathbb{R} \times [a, b]$.

Finally, observe that, from Proposition 5, there are N_T and, for each $[a, b] \subset (0, \varepsilon_1]$, a constant $C_T > 0$ such that

$$\left| (T^{x,t,\varepsilon})^{(j)}(\theta) \right| \leq C_T e^{N_T L(\varepsilon)(1+2\Delta(\varepsilon))|x|} \tag{68}$$

for all $j \in \{1, \dots, p + 1\}$ and all $(x, t, \theta, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [a, b]$.

We start the discussion of the induction argument. Observe that the case $q = 0$ follows directly from (59) and (60). Let $N \in \{1, \dots, p\}$ and assume that properties (Q.1) and (Q.2) are valid for $0 \leq q \leq N - 1$. We will show that this ensures that such properties also hold for $q = N$.

Proceeding exactly as in the proof of Proposition 5, we obtain the following from Faà di Bruno’s formula:

$$\begin{aligned}
 & (\zeta_i^{x,t,\varepsilon} (T^{x,t,\varepsilon}(\theta)))^{(N)} \\
 &= D^{(N)} \zeta_i (\Lambda^{x,t,\varepsilon} (T^{x,t,\varepsilon}(\theta))) \cdot \left((\Lambda^{x,t,\varepsilon})' (T^{x,t,\varepsilon}(\theta)) \right)^N \cdot \left((T^{x,t,\varepsilon})'(\theta) \right)^N \\
 &+ D \zeta_i (\Lambda^{x,t,\varepsilon} (T^{x,t,\varepsilon}(\theta))) \cdot \left((\Lambda^{x,t,\varepsilon})^{(N)} (T^{x,t,\varepsilon}(\theta)) \right) \cdot \left((T^{x,t,\varepsilon})'(\theta) \right)^N \\
 &+ \sum_{j=2}^{N-1} D^{(j)} \zeta_i (\Lambda^{x,t,\varepsilon} (T^{x,t,\varepsilon}(\theta))) \\
 &\cdot B_{N,j} \left((\Lambda^{x,t,\varepsilon})' (T^{x,t,\varepsilon}(\theta)), \dots, (\Lambda^{x,t,\varepsilon})^{(N-j+1)} (T^{x,t,\varepsilon}(\theta)) \right) \left((T^{x,t,\varepsilon})'(\theta) \right)^N \\
 &+ D \zeta_i (\Lambda^{x,t,\varepsilon} (T^{x,t,\varepsilon}(\theta))) \cdot \left((\Lambda^{x,t,\varepsilon})' (T^{x,t,\varepsilon}(\theta)) \right) \cdot \left((T^{x,t,\varepsilon})^{(N)}(\theta) \right) \\
 &+ \sum_{j=2}^{N-1} \sum_{l=1}^j \left[D^{(l)} \zeta_i (\Lambda^{x,t,\varepsilon} (T^{x,t,\varepsilon}(\theta))) \right. \\
 &\cdot B_{j,l} \left((\Lambda^{x,t,\varepsilon})' (T^{x,t,\varepsilon}(\theta)), \dots, (\Lambda^{x,t,\varepsilon})^{(j-l+1)} (T^{x,t,\varepsilon}(\theta)) \right) \left. \right] \\
 &\times B_{N,j} \left((T^{x,t,\varepsilon})'(\theta), \dots, (T^{x,t,\varepsilon})^{(N-j+1)}(\theta) \right).
 \end{aligned} \tag{69}$$

For simplicity, we will denote the summands on the right-hand side of this equation by *I*, *II*, *III*, *IV*, and *V*, respectively. If ε is replaced by $\tilde{\varepsilon}$, we will denote those terms by *I'*, *II'*, *III'*, *IV'*, and *V'*, respectively

Considering the hypothesis of induction combined with (61), (62), (65), (67) and (68), it follows that there is $N_I > 0$ and, for each $[a, b] \subset (0, \varepsilon_1]$, $C_I > 0$ such that

$$\|I - I'\| \leq C_I e^{N_I L(\varepsilon)(1+2\Delta(\varepsilon))|x|} |\varepsilon - \tilde{\varepsilon}|. \tag{70}$$

for all $(x, t, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and all $\varepsilon, \tilde{\varepsilon} \in [a, b]$. Similarly, since $B_{N,j}$ and $B_{j,l}$ are polynomials, there are $N_{III} > 0$ and $N_V > 0$, and, for each $[a, b] \subset (0, \varepsilon_1]$, $C_{III} > 0$ and $C_V > 0$ such that

$$\|III - III'\| \leq C_{III} e^{N_{III} L(\varepsilon)(1+2\Delta(\varepsilon))|x|} |\varepsilon - \tilde{\varepsilon}| \tag{71}$$

and

$$\|V - V'\| \leq C_V e^{N_V L(\varepsilon)(1+2\Delta(\varepsilon))|x|} |\varepsilon - \tilde{\varepsilon}|. \tag{72}$$

for all $(x, t, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and all $\varepsilon, \tilde{\varepsilon} \in [a, b]$.

Regarding II , observe that (64), (67), and the properties of Lipschitz continuity of ζ_i given in hypothesis (iii) guarantee that

$$\begin{aligned} & \left\| D\zeta_i \left(\Lambda^{x,t,\varepsilon} \left(T^{x,t,\varepsilon}(\theta) \right) \right) \cdot \left(\left(\Lambda^{x,t,\varepsilon} \right)^{(N)} \left(T^{x,t,\varepsilon}(\theta) \right) - \left(\Lambda^{x,t,\tilde{\varepsilon}} \right)^{(N)} \left(T^{x,t,\tilde{\varepsilon}}(\theta) \right) \right) \right\| \\ & \leq L(\varepsilon) \left[\left\| \frac{\partial^N f_\varepsilon}{\partial \theta^N} - \frac{\partial^N f_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| + \left\| \frac{\partial^N g_\varepsilon}{\partial \theta^N} - \frac{\partial^N g_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| \right] \\ & \quad + L(\varepsilon) C_\Lambda \left| T^{x,t,\varepsilon}(\theta) - T^{x,t,\tilde{\varepsilon}}(\theta) \right|. \end{aligned}$$

Thus, the hypothesis of induction, combined with (62), (67), (65), (67) and (68), ensures that there is $N_{II} > 0$ and, for each $[a, b] \subset (0, \varepsilon_1]$, $C_{II} > 0$ such that

$$\begin{aligned} \|II - II'\| & \leq C_{II} e^{N_{II} L(\varepsilon)(1+2\Delta(\varepsilon))|x|} |\varepsilon - \tilde{\varepsilon}| \\ & \quad + L(\varepsilon) \left[\left\| \frac{\partial^N f_\varepsilon}{\partial \theta^N} - \frac{\partial^N f_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| + \left\| \frac{\partial^N g_\varepsilon}{\partial \theta^N} - \frac{\partial^N g_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| \right] e^{N_{II} L(\varepsilon)(1+2\Delta(\varepsilon))|x|} \end{aligned} \tag{73}$$

for all $(x, t, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and all $\varepsilon, \tilde{\varepsilon} \in [a, b]$.

Finally, a similar argument ensures that there is $N_{IV} \in \mathbb{N}$ and, for each $[a, b] \subset (0, \varepsilon_1]$, $C_{IV} > 0$ such that

$$\begin{aligned} \|IV - IV'\| & \leq C_{IV} e^{N_{IV} L(\varepsilon)(1+2\Delta(\varepsilon))|x|} |\varepsilon - \tilde{\varepsilon}| \\ & \quad + L(\varepsilon)(1 + 2\Delta(\varepsilon)) \left| \left(T^{x,t,\varepsilon} \right)^{(N)}(\theta) - \left(T^{x,t,\tilde{\varepsilon}} \right)^{(N)}(\theta) \right| \end{aligned} \tag{74}$$

for all $(x, t, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and all $\varepsilon, \tilde{\varepsilon} \in [a, b]$.

By definition of $T^{x,t,\varepsilon}$, it follows that

$$\frac{\partial^N}{\partial \theta^N} \left(\frac{\partial T_{f_\varepsilon, g_\varepsilon}}{\partial x} \right) (t + x, t, \theta, \varepsilon) = \left(\zeta_0^{x,t,\varepsilon} \left(T^{x,t,\varepsilon}(\theta) \right) \right)^{(N)}.$$

Thus, considering inequalities (70)–(74), it follows that there is $\tilde{N}_T \in \mathbb{N}$ and, for each $[a, b] \subset (0, \varepsilon_1]$, $\tilde{C}_T > 0$ such that

$$\begin{aligned} & \left| \left(T^{x,t,\varepsilon} \right)^{(N)}(\theta) - \left(T^{x,t,\tilde{\varepsilon}} \right)^{(N)}(\theta) \right| \\ & \leq \int_0^x L(\varepsilon)(1 + 2\Delta(\varepsilon)) \left| \left(T^{\tau,t,\varepsilon} \right)^{(N)}(\theta) - \left(T^{\tau,t,\tilde{\varepsilon}} \right)^{(N)}(\theta) \right| d\tau \\ & \quad + \frac{1}{1 + 2\Delta(\varepsilon)} \left[\left\| \frac{\partial^N f_\varepsilon}{\partial \theta^N} - \frac{\partial^N f_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| + \left\| \frac{\partial^N g_\varepsilon}{\partial \theta^N} - \frac{\partial^N g_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| \right] e^{\tilde{N}_T L(\varepsilon)(1+2\Delta(\varepsilon))|x|} \\ & \quad + \tilde{C}_T |\varepsilon - \tilde{\varepsilon}| e^{\tilde{N}_T L(\varepsilon)(1+2\Delta(\varepsilon))|x|} \end{aligned}$$

for all $(x, t, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and all $\varepsilon, \tilde{\varepsilon} \in [a, b]$. From Grönwall’s inequality, it follows that

$$\begin{aligned} & \left| (T^{x,t,\varepsilon})^{(N)}(\theta) - (T^{x,t,\tilde{\varepsilon}})^{(N)}(\theta) \right| \\ & \leq \left[\tilde{C}_T |\varepsilon - \tilde{\varepsilon}| + \frac{1}{1 + 2\Delta(\varepsilon)} \left[\left\| \frac{\partial^N f_\varepsilon}{\partial \theta^N} - \frac{\partial^N f_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| + \left\| \frac{\partial^N g_\varepsilon}{\partial \theta^N} - \frac{\partial^N g_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| \right] \right] \\ & e^{(\tilde{N}_T + 1)L(\varepsilon)(1 + 2\Delta(\varepsilon))|x|} \end{aligned} \tag{75}$$

for all $(x, t, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and all $\varepsilon, \tilde{\varepsilon} \in [a, b]$.

Let us prove that property (Q.2) holds for $q = N$. Observe that

$$\frac{\partial^N f_\varepsilon}{\partial \theta^N}(t, \theta) = \int_{-\infty}^\infty J_1(x) (\zeta_1^{x,t,\varepsilon} (T^{x,t,\varepsilon}(\theta)))^{(N)} dx.$$

Thus, proceeding just as above, we obtain, for each $[a, b] \subset (0, \varepsilon_1]$, a constant $\tilde{C} > 0$ such that

$$\begin{aligned} \left\| \frac{\partial^N f_\varepsilon}{\partial \theta^N} - \frac{\partial^N f_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| & \leq \int_{-\infty}^\infty K e^{-\alpha|x|} L(\varepsilon)(1 + 2\Delta(\varepsilon)) \left| (T^{\tau,t,\varepsilon})^{(N)}(\theta) - (T^{\tau,t,\tilde{\varepsilon}})^{(N)}(\theta) \right| d\tau \\ & + \int_{-\infty}^\infty K e^{-\alpha|x|} L(\varepsilon) \left[\left\| \frac{\partial^N f_\varepsilon}{\partial \theta^N} - \frac{\partial^N f_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| + \left\| \frac{\partial^N g_\varepsilon}{\partial \theta^N} - \frac{\partial^N g_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| \right] \\ & \times e^{N_{II}L(\varepsilon)(1 + 2\Delta(\varepsilon))|x|} + \int_{-\infty}^\infty K e^{-\alpha|x|} \tilde{C} |\varepsilon - \tilde{\varepsilon}| e^{\tilde{N}_T L(\varepsilon)(1 + 2\Delta(\varepsilon))|x|}. \end{aligned}$$

Define

$$C_{1,N} := \frac{16K\tilde{C}}{\alpha}, \quad \tilde{N} := \max\{\tilde{N}_T + 1, N_{II}\}.$$

If ε_1 is sufficiently small as to ensure that $2\tilde{N}L(\varepsilon)(1 + 2\Delta(\varepsilon)) \leq \alpha$ for all $\varepsilon \in (0, \varepsilon_1]$, then it follows by integrating and considering (75) that

$$\left\| \frac{\partial^N f_\varepsilon}{\partial \theta^N} - \frac{\partial^N f_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| \leq \frac{8KL(\varepsilon)}{\alpha} \left[\left\| \frac{\partial^N f_\varepsilon}{\partial \theta^N} - \frac{\partial^N f_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| + \left\| \frac{\partial^N g_\varepsilon}{\partial \theta^N} - \frac{\partial^N g_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| \right] + \frac{4K\tilde{C}}{\alpha} |\varepsilon - \tilde{\varepsilon}|.$$

Proceeding similarly for g_ε , we obtain

$$\left\| \frac{\partial^N g_\varepsilon}{\partial \theta^N} - \frac{\partial^N g_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| \leq \frac{8KL(\varepsilon)}{\alpha} \left[\left\| \frac{\partial^N f_\varepsilon}{\partial \theta^N} - \frac{\partial^N f_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| + \left\| \frac{\partial^N g_\varepsilon}{\partial \theta^N} - \frac{\partial^N g_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| \right] + \frac{4K\tilde{C}}{\alpha} |\varepsilon - \tilde{\varepsilon}|.$$

Hence, if ε_1 is also chosen small enough to ensure that $32L(\varepsilon)(1 + 2\Delta(\varepsilon)) < \alpha$, it follows that

$$\left\| \frac{\partial^N f_\varepsilon}{\partial \theta^N} - \frac{\partial^N f_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| + \left\| \frac{\partial^N g_\varepsilon}{\partial \theta^N} - \frac{\partial^N g_{\tilde{\varepsilon}}}{\partial \theta^N} \right\| \leq \frac{16K\tilde{C}}{\alpha} |\varepsilon - \tilde{\varepsilon}| = C_{1,N} |\varepsilon - \tilde{\varepsilon}|,$$

proving that property (Q.2) holds for $q = N$. Thus, the validity of property (Q.1) for this value of q follows immediately considering (75).

Therefore, we have proved by induction that properties (Q.1) and (Q.2) are valid for $q \in \{0, \dots, p\}$. This concludes the proof of the Lemma, because property (Q.2) ensures local Lipschitz continuity of the functions considered.

References

1. Bogoliubov, N.N., Mitropol'sky, Y.A.: *Asymptotic methods in the theory of non-linear oscillations*. Translated from the second revised Russian edition. International Monographs on Advanced Mathematics and Physics. Hindustan Publishing Corp., Delhi, Gordon and Breach Science Publishers, New York, (1961)
2. Bogolyubov, N.: O Nekotorykh Statističeskikh Metodah v Matematičeskoj Fizike (On Some Statistical Methods in Mathematical Physics). Akademiya Nauk Ukrainkoj SSR, Kiev (1945)
3. Buică, A., Llibre, J.: Averaging methods for finding periodic orbits via Brouwer degree. *Bull. Sci. Math.* **128**(1), 7–22 (2004)
4. Cândido, M.R., Novaes, D.D.: On the torus bifurcation in averaging theory. *J. Diff. Equ.* **268**(8), 4555–4576 (2020)
5. Chicone, C.: *Ordinary differential equations with applications*, volume 34 of *Texts in Applied Mathematics*. Springer, New York, second edition edition, (2006)
6. Comtet, L.: *Advanced combinatorics*. D. Reidel Publishing Co., Dordrecht, enlarged edition, The art of finite and infinite expansions (1974)
7. Culver, W.J.: On the existence and uniqueness of the real logarithm of a matrix. *Proc. Am. Math. Soc.* **17**, 1146–1151 (1966)
8. Fatou, P.: Sur le mouvement d'un système soumis à des forces à courte période. *Bull. Soc. Math. France* **56**, 98–139 (1928)
9. Gantmacher, F.R.: *The theory of matrices. Vols. 1, 2*. Chelsea Publishing Co., New York, Translated by K. A. Hirsch (1959)
10. Hájek, P., Johanis, M.: *Smooth analysis in Banach spaces*. De Gruyter Series in Nonlinear Analysis and Applications, vol. 19. De Gruyter, Berlin (2014)
11. Hale, J.K.: Integral manifolds of perturbed differential systems. *Ann. Math.* **2**(73), 496–531 (1961)
12. Hale, J.K.: *Oscillations in Nonlinear Systems*. McGraw-Hill Book Co., Inc, New York-Toronto-London (1963)
13. Hale, J.K.: *Ordinary Differential Equations*, 2nd edn. Robert E. Krieger Publishing Co., Inc, Huntington, N.Y. (1980)
14. Hartman, P.: *Ordinary Differential Equations*. Society for Industrial and Applied Mathematics, second edition, (2002)
15. Herman, M.-R.: Mesure de Lebesgue et nombre de rotation. In *Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976)*, Lecture Notes in Math., Vol. 597, pp. 271–293. Springer, Berlin (1977)
16. Krylov, N., Bogolyubov, N.: *Prilozhenie metodov nelineinoi mekhaniki k teorii statsionarnykh kolebanii (The Application of Methods of Nonlinear Mechanics to the Theory of Stationary oscillations)*. Kiev: Akademiya Nauk Ukrainkoj SSR, Kiev], (1934)
17. Llibre, J., Novaes, D.D., Teixeira, M.A.: Higher order averaging theory for finding periodic solutions via Brouwer degree. *Nonlinearity* **27**, 563–583 (2014)
18. Novaes, D.D.: An equivalent formulation of the averaged functions via Bell polynomials. In *Extended abstracts Spring 2016—nonsmooth dynamics*, volume 8 of *Trends Math. Res. Perspect. CRM Barc.*, pp. 141–145. Birkhäuser/Springer, Cham, (2017)
19. Novaes, D.D.: An averaging result for periodic solutions of Carathéodory differential equations. *Proc. Am. Math. Soc.* (2021). <https://doi.org/10.1090/proc/15810>
20. Novaes, D.D.: Higher order stroboscopic averaged functions: a general relationship with Melnikov functions. *Electron. J. Qual. Theory Differ. Equ.* **2021**(77), 1–9 (2021)
21. Novaes, D.D., Silva, F.B.: Higher order analysis on the existence of periodic solutions in continuous differential equations via degree theory. *SIAM J. Math. Anal.* **53**(2), 2476–2490 (2021)

22. Sanders, J.A., Verhulst, F., Murdock, J.A.: *Averaging methods in nonlinear dynamical systems*, volume 59. Springer, (2007)
23. Teschl, G.: *Ordinary differential equations and dynamical systems*. Graduate Studies in Mathematics, vol. 140. American Mathematical Society, Providence, RI (2012)
24. Verhulst, F.: *Nonlinear differential equations and dynamical systems*. Universitext. Springer-Verlag, Berlin, second edition edition, 1996. Translated from the 1985 Dutch original

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.